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THE WESS-ZUMINO-WITTEN-NOVIKOV THEORY, KNIZHNIK-ZAMOLODCHIKOV EQUATIONS, AND KRICHEVER-NOVIKOV ALGEBRAS, I

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Dedicated to Prof. S.P.Novikov in honour of his 60-th birthday

Abstract. Elements of a global operator approach to the WZWN theory for compact Riemann surfaces of arbitrary genus g are given. Sheaves of representations of affine Krichever-Novikov algebras over a dense open subset of the moduli space of Riemann surfaces (respectively of smooth, projective complex curves) with N marked points are introduced. It is shown that the tangent space of the moduli space at an arbitrary moduli point is isomorphic to a certain subspace of the Krichever-Novikov vector field algebra given by the data of the moduli point. This subspace is complementary to the direct sum of the two subspaces containing the vector fields which vanish at the marked points, respectively which are regular at a fixed reference point. For each representation of the affine algebra $3g-3+N$ equations $(\partial_k + T[e_k])\Phi = 0$ are given, where the elements $\{e_k\}$ are a basis of the subspace, and T is the Sugawara representation of the centrally extended vector field algebra. For genus zero one obtains the Knizhnik-Zamolodchikov equations in this way. The coefficients of the equations for genus one are found in terms of Weierstraß-σ function.

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1. Introduction

In the development of two-dimensional conformal field theory it was realized that the following problem is of special importance: construct a bundle over the *moduli space of punctured Riemann surfaces*[1] with actions of the gauge and the conformal algebras, and with a projectively flat connection. This problem originates in the well-known paper of Knizhnik and Zamolodchikov [14], where the case of genus zero is considered. To each puncture they assigned an irreducible module of the affine Kac-Moody algebra associated to the gauge algebra. Over each punctured Riemann sphere they took the tensor product of these modules as a fibre of the bundle. Expectations of sections of the so obtained bundle were interpreted by them as N-point correlation functions, where N is the number of punctures. Making use of the Sugawara construction they obtained an action of the Virasoro algebra in each fibre and then defined the connection as $\frac{\partial}{\partial z_p} + L_{-1}^{(p)}$, where $p = 1, \dots, N$ is the index of the punctures, z_p is a local coordinate in a neighbourhood of the p-th puncture and $L_{-1}^{(p)}$ is the corresponding (rescaled) Sugawara operator of degree -1. One of the most famous results of the paper [14] is the derivation of equations which have as solutions the sections of degree zero which are horizontal with respect to the connection. The equations are nowadays known as the Knizhnik-Zamolodchikov (KZ) equations, the connection as the KZ connection. The KZ equations for N-point correlation functions look as follows:

$$\left(k\frac{\partial}{\partial z_p} - \sum_{r\neq p}\frac{t_p^a t_r^a}{z_p - z_r}\right) < \phi_1(z_1)\dots\phi_N(z_N) >= 0\ , \qquad p = 1, \dots, N\ . \tag{1.1}$$

Here k denotes some constant depending on the Kac-Moody algebra and the level of the representation. The t_i^a for $i = 1, \dots, N$ are the (anti-hermitian) representation matrices for the a^{th} generator of the (finite-dimensional) gauge algebra in the representation associated to the point z_i (operating on the i^{th} field). In (1.1) a summation over the index a is assumed.

D.Bernard was the first who considered the case of positive genus. He realized that "almost all the special features of a given WZW model are encoded inside its zero modes" [2, p.81], but "The lack of a precise definition of the action of the zero modes $J^a_{0;\ j}$ on the correlation functions is very bad" [3, p.146]. In other words, he immediately met a problem of defining an action of the zero modes of the current operator on correlation functions. The way-out he proposed was to consider correlation functions (as well as corresponding primary fields) as functions of not only the punctures but of g additional parameters which are elements of the finite-dimensional Lie group G (g denotes the genus of the Riemann surface). The additional parameters are called *twists*. From

[1] See Section 4 for the precise definition what me mean by "moduli space of punctured Riemann surfaces".

the recent point of view what he really constructed is a connection over the *moduli space of representations* of the fundamental group of the punctured Riemann surface in the group G. The many interesting ideas contained in the papers [2],[3] stimulated a number of articles [7],[8],[12] in which higher genus Knizhnik-Zamolodchikov equations are interpreted as equations of horizontality with respect to a projectively flat connection over the moduli space of representations. In particular, in [7],[8] the mathematical foundation of the theory and some of its mathematical results are formulated with more precision.

Hitchin [11] proposed his own approach to the problem. He obtained a projectively flat connection on the moduli space of representations by means of geometric quantization of certain integrable systems.

One more direction was started by the important paper of Tsuchiya, Ueno, Yamada [38] . In contrary to the above mentioned works (except the approach of Knizhnik-Zamolodchikov) they constructed a projectively flat connection on the space of punctured Riemann surfaces (more precisely, on the moduli space of stable curves with marked points). In a most direct and consecutive form their approach used the idea of deforming the complex structure on a Riemann surface by means of cutting and twisting little circles around certain points. They chose local coordinates (resp. formal neighbourhoods) at the punctures and then reproduced the Knizhnik-Zamolodchikov construction with respect to the chosen coordinates. To each puncture they assigned a copy of the Kac–Moody loop algebra and a copy of the Virasoro algebra connected with the fixed local coordinate in the neighbourhood of that puncture. They considered as algebra of gauge symmetries of the theory a central extension of the direct sum of the local loop algebras and as algebra of conformal symmetries the direct sum of the local Virasoro algebras. Starting from the local representations a sheaf of representations was constructed over the moduli space. The *conformal blocks* were introduced as a certain quotient sheaf. It turned out to be a finite-dimensional vector bundle over the moduli space. The above mentioned connection was constructed in this bundle. Note that they were able to supply with their techniques a mathematical proof of the Verlinde formula. See also [39] for a pedagogical presentation of this approach.

It was clear from the very beginning that the basic objects of the Wess-Zumino-Witten-Novikov (WZWN) theory on a Riemann surface are of global nature. In fact, the current and the energy-momentum tensor are abelian differentials (of 1-st and 2-d order respectively). The algebra annihilating conformal blocks consists of (Lie algebra valued) meromorphic functions with certain polar behaviour [8]. Nevertheless, most authors used the above mentioned local approach to gauge and conformal symmetries due to Tsuchiya, Ueno and Yamada. In [16-18] Krichever and Novikov introduced their generalizations of affine Kac-Moody and Virasoro algebras and pointed out a new (global) treatment of gauge and conformal symmetries in two-dimensional conformal field theory. Krichever-Novikov algebras were studied and generalized in [5],[23-29],[33-

37]. In [4],[30] the Sugawara construction for these algebras was considered.

The starting point of this article is our observation that it is natural to consider multipoint Krichever-Novikov algebras as algebras of gauge and conformal symmetries in WZWN theory for arbitrary genus[2]. These algebras have a genuine connection with the basic geometrical objects of the theory, namely with Riemann surfaces with punctures. There is no problem of the action of zero modes of the current operator in the Krichever-Novikov set-up. Moreover, Krichever-Novikov vector fields have a natural connection with deformations of punctured Riemann surfaces. As it is shown below (Theorem 4.5) they can be related to the Kuranishi tangent space of the moduli space of punctured Riemann surfaces. In a dual manner the deformations of punctured Riemann surfaces can be described by (meromorphic) quadratic differentials with at most poles of order one at the punctures. Note that the Krichever-Novikov duality (see Proposition 2.3) supplies such a dual description. See also [9] for one more approach for the two-point case.

In Section 2 the necessary setup for the multipoint Krichever-Novikov algebras in the situation of N incoming points (corresponding to the punctures $(P_1, P_2, \ldots, P_N)$ which can be moved) and one outgoing point (corresponding to a fixed reference point P_∞) are recalled. The definition of the Krichever-Novikov vector field algebras, central extensions, affine algebras corresponding to a fixed finite-dimensional Lie algebra, etc. are given. Necessary results are recalled.

In Section 3 Verma modules for the higher genus multipoint affine algebras are introduced and studied. The Sugawara construction for these Krichever-Novikov algebras is introduced along the lines of [30].

In Section 4 the necessary moduli spaces of compact Riemann surfaces with marked points are introduced. As a technical tool we introduce also enlarged moduli spaces, with first order infinitesimal neighbourhoods at the punctures as additional data. Note that in this article we only deal with the generic situation, i.e. the moduli point corresponds to a generic curve and a generic choice of marked points. Hence, a priori our objects will only be defined over a dense open subset of the moduli space. Sheaf versions of the affine Krichever-Novikov algebras and the Verma modules are given. The elements of the Krichever-Novikov vector field algebra define tangent vectors along the moduli space. In Theorem 4.5 an explicit isomorphism of the tangent space with a certain subspace of the vector field algebra is given. Let $\{X_k,\ k = 1, \ldots, 3g-3+N\}$ be a basis of the tangent space and l_k the element of the vector field algebra which corresponds to X_k under this isomorphism. For sheaves of admissible representation of the affine algebra (e.g. for the Verma modules) the following set of $3g-3+N$ equations

$$\nabla_k \Phi := (\partial_k + T[l_k])\,\Phi = 0 \qquad (1.2)$$

[2] Also I. Krichever expressed this idea to one of the authors in a private talk.

is introduced as *formal KZ equations*. Here ∂_k is the derivation in direction of X_k on the moduli space and $T[l_k]$ the operator corresponding under the Sugawara representation to the vector field l_k. Note that $T[l_k]$ operates vertically in the fibre.

In Section 5 the genus zero case is considered. It is shown how to obtain the original KZ equations from (1.2).

In Section 6 the genus one case is studied. Explicit expressions for the coefficients in the KZ equations in terms of Weierstraß-σ function are derived. In an appendix the KN basis for genus one and N marked points is given.

Altogether the proposed approach enables us to avoid some difficulties of the earlier approaches, to make use of advantages of the global Sugawara construction, and to give a transparent treatment of the geometric origin of the coefficients in the Knizhnik-Zamolodchikov equations. It should be noted that even in the case $g = 0$ our approach provides a new derivation of the original KZ equations.

In the forthcoming part II of the article [31] we will study the KZ equations on the sheaves of Verma modules in more detail, develop further the structure theory and discuss conformal blocks and projective flatness.

2. The algebras of Krichever-Novikov type

(a) The general set-up.

Let us recall here the set-up developed in [26], [23–25]. Let M be a compact Riemann surface of genus g, resp. in the language of algebraic geometry a smooth projective curve over $\mathbb{C}$. Let

$$I = (P_1, \dots, P_N), \quad \text{and} \quad O = (Q_1, \dots, Q_L), \qquad (N, L \geq 1)$$

be disjoint tuples of ordered, distinct points ("marked points" "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points* the points in O the *out-points*. Let $A = I \cup O$ as a set. In this article we are mainly dealing with $\#I = N \geq 1$ and $\#O = 1$. Let ρ be a meromorphic differential on M, holomorphic on $M^* := M \setminus A$, with positive residues at the points in I, negative residues at the points in O, and only purely imaginary periods. By giving the residues (obeying the condition "sum over all residues equals zero") there is a unique such ρ. If we choose an additional point $R \in M^*$ then the function

$$u(P) := \operatorname{Re} \int_R^P \rho$$

is well-defined. Its level lines define a fibering of M^*. Every level line cuts the Riemann surface and separates the in-points from the out-points.

Let $\mathcal{K}$ be the canonical line bundle. Its associated sheaf of local sections is the sheaf of holomorphic differentials. Following the common practice we will usually not distinguish between a line bundle and its associated invertible sheaf (and even between the divisor class corresponding to a meromorphic section of the line bundle). For every $\lambda \in \mathbb{Z}$ we consider the bundle $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$. Here we use the usual convention: $\mathcal{K}^0 = \mathcal{O}$ and $\mathcal{K}^{-1} = \mathcal{K}^*$ is the holomorphic tangent line bundle, (resp. the sheaf of holomorphic vector fields). Indeed, after fixing a theta characteristics, i.e. a bundle S with $S^{\otimes 2} = \mathcal{K}$, it is possible to consider $\lambda \in \frac{1}{2}\mathbb{Z}$. Denote by $\mathcal{F}^\lambda$ the (infinite-dimensional) vector space of global meromorphic sections of $\mathcal{K}^\lambda$ which are holomorphic on $M\setminus A$. Special cases, which are of particular interest to us, are the quadratic differentials ($\lambda = 2$), the differentials ($\lambda = 1$), the functions ($\lambda = 0$), and the vector fields ($\lambda = -1$). The space of functions we will also denote by $\mathcal{A}$ and the space of vector fields by $\mathcal{L}$. By multiplying sections with functions we again obtain sections. In this way the space $\mathcal{A}$ becomes an associative algebra and the $\mathcal{F}^\lambda$ become $\mathcal{A}$-modules.

The vector fields in $\mathcal{L}$ operate on $\mathcal{F}^\lambda$ by taking the Lie derivative. In local coordinates

$$\nabla_e(g)_{|} := L_e(g)_{|} := (e(z)\frac{d}{dz}) \, . \, (g(z)\, dz^\lambda) := \left(e(z)\frac{dg}{dz}(z) + \lambda\, g(z)\frac{de}{dz}(z)\right) dz^\lambda \; . \qquad (2.1)$$

Here $e \in \mathcal{L}$ and $g \in \mathcal{F}^\lambda$. To avoid cumbersome notation we used the same symbol for the section and its representing function. If there is no danger of confusion we will do the same in the following. The space $\mathcal{L}$ becomes a Lie algebra with respect to the Lie bracket (2.1) and the $\mathcal{F}^\lambda$ become Lie modules over $\mathcal{L}$. Let us mention that by the action of $\mathcal{L}$ on $\mathcal{A}$ we may define the Lie algebra $\mathcal{D}^1$ of differential operators of degree ≤ 1 and the $\mathcal{F}^\lambda$ become Lie modules over $\mathcal{D}^1$. It is possible to extend these to differential operators of arbitrary degrees, see [26,28,29] for further information. We will not need this additional structure here.

Definition 2.1. The *Krichever-Novikov pairing* (*KN pairing*) is the pairing between $\mathcal{F}^\lambda$ and $\mathcal{F}^{1-\lambda}$ given by

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \;\to\; \mathbb{C}, \qquad \langle f, g\rangle := \frac{1}{2\pi\,\mathrm{i}}\int_{C_\tau} f\cdot g \; , \qquad (2.2)$$

where C_τ is an arbitrary non-singular level line.

Note that in (2.2) the integral does not depend on the level line chosen. Using residues the pairing can be described in a purely algebraic manner as

$$\langle f, g\rangle = \sum_{P\in I} \mathrm{res}_P(f\cdot g) = -\sum_{Q\in O} \mathrm{res}_Q(f\cdot g)\; . \qquad (2.3)$$

(b) The almost-graded structure.

For the Riemann sphere ($g = 0$) with quasi-global coordinate z and $I = (0)$ and $O = (\infty)$ the introduced vector field algebra is the Witt algebra, i.e. the algebra whose universal central extension is the Virasoro algebra. We denote for short this situation as the *classical situation*. Here it is of fundamental importance that this algebra is a graded algebra. For the higher genus case (and for the multi-point situation for $g = 0$) there is no such grading. It was a fundamental observation by Krichever and Novikov [16–18] that a weaker concept, an almost grading will do.

Definition 2.2. (a) Let $\mathcal{L}$ be an (associative or Lie) algebra admitting a direct decomposition as vector space $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$. $\mathcal{L}$ is called an *almost-graded* (*quasi-graded*, *generalized-graded*) algebra if (1) $\dim \mathcal{L}_n < \infty$ and (2) there are constants R and S with

$$\mathcal{L}_n \cdot \mathcal{L}_m \quad \subseteq \bigoplus_{h=n+m-R}^{n+m+S} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}\ . \tag{2.4}$$

The elements of $\mathcal{L}_n$ are called *homogeneous elements of degree n*.
(b) Let $\mathcal{L}$ be an almost-graded (associative or Lie) algebra and $\mathcal{M}$ an $\mathcal{L}$-module with $\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n$ as vector space. $\mathcal{M}$ is called an *almost-graded* (*quasi-graded*, *generalized-graded*) module, if (1) $\dim \mathcal{M}_n < \infty$, and (2) there are constants R' and S' with

$$\mathcal{L}_m \cdot \mathcal{M}_n \quad \subseteq \bigoplus_{h=n+m-R'}^{n+m+S'} \mathcal{M}_h, \qquad \forall n, m \in \mathbb{Z}\ . \tag{2.5}$$

The elements of $\mathcal{M}_n$ are called *homogeneous elements of degree n*.

By a *weak almost grading* we understand an almost grading without the requiring the finite-dimensionality of the homogeneous subspaces.

For the 2-point situation, $I = \{P\}$ and $O = \{Q\}$, Krichever and Novikov introduced an almost graded structure of the algebras and the modules by exhibiting special bases and defining their elements to be the homogeneous elements. By one of the authors its multi-point generalization was given [25,26], again by exhibiting a special basis. (See also Sadov [21] for some results in similar directions.) For every $n \in \mathbb{Z}$, $p = 1, \ldots, N$ a certain element $f^\lambda_{n,p} \in \mathcal{F}^\lambda$ is exhibited. The $f^\lambda_{n,p}$ for $p = 1, \ldots, N$ are a basis of a subspace $\mathcal{F}^\lambda_n$ and it is shown that

$$\mathcal{F}^\lambda = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^\lambda_n\ .$$

The subspace $\mathcal{F}^\lambda_n$ is called the *homogeneous subspace of degree n*.

Proposition 2.3. [25,26] *(a) By the above definition the vector field algebra $\mathcal{L}$ and the function algebra $\mathcal{A}$ are almost graded and the modules $\mathcal{F}^\lambda$ are almost graded modules over them.*
(b) The basis elements fulfil the duality relation with respect to the KN pairing (2.2)

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{m,r}\rangle = \frac{1}{2\pi\,\mathrm{i}}\int_{C_\tau} f^\lambda_{n,p}\cdot f^{1-\lambda}_{m,r} = \delta^m_{-n}\cdot\delta^r_p\ , \tag{2.6}$$

where C_τ is an arbitrary non-singular level line.

By (2.6) we see that the KN pairing is non-degenerate. Let us introduce the following notation:

$$A_{n,p} := f^0_{n,p},\quad e_{n,p} := f^{-1}_{n,p},\quad \omega^{n,p} := f^1_{-n,p},\quad \Omega^{n,p} := f^2_{-n,p}\ . \tag{2.7}$$

The elements $f^\lambda_{n,p}$ have the following property

$$\mathrm{ord}_{P_i}(f^\lambda_{n,p}) = (n+1-\lambda) - \delta^p_i,\quad i = 1,\ldots,N\ .$$

The orders at the points in O we will give for the $(N,1)$ situation only (this is a short hand notation for $\#I = N, \#O = 1$). Let us denote the single element in O by P_∞

For $g = 0$, or $g \ge 2$, $\lambda \ne 0, 1$ and a generic choice for the points in A we have

$$\mathrm{ord}_{P_\infty}(f^\lambda_{n,p}) = -N\cdot(n+1-\lambda) + (2\lambda-1)(g-1)\ . \tag{2.8}$$

By Riemann-Roch type arguments it is shown in [23] that there is up to a scalar multiple only one such $f^\lambda_{n,p}$. After choosing local coordinates z_p at the points P_p the scalar may be fixed by requiring

$$f^\lambda_{n,p|}(z_p) = z_p^{n-\lambda}(1+O(z_p))\,(dz_p)^\lambda\ .$$

Due to the speciality of the occuring divisors there is for a finite number of degrees n a modified prescription at the point P_∞ needed for the remaining cases. In any case this is done without disturbing the orders at I and the KN duality. The general description is given in [26, p.73]. For the functions $A_{n,p}$ there are only modifications necessary if all orders given by the generic rules are nonpositive. For the differentials $\omega^{n,p}$ there are only modifications necessary if all orders given by the generic rules are nonnegative or if there is exactly one pole of order 1 in the prescription. In any case, modification can only appear for a finite number of values of n.

To give an impression of the modification necessary we like to give them for the $(1,1)$ situation and $g \ge 2$. For the 1-differentials modifications are necessary only for

$-g \le n \le 0$. The modified basis elements of the differentials in the $(1,1)$ situation and $g \ge 2$ are for $-g \le n \le -1$

$$\mathrm{ord}_P(w^n) = -n-1, \quad \mathrm{ord}_{P_\infty}(w^n) = g+n,$$

and $\mathrm{ord}_P(w^0) = -1, \quad \mathrm{ord}_{P_\infty}(w^0) = -1,$ with the additional condition that ω^0 has only imaginary periods. In particular, $\omega^0 = \rho$ up to some multiplication with a scalar. For the functions we set for $-g \le n \le -1$

$$\mathrm{ord}_P(A_n) = n, \quad \mathrm{ord}_{P_\infty}(A_n) = -g-n-1,$$

and $A_0 = 1$. To fix the elements A_n for $-g \le n \le -1$ we add suitable multiples of A_0 such that the duality $\langle w^0, A_n \rangle = 0$ is fulfilled. The $g = 1$ situation will be covered in Section 7.

For the basis elements $f^\lambda_{n,p}$ explicit descriptions in terms of rational functions (for $g = 0$), the Weierstraß σ-function (for $g = 1$), and prime forms and theta functions (for $g \ge 1$) are given in [24]. For a description using Weierstraß $\wp$-function, see [20], [27]. The existence of such an description is necessary in our context because we want to consider the above algebras and modules over the configuration space, resp. the moduli space of pointed curves. In particular, by the explicit representation one sees that the elements vary "analytically" when the complex structure of the Riemann surface is deformed

For further reference and as an illustration let us write down the basis elements for $g = 0$. We choose a quasi-global coordinate z such that the point P_∞ is given by $z = \infty$. Let the points P_i be given by $z = z_i$ for $i = 1, \ldots, N$. Clearly,

$$f^\lambda_{n,p}(z) = (z-z_p)^{n-\lambda}\left(\prod_{\substack{i=1\\ i\ne p}}^{N}(z-z_i)\right)^{n-\lambda+1}\left(\prod_{\substack{i=1\\ i\ne p}}^{N}(z_p-z_i)\right)^{-n+\lambda-1} dz^\lambda \ . \tag{2.9}$$

An explicit description for $g = 1$ can be found in Section 7.

The constructed basis coincide with the Virasoro basis in the classical situation, and with the basis for the two-point situation in higher genus given by Krichever and Novikov [16–18] (up to some index shift).

We need a finer description of the almost graded structure. The following is shown in [25,26]

Proposition 2.4. *There exists constants* $K, L \in \mathbb{N}$ *such that for all* $n, m \in \mathbb{Z}$

$$A_{n,p} \cdot A_{m,r} = \delta_p^r\, A_{n+m,p} + \sum_{h=n+m+1}^{n+m+K} \sum_{s=1}^{N} \alpha^{(h,s)}_{(n,p),(m,r)} A_{h,s}\ ,$$

$$[e_{n,p}, e_{m,r}] = \delta_p^r\,(m-n)\, e_{n+m,p} + \sum_{h=n+m+1}^{n+m+L} \sum_{s=1}^{N} \gamma^{(h,s)}_{(n,p),(m,r)} e_{h,s}\ ,$$

with suitable coefficients $\alpha^{(h,s)}_{(n,p),(m,r)}, \gamma^{(h,s)}_{(n,p),(m,r)} \in \mathbb{C}$.

The constants K and L can be explicitly calculated. They depend on the genus g and on the number of points N. Again we give here only the result for the $(N,1)$ situation and $g \neq 1$.

$$L = \begin{cases} 3g, & g \neq 1 \text{ and } N = 1 \\ 3 + \left[\frac{1}{N}(3g-3)\right], & g \neq 1 \text{ and } N > 1 \ , \end{cases}$$

$$K = \begin{cases} g, & g \neq 1,\ N = 1 \\ 2 + \left[\frac{1}{N}(g-2)\right], & g \neq 1 \text{ and } N > 1 \ . \end{cases} \tag{2.10}$$

Here $[x]$ denotes the largest integer $\leq x$. Clearly, the first alternatives are special cases of the second ones. Strictly speaking, the above value of K is the value for generic n and m. For an overall bound it has to be increased by 1 or 2 depending on g and N.

The algebra $\mathcal{A}$ can be decomposed (as vector space) as

$$\begin{gathered} \mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_{(0)} \oplus \mathcal{A}_+, \\ \mathcal{A}_- := \langle A_{n,p} \mid n \leq -K-1, p = 1,\dots,N \rangle \ , \quad \mathcal{A}_+ := \langle A_{n,p} \mid n \geq 1, p = 1,\dots,N \rangle, \\ \mathcal{A}_{(0)} := \langle A_{n,p} \mid -K \leq n \leq 0, p = 1,\dots,N \rangle \ . \end{gathered} \tag{2.11}$$

and the Lie algebra $\mathcal{L}$ as

$$\begin{gathered} \mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_{(0)} \oplus \mathcal{L}_+, \\ \mathcal{L}_- := \langle e_{n,p} \mid n \leq -L-1, p = 1,\dots,N \rangle \ , \quad \mathcal{L}_+ := \langle e_{n,p} \mid n \geq 1, p = 1,\dots,N \rangle, \\ \mathcal{L}_{(0)} := \langle e_{n,p} \mid -L \leq n \leq 0, p = 1,\dots,N \rangle \ . \end{gathered} \tag{2.12}$$

Due to the almost-grading the subspaces $\mathcal{A}_\pm$ and $\mathcal{L}_\pm$ are subalgebras but the subspaces $\mathcal{A}_{(0)}$, and $\mathcal{L}_{(0)}$ in general are not. We use the term *critical strip* for the latter.

Note that $\mathcal{A}_+$, resp. $\mathcal{L}_+$ can be described as the algebra of functions, (resp. vector fields) vanishing at least of order one (resp. of order 2) at the points $P_i, i = 1,\dots,N$. These algebras can be enlarged by allowing all elements which are regular at all P_i. This could be achieved by adding $\{A_{0,p}, p = 1,\dots,N\}$, (resp. $\{e_{0,p}, e_{-1,p}, i = 1,\dots,N\}$ to the set of basis elements. We denote the enlarged algebras by $\mathcal{A}_+^*$, resp. by $\mathcal{L}_+^*$.

On the other hand $\mathcal{A}_-$ and $\mathcal{L}_-$ could also be enlarged by considering all elements which are vanishing of order one (resp. order two) at the point P_∞. Clearly they could further be enlarged to contain all elements which are regular at P_∞. By this process we include elements from the critical strip into these algebras.

In view of Section 4 we will consider this for $\mathcal{L}_-$ in more detail. By (2.8) we see that passing from $f^\lambda_{n,p}$ to $f^\lambda_{n-1,p}$ the order at P_∞ will increase by N. By direct calculation we obtain for the vector fields

$$-N < \mathrm{ord}_{P_\infty}(e_{-L+1,p}) \le 0, \quad 0 < \mathrm{ord}_{P_\infty}(e_{-L,p}) \le N, \quad N < \mathrm{ord}_{P_\infty}(e_{-L-1,p}) \le 2N \ .$$

In particular the elements in $\mathcal{L}_-$ are all vanishing of at least second order at P_∞. Note that on every degree the order of $e_{n,p}$ at P_∞ does not depend on p. Fix such a degree n. Due to the generic choice of the points a suitable linear combination of $e_{n,p}$ and $e_{n,s}$, $s \ne p$ has exact order 1 more than $e_{n,p}$ at P_∞. In this way it is possible to find basis elements $g_{n,i}$ of $\mathcal{F}^\lambda_n$ with

$$\mathrm{ord}_{P_\infty}(g_{n,i}) = \mathrm{ord}_{P_\infty}(e_{n,p}) + i, \quad i = 0, \ldots, N-1 \ .$$

After making a change of basis in this sense we will find basis elements which can be added to the generators of $\mathcal{L}_-$ and obtain a bigger algebra $\mathcal{L}'_-$ containing all vector fields with vanishing order at least two at P_∞. This involves only the elements of the critical strip of degree lower than n. The corresponding basis elements have to be removed from the critical strip. The remaining subspace we will call *reduced critical strip* $\mathcal{L}'_{(0)}$. Clearly $\mathcal{L}'_-$ can be extended to $\mathcal{L}''_-$ containing all vector fields vanishing at least of order one at P_∞ and $\mathcal{L}^*_-$ containing all vector fields regular at P_∞. The dimension of the corresponding critical strips are one (resp. two) less than the dimension of the reduced critical strip. By counting the orders of the basis elements its dimension can be calculated as

$$\dim \mathcal{L}'_0 = N + N + (3g-3) + 1 + 1 = 2N + 3g - 1 \ . \tag{2.13}$$

The first two terms correspond to $\mathcal{L}_0$ and $\mathcal{L}_{-1}$. For $g \ge 2$ the intermediate term comes from the basis vector fields which have poles at the $P_i, i = 1, \ldots, N$ and P_∞. The $1+1$ corresponds to the above constructed basis vector fields with exact order one, resp. two at P_∞. In Section 4 we will explain how the elements of the critical strip (resp. subsets of them) are related to tangent directions in the moduli space.

A similar decomposition is valid for the critical strip of the function algebra. This yields a modified $\mathcal{A}'_-$ and $\mathcal{A}'_{(0)}$. Note that

$$N \le \mathrm{ord}_{P_\infty}(A_{-K-1,p}) < 2N \ .$$

Let us call the subalgebra of functions which are regular at P_∞ by $\mathcal{A}^*_-$.

(c) Central extensions and affine algebras of higher genus.

The function algebra $\mathcal{A}$ (considered as an abelian Lie algebra) can be centrally extended to a Lie algebra $\widehat{\mathcal{A}}$ via the Lie algebra cohomology 2-cocycle

$$\gamma(g,h) := \frac{1}{2\pi \mathrm{i}} \int_{C_\tau} g dh \ . \tag{2.14}$$

More precisely, $\widehat{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ as vector space with Lie algebra structure given by

$$[\hat{g}, \hat{h}] = \gamma(g,h)\, t_1, \quad [t_1, \widehat{\mathcal{A}}] = 0 \ , \tag{2.15}$$

where we used the notation $\hat{g} := (0,g)$, $\hat{h} := (0,h)$, $t_1 = (1,0)$.

To obtain central extensions of the vector field algebra (generalizing the Virasoro central extension) we have first to choose a global holomorphic projective connection R. The defining 2-cocycle is given as

$$\chi_R(e,f) := \frac{1}{24\pi \mathrm{i}} \int_{C_\tau} \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \ . \tag{2.16}$$

This cocycle was introduced for the $(1,1)$ case by Krichever and Novikov. As shown in [26] it can be extended to the multi-point situation. It defines a central extension $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_R$. Another choice of the projective connection (even if we allow meromorphic projective connections with poles only at the points in A) yields a cohomologous cocycle, hence an equivalent central extension. We denote the non-trivial central generator by t_2. The above cocycles fulfil the important following *locality conditions*.

Proposition 2.5. [26] *There are constants T and S such that for all $m,n \in \mathbb{Z}$*

$$\begin{aligned} \gamma(A_{n,r}, A_{m,p}) \neq 0 \implies T \leq |m+n| \leq 0, \\ \chi_R(e_{n,r}, e_{m,p}) \neq 0 \implies S \leq |m+n| \leq 0, \end{aligned}$$

By considering the order of the integrands in (2.14) and (2.16) we see that the cocycles restricted to the subalgebras $\mathcal{A}_+, \mathcal{A}_-, \mathcal{A}'_-$, resp. $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}'_-$ are vanishing.

Again explicit expressions for T and S can be given, but are not of interest here.

By the locality of the cocycle and by setting $\deg(t_1) := \deg(t_2) := 0$ we obtain in this way an almost grading for $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{A}}$. By the vanishing of the cocycles on the subalgebras $\mathcal{A}_\pm$ and $\mathcal{L}_\pm$ they can be identified in a natural way with subalgebras $\widehat{\mathcal{A}}_\pm$ and $\widehat{\mathcal{L}}_\pm$ of $\widehat{\mathcal{A}}$, resp. $\widehat{\mathcal{L}}$.

Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra with a fixed invariant, nondegenerate symmetric bilinear form $(..|..)$, e.g. for the semi-simple case the Cartan-Killing form. The *higher genus loop algebra* or *higher genus current algebra* is defined as

$$\overline{\mathfrak{g}} := g \otimes \mathcal{A}, \quad \text{with Lie product} \quad [x \otimes g, y \otimes h] := [x,y] \otimes g \cdot h \ .$$

It has a central extension $\widehat{\mathfrak{g}} := \mathbb{C} \oplus \overline{\mathfrak{g}}$ with Lie product

$$[\widehat{x \otimes f}, \widehat{y \otimes g}] = \widehat{[x,y] \otimes (fg)} - (x|y) \cdot \gamma(f,g) \cdot t, \qquad [t, \widehat{\mathfrak{g}}] = 0 \ , \tag{2.17}$$

(where we set $\widehat{x \otimes f} := (0, x \otimes f)$). For the proofs, see [30]. This algebra is called the higher genus (multi-point) affine Lie algebra (or Krichever-Novikov algebra of affine type). Again, we can define an almost grading on $\overline{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$ by setting

$$\deg(t) := 0, \quad \deg(\widehat{x \otimes A_{n,p}}) := \deg(x \otimes A_{n,p}) := n$$

and obtain a splitting as above

$$\begin{aligned} \overline{\mathfrak{g}} &= \overline{\mathfrak{g}}_- \oplus \overline{\mathfrak{g}}_{(0)} \oplus \overline{\mathfrak{g}}_+, \quad \text{with} \quad \overline{\mathfrak{g}}_\beta = \mathfrak{g} \otimes \mathcal{A}_\beta, \quad \beta \in \{-, (0), +\}\ , \\ \widehat{\mathfrak{g}} &= \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{g}}_{(0)} \oplus \widehat{\mathfrak{g}}_+ \quad \text{with} \quad \widehat{\mathfrak{g}}_\pm \cong \overline{\mathfrak{g}}_\pm \quad \text{und} \quad \widehat{\mathfrak{g}}_{(0)} = \overline{\mathfrak{g}}_{(0)} \oplus \mathbb{C} \cdot t\ . \end{aligned} \tag{2.18}$$

In particular, $\widehat{\mathfrak{g}}_\pm$ and $\overline{\mathfrak{g}}_\pm$ are subalgebras. The corresponding is true for the enlarged algebras. Of special interest are

$$\widehat{\mathfrak{g}}^*_- = \overline{\mathfrak{g}}^*_- = \mathfrak{g} \otimes \mathcal{A}^*_-, \qquad \widehat{\mathfrak{g}}^*_+ = \overline{\mathfrak{g}}^*_+ \oplus \mathbb{C}t = (\mathfrak{g} \otimes \mathcal{A}^*_+) \oplus \mathbb{C}t\ .$$

Lemma 2.6.

$$1 = \sum_{p=1}^{N} A_{0,p}\ . \tag{2.19}$$

Proof. Using (2.6) we can write

$$1 = \sum_{n \in \mathbb{Z}} \sum_{p=1}^{N} \langle 1, \omega^{n,p} \rangle A_{n,p}\ .$$

Calculating the orders of the integrand we obtain

$$\langle 1, \omega^{n,p} \rangle = \frac{1}{2\pi\,\mathrm{i}} \int_{C_\tau} \omega^{n,p} = 0\ , \quad \text{for } n \neq 0\ , \qquad \langle 1, \omega^{0,p} \rangle = 1\ .$$

The latter relation is due to the normalization. Hence the claim. □

By this we see that the finite-dimensional Lie algebra $\mathfrak{g}$ can naturally be considered as subalgebra of $\overline{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$. It lies in the subspace $\overline{\mathfrak{g}}_0$.

3. Representations of the multi-point Krichever-Novikov algebras

Let us start this section with a definition of what we mean under Verma modules of an affine multi-point Krichever-Novikov algebra $\widehat{\mathfrak{g}}$. The construction presented below is a generalization of that proposed in [37].

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Let $\overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}$ be the linear subspace of elements of degree 0, i.e. $\overline{\mathfrak{g}}_0 = \bigoplus_{p=1}^{N} \mathfrak{g} \otimes A_{0,p}$. Let $\widehat{\mathfrak{g}}_+ = \overline{\mathfrak{g}}_+ \subset \overline{\mathfrak{g}}$ be the linear subspace of elements of a positive degree (see the notation (2.18)). Let Z be the one-dimensional subspace of $\widehat{\mathfrak{g}}$ generated by the central element t. The degree of t was defined to be 0, hence $\widehat{\mathfrak{g}}_0 = \overline{\mathfrak{g}}_0 \oplus Z$. Recall that from Lemma 2.6 it follows that $\mathfrak{g} \subset \widehat{\mathfrak{g}}_0$ (as a Lie subalgebra).

Lemma 3.1. *The direct sum $\widehat{\mathfrak{g}}'_+ := \widehat{\mathfrak{g}}_0 \oplus Z \oplus \widehat{\mathfrak{g}}_+$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$.*

Proof. Recall from Section 2 that $\mathcal{A}_0 \oplus \mathcal{A}_+$ is the (associative) subalgebra of functions which are regular at all the P_i, $i = 1, \ldots, N$. Hence the claim follows. See also Proposition 2.4. □

Consider the direct sum $\mathfrak{g}_{(N)}$ of N copies of the Lie algebra $\mathfrak{g}$: $\mathfrak{g}_{(N)} := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_N$ where an isomorphism $\varphi_p : \mathfrak{g}_p \to \mathfrak{g}$ is fixed for each $p = 1, \ldots, N$. In what follows we shall assume that the maps φ_p have a special structure. Namely, let $G := \exp \mathfrak{g}$ be the associated Lie group. Set $\gamma := (\gamma_1, \ldots, \gamma_N)$ with $\gamma_p \in G$, $p = 1, \ldots, N$ arbitrary and take $\varphi_p := Ad\, \gamma_p$.

Choose and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a corresponding Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a corresponding upper nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$. Set $\mathfrak{h}_p := \varphi_p^{-1}\mathfrak{h}$, $\mathfrak{b}_p := \varphi_p^{-1}\mathfrak{b}$, $\mathfrak{n}_p := \varphi_p^{-1}\mathfrak{n}$ and $\mathfrak{b}_{(N)} := \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_N$. Let V_p be a one-dimensional linear space over $\mathbb{C}$ with a fixed basis vector v_p ($p = 1, \ldots, N$) and set $V := \bigotimes_{p=1}^{N} V_p$. Choose an arbitrary N-tuple $\lambda := \{\lambda_1, \ldots, \lambda_N\}$ with $\lambda_p \in \mathfrak{h}_p^*$, $p = 1, \ldots, N$.

Define for $p = 1, \ldots, N$ the one-dimensional representations of $\mathfrak{b}_p$ on V_p by

$$h_p v_p = \lambda_p(h_p) v_p, \quad n_p v_p = 0, \qquad \text{for} \quad h_p \in \mathfrak{h}_p, \quad n_p \in \mathfrak{n}_p. \tag{3.1}$$

Extend this to an one-dimensional representation of the Lie algebra $\mathfrak{b}_{(N)}$ in the linear space V by decomposing $x_{(N)} := x_1 \oplus \ldots \oplus x_N$ with $x_1 \in \mathfrak{b}_1, \ldots, x_N \in \mathfrak{b}_N$ and setting

$$\begin{aligned} x_{(N)}(v_1 \otimes \ldots \otimes v_N) = &(x_1 v_1) \otimes v_2 \otimes \ldots \otimes v_N + \\ & v_1 \otimes (x_2 v_2) \otimes \ldots \otimes v_N + \ldots + v_1 \otimes v_2 \otimes \ldots \otimes (x_N v_N). \end{aligned} \tag{3.2}$$

Let us denote this representation of the Lie algebra $\mathfrak{b}_{(N)}$ in the linear space V by $\tau_{\lambda,\gamma}$.

Set $\overline{\mathfrak{b}}_0 = \bigoplus_{p=1}^{N} \mathfrak{b} \otimes A_{0,p} \subseteq \overline{\mathfrak{g}}_0$. Clearly, $\widehat{\mathfrak{b}} := \overline{\mathfrak{b}}_0 \oplus Z \oplus \widehat{\mathfrak{g}}_+$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Take λ and γ (arbitrary) N-tuples as defined above and choose $\delta \in \mathbb{C}$. Our next goal is to assign to each triple $(\lambda, \gamma, \delta)$ a one-dimensional representation of the Lie algebra $\widehat{\mathfrak{b}}$. First we define a linear map

$$\varphi : \widehat{\mathfrak{b}} \to \mathfrak{b}_{(N)}, \quad \text{with } \varphi_{|Z} = 0, \quad \varphi_{|\widehat{\mathfrak{g}}_+} = 0 \quad \text{and,}$$
$$x = \bigoplus_{p=1}^{N} x_p \otimes A_{0,p} \ \in \overline{\mathfrak{b}}_0 \quad \mapsto \varphi_1^{-1}(x_1) \oplus \ldots \oplus \varphi_N^{-1}(x_N)$$

Lemma 3.2. *The map φ is a Lie homomorphism.*

Proof. As special cases of the relations in Proposition 2.4 we obtain

$$A_{0,p}A_{0,p} = A_{0,p} + B_p, \qquad B_p \in \mathcal{A}_+ \tag{3.3 }$$
$$A_{0,p}A_{0,q} = B_{p,q} \qquad B_{p,q} \in \mathcal{A}_+, \quad q \neq p\ . \tag{3.4}$$

The cocycle for defining the central extension vanishes on $\mathcal{A}_0 \oplus \mathcal{A}_+$. This implies the claim. □

Let us note that the map can be extended to $\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+ \to \mathfrak{g}_{(N)}$ by the same definition.

On the one-dimensional space V we define the representation $\tau_{\lambda,\gamma,\delta}$ of $\widehat{\mathfrak{b}}$ by setting

$$\tau_{\lambda,\gamma,\delta}(x_0 \oplus x_+ \oplus t) := \tau_{\lambda,\gamma}(\varphi(x_0)) + \delta \cdot Id\ ,$$

with respect to the decomposition of $\widehat{\mathfrak{b}}$.

Definition 3.3. The linear space

$$\widehat{V}_{\lambda,\gamma,\delta} := U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} V \tag{3.5}$$

with its natural structure of a $\widehat{\mathfrak{g}}$-module is called the Verma module of the Lie algebra $\widehat{\mathfrak{g}}$ corresponding to the data $(\lambda, \gamma, \delta)$. As usual $U(\cdot)$ denotes the universal enveloping algebra of the corresponding Lie algebra. $U(\widehat{\mathfrak{b}})$ operates on V via the representation $\tau_{\lambda,\gamma,\delta}$. The N-tuple λ is called the *weight* of the Verma module, the elements $\gamma_p \in \gamma$ $(p = 1, \ldots, N)$ are called the *twists* and the complex number δ is called the *level* of the Verma module. The vector $v_{\lambda,\gamma,\delta} = v_1 \otimes \cdots \otimes v_N$ is called the *highest weight vector* of the module $V_{\lambda,\gamma,\delta}$.

Proposition 3.4. *The $\widehat{\mathfrak{g}}$-module $\widehat{V}_{\lambda,\gamma,\delta}$ is a $\mathfrak{g}$-module and contains the module*

$$V_{\lambda,\gamma} := V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_N} \ ,$$

where for each $p = 1, \dots, N$, V_{λ_p} is the highest weight module of $\mathfrak{g}_p$ of weight λ_p and $\mathfrak{g}$ operates on V_{λ_p} when twisted by the automorphism $Ad\,\gamma_p$.

Proof. Recall that $\mathfrak{g}$ is embedded as a subalgebra into $\widehat{\mathfrak{g}}$ via $x \mapsto x \otimes 1 = \sum_{p=1}^N x \otimes A_{0,p}$. Hence, the module $\widehat{V}_{\lambda,\gamma,\delta}$ is also a $\mathfrak{g}$-module. The $\mathfrak{g}$ submodule generated from the highest weight vector $v_{\lambda,\gamma,\delta}$ is the module $V_{\lambda,\gamma}$. From the definition of the $\mathfrak{g}$-action on the $V_{\lambda,\gamma,\delta}$ follows that the action on $V_{\lambda,\gamma}$ for decomposable tensors $w = w_1 \otimes w_2 \cdots \otimes w_n$ is given as

$$g.w := (\varphi_1(g).w_1) \otimes w_2 \cdots \otimes w_N + \cdots + w_1 \otimes \cdots \otimes (\varphi_N(g).w_N) \ . \quad \square$$

We have inside the representation space further important subspaces. Recall that $\widehat{\mathfrak{g}}'_+$ is a Lie subalgebra (see Lemma 3.1) of $\widehat{\mathfrak{g}}$. The subspace of $\widehat{V}_{\lambda,\gamma,\delta}$ generated as $\widehat{\mathfrak{g}}'_+$-submodule from $v_{\lambda,\gamma,\delta}$ is called the *subspace of degree zero*. It is denoted by $\widehat{V}_{(\lambda,\gamma,\delta),0}$. Clearly, $V_{\lambda,\gamma} \subseteq \widehat{V}_{(\lambda,\gamma,\delta),0}$ Note also that $\widehat{\mathfrak{g}}_+$ annihilates $\widehat{V}_{(\lambda,\gamma,\delta),0}$.

$\widehat{\mathfrak{g}}$ contains also the subalgebra $\overline{\mathfrak{g}}^*_- \cong \widehat{\mathfrak{g}}^*_-$. Note that this subalgebra was defined in Section 2 using $x \otimes f$ with $f \in \mathcal{A}$ and f regular at P_∞. The same is true for $\overline{\mathfrak{g}}'_- \cong \widehat{\mathfrak{g}}'_-$ where we require that the functions have zeros at P_∞. It is possible to define *conformal blocks* as the space of coinvariants

$$\widehat{V}_{\lambda,\gamma,\delta} / \overline{\mathfrak{g}}^*_- \widehat{V}_{\lambda,\gamma,\delta} \ . \tag{3.6}$$

We will postpone the discussion of the conformal blocks, their structure, etc. to the forthcoming part II of this article. Note that it is also possible to replace in (3.6) $\overline{\mathfrak{g}}^*_-$ by $\overline{\mathfrak{g}}'_-$ and a bigger space will be obtained.

Let us now consider more general modules over $\widehat{\mathfrak{g}}$.

Definition 3.5. A module $\widehat{V}$ over the Lie algebra $\widehat{\mathfrak{g}}$ (resp. a representation of $\widehat{\mathfrak{g}}$) is called an *admissible module* (resp. an *admissible representation*) if the central element t operates as $c \cdot Id$ with $c \in \mathbb{C}$ and if for every $v \in \widehat{V}$ and for all $x \in \mathfrak{g}$ one has $x(n)v = 0$ for $n \gg 0$.

It is evident that each Verma module (3.5) is an admissible module.

For each admissible representation of the affine Krichever-Novikov algebra $\widehat{\mathfrak{g}}$ the (affine) Sugawara construction yields a representation of the centrally extended Krichever-Novikov vector field algebra $\widehat{\mathcal{L}}$ (the Virasoro type algebra). This representation is called *Sugawara representation.* The abelian version for the two-point case was introduced in [17]. The nonabelian case was later considered in [4], [30]. In [30] also the multi-point version was given.

Let $\widehat{V}$ be a fixed admissible module. For $u\otimes A_{n,p}$ with $A_{n,p}$ being a basis element of the algebra $\widehat{\mathcal{A}}$ and $u\in\mathfrak{g}$ we will denote the corresponding operator in $\widehat{V}$ by $u(n,p)$ as well as by $u(A_{n,p})$.

Recall that we assume $\mathfrak{g}$ to be a finite dimensional simple Lie algebra. We choose a basis u_i, $i=1,\dots,\dim\mathfrak{g}$ of $\mathfrak{g}$ and the corresponding dual basis u^i, $i=1,\dots,\dim\mathfrak{g}$ with respect to the invariant non-degenerate symmetric bilinear form $(..|..)$. The Casimir element $\Omega^0=\sum_{i=1}^{\dim\mathfrak{g}}u_iu^i$ of the universal enveloping algebra $U(\mathfrak{g})$ is independent of the choice of the basis. In an abuse of notation we denote $\sum_i u_i(n,p)u^i(m,q)$ simply by $u(n,p)u(m,q)$.

We define the higher genus *Sugawara operator* (also called *Segal operator* or *energy-momentum tensor operator*) as

$$T(Q):=\frac{1}{2}\sum_{n,m}\sum_{p,s}:u(n,p)u(m,s):\,\omega^{n,p}(Q)\omega^{m,s}(Q)\ . \tag{3.7}$$

By $:....:$ we denote some normal ordering. The summation here and in the following formulas for the first indices n,m are over $\mathbb{Z}$ and for the second indices p,s over $\{1,\dots,N\}$. The precise form of the normal ordering is not of importance. As an example we may take the following "standard normal ordering" $(x,y\in\mathfrak{g})$

$$:x(n,p)y(m,r):\ :=\begin{cases} x(n,p)y(m,r), & n\le m\\ y(m,r)x(n,p), & n>m\ .\end{cases} \tag{3.8}$$

The expression $T(Q)$ can be considered as formal series of quadratic differentials in the variable Q with operator-valued coefficients. Expanding it over the basis $\Omega^{k,r}$ of the quadratic differentials we obtain

$$T(Q)=\sum_k\sum_r L_{k,r}\cdot\Omega^{k,r}(Q)\ , \tag{3.9}$$

with

$$\begin{gathered} L_{k,r}=\frac{1}{2\pi\mathrm{i}}\int_{C_\tau}T(Q)e_{k,r}(Q)=\frac{1}{2}\sum_{n,m}\sum_{p,s}:u(n,p)u(m,s):\,l_{(k,r)}^{(n,p)(m,s)},\\ \text{where}\qquad l_{(k,r)}^{(n,p)(m,s)}:=\frac{1}{2\pi\mathrm{i}}\int_{C_\tau}\omega^{n,p}(Q)\omega^{m,s}(Q)e_{k,r}(Q)\ . \end{gathered} \tag{3.10}$$

The following theorem was proved in [30].

Theorem 3.6. *Let $\mathfrak{g}$ be a finite dimensional either abelian or simple Lie algebra and $2\mathsf{k}$ be the eigenvalue of its Casimir operator in the adjoint representation. Let V be an admissible representation where the central element t operates as $\mathsf{c}\cdot$identity. If $\mathsf{c}+\mathsf{k}\neq 0$ then the rescaled modes*

$$L^*_{k,r} = \frac{-1}{2(\mathsf{c}+\mathsf{k})}\sum_{n,m}\sum_{p,s} :u(n,p)u(m,s):\, l_{(k,r)}^{(n,p)(m,s)} \ , \tag{3.11}$$

of the Sugawara operator are well-defined operators on V and define a representation of the centrally extended vector field algebra $\widehat{\mathcal{L}}$ (the Virasoro-type algebra).

We call the $L^*_{k,r}$, resp. the $L_{k,r}$ the Sugawara operators too. The representation obtained in this way is the Sugawara representation of the Lie algebra $\widehat{\mathcal{L}}$ corresponding to the given admissible representation $\widehat{V}$ of $\widehat{\mathfrak{g}}$.

4. Moduli of curves with marked points and the general form of the KZ equation

(a) Moduli spaces.

Let $\mathcal{M}_{g,N}$, resp. $\mathcal{M}_{g,N+1}$ be the moduli space of smooth, projective curves of genus g (over $\mathbb{C}$) with N (resp. $N+1$) marked ordered distinct points. Equivalently, it can be described as moduli space of compact Riemann surfaces with marked points. A point in $\mathcal{M}_{g,N+1}$ is given by the equivalence classes of the data $(M, P_1, P_2, \dots, P_N, P_\infty)$ with M a smooth, projective curve and $P_i \in M$, $i = 1, \dots, N, \infty$. Two such tuples are identified if they are isomorphic (under algebraic maps) as curves with marked points. Let us denote the equivalence class by [..]. To avoid special considerations we will mainly assume in this section $g \geq 2$. In this article we are only dealing with the local situation at a generic curve M with generic markings $(P_1, P_2, \dots, P_N, P_\infty)$[3]. Hence, it is enough to consider a small open subset $\widetilde{W}$ around the point $\tilde{b} = [(M, P_1, P_2, \dots, P_N, P_\infty)]$. A generic curve of $g \geq 2$ does not admit nontrivial infinitesimal automorphisms and we may assume that there exists over $\widetilde{W}$ a universal family of curves with marked points. In particular, there is a proper, flat family of smooth curves over $\widetilde{W}$

$$\mathcal{U} \to \widetilde{W} \ , \tag{4.1}$$

such that for the points $\tilde{b} = [(M, P_1, P_2, \dots, P_N, P_\infty)] \in \widetilde{W}$ we have $\pi^{-1}(\tilde{b}) = M$ and that the sections defined as

$$\sigma_i : \widetilde{W} \to \mathcal{U}, \quad \sigma_i(\tilde{b}) = P_i, \quad i = 1, \dots, N, \infty \tag{4.2}$$

[3]Sometimes one understands by a marking of a curve the choice of a symplectic basis for the homology. But in this article a marking refers always to the choice of (ordered) points on the curve.

are holomorphic. By "forgetting the marking" we obtain a (local) map $\tilde{\nu} : \widetilde{W} \subseteq \mathcal{M}_{g,N+1} \to \mathcal{M}_{g,0}$. Note that the family (4.1) is the pullback of the universal family over $\mathcal{M}_{g,0}$ under $\tilde{\nu}$. For more background information, see [39, Sect. 1.2, Sect. 1.3], in particular see Thm. 1.2.9 of [39].

Let us fix a section $\widehat{\sigma}_\infty$ of the universal family of isomorphy classes of curves (without marking). In particular, for every curve there is a point chosen in a manner depending smoothly on the moduli. (Recall, we are only dealing with the local and generic situation.) The analytic subset

$$W' := \{\tilde{b} = [(M, P_1, P_2, \dots, P_N, P_\infty)] \mid P_\infty = \widehat{\sigma}_\infty([M])\} \quad \subseteq \quad \widetilde{W} \tag{4.3}$$

can be identified with an open subset W of $\mathcal{M}_{g,N}$ via

$$\tilde{b} = [(M, P_1, P_2, \dots, P_N, \widehat{\sigma}_\infty([M]))] \ \to \ b = [(M, P_1, P_2, \dots, P_N)] \ . \tag{4.4}$$

(By the genericity the map is 1:1.)

The dimensions of the moduli spaces are well-known (see also further down in this section)

$$\dim_b(\mathcal{M}_{g,N}) = \begin{cases} 3g - 3 + N, & g \geq 2 \\ \max(1, N), & g = 1 \\ \max(0, N-3), & g = 0 \ . \end{cases} \tag{4.5}$$

Note that for $N \geq 3 - 2g$ the first expression is valid for every genus.

Over the points $b \in W$ we can apply the construction of Section 2 and Section 3 and obtain objects over the subset W of $\mathcal{M}_{g,N}$

$$\mathcal{A}_b, \ \widehat{\mathcal{A}}_b, \ \mathcal{L}_b, \ \widehat{\mathcal{L}}_b, \ \widehat{\mathfrak{g}}_b, \ \mathcal{F}_b^\lambda \ , \tag{4.6}$$

depending on the points $b \in W \subseteq \mathcal{M}_{g,N}$. We need a sheaf description of some of the objects. Consider the universal family $\pi : \mathcal{U} \to W$. First let us introduce the notation $S_b = \sum_{i=1}^N P_i$ for the divisor on M corresponding to the moduli point $b = [(M, P_1, P_2, \dots, P_N)]$. Varying b this defines the divisor of sections $S = \sum_{i=1}^N \sigma_i(W)$ in the family $\mathcal{U}$. We have to enlarge the divisors by adding the reference point $P_\infty = \widehat{\sigma}_\infty([M])$, resp. the section $\widehat{\sigma}_\infty$. Denote by $\nu : W \subseteq \mathcal{M}_{g,N} \to \mathcal{M}_{g,0}$ the map obtained by forgetting the marking. We set

$$\widetilde{S}_b = \sum_{i=1}^N P_i + \widehat{\sigma}_\infty(M), \quad \widetilde{S} = \sum_{i=1}^N \sigma_i(W) + \widehat{\sigma}_\infty(\nu(W)) \ . \tag{4.7}$$

Denote by $\mathcal{O}_\mathcal{U}$ the sheaf of regular functions on $\mathcal{U}$. As usual set $\mathcal{O}_\mathcal{U}(k\widetilde{S})$, $k \in \mathbb{Z}$ the sheaf of functions which have zeros of order at least $-k$ along the divisor $\widetilde{S}$. In particular, for $k \in \mathbb{N}$ this says that the functions have poles of order at most k at the divisor $\widetilde{S}$. By $\mathcal{O}_{\mathcal{U}}(*\widetilde{S})$ we understand the sheaf of functions which have poles along the divisor $\widetilde{S}$. Also we set

$$\pi_*\mathcal{O}_{\mathcal{U}}(*\widetilde{S}) := \lim_{k\to\infty} \pi_*(\mathcal{O}_{\mathcal{U}}(k\widetilde{S})) \ . \tag{4.8}$$

It is a locally free $\mathcal{O}_W$-sheaf (with $\mathcal{O}_W$ the structure sheaf of the space W). Its vector space fibre over $b \in W$ can be identified with

$$\mathrm{H}^0(M_b, \mathcal{O}_{M_b}(*\widetilde{S})) = \mathcal{A}_b \ , \quad M_b = \pi^{-1}(b) \ .$$

The sheaf $\pi_*\mathcal{O}_{\mathcal{U}}(*\widetilde{S})$ can be made to a sheaf of commutative associative $\mathcal{O}_W$-algebras by fibre-wise multiplication. We denote it $\mathcal{A}_W$. As a sheaf of abelian Lie algebras it can be centrally extended to the $\mathcal{O}_W$-sheaf

$$\widehat{\mathcal{A}}_W := \pi_*\widehat{\mathcal{O}_{\mathcal{U}}(*\widetilde{S})} := \pi_*\mathcal{O}_{\mathcal{U}}(*\widetilde{S}) \oplus \mathcal{O}_W \cdot t,$$

where t is the central element and the structure is defined as follows. The elements $f, g \in \mathcal{A}_W(U)$ can be represented as functions on $\pi^{-1}(U)$ with poles only along $\widetilde{S}$. Let γ be the cocycle (2.14). Recall that the latter can be given by calculating residues along S. Then with $r, s \in \mathcal{O}_W(U)$

$$[f + r \cdot t, g + s \cdot t] := \gamma(f, g) \cdot t,$$

defines an element of $\widehat{\mathcal{A}}_W(U)$. Note that $\gamma(f, g) \in \mathcal{O}_W(U)$.

This construction can be extended to the affine algebra situation.

Definition 4.1. Given a finite-dimensional Lie algebra $\mathfrak{g}$ the *sheaf of the associated loop algebra* (or current algebra) $\overline{\mathfrak{g}}_W$ and the *sheaf of the associated affine algebra* $\widehat{\mathfrak{g}}_W$ are defined as

$$\overline{\mathfrak{g}}_W := \mathcal{A}_W \otimes \mathfrak{g}, \qquad \widehat{\mathfrak{g}}_W := \overline{\mathfrak{g}}_W \oplus \mathcal{O}_W \cdot t \ , \tag{4.9}$$

where the Lie structure is given by the naturally extended form of (2.17) (resp. without its central term for $\overline{\mathfrak{g}}_W$).

Clearly, these are $\mathcal{O}_W$-sheaves. Let $b \in W$ and let $\mathcal{O}_{W,b}$ be the local ring at b and M_b its maximal ideal. Set $\mathbb{C}_b \cong \mathcal{O}_{W,b}/M_b$, then we obtain the following canonical isomorphisms

$$\mathbb{C}_b \otimes \mathcal{A}_W \cong \mathcal{A}_b, \qquad \mathbb{C}_b \otimes \widehat{\mathcal{A}}_W \cong \widehat{\mathcal{A}}_b, \qquad \mathbb{C}_b \otimes \overline{\mathfrak{g}}_W \cong \overline{\mathfrak{g}}_b, \qquad \mathbb{C}_b \otimes \widehat{\mathfrak{g}}_W \cong \widehat{\mathfrak{g}}_b \ .$$

Definition 4.2. A sheaf $\mathfrak{V}$ of $\mathcal{O}_W$-modules is called a *sheaf of representations* for the affine algebra $\widehat{\mathfrak{g}}_W$ if the $\mathfrak{V}(U)$ are modules over $\widehat{\mathfrak{g}}_W(U)$.

For a sheaf of representations $\mathfrak{V}$ we obtain that $\mathfrak{V}_b$ is a module over $\widehat{\mathfrak{g}}_b$ for every point in W.

We have to increase the moduli spaces by considering also first order infinitesimal neighbourhoods around the points $P_i, i = 1, \ldots, N, \infty$. We denote this moduli space by $\mathcal{M}^{(1)}_{g,N}$, resp. $\mathcal{M}^{(1)}_{g,N+1}$. The elements of $\mathcal{M}^{(1)}_{g,N+1}$ are given as

$$\tilde{b}^{(1)} = [(M, P_1, P_2, \ldots, P_N, P_\infty, z_1, \ldots, z_N, z_\infty)] \ ,$$

where for $i = 1, \ldots, N, \infty$ the additional data z_i is a coordinate at P_i with $z_i(P_i) = 0$. Two such tuples $\tilde{b}^{(1)}$ and $\tilde{b}^{(1)'}$ are identified if they are equivalent as truncated elements in $\mathcal{M}_{g,N+1}$ and (after this identification) we have

$$z_i' = z_i + O(z_i^2), \qquad i = 1, \ldots, N, \infty \ .$$

The additional degrees of freedom in the moduli space at a point $\tilde{b}$ is given by multiplying fixed coordinates in $\tilde{b}$ at the P_i's by non-zero constants. Denote the corresponding space lying above $\widetilde{W}$ by $\widetilde{W}^{(1)}$.

Again, after fixing a first order infinitesimal neighbourhood around the section $\widehat{\sigma}_\infty$, we can identify the subspace defined similarly to (4.3) with an open subspace of $\mathcal{M}^{(1)}_{g,N}$ containing a neighbourhood of the point at which we make our consideration. Denote this subspace by $W^{(1)}$. Clearly we have N degrees of additional freedom and hence it follows from (4.5)

$$\dim_{b^{(1)}}(\mathcal{M}^{(1)}_{g,N}) = \begin{cases} 3g - 3 + 2N, & g \geq 2 \\ 2N, & g = 1 \\ \max(0, 2N-3), & g = 0 \ . \end{cases} \tag{4.10}$$

The map $\eta : W^{(1)} \to W$ obtained by forgetting the coordinates is a surjective analytic map. Hence by pulling back via η the objects (4.6) and the sheaves $\mathcal{A}_W, \widehat{\mathcal{A}}_W, \overline{\mathfrak{g}}_W, \widehat{\mathfrak{g}}_W$ we obtain sheaves $\mathcal{A}_{W^{(1)}}, \widehat{\mathcal{A}}_{W^{(1)}}, \overline{\mathfrak{g}}_{W^{(1)}}, \widehat{\mathfrak{g}}_{W^{(1)}}$ over $W^{(1)}$. Moreover, over $W^{(1)}$ also the basis elements $f^\lambda_{n,p}$ are well-defined. This was the reason for enlarging the moduli space. Recall that for fixing the basis elements a choice of coordinates z_i around the points P_i were necessary. But note also that only the class of z_i under the equivalence in $\mathcal{M}^{(1)}_{g,N}$ is of importance. Due to the explicit description [24] they depend analytically on the moduli. Pulling back a sheaf of representation $\mathfrak{V}$ over W we obtain a sheaf of representation $\mathfrak{V}^{(1)} = \eta^*\mathfrak{V}$ of $\widehat{\mathfrak{g}}_{W^{(1)}}$. More generally, we can define sheaves of representations over

$W^{(1)}$ directly. In particular for these sheaves of representations operators depending on the KN basis are well-defined.

We want to study how a different choice of coordinates can be expressed on $\mathcal{M}_{g,N}$. Take $b \in \mathcal{M}_{g,N}$ and choose coordinates z_i at the points P_i. Only the coordinate classes are of importance. Hence we can express this as choosing a lift $W \to W^{(1)}$ of η. From the construction of the KN basis elements in Section 2 the following lemma is immediate.

Lemma 4.3. *Let $z'_p = \alpha_p \cdot z_p + O(z_p^2)$ be another coordinate at P_p. Let $f^\lambda_{n,p}$ ($f^{\lambda\prime}_{n,p}$) be a KN basis element of $\mathcal{F}^\lambda$ w.r.t. z_p (w.r.t. z'_p) then*

$$f^{\lambda\prime}_{n,p} = (\alpha_p)^n f^\lambda_{n,p}, \quad \text{and } f^{\lambda\prime}_{n,s} = f^\lambda_{n,s}, \quad s \neq p\ . \tag{4.11}$$

Note that the α_p are nonvanishing local functions on $\mathcal{M}_{g,N}$. This behaviour has some important consequences.

(1) The grading of $\widehat{\mathfrak{g}}$ is given with respect to the basis elements $A_{n,p}$ in $\mathcal{A}$. From (4.11) it follows that the degree is not changed by passing from one system of coordinates to another. Clearly, this globalizes over W. Hence we can equip $\mathcal{A}_W, \widehat{\mathcal{A}}_W, \overline{\mathfrak{g}}_W$ and $\widehat{\mathfrak{g}}_W$ with an almost grading. This allows to define a sheaf of admissible representations to be a sheaf of representations where all representations are admissible. In addition we will usually require (if nothing else is said) that the central element t operates as $c \cdot id$ with c a function on W. This function is called the level function. Very often we will even assume c to be a constant c, which is just called the level of the representation.

(2) Due to the possibility to globalize the grading to $\widehat{\mathfrak{g}}_W$ it is possible to define the *Verma module sheaf* in a straightforward manner extending Definition 3.3

$$\widehat{V}_{(\lambda,\gamma,\delta),W} := U(\widehat{\mathfrak{g}}_W) \otimes_{U(\widehat{\mathfrak{b}}_W)} V \tag{4.12}$$

On first sight it looks as if the Verma module sheaf is only defined over $W^{(1)}$ because the basis elements $A_{0,p}$ are involved. But from (4.11) it follows that they are indeed independent of the coordinate classes. The Verma module sheaves are sheaves of admissible representations. Note, it is even possible to vary the data $(\lambda, \gamma, \delta)$ over the moduli.

The subspace of degree zero defines a subsheaf. The same is true for the subspace which is annihilated by $\widehat{\mathfrak{g}}_+$ and for the subspace $\overline{\mathfrak{g}}^*_- \widehat{V}_{\lambda,\gamma,\delta}$, see (3.6). One possible way to define the sheaf of conformal blocks is to define the quotient sheaf with respect to the latter subsheaf. The discussion of these objects is postponed to the forthcoming part II of this article.

(3) For every sheaf of admissible representations over $W^{(1)}$ the Sugawara operators are well-defined. Note that the individual operators $u(n,p)$ (see Section 3) depend on

the coordinates. Let us consider a sheaf of admissible representations over W and choose coordinates. From (4.11) we know how the operators transform if we choose a different coordinate $z_p' = \alpha_p z_p + O(z_p^2)$. We obtain $(u(n,p))' = (\alpha_p)^n u(n,p)$. The factor will be cancelled by the contribution from $\omega^{n,p\prime} = (\alpha_p)^{-n}\omega^{n,p}$. Hence it follows from (3.7) for the Sugawara operators that $T(Q)' = T(Q)$. The individual operators $L_{k,s}$ depend indeed on the coordinates. They are only well-defined over $W^{(1)}$. They transform as the vector fields $e_{k,s}$ do, as can be seen from (3.9), (3.10). If we assume the level function c to obey the condition $(c+\mathsf{k}) \neq 0$ (see Theorem 3.6) and if we assign to the vector field $l \in L$ the operator

$$T[l] := \frac{-1}{\mathsf{c}+\mathsf{k}} \cdot \frac{1}{2\pi\,\mathrm{i}} \int_{C_\tau} T(Q) l(Q) \tag{4.13}$$

we see that this operator does not depend on the coordinates. Clearly this is the Sugawara representation. We get $T[e_{n,p}] = L^*_{n,p}$, see (3.10).

Let us now consider the tangent space at the moduli spaces. The Kodaira-Spencer map for a versal family of complex analytic manifolds $Y \to B$ over the base B at the base point $b \in B$,

$$T_b(B) \to \mathrm{H}^1(Y_b, T_{Y_b}) \tag{4.14}$$

is an isomorphism (e.g. [15]). Here $T_b(B)$ denotes the tangent space of B at the point b, Y_b is the fibre over b and T_{Y_b} the (holomorphic) tangent sheaf of Y_b. We are in the local generic situation where we have a universal family. Hence we can employ (4.14). Let M be the curve fixed by b, resp $b^{(1)}$. We obtain

$$T_{[M]}(\mathcal{M}_{g,0}) \cong \mathrm{H}^1(M, T_M)\ .$$

More generally, (see also [39], [38])

$$T_b(\mathcal{M}_{g,N}) \cong \mathrm{H}^1(M, T_M(-S_b)), \qquad T_{b^{(1)}}(\mathcal{M}_{g,N}^{(1)}) \cong \mathrm{H}^1(M, T_M(-2S_b))\ . \tag{4.15}$$

The first order vanishing condition at the points $P_1, \ldots, P_N$ comes from the fact that the vector fields which do not generate a non-trivial complex deformation of the curve should also not move the points to be a trivial deformation of the marked curve. The second order vanishing condition corresponds to the fact that it should additionally not change the first order infinitesimal neighbourhood. Via Serre duality [10], [22] we have

$$\begin{gathered} \mathrm{H}^1(M, T_M) \cong \mathrm{H}^0(M, \mathcal{K}^2), \qquad \mathrm{H}^1(M, T_M(-S_b)) \cong \mathrm{H}^0(M, \mathcal{K}^2(S_b)), \\ \mathrm{H}^1(M, T_M(-2S_b)) \cong \mathrm{H}^0(M, \mathcal{K}^2(2S_b))\ . \end{gathered} \tag{4.16}$$

Note that for the Kodaira-Spencer mapping there exists also a sheaf version [39, Cor. 1.2.5]

$$T_W \cong R^1\pi_* T_{\mathcal{U}/W}(-S)\ . \tag{4.17}$$

(b) Krichever-Novikov algebras and tangent vectors of the moduli spaces.

We want to show that the elements of the cohomology groups in (4.15) can be identified with elements of the critical strip of the Krichever-Novikov vector field algebra. Hence the latter can be identified with tangent vectors to the moduli spaces.

Let M be the Riemann surface we are dealing with and let U_∞ be a coordinate disc around P_∞, such that $P_1,\ldots,P_N \notin U_\infty$. Let $U_1 = M \setminus \{P_\infty\}$. Because U_1 and U_∞ are affine (resp. Stein) [10,p.297] we get $\mathrm{H}^1(U_j,F)=0$, $j=1,\infty$ for every coherent sheaf F. Hence, the sheaf cohomology can be given as Cech cohomology with respect to the covering $\{U_1,U_\infty\}$. Set $U_\infty^* = U_1\cap U_\infty = U_\infty\setminus\{P_\infty\}$. The 2-cocycles can be given by $s_{1,\infty}\in F(U_\infty^*)$ $(s_{0,0}=s_{\infty,\infty}=0)$, hence by arbitrary sections over the punctured coordinate disc U_∞^*

Coming back to the holomorphic tangent bundle T_M. For any element f of the KN vector field algebra its restriction to U_∞^* is holomorphic and defines an element of $\mathrm{H}^1(M,T_M)$. Note that it defines also an element of $\mathrm{H}^1(M,T_M(D))$, where D is any divisor supported outside of U_∞^*. We introduce the map

$$\theta_D:\mathcal{L}\ \to\ \mathrm{H}^1(M,T_M(D)),\qquad f\mapsto\theta_D(f):=[f_{|U_\infty^*}]\ . \tag{4.18}$$

If the divisor D is clear from the context we will suppress it in the notation. For us only the divisors $D=-kS_b$ with $k\in\mathbb{N}_0$ are of importance.

Recall the discussion in Section 2 on the reduced critical strip. It can be decomposed as

$$\mathcal{L}'_{(0)}=\langle e_{0,p},p=1,\ldots,N\rangle\oplus\langle e_{-1,p},p=1,\ldots,N\rangle\oplus\mathcal{L}^*_{(0)}\oplus\mathcal{L}^\infty_{(0)}\ . \tag{4.19}$$

Here $\mathcal{L}^*_{(0)}$ is the subspace generated by the basis elements (after the explained change of basis elements) with poles at the P_i, $i=1,\ldots,N,\infty$ and $\mathcal{L}^\infty_{(0)}$ is the two-dimensional space generated by the basis vector fields of exact order one resp. order zero at P_∞. Note that (assuming $g\ge 2$)

$$\dim\mathcal{L}^*_{(0)}=3g-3\ .$$

Recall that $\mathcal{L}^*_-$ was introduced as the algebra of all vector fields (in $\mathcal{L}$) regular at P_∞.

Proposition 4.4. *Set $S_b=\sum_{i=1}^N P_i$. The map*

$$\theta_{-kS_b}:\mathcal{L}\ \to\ \mathrm{H}^1(M,T_M(-kS_b),\qquad k\ge 0 \tag{4.20}$$

is a surjective map. It gives an isomorphism

$$\mathcal{L}_{k-2}\oplus\mathcal{L}_{k-3}\cdots\oplus\mathcal{L}_{-1}\oplus\mathcal{L}^*_{(0)}\ \cong\ \mathrm{H}^1(M,T_M(-kS))\ . \tag{4.21}$$

The kernel of the map is given as

$$\ker\theta_{-kS_b} = \mathcal{L}_-^* \oplus \bigoplus_{n\ge k-1} \mathcal{L}_n \ . \tag{4.22}$$

Proof. The map θ_{-kS_b} is linear. Using Serre duality, we calculate $\dim \mathrm{H}^1(M, T_M(-kS_b)) = \dim \mathrm{H}^0(M, K^2(kS_b))$. But $\deg(K^2(kS_b)) = 2(2g-2) + kN \ge 2g-1$ for $g \ge 2$. By Riemann-Roch (in the non-special region) [22] we obtain

$$\dim \mathrm{H}^1(M, T_M(-kS_b)) = \dim \mathrm{H}^0(M, K^2(kS_b)) = 3(g-1) + kN \ .$$

This coincides with the dimensions of the spaces on the l.h.s. of (4.21). First we prove (4.22). Assume $\theta_{-kS_b}(f) = 0$. Hence, we can write

$$f_{|U_\infty^*} = f_{1|U_\infty^*} - f_{0|U_\infty^*}$$

with f_0 defined and regular on U_∞ and f_1 defined outside of P_∞ and with order at least k at every point $P_i, i = 1, \dots, N$. Because f_0 is regular at P_∞ the vector field f_1 has to have the same singular part as f at P_∞. In particular it can be extended to a global meromorphic vector field. This implies $f_1 \in \mathcal{L}$. From $\mathrm{ord}_{P_i}(f_1) \ge k$, $i = 1, \dots, N$ it follows $f_1 \in \bigoplus_{n\ge k-1} \mathcal{L}_n$. Now $f_{0|U_\infty^*} = (f_1 - f)_{|U_\infty^*}$. But $f_1 - f$ is a globally defined meromorphic vector field with poles at most at $\{P_1, \dots, P_N, P_\infty\}$. Hence the same is true for the extension of f_0 (which we denote by the same symbol). In particular $f_0 \in \mathcal{L}$. Due to the regularity at P_∞ we get $f \in \mathcal{L}_-^*$. This shows $\subseteq$.
For $f \in \bigoplus_{n\ge k-1} \mathcal{L}_n$ we set $f_1 = f_{|U_1}$, $f_0 = 0$, and for $f \in \mathcal{L}_-^*$ we set $f_1 = 0$, $f_0 = -f_{|U_\infty}$. We see that their cohomology classes will vanish. Hence, (4.22)
From (4.22) it follows that θ_{-kS_b} is injective if restricted to the complementary space. From the equality of the dimension follows surjectivity and (4.21). $\square$

Note that the spaces $\ker\theta_{-kS_b}$ are invariantly defined. The following (linear) isomorphism are of special interest for us

$$\begin{gathered}\mathrm{H}^1(M, T_M) \cong \mathcal{L}_{(0)}^*, \qquad \mathrm{H}^1(M, T_M(-S_b)) \cong \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^*, \\ \mathrm{H}^1(M, T_M(-2S_b)) \cong \mathcal{L}_0 \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^* \ .\end{gathered} \tag{4.23}$$

Using the Kodaira-Spencer map (4.15) we obtain

Theorem 4.5. *The tangent spaces of the moduli spaces $\mathcal{M}_{g,0}$, $\mathcal{M}_{g,N}$ and $\mathcal{M}_{g,N}^{(1)}$ at the points which correspond to the curve M with marked points $I = (P_1, P_2, \dots, P_N)$ and $O = (P_\infty = \widehat{\sigma}_\infty(M))$ can be identified with the following subspaces of the critical strip of the Krichever-Novikov vector field algebra assigned to this marked curve:*

$$T_{[M]}\mathcal{M}_{g,0} \cong \mathcal{L}_{(0)}^*, \qquad T_b\mathcal{M}_{g,N} \cong \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^*, \qquad T_{b^{(1)}}\mathcal{M}_{g,N}^{(1)} \cong \mathcal{L}_0 \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_{(0)}^* \ , \tag{4.24}$$

where $b=[(M,P_1,P_2,\dots,P_N)]$ *and* $b^{(1)}=[(M,P_1,P_2,\dots,P_N,z_1,\dots,z_N)]$.

Because the vector fields in $\ker\theta\subset\mathcal{L}$ are not corresponding to deformations in the moduli we sometimes call this vector fields *vertical vector fields*. They should not be confused with the sections of the relative tangent sheaf of the universal family. The following should also be kept in mind. As already remarked, $\ker\theta$ is invariantly defined. But the definition of the critical strip, hence of the complementary subspace to $\ker\theta$, is only fixed by the order prescription for the basis. A different prescription (which involves changing the required orders) will yield a different identification of tangent vectors on the moduli space with vector fields in the fibre.

This was a description of the connection between the Krichever-Novikov basis elements and the infinitesimal moduli parameter using Cech cohomology. For another (but nevertheless equivalent) approach in the $N=1$ case, see [9].

Let us close this subsection in discussing the necessary modifications for genus 0 and 1. In these cases there are global holomorphic vector fields. Hence the decomposition of the critical strip (4.19) and its identification (4.23) are not valid anymore. The space $\mathcal{L}^*_{(0)}\oplus\mathcal{L}^\infty_{(0)}$ (resp. for $g=1$ a part of it) already appears as subspace of $\mathcal{L}_0\oplus\mathcal{L}_{-1}$. This is in complete conformity with the corrected dimension of the moduli space.

Let us first consider $g=0$. It is always possible to move three distinct points to the triple $(0,1,\infty)$ by an automorphism of $\mathbb{P}^1$. If this is done there are no further automorphisms. Hence the moduli space $\mathcal{M}_{0,N}$ has a non-zero dimension exactly for $N\ge 4$. Its dimension is $\min(0,N-3)$. It is quite useful to map the reference point P_∞ always to ∞. But one should pay attention to the fact that the map (4.4) is not 1:1 anymore. In other words, replacing W' by W is not possible. Note that $\dim W'=N-2=\dim\mathcal{M}_{0,N}+1$ for $N\ge 3$. In this case it is even better to work with the configuration space (and this is usually done)

$$\widehat{W}:=\{(P_1,P_2,\dots,P_N)\mid P_i\in\mathbb{C},\ P_i\ne P_j,\ \text{for } i\ne j\}$$

of N points and study the remaining invariance at the end.

For $g=1$ the situation is similar. For a generic elliptic curve E we always have the translations by points of E as automorphisms. After fixing a point as the zero of the group law on E (which might be chosen as the reference point P_∞) this automorphisms are not possible anymore. The only non-trivial automorphism which remains for the generic curve is the involution $x\to -x$. Again the map (4.4) is not 1:1 anymore. For example, for $N=1$ we have $\dim W'=2$ but $\dim\mathcal{M}_{1,1}=1$. We have to work inside $\mathcal{M}_{1,N+1}$. Again, it is useful to work with the "configuration space" picture. Note that for higher genus in the generic situation the moduli space locally coincides with the "configuration space".

(c) The formal KZ equations.

Let $\mathfrak{V}$ be a sheaf of admissible representations of the affine algebra $\widehat{\mathfrak{g}}$ over W, resp. over $W^{(1)}$ as introduced above. We assume the level c to be constant and obeying the condition $(\mathsf{c}+\mathsf{k})\neq 0$. By Theorem 4.5 the elements of the (fixed) critical strip (resp. of a subspace of the critical strip) correspond to tangent vectors along the moduli space $\mathcal{M}_{g,N}$, resp. along $\mathcal{M}_{g,N}^{(1)}$. In particular, the maps θ introduced in (4.18) are isomorphisms if restricted to the subspaces. Denote the tangent vectors by X_k, $k=1,\dots,3g-3+N$ (resp. $k=1,\dots,3g-3+2N$) and set $l_k=\theta^{-1}(X_k)$ for the corresponding element of the critical strip. Assume X_k operates linearly as operator ∂_k on the space of sections. Assume further that ∂_k operates as derivation

$$\partial_k(s\,\Phi)=(X_k(s))\Phi+s\,\partial_k(\Phi),$$

for Φ a section of $\mathfrak{V}$ and s a local function on the base W, resp. on $W^{(1)}$.

Let us concentrate on sheaves of representations which are defined over $\mathcal{M}_{g,N}$. Examples are the Verma module sheaves. As seen in Section 3 the elements of (the centrally extended) vector field algebra operate vertically on the fibre of the representation sheaf via the Sugawara representation. Recall the definition (4.13) of the the operator $T[l]$ which is defined for every vector field $l\in\mathcal{L}$. The operator does not depend on the coordinates.

We define for sections Φ of $\mathfrak{V}$ and for every k the operator

$$\nabla_k\Phi := (\partial_k+T[l_k])\,\Phi\ . \tag{4.25}$$

Definition 4.6. The *formal KZ equations* are defined as the set of equations

$$\begin{aligned}\nabla_k\,\Phi&=0\ ,\\ \text{for}\quad k&=1,\dots,3g-3+N\ .\end{aligned} \tag{4.26}$$

Using (3.10) and $l_k^{(n,p)(m,s)}:=\frac{1}{2\pi\mathrm{i}}\int_{C_\tau}\omega^{n,p}\omega^{m,s}l_k$ we can rewrite this as

$$\left(\partial_k-\frac{1}{\mathsf{c}+\mathsf{k}}\sum_{\substack{n,m\\p,s}}l_k^{(n,p)(m,s)}\ {:}u(n,p)u(m,s){:}\right)\ \Phi=0\ ,\quad k=1,\dots,3g-3+N\ . \tag{4.27}$$

In view of the decomposition (4.24) the set of equations divides up into two subsets of different interpretation. We have N equations for moving the points. They are related to $e_{-1,p}$, $p=1,\dots,N$. The other ones corresponding to the $3g-3$ elements $l_k\in\mathcal{L}^*_{(0)}$ (for $g\geq 2$). They are responsible for changing the complex structure of the curve.

Sometimes we will consider sections which take their values in certain subspaces. For the Verma module sheaf for example the subsheaf consisting of the elements in $\mathfrak{V}$ of degree zero, or the subspace of the elements annihilated by $\widehat{\mathfrak{g}}_+$ (in its sheaf versions). Or we will consider induced actions on quotient sheaves in such cases when the operator ∇_k maps the subspaces which are factored out to themselves. With this situation we will deal in the forthcoming part II of this article.

For calculations it is often useful to express (4.25), resp. (4.26) in terms of the KN vector field basis $e_{n,p}$ and set $X_{n,p} = \theta(e_{n,p})$. But the definition of $e_{n,p}$ depends on the coordinates like $e'_{n,p} = (\alpha_p)^n e_{n,p}$. We obtain also $X'_{n,p} = \theta(e'_{n,p}) = (\alpha_p)^n X_{n,p}$ and $T[e'_{n,p}] = (\alpha_p)^n T[e_{n,p}]$. For the operators we obtain $\nabla'_{n,p} = (\alpha_p)^n \nabla_{n,p}$. In particular, the set of formal KZ equations expressed in terms of the KN basis elements will yield an equivalent set under coordinate transformations. Note also that the KN basis elements are fixing the tangent directions.

The corresponding set of formal KZ equations for sheaves of representations over $\mathcal{M}^{(1)}_{g,N}$ are obtained by adding N additional equations corresponding to the vector fields $e_{0,p}$, $p = 1, \ldots, N$. They deform local coordinates at the marked points. In this case the operators $\nabla_{n,p}$ are well-defined.

Keep in mind that (for example) it is not possible to talk about changing the complex structure and "fixing the points". Neither is it possible to move the points (in $\mathcal{M}^{(1)}_{g,N}$) and "fix the coordinates"). Nevertheless, if we have global coordinates we have for every movement of the points also a definite change of coordinates. In other words, we are considering a special subspace of $W^{(1)} \subseteq \mathcal{M}^{(1)}_{g,N}$ isomorphic to $W \subseteq \mathcal{M}_{g,N}$. In this case it is possible to consider the sheaves of representation and the corresponding Sugawara operators which are defined over $\mathcal{M}^{(1)}_{g,N}$ as sheaves of representations over $\mathcal{M}_{g,N}$ and drop the corresponding part of equations. But let us stress the fact that this depends on the coordinate prescription given.

Such coordinates exist for $g = 0$ (the quasi-global coordinate z) and for $g = 1$ (the coordinate on the simply-connected covering). In Section 5 and Section 6 we are exactly dealing with this situation. For higher genus all uniformisations have certain disadvantages. We could either realize the curve as upper half-plane modulo a Fuchsian group or embed it into its Jacobian torus. In the first case we do not have a nice behaviour under deformation of the complex structure. In the second case we have multi-dimensional coordinates.

5. The case of lower genus: $g = 0$

Let us show how to obtain the original Knizhnik-Zamolodchikov equations (1.1) from (4.27) for $g = 0$. Let z_i $(i = 1, \dots, N)$ be the N moving points and fix the reference point z_∞ to be ∞. For definiteness take as representation sheaf a Verma module sheaf as introduced in (4.12), (3.5). We set $\alpha(i) := \prod_{\substack{l=1\\ l\neq i}}^{N} (z_i - z_l)^{-1}$ for $i = 1, \dots, N$. For every point z_i $(i = 1, \dots, N)$ the KN basis elements of degree (-1) for the quadratic differentials, resp. for the vector fields are given as

$$\Omega^i(z) := \Omega^{-1,i} = \frac{\mathrm{d}z^2}{z - z_i}, \quad \text{resp.} \quad e_i(z) := e_{-1,i} = \big(\alpha(i) \prod_{j\neq i} (z - z_j)\big) \frac{\partial}{\partial z} \,. \tag{5.1}$$

The vector field e_i evaluates to $\frac{\partial}{\partial z}$ at the point z_i and vanishes at all other points z_j, $j \neq i$. Therefore e_i corresponds to the basic direction ∂_i on the configuration space which is responsible for moving the point z_i. On the other hand, e_i is exactly the Krichever-Novikov dual vector field to the quadratic differential Ω^i. This follows from calculating the residues, see (2.6) .

The coefficients $l_k^{(m,i)(n,j)}$ are given by

$$l_k^{(m,i)(n,j)} = \frac{1}{2\pi \,\mathrm{i}} \int_{C_\tau} \omega^{(m,i)} \omega^{(n,j)} e_k \,. \tag{5.2}$$

Recall from (2.9)

$$\omega^{(m,i)} = \frac{\alpha(i)^{-m}\, dz}{(z - z_i) \prod_{s=1}^{N} (z - z_s)^m} \,.$$

Therefore the integrand in (5.2) equals to

$$\frac{\alpha(i)^{-m} \alpha(j)^{-n} \alpha(k)\, dz}{(z - z_i)(z - z_j)(z - z_k) \prod_{s=1}^{N} (z - z_s)^{m+n-1}} \,. \tag{5.3}$$

The coefficients (5.2) can be obtained by summation of the residues at the points $z_1, \dots, z_N$ (or alternatively by the negative of the residue at the point z_∞). Note that if $(m + n \leq -2)$ or $(m + n = -1$ and not $i = j = k)$ then all residues at the points $z_1, \dots, z_N$ vanish. If $m + n > 0$ the residue vanishes at z_∞. Here we consider only the case when Φ in (4.27) is a vector which is annihilated by the subalgebra $\widehat{\mathfrak{g}}_+$; note that the KZ equations were originally obtained under the same assumption in [14]. But, if

$m+n=0$ and $m,n\neq 0$ then either m or n is positive and hence $:u(m,i)u(n,j):\Phi=0$, because by the normal ordering the elements of positive degree will appear on the right. The nonzero coefficients in (4.27) for a given $k=1,\dots,N$ are as follows:

$$
\begin{aligned}
l_k^{(0,i)(0,i)} &= \frac{\alpha(i)^{-1}\alpha(k)}{z_i-z_k},\ (i\neq k), \qquad l_k^{(0,i)(0,k)} = l_k^{(0,k)(0,i)} = \frac{1}{z_k-z_i},\ (i\neq k),\\
l_k^{(0,k)(0,k)} &= \sum_{i\neq k}\frac{1}{z_k-z_i}, \qquad l_k^{(-1,k)(0,k)} = l_k^{(0,k)(-1,k)} = \alpha(k)^2\ .
\end{aligned}
\tag{5.4}
$$

In the remainder of this section we will show that after modifying the vector fields e_k by adding vertical vector fields (i.e. vector fields which have zeros at all the points z_i and hence are not moving these points), applying a certain factorization process, and calculating the structure constants with respect to this basis, that all coefficients can be eliminated except the one with $m=n=0$ and nonequal upper indices. Hence the equation (4.27) (now with respect to the modified basis) will have the following form:

$$
\left(\partial_i - \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{j\neq i}\frac{:u(0,i)u(0,j):+:u(0,j)u(0,i):}{z_i-z_j}\right)\Phi=0\ ,\quad i=1,\dots,N. \tag{5.5}
$$

Note that the coefficients $l_k^{(0,i)(0,j)}$ $(i\neq j)$ can also be obtained from the expansion

$$
\begin{aligned}
\omega^{0,i}(z)\omega^{0,j}(z) &= \frac{\mathrm{d}z^2}{(z-z_i)(z-z_j)} = \frac{1}{z_i-z_j}\left(\frac{\mathrm{d}z^2}{z-z_i}-\frac{\mathrm{d}z^2}{z-z_j}\right)\\
&= \frac{1}{z_i-z_j}(\Omega^i-\Omega^j)\ (i\neq j),
\end{aligned}
$$

which implies

$$
l_k^{(0,i)(0,j)} = \begin{cases} 0, & k\neq i \text{ and } k\neq j\\ \frac{1}{z_k-z_j}, & k=i\\ -\frac{1}{z_i-z_k}, & k=j\ . \end{cases}
$$

Let us improve the vector fields $e_k, k=1,\dots,N$ in order to eliminate the contribution of the terms with $i=j$, $m=n=0$ (without changing the expressions for $i\neq j$). We set $\tilde\Omega^i:=\omega^{0,i}\omega^{0,i}=\frac{dz^2}{(z-z_i)^2}$. Let us denote the Krichever-Novikov pairing (2.2) between 2-differentials and vector fields by the angle brackets $\langle\cdot,\cdot\rangle$. Then

$$
\langle\tilde\Omega^i,e_k\rangle = \frac{1}{z_k-z_i}\prod_{s\neq i,k}\frac{z_i-z_s}{z_k-z_s},\ (k\neq i)\ , \qquad \langle\tilde\Omega^k,e_k\rangle = \sum_{s\neq k}\frac{1}{z_k-z_s}. \tag{5.6}
$$

Let us pass from the set of vector fields $\{e_k\,|\,k=1,\dots,N\}$ to the set of vector fields $\{e'_k\,|\,k=1,\dots,N\}$ where

$$
e'_k := e_k+\sum_{i=1}^N\lambda_{ki}E_i, \qquad \lambda_{ki}\in\mathbb{C}\ , \tag{5.7}
$$

and $\{E_s \mid s=1,\dots,N\}$ are the Krichever-Novikov basis elements of degree zero for the vector fields, i.e

$$E_i := e_{0,i} = (z-z_i)\prod_{s\neq i}\frac{(z-z_s)^2}{(z_i-z_s)^2}\frac{\partial}{\partial z}. \tag{5.8}$$

Lemma 5.1. *(a) E_i is a vertical vector field ($i=1,\dots,N$).*
(b) $\langle\tilde\Omega^i, E_j\rangle = \delta_{ij}$ ($i,j=1,\dots,N$)
(c) $\langle\Omega^i, E_j\rangle = 0$ ($i,j=1,\dots,N$)

Proof. (a) follows from the fact that the vector fields E_k have zeros at all the points $\{z_s \mid s=1,\dots,N\}$.

(b) By definition

$$\langle\tilde\Omega^i, E_j\rangle = \sum_{p=1}^{N} res_{z_p}\frac{(z-z_j)}{(z-z_i)^2}\prod_{s\neq j}\frac{(z-z_s)^2}{(z_j-z_s)^2}\,dz.$$

If $i\neq j$ then the 1-form on the right hand side of the latter relation is holomorphic and all residues equal to zero. If $i=j$ then all the residues are zero except at the point z_i and there it is equal to 1.

(c) All 1-forms $\Omega^i E_j$, $i,j=1,\dots,N$ are holomorphic which proves the claim, see also (2.6). □

By Lemma 5.1(a) the vector fields $\{e'_k \mid k=1,\dots,N\}$ correspond to the basic infinitesimal deformations ∂_i too, as well as the vector fields $\{e_k \mid k=1,\dots,N\}$ do. By Lemma 5.1(c) the coefficients $l_k^{(m,i)(n,j)}$ ($i\neq j$) remain the same under replacing the set of vector fields $\{e_k \mid k=1,\dots,N\}$ with the set $\{e'_k \mid k=1,\dots,N\}$.

Now let us find the coefficients λ_{ki} in (5.7) in such a way that

$$\langle e'_k, \tilde\Omega^i\rangle = 0,\quad i,k=1,\dots,N. \tag{5.9}$$

This means that $l_k^{(0,i)(0,i)} = 0$, ($i=1,\dots,N$) if these coefficients are calculated with respect to the vector fields $\{e'_k \mid k=1,\dots,N\}$. Recall that $\tilde\Omega^i = \omega^{0,i}\omega^{0,i}$.

The equation (5.9) means

$$\langle e_k + \sum_{s=1}^{N}\lambda_{ks}E_s, \tilde\Omega^i\rangle = 0\ ,\qquad i,k=1,\dots,N. \tag{5.10}$$

By Lemma 5.1(b) this is equivalent to $\langle e_k,\tilde\Omega^i\rangle + \lambda_{ki} = 0$. Hence (5.9) if and only if $\lambda_{ki} = -\langle e_k,\tilde\Omega^i\rangle$.

Taking into account (5.6) this enables us to give explicit expressions for the adjusted e'_k's:

$$e'_k = e_k + \sum_{i\neq k} \frac{\prod_{s=1}^{N}(z-z_s)^2}{z_k - z_i} \cdot \alpha(k)\left(\frac{\alpha(i)}{z-z_i} - \frac{\alpha(k)}{z-z_k}\right)\frac{\partial}{\partial z},$$

where $i,k = 1,\dots,N$. Note that there might occur now nonvanishing coefficients with $m+n=1$. But again under the assumption that $\widehat{\mathfrak{g}}_+$ annihilates Φ and with the normal ordering the corresponding operator terms will not contribute to the final equation.

At last the term $l_k^{(0,k)(-1,k)}(:u(0,k)u(-1,k): + :u(-1,k)u(0,k):)$ of the Knizhnik-Zamolodchikov equation can be eliminated by either considering elements only of degree zero in the result or by passing to the quotient sheaf $\widehat{V}_{\lambda,\gamma,\delta}/\overline{\mathfrak{g}}^*_- \widehat{V}_{\lambda,\gamma,\delta}$ (see (3.6)) . Of course, one has to assume $:u(0,k)u(-1,k): = :u(-1,k)u(0,k): = u(-1,k)u(0,k)$ for doing that. But this is true in the standard normal ordering (3.8).

¿From (5.5) the exact form of (1.1) can be obtained by taking Φ to be from the the $\mathfrak{g}$-module $V_{\lambda,\gamma}$ (see Proposition 3.4) which is a tensor product of the individual representations V_{λ_i}. If we assume that the u_a are a selfdual basis of $\mathfrak{g}$ and take the standard normal ordering then

$$u_a(0,i)\Phi = t_i^a\Phi, \quad \text{and further} \quad u_a(0,j)u_a(0,i)\Phi = t_j^a t_i^a\,\Phi = t_i^a t_j^a\,\Phi \quad \text{for } j\neq i\ .$$

Hence,

$$\left(\frac{\partial}{\partial z_i} - \frac{2}{\mathsf{c}+\mathsf{k}}\sum_{j\neq i}\frac{t_i^a t_j^a}{z_i - z_j}\right)\Phi = 0\ , \qquad i = 1,\dots,N\ .$$

6. The case of lower genus: $g = 1$

The purpose of this section is to obtain explicit expressions for the coefficients of the KZ equations via the Weierstrass σ-function.

1. Let us take the following set of vector fields corresponding to the motion of the points $z_1,\dots,z_N$: $e_k(z) = A_{0,k}(z)\frac{\partial}{\partial z}$ ($k = 1,\dots,N$, see Section 7), or explicitly

$$e_k(z) = \prod_{s\neq k}\frac{\sigma(z-z_s)}{\sigma(z_k - z_s)}\cdot\frac{\sigma(z_k - z_0)^N}{\sigma(z-z_0)^N}\cdot\frac{\sigma(z + \sum_{s\neq k} z_s - Nz_0)}{\sigma(\sum_{s=1}^N z_s - Nz_0)}\frac{\partial}{\partial z}. \tag{6.1}$$

Here z_0 is the fixed reference point. Our first goal is to find the contribution of certain terms of the form $l_k^{(m,i)(n,j)}\ :u(m,i)u(n,j):$ into the KZ equations. Note that by the duality relations $\omega^{m,i} = A_{-m-1,i}dz$. We will consider several cases. As in genus zero let Φ be an element of the representation space which is annihilated by the subalgebra $\widehat{\mathfrak{g}}_+$.

1.1. $m \neq 0$, $n \neq 0$.
Define $\tilde{\omega}^{m,i}$, $\tilde{\omega}^{n,j}$, $\tilde{e}_k$ by the relations $\omega^{m,i} = \sigma(z-z_i)^{-1}\tilde{\omega}^{m,i}$, $\omega^{n,j} = \sigma(z-z_j)^{-1}\tilde{\omega}^{n,j}$, $e_k = \sigma(z-z_k)^{-1}\tilde{e}_k$. Then $ord_{z_s}\tilde{\omega}^{m,i} = -m$, $ord_{z_s}\tilde{\omega}^{n,j} = -n$, $ord_{z_s}\tilde{e}_k = 1$ ($s = 1,\dots,N$), $ord_{z_0}\tilde{\omega}^{m,i} = Nm$, $ord_{z_0}\tilde{\omega}^{n,j} = Nn$, $ord_{z_0}\tilde{e}_k = -N$. Set $\tilde{\omega}_k^{(m,i)(n,j)} := \tilde{\omega}^{m,i}\tilde{\omega}^{n,j}\tilde{e}_k$. Then one has

$$\omega^{m,i}\omega^{n,j}e_k = \frac{\tilde{\omega}_k^{(m,i)(n,j)}}{\sigma(z-z_i)\sigma(z-z_j)\sigma(z-z_k)}. \tag{6.2}$$

and $ord_{z_s}\tilde{\omega}_k^{(m,i)(n,j)} = -m-n+1$ for all $s = 1,\dots,N$. If $m+n \geq 1$ then (6.2) is holomorphic at z_0. If $m+n = 0$ then either $m > 0$ or $n > 0$. In both cases $:u(m,i)u(n,j)\colon \Phi = 0$. By the same reason for the case $m+n = -1$ one would have $m = 0$, $n = -1$ (or vice versa). But in this subsection we assume $m \neq 0$, $n \neq 0$, so this case will not appear. It remains $m+n \leq -2$. Then for the order of the numerator of (6.2) one has $ord_{z_s}\tilde{\omega}_k^{(m,i)(n,j)} = -m-n+1 \geq 3$ ($s = 1,\dots,N$), hence the 1-form is holomorphic at all the marked points even if some or all of the points z_i, z_j, z_k coincide.

Hence the case under consideration does not contribute to the KZ equations.

1.2. $m = 0$, $n \neq 0$ (or vice versa).
By (7.4) $\omega^{0,i} = \hat{\omega}^{0,i} - \sum_{s=1}^{N}\gamma_{i,s}\omega^{-1,s}$ where $\hat{\omega}^{0,i} = A'_{-1,i}dz$ (see (7.7)). The order of the sum on the right hand of the latter relation at any moving point is determined by the order of its first summand because all other terms have some bigger orders. If $n > 0$ then $:u(0,i)u(n,j)\colon \Phi = 0$. Using the definition of e_k and $\omega^{m,i} = A_{-m-1,i}dz$ one obtains $\hat{\omega}^{0,i}\omega^{n,j}e_k = A'_{-1,i}A_{-1-n,j}A_{0,k}\,dz$. By the explicit expressions (6.1) , (7.6)-(7.8) one obtains the following: if $n \leq -2$ then $-n+1 \geq 3$ and $\hat{\omega}^{0,i}\omega^{n,j}e_k$ is holomorphic at any marked point. As $n \neq 0$ it remains only to consider the case $n = -1$. In this case $\hat{\omega}^{0,i}\omega^{n,j}e_k$ has a pole of order at most one at the moving points. And this will occur only if $i = j = k$. In particular, the terms with γ's are holomorphic and do not contribute to the result. An easy calculation of the residues gives

$$l_k^{(0,k)(-1,k)} = 1 \ \ (k = 1,\dots,N). \tag{6.3}$$

In the same way we obtain the symmetric expression $l_k^{(-1,k)(0,k)} = 1$.

1.3. $m = n = 0$.
To find $l_k^{(0,i)(0,j)}$ one has to consider 1-forms $\omega^{0,i}\omega^{0,j}e_k$ where $\omega^{0,i} = \hat{\omega}^{0,i} - \sum_{s=1}^{N}\gamma_{is}\omega^{-1,s}$,

$\omega^{0,j} = \hat\omega^{0,j} - \sum_{r=1}^{N} \gamma_{jr}\omega^{-1,r}$. The terms $\omega^{-1,s}\omega^{-1,r}e_k$ are holomorphic because all their factors are. A term of the form $\omega^{-1,s}\hat\omega^{0,j}e_k$ can have a pole (which is necessarily of order 1) only if $s = j = k$ (and is holomorphic otherwise). So one has

$$l_k^{(0,i)(0,j)} = \langle \hat\omega^{0,i}\hat\omega^{0,j}, e_k\rangle - (\gamma_{jk}\delta_i^k + \gamma_{ik}\delta_j^k)\langle \omega^{-1,k}\hat\omega^{0,k}, e_k\rangle \ .$$

The second scalar product was found already and equals $l_k^{(0,k)(-1,k)}$ which is simply 1. For the first term one has $\hat\omega^{0,i}\hat\omega^{0,j}e_k = A'_{-1,i}A'_{-1,j}A_{0,k}\,dz$. In case $i \neq j$ this 1-form has a residue at the point z_i if $i = k$ or at the point z_j if $j = k$ (if i, j, k are pairwise different then the 1-form is holomorphic). For $i = k \neq j$ one has

$$l_k^{(0,k)(0,j)} = \frac{1}{\sigma(z_k - z_j)}\frac{\sigma(z_k - w_1)\sigma(z_k - w_2)\sigma(z_j - z_0)}{\sigma(z_j - w_1)\sigma(z_j - w_2)\sigma(z_k - z_0)} - \gamma_{jk} \tag{6.4}$$

and an analogous expression for $i \neq k = j$. There is $w_1 + w_2 = z_i + z_0$ in (6.4) and w_1 is the same as in (7.8). Note that in case $i = k \neq j$ this coefficient comes with the operator-valued factor $:u(0,k)u(0,j):$ and with the factor $:u(0,i)u(0,k):$ in case $i \neq k = j$.

In the same way as in case $g = 0$ we can annihilate the contribution of $\omega^{0,i}\omega^{0,i}$ by adding certain vector fields of degree 0 to e_k. So we come to the following form of the KZ equations corresponding to moving the points ($k = 1, \dots, N$):

$$\begin{aligned}\partial_k\Phi - \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{i\neq k} l_k^{(0,k)(0,i)}\,(:u(0,k)u(0,i): + :u(0,i)u(0,k):)\,\Phi \\ - \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{i=1}^{N}(:u(0,i)u(-1,i): + :u(-1,i)u(0,i):)\,\Phi = 0 \ .\end{aligned} \tag{6.5}$$

2. Now let us consider the KZ equation corresponding to the deformation of the complex structure (i.e. to a change of the moduli parameter). Consider the following vector field:

$$e_0 = \sigma(z - E)^{N+1}\sigma(z - z_0)^{-1}\prod_{s=1}^{N}\sigma(z - z_s)^{-1}\frac{\partial}{\partial z} \ ,$$

where $E = (N+1)^{-1}(z_0 + z_1 + ... + z_N)$.

The vector field e_0 has simple poles at all the points $z_1, \dots, z_N, z_0$. It follows from the following lemma that the corresponding tangent vector on the moduli space is non-trivial and that it is independent of the ones generated by the vector fields $e_1, \dots, e_N$. Hence, it corresponds indeed to a deformation of the complex structure. In the lemma vertical vector fields are considered with respect to the moduli space $\mathcal{M}_{g,N}$.

Lemma 6.2. *For points $z_1, \dots, z_n, z_0$ in generic position the vector field e_0 cannot be expressed as a linear combination of the vector fields $e_1, \dots, e_N$ and vertical vector fields.*

Proof. As the vector fields $e_1, \dots, e_N$ are regular at the points $z_1, \dots, z_N$ their residues at the point z_0 are zero (on the elliptic curve one can speak about residues of a vector field). All vertical vector fields have zero residues at z_0 too. This is evident for the ones which are regular at the point z_0. As for the ones which have zeroes at the points $z_1, \dots, z_N$ they also have zero residues at the point z_0 because the sum of all their residues vanishes. But the vector field e_0 has a simple pole at z_0 with nonzero residue for a generic point of the moduli space under consideration. This proves the lemma. □

Let us consider several cases as earlier.

2.1. $m, n \neq 0$.
Define $\tilde{\omega}_0^{(m,i)(n,j)}$ from the relation

$$\omega^{m,i}\omega^{n,j}e_0 = \frac{\tilde{\omega}_0^{(m,i)(n,j)}}{\sigma(z-z_i)\sigma(z-z_j)}, \tag{6.6}$$

Then $ord_{z_s}\tilde{\omega}_0^{(m,i)(n,j)} = -m-n-1$ $(s = 1, \dots, N)$. As the order of the denominator of (6.6) could be at most two the 1-form (6.6) is holomorphic at the points $z_1, \dots, z_N$ as soon as $-m-n-1 \geq 2$, i.e. if $m+n \leq -3$. If $m > 0$ or $n > 0$ then $:u(m,i)u(n,j): \Phi = 0$. So one has either $m+n = -1$ or $m+n = -2$. As $m, n \neq 0$ the first case does not occur and in the second case only $m = n = -1$ remains. Also $i = j$ in this case because otherwise (6.6) is holomorphic. Using explicit formulas one obtains

$$l_0^{(-1,i)(-1,i)} = \sigma(z_i - E)^{N+1}\sigma(z_i - z_0)^{-1}\prod_{s\neq i}\sigma(z_i - z_s)^{-1} \tag{6.7}$$

for $i = 1, \dots, N$.

2.2. $m = 0, n \neq 0$ (or vice versa).
In this case one has to put $m = 0$ in (6.6). Then $ord_{z_s}\tilde{\omega}^{(0,i)(n,j)} = -(n+1)$ $(s = 1, \dots, N)$. If $n > 0$ then $:u(0,i)u(n,j): \Phi = 0$. If $-(n+1) \geq 2$, i.e. $n \leq -3$ the 1-form (6.6) is holomorphic. Hence one has only to consider $n = -1$ and $n = -2$.

If $n = -1$ then $\omega^{0,i}\omega^{-1,j}e_0 = \hat{\omega}^{0,i}\omega^{-1,j}e_0 - \sum_{s=1}^{N}\gamma_{i,s}\omega^{-1,s}\omega^{-1,j}e_0$. If $s \neq j$ then the 1-forms $\omega^{-1,s}\omega^{-1,j}e_0$ are holomorphic at all moving points (see above). If $s = j$ then the contribution of this 1-form equals to $l_0^{(-1,j)(-1,j)}$ (see Subsection 2.1). Furthermore

$$\hat{\omega}^{0,i}\omega^{-1,j}e_0 = \frac{F(z)}{\sigma(z-z_i)\sigma(z-z_j)}dz,$$

where

$$F(z) := A'_{-1,i}(z)A_{0,j}(z)\sigma(z-z_0)^{-1}\sigma(z-E)^{N+1}\sigma(z-z_i)\prod_{s\neq j}\sigma(z-z_s)^{-1}\ ,$$

and $A'_{-1,i}(z)$, $A_{0,j}(z)$ are introduced in Section 7. As follows from Section 7that $F(z)$ has order zero at the points $z_1,\ldots,z_N$. If $i\neq j$ then there are two residues at the point z_i and at the point z_j. So we obtain

$$l_0^{(0,i)(-1,j)} = \frac{F(z_i)-F(z_j)}{\sigma(z_i-z_j)} - \gamma_{ij}l_0^{(-1,j)(-1,j)} \qquad \text{for} \quad i\neq j. \tag{6.8}$$

This residue contributes to the KZ equation with the operator-valued coefficient $:u(0,i)u(-1,j):$. Furthermore $l_0^{(-1,i)(0,j)} = l_0^{(0,j)(-1,i)}$ but it comes with the operator-valued coefficient $:u(-1,i)u(0,j):$. In the case $i=j$ poles of 2-d order arise and one has

$$l_0^{(0,i)(-1,i)} = \frac{d}{dz}\left((z-z_i)^2F(z)\right)\Big|_{z=z_i} - \gamma_{ii}l_0^{(-1,i)(-1,i)}.$$

Consider now $n=-2$. Then the order of the numerator of (6.6) equals 1 at all the moving points. So the residue can be nonzero only if $i=j$. The additional terms $\gamma_{is}\omega^{-1,s}\omega^{-2,i}e_0$ are holomorphic and will not contribute into the result. By explicit formulas one has

$$l_0^{(0,i)(-2,i)} = \sigma(z_i-E)^{N+1}\sigma(z_i-z_0)^{-1}\prod_{s\neq i}\sigma(z_i-z_s)^{-1}\ . \tag{6.9}$$

2.3. $m=n=0$.
One has

$$\omega^{0,i}\omega^{0,j}e_0 = \hat\omega^{0,i}\hat\omega^{0,j}e_0 - \sum_{s=1}^N\gamma_{is}\omega^{-1,s}\hat\omega^{0,j}e_0 - \sum_{r=1}^N\gamma_{jr}\omega^{-1,r}\hat\omega^{0,i}e_0 + \sum_{s,r=1}^N\gamma_{is}\gamma_{jr}\omega^{-1,s}\omega^{-1,r}e_0.$$

Contributions of the terms with γ's can be expressed via the coefficients calculated in the Subsections 2.1 and 2.2. For the first term of the above sum one has $\hat\omega^{0,i}\hat\omega^{0,j}e_0 = F_{ij}(z)\,dz$, where

$$F_{ij}(z) = C_{-1,i}^{-1}C_{-1,j}^{-1}\frac{\sigma(z-w_1)^2\sigma(z-w_2)^2}{\sigma(z-z_i)\sigma(z-z_j)}\prod_{s=1}^N\sigma(z-z_s)^{-1} \times\sigma(z-E)^{N+1}\sigma(z-z_0)^{-3} \tag{6.10}$$

and the coefficients are given by (7.8). It has poles of 2-d order and if $i=j$ then even of 3-d order at the points z_i, z_j. The corresponding coefficients look as follows:

$$l_0^{(0,i)(0,j)} = \langle \hat\omega^{0,i}\hat\omega^{0,j}, e_0\rangle - \sum_{s=1}^{N}(\gamma_{is} l_0^{(0,j)(-1,s)} + \gamma_{js} l_0^{(0,i)(-1,s)} + \gamma_{is}\gamma_{js} l_0^{(-1,s)(-1,s)}) \quad (6.11)$$

where $\langle \hat\omega^{0,i}\hat\omega^{0,j}, e_0\rangle = \frac{d}{dz}\left((z-z_i)^2 F_{ij}(z)\right)\Big|_{z=z_i} + \frac{d}{dz}\left((z-z_j)^2 F_{ij}(z)\right)\Big|_{z=z_j}$ $(i \neq j)$, $\langle \hat\omega^{0,i}\hat\omega^{0,i}, e_0\rangle = \frac{1}{2}\frac{d^2}{dz^2}\left((z-z_i)^3 F_{ij}(z)\right)\Big|_{z=z_i}$. All the terms of the KZ equation which contain $u(n,j)$, $n<-1$ can be eliminated by factorization over $\overline{\mathfrak{g}}_-^*$ as in Section 5 and we obtain the equation corresponding to the deformation of the complex structure in the following form:

$$\begin{aligned} \partial_0\Phi &- \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{i,j=1}^{N} l_0^{(0,i)(0,j)} :u(0,i)u(0,j): \Phi \\ &- \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{i,j=1}^{N} l_0^{(0,i)(-1,j)}(:u(0,i)u(-1,j): + :u(-1,j)u(0,i):)\Phi \\ &- \frac{1}{\mathsf{c}+\mathsf{k}}\sum_{i=1}^{N} l_0^{(-1,i)(-1,i)} :u(-1,i)u(-1,i): \Phi \quad = \quad 0 \ . \end{aligned} \quad (6.12)$$

7. Appendix: KN Basis for the elliptic case

In the elliptic case (i.e. $g=1$) the canonical bundle $\mathcal{K}$ and hence all its powers are trivial. This implies that for all weights λ there is a change of the prescription for certain basis elements necessary. The adopted prescription is given in this appendix. Furthermore, explicit expressions in terms of Weierstraß-σ function are given.

By the triviality of the canonical bundle we have the following relation between the basis elements of $\mathcal{F}^\lambda$

$$f_{n,p}^\lambda = A_{n-\lambda,p}\,dz^\lambda \ . \quad (7.1)$$

In particular, by fixing a basis of $\mathcal{F}^0$ we obtain one for every $\mathcal{F}^\lambda$.
The standard prescription for the $(N,1)$ situation with incoming points $\{P_1, P_2, \ldots, P_N\}$ and outgoing point $\{P_\infty\}$ is

$$\mathrm{ord}_{P_p}(A_{n,p}) = n, \quad \mathrm{ord}_{P_i}(A_{n,p}) = n+1, \text{ for } i \neq p, \qquad \mathrm{ord}_{P_\infty}(A_{n,p}) = -N(n+1) \ .$$

For the $(1,1)$ situation we set as above $A_0 = 1$ and $A'_{-1}dz = \rho = (\omega^0)'$. If we want to have duality we have to modify these elements. With

$$c = \frac{1}{2\pi\,\mathrm{i}} \int_{C_\tau} {A'_{-1}}^2 dz \tag{7.2}$$

we set

$$A_{-1} := A'_{-1} - \frac{c}{2} A_0 \tag{7.3}$$

and obtain the required relation e.g. $\langle A_{-1}, \omega^0 \rangle = 0$. Note that $\omega^0 = A_{-1}dz$ is not obliged to have purely imaginary periods.

For the $(N,1)$ situation with $N > 1$ we have the standard prescription for $A_{0,p}$ and hence for $\omega^{-1,p}$ and the following modified prescription for $n = -1$

$$\mathrm{ord}_{P_p}(A'_{-1,p}) = -1, \quad \mathrm{ord}_{P_i}(A'_{-1,p}) = 0, \text{ for } i \neq p, \quad \mathrm{ord}_{P_\infty}(A'_{-1,p}) = -1 \ .$$

By adding certain linear combinations of $A_{0,s}$ to $A'_{-1,p}$ we obtain again a basis which fulfils the duality. In more detail, set

$$\gamma_{r,p} := \frac{1}{2} \frac{1}{2\pi\,\mathrm{i}} \int_{C_\tau} A'_{-1,r} A'_{-1,p} dz \ . \tag{7.4}$$

The adjusted basis elements are given as

$$A_{-1,r} := A'_{-1,r} - \sum_{s=1}^{N} \gamma_{r,s} A_{0,s} \ . \tag{7.5}$$

Explicit representations of the basis elements for $g = 1$ in terms of Weierstraß σ-function are as follows [24]:

Let the torus be given as $T = \mathbb{C} \bmod L$ with the normalized lattice $L = \langle 1, \tau \rangle_{\mathbb{Z}}$. Let $z_i \in \mathbb{C}$, $i = 1, .., N$ be fixed lifts of the points $P_i \in T$ to $\mathbb{C}$, i.e. $z_i \bmod L = P_i$ and z_0 be a fixed lift of the point $P_\infty \in T$. Let $n \in \mathbb{Z}$ be fixed. For $p = 1, \ldots, N$ set

$$b_{n,p} := b_{n,p}(z_1, \ldots, z_N, z_0) := -(n+1) \sum_{i=1}^{N} z_i + z_p + N(n+1) z_0 \ .$$

If $n \neq -1$ for $N > 1$ or $n \neq 0, -1$ for $N = 1$ then for generic choices of the points z_i and the point z_0 the $b_{n,p}$ does not coincide $\bmod L$ with them, even if we move them locally. But note that the points $b_{n,p}$ depend on the variation of the points z_i and z_0. At the point P_i we choose as coordinate $z_i = (z - z_i), i = 0, 1, \ldots, N$.

Proposition 7.1. [26, p.53] *Let the points $P_i, i = 1, \dots, N$ and P_∞ be generic. Then the basis element $A_{n,p}$ for ($n \neq 0, -1$ and $N = 1$) and for ($n \neq -1$ and $N > 1$) is given as*

$$A_{n,p}(z) = C_{n,p}^{-1}\left(\prod_{i=1}^{N}\sigma(z - z_i)\right)^{n+1}\sigma(z - z_p)^{-1}\sigma(z - z_0)^{-N(n+1)}\sigma(z - b_{n,p}) \ , \qquad (7.6)$$

with

$$C_{n,p} := \left(\prod_{i \neq p}\sigma(z_p - z_i)\right)^{n+1}\sigma(z_p - z_0)^{-N(n+1)}\sigma(z_p - b_{n,p}) \ .$$

In particular, it is a well-defined function on the torus.

The point $b_{n,p}$ mod L is the additional zero of the basis element $A_{n,p}$. Note that the constants $C_{n,p}$ depend also on the points $z_0, \dots, z_N$.

For the exceptional values one obtains the following results.
For $n = 0$: If $N = 1$ one sets $A_0 := 1$. For $N > 1$ the above formula is already the correct formula. It specialises to

$$A_{0,p} = C_{0,p}^{-1}\left(\prod_{i \neq p}\sigma(z - z_i)\right)\sigma(z - z_0)^{-N}\sigma(z + \sum_{i \neq p} z_i - Nz_0) \qquad (7.7)$$

with

$$C_{0,p} := \left(\prod_{i \neq p}\sigma(z_p - z_i)\right)\sigma(z_p - z_0)^{-N}\sigma(z_p + \sum_{i \neq p} z_i - Nz_0) \ .$$

For $n = -1$: In this case the order at the point P_∞ is set to -1. We have to find elements $w_1, w_2 \in \mathbb{C}$ such that $w_1 + w_2 = z_p + z_0$ but $w_1, w_2 \neq z_i$ mod L, i $= 0, \dots, N$. It is possible to choose w_1 generic and this will fix w_2.

Proposition 7.2. [26] *For generic choice of points we have*

$$A_{-1,p} = C_{-1,p}^{-1}\sigma(z - z_p)^{-1}\sigma(z - z_0)^{-1}\sigma(z - w_1)\sigma(z - (z_p + z_0 - \omega_1)) \qquad (7.8)$$

with

$$C_{-1,p} := \sigma(z_p - z_0)^{-1}\sigma(z_p - w_1)\sigma(\omega_1 - z_0) \ .$$

The arbitrariness of the choice in w_1 comes from the fact that the element is only fixed by the orders at P_i and P_∞ up to addition of an arbitrary constant.

Now $(\omega^{0,p})' = A'_{-1,p}dz$ and $\omega^{-1,p} = A_{0,p}dz$. But note that the duality condition is not yet fulfilled. In particular we do not have $\langle\omega^{0,p}, A'_{-1,s}\rangle = 0$. To obtain this condition we have to add combinations of the elements $A_{0,r}$ to the $A'_{-1,s}$, as it is expressed by (7.4), (7.5).

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