Discussion Paper No. 05-07

Decomposing Integrated Assessment Climate Change

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Nontechnical Summary

Integrated assessment modeling emerged in the mid-eighties as a new paradigm for interfacing science and policy concerning complex environmental issues. As to climate policy analysis, integrated assessment models (IAMs) aim to represent the causal chain through which (i) economic activities trigger anthropogenic greenhouse gas emissions, (ii) emissions of greenhouse gases translate into atmospheric concentration, temperature shift, and climate change, and (iii) climate change feeds back via the ecosystem to the economy.

In order to derive "ideal" climate policies˜ usually defined from an economic efficiency point of view˜ IAMs are typically phrased as mathematical optimization programs. One shortcoming of the explicit optimization approach is that computational tractability demands highly simplified formulations of both the economic and environmental sub-models. A more subtle disadvantage of IAMs cast as optimization problems is that they cannot directly incorporate second-best effects such preexisting tax distortions. Thus, "optimal" policies emerging from IAMs cast as mathematical programs are only optimal in a perfect undistorted economy.

In this paper, we present a decomposition approach to integrated assessment modeling that overcomes the central shortcomings of the optimization approach. Our decomposition of IAMs is based on a linear approximation to the climate sub-model and provides a convenient framework for the formulation of the economic sub-model as a mixed complementarity problem. This offers considerable advantages as compared to conventional mathematical programming. First, the complementarity formulation cum decomposition permits more precise terminal approximation using state-variable targeting for the economic sub-model. It also facilitates more accurate cost-benefit calculus based on a climate sub-model operating over a longer time horizon. From a computational point of view, the drastic reduction in model periods compared to mathematical programming permits more scope for policy-relevant details. Second, the MCP formulation provides a means of incorporating second-best effects so that relevant complexities such as distortionary taxes or other market failures (e.g. knowledge spillovers) can be accounted for in the policy design process.

Beyond the specific advantages of the complementarity approach over mathematical programming, our decomposition allows the separation of components from different disciplines through a consistent, well-defined interface. The economic model generates emission paths, and the climate model returns temperature profiles and their partial derivatives with respect to emissions. In this way, modelers in each discipline can focus on their specific expertise. Furthermore, the decomposition permits assessment of the relative importance of the various model components – it becomes e.g. fairly easy to ex-change the bio-/geo-physical modules and track down the sensitivity of results with respect to alternative formulations of bio/-geo-physical relationships.

Decomposing Integrated Assessment of Climate Change

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Abstract

We present a decomposition approach for integrated assessment modeling of climate policy based on a linear approximation of the climate system. Our objective is to demonstrate the usefulness of decomposition for integrated assessment models posed in a complementarity format. First, the complementarity formulation *cum* decomposition permits a precise representation of post-terminal damages thereby substantially reducing the model horizon required to produce an accurate approximation of the infinite-horizon equilibrium. Second, and central to the economic assessment of climate policies, the complementarity approach provides a means of incorporating second-best effects that are not easily represented in an optimization model.

JEL classification: C61, C63, D58, D61

Keywords: integrated assessment; decomposition; terminal constraints; optimal taxation

1 Introduction

Integrated assessment modeling emerged in the mid-eighties as a new paradigm for interfacing science and policy concerning complex environmental issues. An integrated assessment model provides a framework combining complementary knowledge from various disciplines in order to derive insights into key questions of policy formulation. Integrated assessment models (IAMs) link mathematical representations of the natural system and the socio-economic system to capture cause-effect chains including feedback.[1]

Weyant et al. (1996) distinguish two broad classes of IAMs: *policy optimization models* which seek optimal policies, and *policy simulation models* which assess specific policy measures. Policy optimization models are normative in the sense that they strive to derive an "ideal" policy, usually defined from an economic efficiency point of view.[2] The level of modeling detail in optimization models is constrained by the need to keep the optimization algorithm tractable. Therefore, these models tend to be based on compact representations of both the socio-economic and natural science systems. A prominent example of an optimizing IAM is the Dynamic Integrated Climate Economy (DICE) model by Nordhaus (1994) which incorporates stylized representations of both the global economy and the global carbon cycle. Policy evaluation models – often referred to as simulation models – typically are used to evaluate the impact of an exogenously specified policy. Avoiding optimization, policy evaluation models tend to be descriptive and can contain much greater modeling detail on bio-/geo-physical, economic or social aspects. An early example of this type of model is the Integrated Model to Assess the Greenhouse Effect (IMAGE) by Rotmans (1990). The present paper focuses exclusively on policy optimization formulations.

In terms of policy design, optimization models are typically phrased as nonlinear mathematical programs (NLP) which permit derivation of best-response policies. Policy responses in these models can be traced to the rational behavior of economic agents. In contrast, the impacts simulated in policy evaluation models tend to be more like "black boxes" (Kelly and Kolstad 1999). One shortcoming of the optimization approach is that computational tractability demands highly simplified formulations of both the economic and environmental sub-models. A more subtle disadvantage of IAMs cast as nonlinear programs is that they cannot directly incorporate second-best effects such as preexisting tax distortions. Thus, "op-

[1]An early example of formal integrated assessment is the RAINS model of acidification in Europe (Alcamo et al. 1985). Over the past years, a variety of models have been developed for the integrated assessment of climate change – for surveys see Weyant et al. (1996), Parson and Fisher-Vanden (1997), or Kelly and Kolstad (1999).

[2]Policy instrument variables such as emission control rates or emission taxes are derived given explicit policy goals, e.g. maximizing social welfare or minimizing the social costs of meeting exogenous environmental targets.

timal" policies emerging from IAMs in NLP format are only optimal in a perfect, undistorted economy.

We present a decomposition approach to integrated assessment modeling of climate change that enables us to conveniently formulate the economic sub-model as a mixed complementarity problem (MCP – see Rutherford 1995). The MCP formulation overcomes two central shortcomings of the conventional nonlinear optimization approach. First, we can use superior terminal methods for approximating the infinite horizon in the economic model, which drastically reduces the number of model periods vis-à-vis a NLP approach, thereby increasing the scope for policy-relevant details on other model dimensions. Second, the MCP framework provides a means of incorporating second-best effects so that relevant complexities such as distortionary taxes or other market failures (e.g. knowledge spillovers) can be accounted for in the policy design process. As an added benefit – independent of the concrete mathematical MCP or NLP representation – our decomposition permits a convenient division of work between expert modelers in different disciplines.

The remainder of this paper is as follows. In section 2, we lay out the generic decomposition approach and provide the MCP formulation of terminal constraints for approximating the infinite horizon of the decomposed IA problem. In section 3, we first demonstrate the usefulness of the decomposed MCP framework for approximating the infinite horizon of the DICE model which has served for several years as a prototype IAM in the field of climate change. We then extend the basic DICE setting for public goods funded through distortionary taxation in order to illustrate the importance of a second-best setting for the derivation and design of climate policies. In section 4, we conclude. For the sake of brevity, we abstain from presenting a detailed description of the models' algebra. The interested reader can download this information together with the programming code for the numerical models from `ftp://ftp.zew.de/pub/zew-docs/div/iam.pdf`.

2 Decomposition

Figure 1 illustrates the generic structure of IAMs for climate policy analysis. These models aim to represent the causal chain through which (i) economic activities trigger anthropogenic greenhouse gas emissions, (ii) emissions of greenhouse gases translate into atmospheric concentration, temperature shift, and climate change, and (iii) climate change feeds back via the ecosystem to the economy.

Policy optimization models of climate change adopt a cost-benefit perspective in which the current marginal costs of controlling greenhouse gas emissions are balanced against the

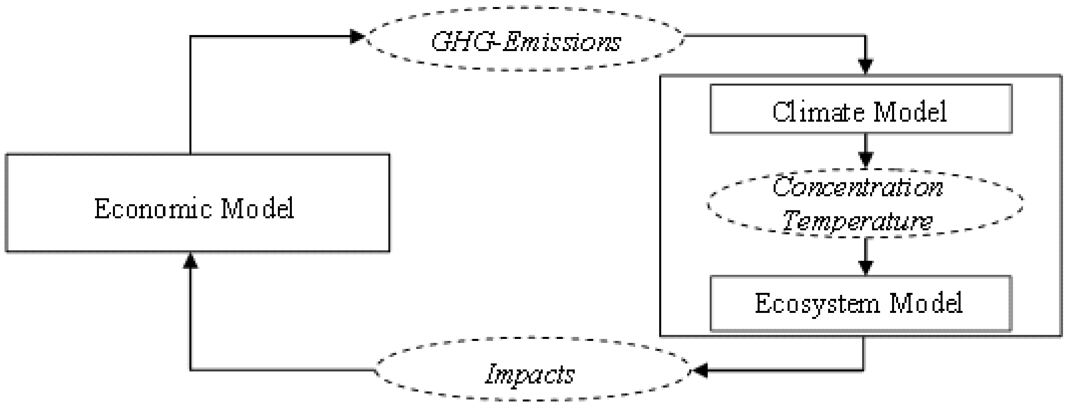

Figure 1: *Schematic Structure of Integrated Assessment Models for Climate Change*

future marginal damages induced by those emissions. Climate change impacts are portrayed by parametric relationships between economic losses and the global mean temperature (i.e., a "damage function").

In simple formal terms, the climate policy problem can be stated as a nonlinear optimization problem (NLP) of a single infinitely-lived agent:

$$\sum_{t=0}^{\infty} \rho_t U(C_t, D_t)$$

s.t.

$$\begin{aligned} C_t &= F(K_t, D_t, E_t) - I_t \\ K_{t+1} &= (1-\delta)K_t + I_t \\ D_t &= H(S_t) \\ S_{t+1} &= G(S_t, E_t) \\ K_0 = \bar{K}_0, & \quad S_0 = \bar{S}_0 \end{aligned}$$

where ρ_t is the discount factor in period t, U denotes intertemporal utility, C_t represents consumption in period t, F characterizes production in period t as a function of capital, damages (with potentially adverse effects on productivity), and emissions, D_t denotes damages of climate change in period t, K_t is the capital stock in period t (with $K_0 = \bar{K}_0$ exogenously specified), E_t denotes the emissions in period t, I_t is investment in period t, H describes the functional relationship between the climate state and damages, S_t is a vector of the climate state (with $S_0 = \bar{S}_0$ as the initial climate state), and G characterizes the motion of the climate state as a function of the previous climate state and anthropogenic emissions used as production input. Note that we can merge the relationships $D_t = H(S_t)$ and $S_{t+1} = G(S_t, E_t)$ into a single equivalent equation

$$D_t = \Gamma_t(S_0, E_0, E_1, ..., E_{t-1}),$$

where Γ_t renders damages in period t as a function of the initial climate state and emissions in previous periods.

Our decomposition is based on a linear approximation of the climate response, i.e. climate impacts D_t, to anthropogenic activities, i.e. emissions, of the economic system:

$$D_t \approx \bar{D}_t + \sum_{\tau=0}^{t} \frac{\partial \Gamma_t}{\partial E_\tau}(E_\tau - \bar{E}_\tau)$$

where $\bar{D}_t$ is the reference level value for climate impacts in period t, $\bar{E}_\tau$ is the reference level value for emissions in period τ, $\frac{\partial \Gamma_t}{\partial E_\tau}$ denotes the gradient of climate impacts in period t to anthropogenic emissions in period τ.

In our implementation, we have evaluated the Jacobian $\frac{\partial \Gamma_t}{\partial E_\tau}$ for the climate sub-model using numerical differencing:[3]

$$\frac{\partial \Gamma_t}{\partial E_\tau} = \frac{\bar{D}_t - \Gamma_t(S_0, E_0, ..., \bar{E}_\tau + \epsilon, ..., \bar{E}_t)}{\epsilon}.$$

The climate model is nonlinear, so iterative refinement of the linear approximation is required. For our concrete numerical implementation of the DICE model, we find that this diagonalization procedure quickly converges.

A central advantage of the decomposition relates to the different nature of dynamics in the economic and the climate sub-models. Due to intertemporal optimization by economic agents, the economic sub-model must typically be solved simultaneously: current investment depends on future returns to capital, future economic damages, etc. In contrast, the climate sub model may be evaluated *recursively* given emission paths from the economic model. This permits us to solve the climate equations "offline". The decomposition is effective provided that the climate system Jacobian is stable. Our computational experience to date suggests that this is the case, and this permits us to avoid integrating the complex system of climate system equations within the intertemporal economic model. Our decomposition then results in a sparse economic policy model based on simple but accurate reduced-form representation of climate impacts: We replace the explicit representation S_t of the climate sub-model by a linear approximation of climate impacts D_t.

[3]Numerical differencing may pose high computational costs if the underlying climate model is computationally intensive. In those cases, another method of sensitivity analysis may be appropriate.

The reduced-form representation of the climate sub-model in our decomposition approach allows us to conveniently formulate the economic policy problem as a mixed complementarity problem (MCP). The MCP framework exploits the complementarity features of economic equilibrium, thereby including the NLP representation of economic equilibrium as a special case (Mathiesen 1985, Rutherford 1995).[4] As compared to the conventional representation of the climate policy problem in terms of a nonlinear program, the MCP formulation of the economic sub-model offers considerable advantages. First, we are better able to approximate the infinite horizon by state-variable targeting for the economic sub-model and cost-benefit calculus through the climate sub-model. Second, the MCP formulation relaxes the integrability constraints imposed by the NLP framework, thereby accommodating second-best settings that reflect initial inefficiencies.

Terminal Constraints

Approximation of an infinite horizon economy within a finite horizon numerical model requires "terminal constraints". For example, in the steady state, gross investment is proportional to the capital stock through the growth rate of the labor force and the capital depreciation rate. A typical terminal constraint for investment might then require sufficient investment to cover growth plus depreciation:

$$I_T = (\chi + \delta)K_T$$

where χ denotes the steady-state growth rate.

Optimization models phrased as nonlinear programs use such (integrable) constraints on investment in the terminal period together with an adjustment term in the utility function to account for the "consumption" value of the terminal capital stock. After a policy shock, however, the "true" value of the capital stock in the terminal period is unknown. In this context, the NLP formulation typically imposes the long run steady-state value of the capital stock with the requirement that the model horizon must be sufficiently long to converge to the steady state. A complementarity formulation, on the other hand, allows including post-terminal capital stock as an endogenous variable. Using *state variable targeting* for this variable, the growth of investment in the terminal period is related to the growth rate of capital or any other "stable" quantity variable in the model (Lau, Pahlke, and Rutherford 2002):

$$\frac{I_T}{I_{T-1}} = \frac{K_T}{K_{T-1}}.$$

[4]By forming the Lagrangian and differentiating, a nonlinear program can be posed as a complementarity problem.

Beyond state variable targeting to determine the post-terminal capital stock, the decomposition *cum* MCP accommodates the precise approximation of post-terminal damages from emissions reflected by the terminal value of the climate state S_T. The complementarity model formulation has explicit price indices representing the cost of abatement and the benefits offered through abatement. A linear approximation to the climate model portrays the time profile of marginal benefits associated with emission reductions at different points in time through the economic model. Thus, we can compare the benefits associated with cutbacks in emissions in the later periods of the model with the benefits of those cutbacks in periods which lie beyond the terminal period of the model:[5]

$$-p_t \frac{\partial F}{\partial E_t} = \sum_{\tau=t}^{\infty} \frac{\partial \Gamma_\tau}{\partial E_t} p_\tau^D = \sum_{\tau=t}^{T} \frac{\partial \Gamma_\tau}{\partial E_t} p_\tau^D + \sum_{\tau=T+1}^{\infty} \frac{\partial \tilde{\Gamma}_\tau}{\partial E_t} \tilde{p}_\tau^D$$

where p_t is the price of macro good production in period t, and p_τ^D is the price (cost) of damage in period τ.

Post-terminal damages are calculated on the basis of the climate sub-model which is solved for several decades beyond the terminal period of the economic sub-model. Extrapolating present value prices and quantities into the post-terminal period then permits us to relate marginal emission throughout the time horizon of the economic sub-model to damages occurring after the terminal period of the economic sub-model. The valuation of post-terminal damages is based on a geometric extrapolation of post-terminal prices $\tilde{p}_\tau^D$, and post-terminal climate $\tilde{\Gamma}_\tau$ is calculated on the basis of post-terminal emission paths which are extrapolated from the economic sub-model.

Integrability Constraints

First-order conditions of mathematical programs only correspond to equilibrium conditions for the case of integrability that implies efficient allocation (Pressman 1970 or Takayama and Judge 1971)[6]. Thus, IAMs of climate change cast as nonlinear optimization models are forced to provide a highly stylized representation of the economy in order to avoid "non-integrabilities" that can not be handled in the single optimization framework.[7] In contrary,

[5] In contrast, the optimization formulation of IAMs for climate change employs "transversality" adjustment terms to reflect post-terminal damages, but the specification of the values for these penalties remains ad-hoc (Nordhaus 1994).

[6] In practical terms, integrability refers to a situation where the shadow prices of programming constraints coincide with market prices.

[7] Integrability problems may be relaxed in the optimization context by adding a correction term to the objective and solving a sequence of nonlinear programs to obtain a market equilibrium (see e.g. Rutherford 1999).

the MCP formulation of economic problems permits the incorporation of "non-integrabilities" to reflect inefficiencies of market allocation induced by distortionary taxes, institutional price constraints, spillovers, etc.[8]

3 Illustration

We illustrate the advantages of our decomposed MCP formulation using the DICE model by Nordhaus (1994) that is originally formulated as a nonlinear program. Because of its simplicity and relative transparency, DICE and its multiregional extension, RICE (Nordhaus and Yang 1996), have been widely used for the integrated assessment of climate change. DICE is based on Ramsey's model of saving and investment. A single world producer-consumer chooses between current consumption, investment in productive capital, and costly measures to reduce current emissions and slow climate change. Population growth and technological change (productivity growth) are both exogenous. The representative consumer maximizes the discounted utility of consumption over an infinite horizon subject to a Cobb-Douglas production function which includes damages from climate change as a quadratic function of global mean temperature. In the absence of abatement measures, anthropogenic emissions occur in direct proportion to output. Emissions per unit output are assumed to decline exogenously at a fixed rate and can be further reduced by costly emission-control measures. Within a simple reduced form "two-box" (ocean and atmosphere) climate sub-model based on Schneider and Thompson (1981), emissions accumulate and increase the stock of greenhouse gases in the atmosphere. As this stock grows, it increases the amount of solar radiation trapped by the earth's atmosphere which in turn triggers an increase in global mean temperature.

For our illustrative application of the decomposition approach, we distinguish two alternative mathematical formulations of the DICE integrated assessment model: the familiar implementation as a nonlinear mathematical program (NLP) and the model's representation as a mixed complementarity problem (MCP).

In order to evaluate the sensitivity of the optimal policy with respect to the model horizon, we run both models for horizons of 5, 10, 20, and 40 periods (with each period representing a 10-year time interval). As is evident in Figure 2, the MCP model is virtually insensitive to the model horizon, whereas the NLP model shows a drastic sensitivity, in particular for the

[8] Other important examples of non-integrabilities include individual demand functions which do not only depend on prices but also on the initial endowments (Chipman 1974).

first few decades. Furthermore, the differences in optimal emission control rates[9] between the two model formulations differ substantially, particularly for short time horizons. In practical terms, the precise terminal approximation of the MCP approach offers a major improvement in the range and details of policy analysis that can be covered: Since the economic sub-model only requires a short-term horizon, one can elaborate on policy-relevant complexities.

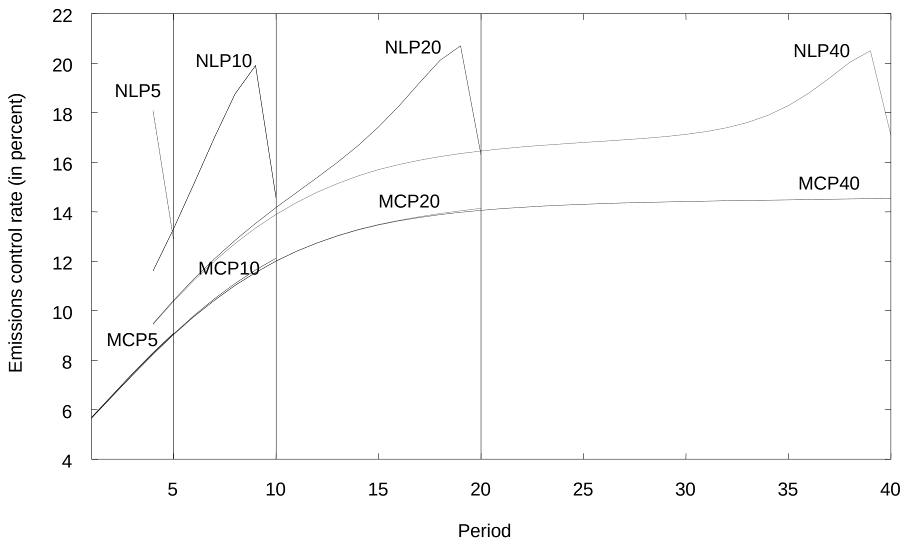

Figure 2: *Sensitivity of Emission Control Rate with respect to the Model Horizon - NLP Model vs. MCP Model*

Another key advantage of the decomposed MCP framework for applied policy analysis is the ease with which it can incorporate second-best effects. We illustrate the importance of market distortions by considering a simple extension of the DICE model in which a public good provided in each period is funded through a distortionary tax on capital earnings. In the reference simulation, we hold the capital tax fixed at an exogenous rate and compute the "optimal" abatement profile together with the resulting level of public goods provision. In the counterfactual simulation, we endogenize the capital tax rate through an equal-yield constraint (keeping public good provision at the reference level) and evaluate the marginal utility of perturbations of the "optimal" abatement profile for each model period.

As has been observed by several authors (Goulder, 1995) preexisting tax distortions affect the economic cost of climate policy instruments. When the government applies emission

[9]The key policy instrument in the DICE model is the emissions control rate, the fraction of emissions which are mitigated relative to the uncontrolled level.

restrictions, these raise revenue which may be used to reduce other taxes. In this case, where revenues from carbon permit sales are used to replace distortionary taxes, the "optimal" abatement profile is too low. This occurs because the marginal benefit calculus is implicitly based on a marginal cost of public funds equal to 1, whereas distortionary financing of public provision implies that the marginal cost of public funds is greater than one. The larger the baseline tax rate on capital in our example, the larger is the marginal benefit of increasing stringency of environmental restrictions. Figure 3 illustrates our reasoning for alternative capital tax rates of 5%, 10%, 25% and 50%.

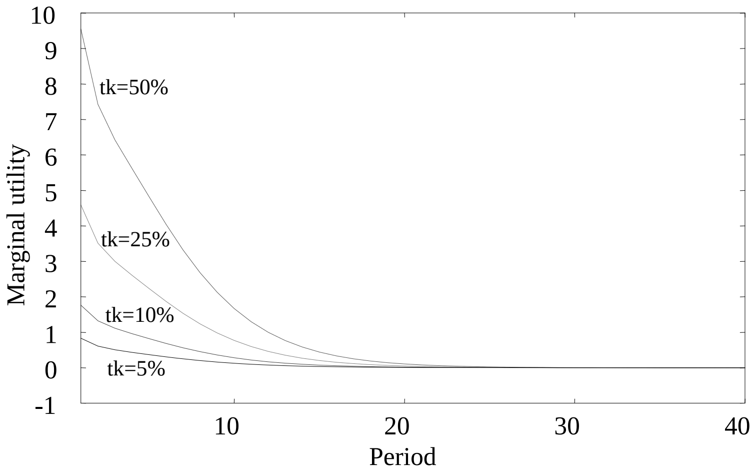

Figure 3: *Marginal Utility of 1% Additional Abatement For Alternative Capital Tax Rates*

4 Conclusions

In this paper, we have presented a new approach to integrated assessment modeling of climate change. Our decomposition of IAMs is based on a linear approximation to the climate sub-model and provides a convenient framework for the complementarity formulation of the economic sub-model. This offers considerable advantages as compared to traditional nonlinear programming. First, the complementarity formulation *cum* decomposition permits more precise terminal approximation using state-variable targeting for the economic sub-model. It also permits more accurate cost-benefit calculus based on a climate sub-model operating over a longer time horizon. From a computational point of view, the reduction in model periods vis-à-vis nonlinear programming permits more scope for policy-relevant details. Second, the MCP formulation provides a convenient means of incorporating second-best effects that may substantially alter policy conclusions based on the assumptions of perfect undistorted economies.

Beyond the specific advantages of the complementarity approach over nonlinear programming, our decomposition allows the separation of components from different disciplines through a consistent, well-defined interface. The economic model generates emission paths, and the climate model returns temperature profiles and their partial derivatives with respect to emissions. In this way, modelers in each discipline can focus on their specific expertise. Furthermore, the decomposition permits assessment of the relative importance of the various model components – it becomes e.g. fairly easy to ex-change the natural science modules and track down the sensitivity of results with respect to alternative formulations of natural science relationships.

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Appendix A: Algebraic Summary

We use the DICE model by Nordhaus (1994) originally formulated as a nonlinear program in order to illustrate the advantages of the decomposed mixed complementarity framework. DICE is based on Ramsey's model of saving and investment. A single world producer-consumer chooses between current consumption, investment in productive capital, and costly measures to reduce current emissions and slow climate change. Population growth and technological change (productivity growth) are both exogenous. The representative consumer maximizes the discounted utility of consumption over an infinite horizon, subject to a Cobb-Douglas production function which includes damages from climate change as a quadratic function of global mean temperature. In the absence of abatement measures, anthropogenic emissions occur in direct proportion to output. Emissions per unit output are assumed to decline exogenously at a fixed rate and can be further reduced by costly emission-control measures. Within a simple reduced form "two-box" (ocean and atmosphere) climate sub-model, emissions accumulate and increase the stock of greenhouse gases in the atmosphere. As this stock grows, it increases the amount of solar radiation trapped by the earth's atmosphere which in turn triggers an increase in global mean temperature.

In section A.1, we start with the original implementation of DICE as a nonlinear program (NLP) that integrates stylized representations of the global economy and the climate system. In section A.2, we proceed with the decomposition of the integrated economy-climate model while maintaining the NLP formulation of the economic sub-model. In section A.3, we re-cast the NLP formulation of the economic sub-model as a mixed complementarity problem (MCP) thereby making use of state-variable targeting for the economic sub-model and cost-benefit calculus through the climate sub-model to better approximate the infinite horizon. In section A.4, we lay out a simple public finance extension of the basic DICE model to account for pre-existing market distortions within the MCP framework.

The programming code which goes along with our algebraic model descriptions is readily available in Appendix B.

A.1 Integrated NLP Formulation

The standard assumptions for the Ramsey model imply that the optimal allocation of resources by a central planner who maximizes the utility of the representative agent is identical to the optimal allocation of resources in an undistorted decentralized economy. The model can be interpreted in an optimizing NLP framework as the outcome of idealized competitive markets.

In the NLP setting, the representative agent explicitly maximizes the discounted value of "utility" from consumption subject to a number of economic and geophysical constraints (see Appendix B.2 for the programming code of the standard NLP formulation of DICE).

Objective function

The economic objective function in DICE is defined as:

$$\sum_{t=1}^{T} \rho_t L_t \log\left(C(t)/L(t)\right) \tag{1}$$

where:

C_t:=	is consumption in period t,
L_t:=	is the exogenous labor supply in period t (population growth), and
ρ_t :=	denotes the discount factor.

Economic constraints

The economic model consists of equations describing technology, abatement options, output markets, emissions, and capital accounting. Gross economic output is given by a standard Cobb-Douglas function:

$$Q_t = a_l L_t^{1-\gamma} K_t^{\gamma} \tag{2}$$

where:

Q_t:=	denotes gross economic output,
a_l:=	represents the level of total factor productivity,
K_t:=	is the capital stock in period t (with $K_0 = \overline{K}_0$ exogenously specified), and
γ :=	is the capital value share (capital elasticity in output).

Abatement options are described by a geometric control cost function:

$$A_t = b_1 \Upsilon_t^{b_2} \tag{3}$$

where:

A_t:=	is the abatement level in period t,
Υ_t :=	denote the emission control rate in period t, and
b_1,b_2:=	are the exogenous parameters of the abatement cost function.

Total emissions are directly linked to gross output. The emission control rate Υ_t describes the endogenous relationship between emissions and gross output:

$$\Upsilon_t = 1 - \frac{E_t}{\sigma_t Q_t} \tag{4}$$

where:

E_t:= denotes the emissions in period *t*,

σ_t := is an exogenous efficiency improvement factor which scales down the emission intensity of macro production over time.

Output net of abatement and damage costs (both of which measured as loss in output) equals:

$$Y_t = Q_t - A_t Q_t - D_t Y_t \tag{5}$$

where:

Y_t:= represents net output in period *t*, and

D_t:= denotes damages of climate change in period *t*.

In each period, net economic output is divided between consumption and investment:

$$Q_t = C_t + I_t \tag{6}$$

The capital stock is determined by the balance between depreciation and capital investment:

$$K_t = (1-\delta) K_{t-1} + I_t \tag{7}$$

where:

δ := denotes the capital depreciation rate.

Geophysical constraints

The climate sub-model in DICE contains four stylized geophysical relationships that link together the different forces affecting climate change: emission accumulation and transportation (carbon cycle), radiative forcing, and temperature-climate relationships for the atmosphere and lower oceans.

Emission accumulation and transportation is defined as:

$$M_t = 590 + \beta E_t + (1-\delta_M)\,(M_{t-1} - 590) \tag{8}$$

where:

M_t:= denotes the atmospheric concentration of CO_2 emission,

β := is the marginal atmospheric retention rate, and

δ_M := represents the carbon transfer rate to the deep ocean.

Radiative forcing is a function of CO_2 emission concentration and other non-CO_2 greenhouse gases:

$$F_t = 4.1\big(\log(M_t / 590) / \log(2)\big) + O_t \tag{9}$$

where:

F_t:= is radiative forcing (i.e. the increase of surface warming in watts per square meter), and

O_t:= represent other greenhouse gases (most notably CH_4 and N_2O) that are taken as exogenous.

Radiative forcings warm the atmospheric layer, which in turn warms the upper ocean, thereby gradually warming the deep oceans. Due to thermal inertia of different layers there are time lags in climate change. The links between radiative forcing and temperature changes in the atmosphere and the deeper oceans are given as:

$$\mathrm{T}_t^E = \mathrm{T}_{t-1}^E + c_1\left(F_{t-1} - c_2\mathrm{T}_{t-1}^E - c_3\left(\mathrm{T}_{t-1}^E - \mathrm{T}_{t-1}^L\right)\right) \tag{10}$$

$$\mathrm{T}_t^L = \mathrm{T}_{t-1}^L + c_4\left(\mathrm{T}_{t-1}^E - \mathrm{T}_{t-1}^L\right) \tag{11}$$

where:

T_t^E := is the temperature in the atmosphere,

T_t^L := is the temperature in the lower oceans, and

c_1, c_2, c_3, c_4 := are geophysical parameters of climate dynamics.

Economic-geophysical linkage constraint

The interface between the economic system sub-model and the climate system sub-model is given by an assumed quadratic relationship between atmospheric temperature and climate change damage:

$$D_t = \upsilon\left(\mathrm{T}_t^E\right)^2 \tag{12}$$

where:

υ := denotes a damage coefficient which is calibrated based on the damage level assumed to be associated with CO_2 doubling.

Terminal constraints

Approximation of an infinite horizon economy within a finite horizon numerical model requires “terminal constraints”. For example, in the steady state, gross investment is proportional to the capital stock through the growth rate of the labor force and the capital depreciation rate. A typical terminal constraint for investment might then require sufficient investment to cover growth plus depreciation:

$$I_T = \left(\chi + \delta_M\right)K_T \tag{13}$$

where:

χ := denotes the growth rate of the labor force.

DICE uses this (integrable) constraint on investment in the terminal period together with an adjustment term in the utility function to account for the “consumption” value of terminal capital stock. In addition, adjustment terms are incorporated to reflect post-terminal damages from emission concentrations and temperature. The adjusted objective function then reads as:

$$\left[\sum_{t=1}^{T}\rho_t L_t \log\left(C(t)/L(t)\right)\right] + \rho_t\left(\phi^K K^T + \phi^M M_T + \phi^{\mathrm{T}^E}\mathrm{T}_T^E\right) \tag{1'}$$

where:

$\phi^K :=$ is the (positive) "transversality" coefficient for capital,

$\phi^M :=$ is the (negative) "transversality" coefficient for emission concentration, and

$\phi^{\mathrm{T}^E} :=$ is the (negative) "transversality" coefficient for temperature.

A.2 Decomposed NLP formulation

Our first extension of Nordhaus' model involves decomposition of the integrated economy-climate model is based on a linear approximation of the climate model (see Appendix B.3 for the programming code). The decomposition replaces the climate equations in the economic model with a reduced form representation of climate impacts (see equation 12):

$$D_t \approx \overline{D_t} + \sum_{\tau=1}^{t}\frac{\partial \Gamma_t}{\partial E_\tau}\left(E_\tau - \overline{E_\tau}\right) \tag{14}$$

where:

$\overline{D_t} :=$ is the reference level value for climate impacts in period *t*,

$\overline{E_\tau} :=$ is the reference level value for emissions in period τ, and

$\frac{\partial \Gamma_t}{\partial E_\tau} :=$ denotes the gradient of climate impacts in period *t* to anthropgenic

emissions in period τ .

The central idea of the decomposition is that the local dependence of climate impacts in period *t* on emissions in period *t* can be calculated as a diagonalization procedure in the climate sub-model using numerical differencing:

$$\frac{\partial \Gamma_t}{\partial E_\tau} = \frac{\overline{D_t} - \Gamma_t}{\varepsilon} \tag{15}$$

where:

$\varepsilon :=$ is a sufficiently small emission interval for numerical differencing.

Introduction of the linear climate model requires that we account for the local dependence of the transversality terms in the objective function on emissions, and we can calculate the gradient of the transversality terms as:

$$\frac{\partial \Omega_T}{\partial E_\tau} = \frac{\left(\phi^M M_T + \phi^{\mathrm{T}^E}\mathrm{T}_T^E\right) - \left(\phi^M \overline{M_T} + \phi^{\mathrm{T}^E}\overline{\mathrm{T}_T^E}\right)}{\varepsilon} \tag{16}$$

where:

$\dfrac{\partial \Omega_T}{\partial E_\tau}$:= denotes the local dependence of the transversality terms in the terminal period on emissions in period τ .

Thus, we obtain the adjusted objective function:

$$\left[\sum_{t=1}^{T} \rho_t L_t \log\left(C(t)/L(t)\right)\right] + \rho_t \left(\phi^K K^T + \sum_{t=1}^{T} \frac{\partial \Omega_T}{\partial E_t}\left(E_t - \overline{E_t}\right)\right) \tag{1''}$$

Altogether, the decomposed model consists of an economic sub-model comprising equations (1’’), (2)-(7), (13), and (14), and the climate sub-model compromising equations (8)-(12), (15), and (16). We solve the decomposed model iteratively, by first solving the economic model and then using the resulting emissions profile to evaluate the climate model and its derivatives. Successive solutions converge rapidly as the partial derivatives of temperature with respect to emissions turn out to be very stable.

A.3 Decomposed MCP formulation

Next, we provide the algebraic formulation of the decomposed MCP approach to DICE (see Appendix B.4 for the programming code). Following Mathiesen (1985), the economic sub-model can be characterized by two classes of equilibrium conditions that reflect the first-order conditions of the NLP: (i) zero profit conditions for constant returns activities, and (ii) market clearance conditions for goods and factors. The decision variables are two vectors: (i) activity levels for constant returns production, and (ii) prices for goods (services) and factors. In equilibrium, each of these variables is linked to one inequality condition: (i) an activity level to a zero profit condition, and (ii) a price to a market clearance condition.[1] The primal constraints of the NLP economic sub-model constitute the market-clearance conditions for the MCP whereas the shadow prices (dual variables) of these constraints coincide with market prices. Differentiation of the NLP Langragian with respect to the primal variables (activity levels) renders the zero-profit conditions of the MCP for consumption, capital accumulation, investment, net output, gross output, abatement, emissions, damage, and emission control. We indicate the associated complementary variable to each equilibrium condition using the “perp” operator, “$\perp$ ”.

– *consumption*:

$$\rho_t L_t / C(t) = p_t^C \qquad \perp C_t \tag{17}$$

where:

[1] In a model with multiple agents, we must add an additional class of income balances that relate factor income to expenditure of agents (with associated income variables).

$p_t^C :=$ is the price of consumption in period t.

– *capital accumulation*:

$$p_t^K K_t = \gamma p_t^Q Q_t + p_{t+1}^K \left(1-\delta\right) K_t \qquad \perp K_t \qquad (18)$$

where:

$p_t^Q :=$ denotes the price of gross output in period t, and

$p_t^K :=$ is the price of capital in period t.

– *investment*:

$$p_t^C = p_{t+1}^K \qquad \perp I_t \qquad (19)$$

– *net output*:

$$p_t^Y \left(1+D_t\right) = p_t^C \qquad \perp Y_t \qquad (20)$$

where:

$p_t^Y :=$ represents the price of net output in period t.

– *gross output*:

$$p_t^Q = p_t^Y \left(1-A_t\right) - p_t^E \sigma_t \left(1-\Upsilon_t\right) \qquad \perp Q_t \qquad (21)$$

where:

$p_t^E :=$ is the price of emissions.

– *abatement*:

$$p_t^A + p_t^Y Q_t = 0 \qquad \perp A_t \qquad (22)$$

where:

$p_t^A :=$ denotes the price of abatement.

– *damage*:

$$p_t^D + p_t^Y Y_t = 0 \qquad \perp D_t \qquad (23)$$

where:

$p_t^D :=$ is the price of damage in period t.

– *emissions:*

$$-p_t^E = \sum_{\tau=1}^{T} p_\tau^D \frac{\partial \Gamma_\tau}{\partial E_t} + p_T^D \chi_t \qquad \perp E_t \qquad (24)$$

where:

$\chi_t :=$ is the (parameterized) post-terminal damage of emissions in period t (see below (16')).

– *emission control*:

$$-p_t^E \sigma_t Q_t = p_t^A b_1 b_2 \Upsilon_t \left(b_2 - 1\right) \qquad \perp \Upsilon_t \qquad (25)$$

Terminal Constraints

In the complementarity formulation, the post-terminal capital stock enters as an endogenous variable. Using *state variable targeting* for this variable, we can relate the growth of investment in the terminal period to the growth rate of capital or any other "stable" quantity variable in the model (Lau, Pahlke, and Rutherford 2002):

$$I_T / I_{T-1} = K_T / K_{T-1} \qquad \perp K_T \qquad (26)$$

Furthermore, we need a constraint that defines the price of the post-terminal capital:

$$I_t + K_T \left(1 - \delta\right) = KT \qquad \perp p_T^K \qquad (27)$$

where:

$KT :=$ represents the post-terminal capital stock.

The complementarity model formulation has explicit price indices representing the cost of abatement and the benefits offered through abatement. A linear approximation to the climate model portrays the time profile of marginal benefits associated with emission reductions at different points in time through the economic model. Thus, we can compare the benefits associated with cutbacks in emissions in the later periods of the model with the benefits of those cutbacks in periods which lie beyond the terminal period of the model.[2] Post-terminal damages are calculated on the basis of the climate sub-model which is solved for several decades beyond the terminal period of the economic sub-model. Extrapolating present value prices and quantities into the post-terminal period then permits us to relate marginal emission throughout the time horizon of the economic sub-model to damages occurring after the terminal period of the economic sub-model. The valuation of post-terminal damages is based on a geometric extrapolation of post-terminal prices, and post-terminal climate is calculated on the basis of post-terminal emission paths which are extrapolated from the economic sub-model:

[2] In contrast, the NLP version of DICE employs "transversality" coefficients for carbon stocks, but the specification of the values for these penalties remains ad-hoc.

$$\chi_t = \sum_{\tau=T}^{TC} \frac{\partial \Gamma_\tau}{\partial E_t} \frac{\overline{p_\tau^D}}{\overline{p_T^D}} \qquad (16')$$

where:

$\overline{p}_\tau^D$:=	is the reference price of damage in period τ, and
TC:=	denotes the extended time horizon of the climate sub-model beyond the terminal period *T* of the economic sub-model.

The decomposed MCP formulation of DICE combines equations (2)-(7), (13), (14), and (17)-(27) for the economic sub-model and equations (8)-(12), (15), and (16') for the climate sub-model.

A.4 Decomposed MCP formulation with Distortionary Public Funding

Our final model version extends the MCP formulation of DICE's economic sub-model with a public sector which finances the provision of a public good model through distortionary taxation of capital earnings (see Appendix B.5 for the programming code). The extended MCP model *cum* decomposition can then be used to illustrate the importance of initial market distortions for the formulation of climate response policies.

The modifications and extensions of the initial MCP setting without public good provision involve:

– *capital accumulation (zero-profit condition)*:

$$p_t^K K_t = \gamma p_t^Q Q_t / \left(1 + t_k\right) + p_{t+1}^K \left(1 - \delta\right) K_t \qquad \perp K_t \qquad (18')$$

where:

t_k :=	denotes the tax rate on capital earnings (as the equal-yield instrument).

– *equal-yield constraint for public good provision:*

$$G = \overline{G} \qquad \perp t_k \qquad (28)$$

where:

G:=	is the level of public good provision (likewise: government demand), and
$\overline{G}$:=	denotes a fixed target level (index) of public good provision.

– *explicit definition of rents on emissions:*

$$\zeta_t = p_t^E E_t - p_t^Y A_t Q_t \qquad \perp \zeta_t \qquad (29)$$

where:

$\zeta_t :=$ denotes the rents on emissions in period *t*.

– *government budget constraint:*

$$G\sum_{t=0}^{T}\left(p_t^C L_t / L_0\right) = t_k\left(\gamma p_t^Q Q_t / \left(1+t_k\right)\right) + \zeta_t \qquad \perp G \qquad (30)$$

The decomposed MCP formulation with distortionary taxation combines equations (2)-(7), (13), (14), (17), (18’), and (19)-(30) for the economic sub-model and equations (8)-(12), (15), and (16’) for the climate sub-model.

Appendix B: GAMS Programming Code

Numerically, the algebraic models are implemented in GAMS (www.gams.com). Below, we present the original input data of DICE (see Appendix B.1) together with the GAMS programming code for all the model variants (see Appendices B.2-B.5) that have been laid out in Appendix A.

B1: Data input for DICE 94 model (File: dicedata.gms)

```
*      Reference:   Nordhaus, W.D. (1994): Managing the Global Commons:
*      The Economics of Climate Change, The MIT Press, Cambridge, MA.
$if not set t $set t 40
$if not defined t  SET    t Time periods /1*%t%/
$if not defined tc ALIAS (t,tc);

SETS   tfirst(t)     First period,
       tlast(t)      Last period;
SCALARS
       bet           Elasticity of marginal utility /0/
       r             Rate of social time preference per year /.03/
       gl0           Growth rate of population per decade /.223/
       dlab          Decline rate of population growth per dec /.195/
       deltam        Removal rate carbon per decade /.0833/
       ga0           Initial growth rate for technology per decade /.15/
       dela          Decline rate of technology per decade /.11/
       sig0          CO2-equiv-GWP ratio /.519/
       gsigma        Growth of sigma per decade /-.1168/
       dk            Depreciation rate on capital per year /.10/
       gamma         Capital elasticity in output /.25/
       m0            CO2-equiv concentrations 1965 billion tons carbon /677/
       tl0           Lower stratum temperature (C) 1965 /.10/
       t0            Atmospheric temperature (C) 1965 /.2/
       atret         Marginal atmospheric retention rate /.64/
       q0            1965 gross world output trillions 1989 US dollars /8.519/
       L0            1965 world population millions /3369/
       k0            1965 value capital billions 1989 US dollars /16.03/
       c1            Coefficient for upper level /.226/
       lam           Climate feedback factor /1.41/
       c3            Coefficient trans upper to lower stratum /.440/
       c4            Coeff of transfer for lower level /.02/
       a0            Initial level of total factor productivity /.00963/
       a1            Damage coeff for co2 doubling (fraction GWP) /.0133/
       b1            Intercept control cost function /.0686/
       b2            Exponent of control cost function /2.887/
       phik          Transversality coef. capital /140 /
       phim          Transversality coef. carbon ($ per ton) /-9/
       phite         Transversality coef. temp (billion $ per degree C) /-7000 /
PARAMETERS
       L(tc)         Level of population and labor
       al(tc)        Level of total factor productivity (TFP)
       sigma(tc)     Emissions-output ratio
       rr(tc)        Discount factor
       ga(tc)        Growth rate of TFP from 0 to T
       forcoth(tc)   Exogenous forcings from other greenhouse gases
       gl(tc)        Growth rate of labor 0 to T
       gsig(tc)      Cumulative improvement of energy efficiency;

tfirst(t) = yes$(ord(t) eq 1);  tlast(t)= yes$(ord(t) eq card(t));
gl(tc) = (gl0/dlab)*(1-EXP(-dlab*(ORD(tc)-1)));
L(tc)=L0*EXP(gl(tc))*.9;
ga(tc)= (ga0/dela)*(1-EXP(-dela*(ORD(tc)-1)));
al(tc) =a0*EXP(ga(tc));
gsig(tc) = (gsigma/dela)*(1-EXP(-dela*(ORD(tc)-1)));
sigma(tc)=sig0*EXP(gsig(tc));  rr(tc) = (1+r)**(10*(1-ORD(tc)));
forcoth(tc) = 1.42;
forcoth(tc)$(ORD(tc) lt 15) = .2604+.125*ORD(tc)-.0034*ORD(tc)**2;
```

B2: NLP Model Formulation

```
$title     DICE version 1994 –NLP formulation

$include dicedata

VARIABLES
       C(t)          Consumption trillion US dollars
       K(t)          Capital stock trillion US dollars
       I(t)          Investment trillion US dollars
       D(t)          Damage
       A(t)          Abatement cost
       Y(t)          Output net abatement and damage costs
       Q(t)          Gross Output
       E(t)          CO2-equiv emissions billion t
       M(t)          CO2-equiv concentration billion t
       MIU(t)        Emission control rate GHGs
       FORC(t)       Radiative forcing - W per m2
       TE(t)         Temperature - atmosphere C
       TL(t)         Temperature - lower ocean C
       UTILITY       Maximand;

POSITIVE VARIABLES MIU, TE, M, Y, C, K, I;

EQUATIONS
       UTIL          Objective function
       YY(t)         Output
       AA(t)         Abatement
       DD(t)         Damage
       QQ(t)         Underlying production function
       CC(t)         Consumption
       KK(t)         Capital balance
       KC(t)         Terminal condition of K
       EE(t)         Emissions process
       FORCE(t)      Radiative forcing equation
       MM(t)         CO2 distribution equation
       TTE(t)        Temperature-climate equation for atmosphere
       TLE(t)        Temperature-climate equation for lower oceans;

CC(t)..       C(t) =E= Y(t) - I(t);
YY(t)..       Y(t) =E= Q(t) - A(t)*Q(t) - D(t)*Y(t);
AA(t)..       A(t) =E= b1 * MIU(t)**b2;
DD(t)..       D(t) =E= (a1/9)*SQR(TE(t));
QQ(t)..       Q(t) =E= al(t) * L(t)**(1-gamma) * K(t)**gamma;
KK(t)..       K(t) =L= (1-dk)**10 * K(t-1) + 10 * I(t-1) + (k0*0.9)$tfirst(t);
KC(tlast)..   dk * K(tlast) =L= I(tlast);
EE(t)..       E(t) =G= 10 * sigma(t) * (1-MIU(t)) * Q(t);
FORCE(t)..    FORC(t) =E= 4.1*(LOG(M(t)/590)/LOG(2)) + forcoth(t);
MM(t)..       M(t) =E= 590 + atret*E(t) + (1-deltam)*(M(t-1)-590) + m0$tfirst(t);
TTE(t)..      TE(t) =E= TE(t-1)+c1*(FORC(t-1)-lam*TE(t-1)
                        -c3*(TE(t-1)-TL(t-1))) + t0$tfirst(t);
TLE(t)..      TL(t) =E= TL(t-1)+c4*(TE(t-1)-TL(t-1)) + tl0$tfirst(t);
UTIL..        UTILITY =E= SUM(t, 10 *rr(t)*L(t)*LOG(C(t)/L(t))/0.55)
       + SUM(tlast, rr(tlast)*(phik*K(tlast)+phim*M(tlast)+phite*TE(tlast)));

*      Assign a naive starting point which is in the domain of the functions:

C.L(t) = 1; K.L(t) = 1; I.L(t) = 1; Y.L(t) = 1; Q.L(t) = 1; E.L(t) = 1;
M.L(t) = 1; MIU.L(t) = 1; FORC.L(t) = 1; TE.L(t) = 1; TL.L(t) = 1; UTILITY.L = 1;

* Upper and Lower Bounds for economic reasons or stability

MIU.UP(t) = 0.99; MIU.LO(t) = 0.01; K.LO(t) = 1; TE.UP(t) = 20; M.LO(t) = 600;
C.LO(t) = 2;



*      Initial values:
```

```
MIU.fx('1')=0.; MIU.fx('2')=0.; MIU.fx('3')=0.;

model CO2 /all/;
solve CO2 maximizing UTILITY using NLP;
```

B3: Decomposed NLP Model Formulation

```
$title DICE version 1994 – Decomposed NLP formulation

$include dicedata

alias (t,tp);

PARAMETER
      dref(t)       Reference values of damage
      grad(tp,t)    Local dependence of D(tp) on E(t),
      eref(t)       Reference values of emissions,
      gradt(tp)     Local dependence of transversality terms on emissions;

VARIABLES
      C(t)          Consumption trillion US dollars
      K(t)          Capital stock trillion US dollars
      I(t)          Investment trillion US dollars
      D(t)          Damage
      A(t)          Abatement cost
      Y(t)          Output net abatement and damage costs
      Q(t)          Gross Output
      E(t)          CO2-equiv emissions billion t
      MIU(t)        Emission control rate GHGs
      UTILITY       Maximand;

POSITIVE VARIABLES MIU, E, Y, C, K, I;

EQUATIONS
      UTIL          Objective function
      YY(t)         Output
      AA(t)         Abatement
      QQ(t)         Underlying production function
      CC(t)         Consumption
      KK(t)         Capital balance
      KC(t)         Terminal condition of K
      EE(t)         Emissions process,
      DD(t)         Linear climate model;

CC(t)..       C(t) =E= Y(t) - I(t);
YY(t)..       Y(t) =E= Q(t) - A(t)*Q(t) - D(t)*Y(t);
AA(t)..       A(t) =E=  b1 * MIU(t)**b2;
QQ(t)..       Q(t) =E= al(t) * L(t)**(1-gamma) * K(t)**gamma;
KK(t)..       K(t) =L= (1-dk)**10 * K(t-1) + 10 * I(t-1) + (0.9*k0)$tfirst(t);
KC(tlast)..   dk * K(tlast) =L= I(tlast);
EE(t)..       E(t) =G= 10 * sigma(t) * (1-MIU(t)) * Q(t);
DD(t)..       D(t) =E= dref(t) + sum(tp, grad(t,tp)*(E(tp)-eref(tp)));
UTIL..        UTILITY =E= SUM(t, 10 *rr(t)*L(t)*LOG(C(t)/L(t))/0.55)
        + SUM(tlast, rr(tlast)*((phik*K(tlast))+sum(t,gradt(t)*(E(t)-eref(t)))));

*      Assign a naive starting point which is in the
*      domain of the functions:

K.L(t) = k0 * L(t)/L0;  K.LO(t) = K.L(t)/100;
Q.L(t) = al(t) * K.L(t)**gamma * L(t)**(1-gamma);
MIU.L(t) = 0; E.L(t) = 10 * sigma(t) * (1-MIU.L(t)) * Q.L(t);
Y.L(t) = Q.L(t); C.L(t) = 1; A.L(t) = 0; D.L(t) = 0;
I.L(t) = dk*K.L(T)/10; MIU.L(t) = 1; UTILITY.L = 1;



* Upper and Lower Bounds for economic reasons or stability
```

```
MIU.UP(t) = 0.99; MIU.LO(t) = 0.01; K.LO(t) = 1; C.LO(t) = 2;

*      Initial values:

MIU.fx('1')=0; MIU.fx('2')=0; MIU.fx('3')=0;

MODEL dice /all/;

parameters
       m(t)         CO2-equiv concentration billion t
       forc(t)      Radiative forcing - W per m2
       te(t)        Temperature - atmosphere C
       tl(t)        Temperature - lower ocean C
       termv        Terminal value of atmophere
       deltaE       Difference iterval /0.001/;

m(tfirst) = M0; te(tfirst) = T0;  tl(tfirst) = TL0;

*--------------------------------------------------------------------------------
$onecho >climatemodel.gms
loop(t,
       forc(t)      = 4.1*(LOG(m(t)/590)/LOG(2)) + forcoth(t);
       m(t+1)       = 590 + atret*eref(t) + (1-deltam)*(m(t)-590);
       te(t+1)      = te(t)+c1*(forc(t)-lam*te(t)-c3*(te(t)-tl(t)));
       tl(t+1)      = tl(t)+c4*(te(t)-tl(t));
       dref(t)      = (a1/9) * sqr(te(t));
);
$offecho
*--------------------------------------------------------------------------------

*--------------------------------------------------------------------------------
$onecho >gradients.gms
       eref(t) = E.L(t);
$include climatemodel
       D.L(t) = dref(t); termv  = sum(tlast, phim*m(tlast)+phite*te(tlast));
       grad(t,tp) = 0;     gradt(t) = 0;
       loop(tp,eref(tp) = eref(tp) + deltaE;
$include climatemodel
         grad(t,tp) = (dref(t)-D.L(t)) / deltaE;
         gradt(tp) = (sum(tlast, phim*m(tlast)+phite*te(tlast))-termv) / deltaE;
         eref(tp) = eref(tp) - deltaE;);
       dref(t) = D.L(t);
$offecho
*--------------------------------------------------------------------------------

set    diagitr             Diagonalization iterations /diag1*diag5/;
loop(diagitr,
       eref(t) = e.l(t);
$include gradients
       SOLVE dice maximizing UTILITY using NLP;
);
```

B4: Decomposed MCP Formulation with Adjusted Terminal Constraints

```
$title DICE version 1994 -- MCP Formulation with Adjusted Terminal Constraints

scalar nlpsol /0/;

$if not set tc $set tc 60
$if not set t  $set t  40

set    tc     /1*%tc%/,
       t(tc)  /1*%t%/;

$include dicedata

alias (t,tp);
```

```
PARAMETER
      dref(tc)     Reference values of damage
      pdref(tc)    Reference present value of damage
      grad(tc,t)   Local dependence of D(tp) on E(t),
      eref(tc)     Reference values of emissions
      xi(t)        Post-terminal damage value;

VARIABLES
      C(t)         Consumption trillion US dollars
      K(t)         Capital stock trillion US dollars
      I(t)         Investment trillion US dollars
      Y(t)         Output net abatement and damage costs
      D(tc)        Damage
      A(t)         Abatement cost
      Q(t)         Gross Output
      E(t)         CO2-equiv emissions billion t
      MIU(t)       Emission control rate GHGs

      PY(t)        Output
      PQ(t)        Underlying production function
      PC(t)        Consumption
      PA(t)        Shadow price on abatement cost coefficent
      PD(t)        shadow price on damage coefficent
      PK(t)        Capital balance
      PE(t)        Emissions process
      KT           Terminal Capital stock,
      PKT          Shadow price on terminal capital;

POSITIVE VARIABLES MIU, Y, C, K, I, PE;

EQUATIONS
      YY(t)        Output,
      AA(t)        Abatement,
      DD(t)        Damage (linear climate model),
      QQ(t)        Underlying production function,
      CC(t)        Consumption,
      KK(t)        Capital balance,
      EE(t)        Emissions process,

      EQ_C(t)      Consumption trillion US dollars,
      EQ_K(t)      Capital stock trillion US dollars,
      EQ_I(t)      Investment trillion US dollars,
      EQ_Y(t)      Output net abatement and damage costs,
      EQ_Q(t)      Gross Output,
      EQ_A(t)      Abatement,
      EQ_D(t)      Damage,
      EQ_MIU(t)    Emission control rate GHGs
      EQ_E(t)      CO2-equiv emissions billion t
      EQ_PKT        Equilibrium for terminal capital market,
      EQ_KT        Equilibrium for terminal capital stock;


CC(t)..       C(t) =E= Y(t) - I(t);
YY(t)..       Y(t) =E= Q(t)*(1-A(t)) - D(t)*Y(t);
AA(t)..       A(t) =E=  b1 * MIU(t)**b2;
QQ(t)..       Q(t) =E= al(t) * L(t)**(1-gamma) * K(t)**gamma;
KK(t)..       K(t) =L= (1-dk)**10 * K(t-1) + 10 * I(t-1) + (k0*0.9)$tfirst(t);
EE(t)..       E(t) =G= 10 * sigma(t) * (1-MIU(t)) * Q(t);
DD(t)..       D(t) =E= dref(t) + SUM(tp, grad(t,tp)/1e6*(E(tp)-eref(tp)));
EQ_C(t)..     10 * rr(t) * L(t) / (0.55*C(t)) =E= PC(t);
EQ_K(t)..     K(t) * PK(t)  =G=
         gamma * PQ(t) * Q(t) + (PK(t+1)+PKT$tlast(t)) * (1-dk)**10 * K(t);
EQ_I(t)..     PC(t) =E= 10 * (PK(t+1) + PKT$tlast(t));
EQ_Y(t)..     PY(t) * (1+D(t)) =E= PC(t);
EQ_Q(t)..     PQ(t) =E= PY(t)*(1-A(t)) - PE(t)*10*sigma(t)*(1-MIU(t));
EQ_E(t)..     -PE(t) =E= SUM(tp, PD(tp)*grad(tp,t)/1e6) +
                      SUM(tlast, PD(tlast)*xi(t));
EQ_A(t)..    PA(t) + PY(t)*Q(t) =E= 0;
EQ_D(t)..    PD(t) + PY(t)*Y(t) =E= 0;
```

```
EQ_MIU(t)..  -PE(t)*10*sigma(t)*Q(t) =E= PA(t) * b1 * b2 * MIU(t)**(b2-1);
EQ_PKT..      SUM(tlast, 10 * I(tlast) + K(tlast) * (1-dk/100)**10) =E= KT;
EQ_KT..       SUM(tlast(t), I(t)/I(t-1) - Y(t)/Y(t-1)) =E= 0;

MODEL DICEMCP /CC.PC, YY.PY, AA.PA, QQ.PQ, KK.PK, EE.PE, DD.PD, EQ_C.C,
     EQ_K.K, EQ_I.I, EQ_Y.Y, EQ_Q.Q, EQ_E.E, EQ_A.A, EQ_D.D, EQ_MIU.MIU,
     EQ_KT.KT, EQ_PKT.PKT /;

PARAMETERS
       m(tc)          CO2-equiv concentration billion t,
       forc(tc)       Radiative forcing - W per m2,
       te(tc)         Temperature - atmosphere C,
       tl(tc)         Temperature - lower ocean C,
       deltaE         Difference iterval /0.01/;

m(tfirst) = M0; te(tfirst) = T0; tl(tfirst) = TL0;

*--------------------------------------------------------------------------------
$onecho >climatemodel.gms
loop(tc,forc(tc) = 4.1*(LOG(m(tc)/590)/LOG(2)) + forcoth(tc);
       m(tc+1)        = 590 + atret*eref(tc) + (1-deltam)*(m(tc)-590);
       te(tc+1) = te(tc)+c1*(forc(tc)-lam*te(tc)-c3*(te(tc)-tl(tc)));
       tl(tc+1) = tl(tc)+c4*(te(tc)-tl(tc));
       dref(tc) = (a1/9) * sqr(te(tc)););
$offecho
*--------------------------------------------------------------------------------

*--------------------------------------------------------------------------------
$onecho >gradients.gms
       eref(t) = E.L(t);
       pdref(t) = PD.L(t);
       LOOP((tlast,tc)$(not t(tc)),
         eref(tc)  = eref(tlast)  * (sigma(tc)*L(tc))/(sigma(tlast)*L(tlast));
         pdref(tc) = pdref(tlast) * (L(tc)*rr(tc))   /(L(tlast)*rr(tlast));
       );
$include climatemodel
       D.L(tc) = dref(tc);
       grad(tc,tp) = 0;
       loop(tp, eref(tp) = eref(tp) + deltaE;
$include climatemodel
         grad(tc,tp) = (dref(tc)-D.L(tc))*1e6 / deltaE;
         eref(tp) = eref(tp) - deltaE;);
       dref(tc) = D.L(tc);
       loop(tlast,
         xi(t) = sum(tc$(not t(tc)), grad(tc,t)/1e6*pdref(tc)) / pdref(tlast);
       );

$offecho
*--------------------------------------------------------------------------------

*      Begin with a replication of the NLP solution:

PARAMETER    elog   Log of emissions,
             delta  Change in key variables;

IF (nlpsol,
       execute_load 'nlpsol.gdx',eref,C,K,I,D,A,Y,Q,E,MIU,YY,AA,QQ,CC,KK,EE,DD;
       PC.L(t) = CC.M(t);
       PY.L(t) = YY.M(t);
       PA.L(t) = AA.M(t);
       PD.L(t) = DD.M(t);
       PQ.L(t) = QQ.M(t);
       PK.L(t) = KK.M(t);
       PE.L(t) = -EE.M(t);
       elog(tc,"NLPSOL") = na;
       elog(t,"NLPSOL") = eref(t);
       LOOP(tfirst,
         E.L(t) = E.L(tfirst) * (sigma(t)*L(t))/(sigma(tfirst)*L(tfirst)););
       delta(t,"E") = E.L(t);
```

```
      delta(t,"I") = I.L(t);
      delta(t,"C") = C.L(t);
ELSE
      LOOP(tfirst,
        K.L(t) = k0*0.9 * L(t)/L(tfirst);
        Q.L(t) = al(t) * L(t)**(1-gamma) * K.L(t)**gamma;
        I.L(t) = (K.L(t+1) - (1-dk)**10*K.L(t)) / 10;
        MIU.l(t) = 0.1;
        A.L(t) =  b1 * MIU.L(t)**b2;
        D.L(t) = 0;
        Y.L(t) = Q.L(t)*(1-A.L(t)) - D.L(t)*Y.L(t);
        C.l(t) = Y.L(t) - I.L(t);
        PC.L(t) = 10 * rr(t) * L(t) / (0.55*C.L(t));
        PY.L(t) = PC.L(t) / (1+D.L(t));
        PQ.l(t) = PY.l(t);
        PK.l(t) = PY.l(t);
        PA.l(t) = -PY.L(t)*Q.L(t);
        PD.l(t) = -PY.L(t)*Y.L(t);
        PE.l(t) = -PA.L(t)*b1*b2*MIU.L(t)**(b2-1)/(10*sigma(t)*Q.L(t))
      );
);

MIU.UP(t) = 0.99;  MIU.LO(t) = 0.01;

KT.L = sum(tlast, K.L(tlast)); PKT.L = sum(tlast, PK.L(tlast)); PKT.UP = +INF;

set   diagitr      Diagonalization iterations /iter0*iter4/;

LOOP(diagitr,
$INCLUDE gradients
      SOLVE DICEMCP USING MCP;
);
```

B5: Decomposed MCP Formulation with Distortionary Taxation

```
$title DICE version 1994 -- MCP Formulation with Distortionary Taxation

$if not set tk $set tk 0.25
scalar nlpsol /0/, tk0/%tk%/;

scalar g0     Baseline government /1/;

$if not set tc $set tc 60
$if not set t  $set t  40

set   tc     /1*%tc%/,
      t(tc)  /1*%t%/;

$include dicedata

alias (t,tp);

PARAMETER
      dref(tc)      Reference values of damage
      pdref(tc)     Reference present value of damage
      grad(tc,t)    Local dependence of D(tp) on E(t),
      eref(tc)      Reference values of emissions
      xi(t)         Post-terminal damage value;

VARIABLES
      C(tc)         Consumption trillion US dollars
      G             Government demand
      K(t)          Capital stock trillion US dollars
      I(t)          Investment trillion US dollars
      Y(t)          Output net abatement and damage costs
      D(tc)         Damage
      A(t)          Abatement cost
      Q(t)          Gross Output
      E(t)          CO2-equiv emissions billion t
```

```
       MIU(t)        Emission control rate GHGs

       PY(t)         Output
       PQ(t)         Underlying production function
       PC(t)         Consumption
       PA(t)         Shadow price on abatement cost coefficent
       PD(t)         shadow price on damage coefficent
       PK(t)         Capital balance
       PE(t)         Emissions process
       TK            Capital tax rate
       KT            Terminal Capital stock,
       PKT           Shadow price on terminal capital
       RENT(t)       Emission rents;

POSITIVE VARIABLES MIU, Y, C, K, I, PE;

EQUATIONS
       YY(t)         Output,
       AA(t)         Abatement,
       DD(t)         Damage (linear climate model),
       QQ(t)         Underlying production function,
       CC(t)         Consumption,
       KK(t)         Capital balance,
       EE(t)         Emissions process,

       EQ_C(t)       Consumption trillion US dollars,
       EQ_K(t)       Capital stock trillion US dollars,
       EQ_I(t)       Investment trillion US dollars,
       EQ_Y(t)       Output net abatement and damage costs,
       EQ_Q(t)       Gross Output,
       EQ_A(t)       Abatement,
       EQ_D(t)       Damage,
       EQ_MIU(t)     Emission control rate GHGs
       EQ_E(t)       CO2-equiv emissions billion t
       EQ_PKT        Equilibrium for terminal capital market,
       EQ_KT         Equilibrium for terminal capital stock
       EQ_G          Government budget,
       EQ_TK         Capital tax rate
       EQ_RENT(t)    Defining equation for emission rents;

CC(t)..       C(t) + G * L(t)/L0 =E= Y(t) - I(t);
YY(t)..       Y(t) =E= Q(t)*(1-A(t)) - D(t)*Y(t);
AA(t)..       A(t) =E=  b1 * MIU(t)**b2;
QQ(t)..       Q(t) =E= al(t) * L(t)**(1-gamma) * K(t)**gamma;
KK(t)..       K(t) =L= (1-dk)**10 * K(t-1) + 10 * I(t-1) + (k0*0.9)$tfirst(t);
EE(t)..       E(t) =G= 10 * sigma(t) * (1-MIU(t)) * Q(t);
DD(t)..       D(t) =E= dref(t) + SUM(tp, grad(t,tp)/1e6*(E(tp)-eref(tp)));
EQ_C(t)..     10 * rr(t) * L(t) / (0.55*C(t)) =E= PC(t);
EQ_K(t)..     K(t) * PK(t)  =G=
         gamma*PQ(t)*Q(t)/(1+TK) + (PK(t+1)+PKT$tlast(t)) * (1-dk)**10 * K(t);
EQ_I(t)..     PC(t) =E= 10 * (PK(t+1) + PKT$tlast(t));
EQ_Y(t)..     PY(t) * (1+D(t)) =E= PC(t);
EQ_Q(t)..     PQ(t) =E= PY(t)*(1-A(t)) - PE(t)*10*sigma(t)*(1-MIU(t));
EQ_E(t)..    -PE(t) =E= SUM(tp, PD(tp)*grad(tp,t)/1e6) +
                        SUM(tlast, PD(tlast)*xi(t));
EQ_A(t)..     PA(t) + PY(t)*Q(t) =E= 0;
EQ_D(t)..     PD(t) + PY(t)*Y(t) =E= 0;
EQ_MIU(t)..   -PE(t)*10*sigma(t)*Q(t) =E= PA(t) * b1 * b2 * MIU(t)**(b2-1);
EQ_PKT..      SUM(tlast, 10 * I(tlast) + K(tlast) * (1-dk/100)**10) =E= KT;
EQ_KT..       SUM(tlast(t), I(t)/I(t-1) - Y(t)/Y(t-1)) =E= 0;
EQ_TK..       G =e= g0;
EQ_G..        G * SUM(t,L(t)/L0*PC(t)) =E=
              TK * sum(t, gamma*PQ(t)*Q(t)/(1+TK)) + SUM(t, RENT(t));
EQ_RENT(t).. RENT(t) =e= PE(t)*E(t) - PY(t)*A(t)*Q(t);



MODEL DICEMCP /CC.PC, YY.PY, AA.PA, QQ.PQ, KK.PK, EE.PE, DD.PD, EQ_C.C,
     EQ_K.K, EQ_I.I, EQ_Y.Y, EQ_Q.Q, EQ_E.E, EQ_A.A, EQ_D.D, EQ_MIU.MIU,
```

```
     EQ_KT.KT, EQ_PKT.PKT, EQ_G.G, EQ_TK.TK, EQ_RENT.RENT /;

PARAMETERS
       m(tc)         CO2-equiv concentration billion t,
       forc(tc)      Radiative forcing - W per m2,
       te(tc)        Temperature - atmosphere C,
       tl(tc)        Temperature - lower ocean C,
       deltaE        Difference iterval /0.01/;

m(tfirst) = M0;
te(tfirst) = T0;
tl(tfirst) = TL0;

*--------------------------------------------------------------------------------
$onecho >climatemodel.gms
loop(tc,forc(tc) = 4.1*(LOG(m(tc)/590)/LOG(2)) + forcoth(tc);
       m(tc+1)       = 590 + atret*eref(tc) + (1-deltam)*(m(tc)-590);
       te(tc+1) = te(tc)+c1*(forc(tc)-lam*te(tc)-c3*(te(tc)-tl(tc)));
       tl(tc+1) = tl(tc)+c4*(te(tc)-tl(tc));
       dref(tc) = (a1/9) * sqr(te(tc)););
$offecho
*--------------------------------------------------------------------------------

*--------------------------------------------------------------------------------
$onecho >gradients.gms
       eref(t) = E.L(t);
       pdref(t) = PD.L(t);
       LOOP((tlast,tc)$(not t(tc)),
         eref(tc)  = eref(tlast)  * (sigma(tc)*L(tc))/(sigma(tlast)*L(tlast));
         pdref(tc) = pdref(tlast) * (L(tc)*rr(tc))   /(L(tlast)*rr(tlast));
       );
$include climatemodel
       D.L(tc) = dref(tc);
       grad(tc,tp) = 0;
       loop(tp, eref(tp) = eref(tp) + deltaE;
$include climatemodel
         grad(tc,tp) = (dref(tc)-D.L(tc))*1e6 / deltaE;
         eref(tp) = eref(tp) - deltaE;);
       dref(tc) = D.L(tc);
       loop(tlast,
         xi(t) = sum(tc$(not t(tc)), grad(tc,t)/1e6*pdref(tc)) / pdref(tlast);
       );

$offecho
*--------------------------------------------------------------------------------

*      Begin with a replication of the NLP solution:

PARAMETER    elog   Log of emissions,
             delta  Change in key variables;

IF (nlpsol,
       execute_load 'nlpsol.gdx',eref,C,K,I,D,A,Y,Q,E,MIU,YY,AA,QQ,CC,KK,EE,DD;
       PC.L(t) = CC.M(t);
       PY.L(t) = YY.M(t);
       PA.L(t) = AA.M(t);
       PD.L(t) = DD.M(t);
       PQ.L(t) = QQ.M(t);
       PK.L(t) = KK.M(t);
       PE.L(t) = -EE.M(t);
       elog(tc,"NLPSOL") = na;
       elog(t,"NLPSOL") = eref(t);
       LOOP(tfirst,
         E.L(t) = E.L(tfirst) * (sigma(t)*L(t))/(sigma(tfirst)*L(tfirst)););
       delta(t,"E") = E.L(t);
       delta(t,"I") = I.L(t);
       delta(t,"C") = C.L(t);
ELSE
       LOOP(tfirst,
```

```
        K.L(t) = k0*0.9 * L(t)/L(tfirst);
        Q.L(t) = al(t) * L(t)**(1-gamma) * K.L(t)**gamma;
        I.L(t) = (K.L(t+1) - (1-dk)**10*K.L(t)) / 10;
        MIU.l(t) = 0.1;
        A.L(t) =  b1 * MIU.L(t)**b2;
        D.L(t) = 0;
        Y.L(t) = Q.L(t)*(1-A.L(t))/ (1 + D.L(t));
        C.l(t) = Y.L(t) - I.L(t);
        PC.L(t) = 10 * rr(t) * L(t) / (0.55*C.L(t));
        PY.L(t) = PC.L(t) / (1+D.L(t));
        PQ.l(t) = PY.l(t);
        PK.l(t) = PY.l(t);
        PA.l(t) = -PY.L(t)*Q.L(t);
        PD.l(t) = -PY.L(t)*Y.L(t);
        PE.l(t) = -PA.L(t)*b1*b2*MIU.L(t)**(b2-1)/(10*sigma(t)*Q.L(t))
      );
);
MIU.UP(t) = 0.99;
MIU.LO(t) = 0.01;

KT.L = sum(tlast, K.L(tlast));
PKT.L = sum(tlast, PK.L(tlast)); PKT.UP = +INF;

PARAMETER    elog   Log of emissions,
             delta  Change in key variables;


TK.FX = tk0;

set    diagitr      Diagonalization iterations /iter0*iter4/;

LOOP(diagitr,
$INCLUDE gradients
      elog(tc,diagitr) = eref(tc);
      SOLVE DICEMCP USING MCP;
);
$INCLUDE gradients
elog(tc,"sol") = eref(tc);
*.$libinclude plot elog

PARAMETER    budget(t,*)  Public budget;

budget(t,"TK") = TK.L * gamma*PQ.L(t)*Q.L(t)/(1+TK.L);
budget(t,"RENT") = RENT.L(t);
*.$LIBINCLUDE PLOT budget

*      Now endogenize the capital tax rate and consider look at
*      the marginal utility of emission reductions:

PARAMETER    u0     Reference utility,
             du(t)  Marginal utility of emission reductions;

g0 = G.L; TK.UP = +inf; TK.LO = -inf; E.fx(t) = E.L(t);
loop(tc$(not t(tc)),  C.L(tc) = C.L(tc-1) * L(tc)/L(tc-1););
u0 = sum(tc, 10 * rr(tc)*L(tc)*LOG(C.l(tc)/L(tc))/0.55);
alias (t,tloop);
LOOP(tloop,
      E.FX(tloop) = E.L(tloop)/1.01;
      SOLVE DICEMCP USING MCP;
      E.FX(tloop) = E.L(tloop)*1.01;
      du(tloop) = sum(tc, 10 * rr(tc)*L(tc)*LOG(C.l(tc)/L(tc))/0.55) - u0;
);
$setglobal domain t
$libinclude plot du
```