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On Fedosov's approach to Deformation Quantization with Separation of Variables

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Abstract

The description of all deformation quantizations with separation of variables on a Kähler manifold from [8] is used to identify the Fedosov star-product of Wick type constructed by M. Bordemann and S. Waldmann in [3]. This star-product is shown to be the one with separation of variables which corresponds to the trivial deformation of the Kähler form in the sense of [8]. To this end a formal Fock bundle on a Kähler manifold is introduced and an associative multiplication on its sections is defined.

Introduction

For a given vector space E we call formal vectors the elements of the space $E[\nu^{-1},\nu]]$ of formal Laurent series in a formal parameter ν with a finite principle part and coefficients in E. Thus we consider the field of formal numbers $\mathbf{K} = \mathbf{C}[\nu^{-1},\nu]]$, formal functions, forms and differential operators.

Deformation quantization of a Poisson manifold $(M, \{\cdot,\cdot\})$, as defined by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [2], is a structure of associative algebra on the space of formal functions $\mathcal{F} = C^\infty(M)[\nu^{-1},\nu]]$. The product $*$ in this algebra (called a star-product) is a $\mathbf{K}$-linear ν-adically continuous product given on functions $f, g \in C^\infty(M)$ by the formula

$$f * g = \sum_{r=0}^{\infty} \nu^r C_r(f,g). \tag{1}$$

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In (1) C_r are bidifferential operators such that $C_0(f,g) = fg,\ C_1(f,g) - C_1(g,f) = i\{f,g\}$. The constant 1 is assumed to be the unit in the algebra $(\mathcal{F}, *)$.

Two star-products $*_1$ and $*_2$ are called equivalent if there exists an isomorphism of algebras $B : (\mathcal{F}, *_1) \to (\mathcal{F}, *_2)$ given by a formal differential operator $B = 1 + \nu B_1 + \nu^2 B_2 + \ldots$.

The problem of existence and classification up to equivalence of star-products on Poisson manifolds was first solved for symplectic manifolds (the main references are [5,6,7,12,13]; for a historical account see [14]). In the general case it was solved by Kontsevich [10].

Let M be a Kähler manifold, endowed with a Kähler $(1,1)$-form ω_{-1} and the corresponding Poisson bracket. In [8] we gave a simple geometric description of all star-products on M which have the following property of separation of variables: in a local holomorphic chart the operators C_r from (1) act on the first argument by antiholomorphic derivatives, and on the second argument by holomorphic ones. We have shown that these star-products are naturally parametrized by geometric objects, the formal deformations of the Kähler form $(1/\nu)\omega_{-1}$.

The interest in deformation quantization with separation of variables is explained by the fact that the Wick star-product on $\mathbf{C}^n$ and the star-products obtained from Berezin's quantization on Kähler manifolds in [4,11,9] have the property of separation of variables.

In [3] Bordemann and Waldmann constructed a star-product with separation of variables on an arbitrary Kähler manifold (M, ω_{-1}), using the geometric approach developed by Fedosov in [6,7]. The goal of this letter is to identify the star-product obtained in [3], using the parametrization from [8]. We show that this star-product corresponds to the trivial deformation of the Kähler form $(1/\nu)\omega_{-1}$.

1. Deformation quantizations with separation of variables

For an open subset $U \subset M$ set $\mathcal{F}(U) = C^\infty(U)[\nu^{-1}, \nu]]$. Since the star-product (1) is given by formal bidifferential operators, it can be localized to any open subset $U \subset M$. We denote its restriction to $\mathcal{F}(U)$ also by $*$.

Denote by $\mathcal{L}^*(U)$ and $\mathcal{R}^*(U)$ the sets of all operators of left and right star-multiplication in the algebra $(\mathcal{F}(U), *)$ respectively. All these operators are formal differential ones. The subalgebras $\mathcal{L}^*(U)$ and $\mathcal{R}^*(U)$ of the algebra of formal differential operators on U are commutants of each other.

Now let (M, ω_{-1}) be a Kähler manifold with the Kähler $(1,1)$-form ω_{-1}.

Consider a star-product $*$ on M with the following property of separation of variables. For an arbitrary local coordinate chart $U \subset M$ with holomorphic coordinates $\{z^k\}$ (and antiholomorphic coordinates $\{\bar{z}^l\}$) assume that the operators from $\mathcal{L}^*(U)$ contain only holomorphic derivatives and the operators from $\mathcal{R}^*(U)$ contain only antiholomorphic ones. This is equivalent to the fact that the operators from $\mathcal{L}^*(U)$ and $\mathcal{R}^*(U)$ commute with the point-wise multiplication operators by antiholomorphic and holomorphic functions on U respectively. It means that, given a holomorphic function a and antiholomorphic function b on U, the point-wise multiplication operators by a and b belong to $\mathcal{L}^*(U)$ and $\mathcal{R}^*(U)$ respectively. Therefore $L^*_a = a$ and $R^*_b = b$, so that for $f \in \mathcal{F}(U)$ $a * f = af, \; f * b = bf$ holds. This property was used for the definition of quantization with separation of variables in [8].

It was shown in [8] that the star-products with separation of variables on (M, ω_{-1}) are in 1—1 correspondence with the formal deformations of the Kähler form $(1/\nu)\omega_{-1}$, i.e., with the formal forms $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \ldots$ such that all $\omega_r, \; r \geq 0$, are closed but not necessarily nondegenerate $(1,1)$-forms on M.

Given an arbitrary formal deformation ω of the Kähler form $(1/\nu)\omega_{-1}$, one can recover the corresponding star-product with separation of variables as follows. On each contractible coordinate chart $(U, \{z^k\})$ on M choose a formal potential $\Phi = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \ldots$ of the form ω, so that $\omega = i\partial\bar{\partial}\Phi$. Then $L^*_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \partial/\partial z^k$ and $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$. Moreover, the set $\mathcal{L}^*(U)$ consists of all formal differential operators which commute with all $R^*_{\bar{z}^l} = \bar{z}^l$ and $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$, and, respectively, $\mathcal{R}^*(U)$ is the commutant of the set of all operators $L^*_{z^k} = z^k$ and $L^*_{\partial\Phi/\partial z^k}$. This completely determines the star-product.

Remark. In [3] star-products with separation of variables on Kähler manifolds are called star-products of Wick type, since the Wick star-product is the simplest one of this kind. However, one can consider star-products with separation of variables on an arbitrary symplectic manifold endowed with a pair of transversal Lagrangean polarizations (see [1]). In the Kähler case these are the holomorphic and antiholomorphic polarizations.

2. The formal Wick algebras bundle and the formal Fock bundle

Consider $\mathbf{C}^n$ with holomorphic coordinates $\{\zeta^k\}$ (and antiholomorphic coordinates $\{\bar{\zeta}^l\}$) endowed with a Hermitian (1,1)-form $ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l$ (here g_{kl} are constants). Denote by $\circ$ the Wick star-product on $(\mathbf{C}^n, ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l)$. This is the star-product with separation of variables, corresponding to the trivial deformation of the (1,1)-form $(1/\nu) i g_{kl} d\zeta^k \wedge d\bar{\zeta}^l$. The Wick star-product of functions $f, g \in C^\infty(\mathbf{C}^n)$ is given by the well-known explicit formula

$$f \circ g = \sum_{r=0}^{\infty} \frac{\nu^r}{r!} g^{l_1 k_1} \dots g^{l_r k_r} \frac{\partial^r f}{\partial \bar{\zeta}^{l_1} \dots \partial \bar{\zeta}^{l_r}} \frac{\partial^r g}{\partial \zeta^{k_1} \dots \partial \zeta^{k_r}},$$

where (g^{lk}) is the matrix inverse to (g_{kl}). Here, as well as in the rest of the letter we use Einstein's summation convention.

Introduce the following gradings on the variables $\nu, \zeta^k, \bar{\zeta}^l$: $deg_\nu(\nu) = 1$, $\deg_\nu(\zeta) = \deg_\nu(\bar{\zeta}) = 0$; $deg_s'(\zeta) = 1$, $deg_s'(\nu) = deg_s'(\bar{\zeta}) = 0$; $deg_s''(\bar{\zeta}) = 1$, $deg_s''(\nu) = deg_s''(\zeta) = 0$; $deg_s = deg_s' + deg_s''$; $Deg' = deg_\nu + deg_s'$; $Deg'' = deg_\nu + deg_s''$; $Deg = Deg' + Deg'' = 2deg_\nu + deg_s$.

The Wick product $\circ$ is a graded product on polynomials in $\nu, \zeta^k, \bar{\zeta}^l$ with respect to the gradings Deg', Deg'' and Deg. The total grading Deg is analogous to the one on the formal Weyl algebra used by Fedosov.

The "normal ordering" procedure establishes a 1—1 correspondence between the polynomials from $\mathbf{K}[\zeta^k, \bar{\zeta}^l]$ and holomorphic differential operators on $\mathbf{C}^n$ with coefficients in $\mathbf{K}[\zeta^k]$. Set $\hat{\zeta}^k = \zeta^k$, $\hat{\bar{\zeta}}^l = \nu g^{lk} \partial / \partial \zeta^k$. The "normal ordering" relates to a polynomial $\phi(\zeta, \bar{\zeta}) = \phi_{\alpha,\beta} \zeta^\alpha \bar{\zeta}^\beta$ the operator $\hat{\phi} = \phi_{\alpha,\beta} \hat{\zeta}^\alpha \hat{\bar{\zeta}}^\beta$. Here $\alpha = (k_1, \dots, k_p), \beta = (l_1, \dots, l_q)$ are multi-indices, $\zeta^\alpha = \zeta^{k_1} \dots \zeta^{k_p}$, $\bar{\zeta}^\beta = \bar{\zeta}^{l_1} \dots \bar{\zeta}^{l_q}$, $\hat{\zeta}^\alpha = \hat{\zeta}^{k_1} \dots \hat{\zeta}^{k_p}$, $\hat{\bar{\zeta}}^\beta = \hat{\bar{\zeta}}^{l_1} \dots \hat{\bar{\zeta}}^{l_q}$ and $\phi_{\alpha,\beta} \in \mathbf{K}$. The polynomial ϕ is called the Wick symbol of the operator $\hat{\phi}$. The operator product transferred to Wick symbols provides the Wick product $\circ$.

The Wick product $\circ$ can be extended to the space W of formal series in $\nu^{-1}, \nu, \zeta^k, \bar{\zeta}^l$ with a finite principal part in ν,

$$w = \sum_{r \geq r_0, p, q \geq 0} \nu^r \sum_{\alpha, \beta, |\alpha| = p, |\beta| = q} w_{r,\alpha,\beta} \zeta^\alpha \bar{\zeta}^\beta.$$

Here $r_0 \in \mathbf{Z}$, $\alpha = (k_1, \dots, k_p), \beta = (l_1, \dots, l_q)$ are multi-indices, $\zeta^\alpha = \zeta^{k_1} \dots \zeta^{k_p}$, $\bar{\zeta}^\beta = \bar{\zeta}^{l_1} \dots \bar{\zeta}^{l_q}$, and the terms of the series are ordered by increasing degrees $Deg = p + q + 2r$. Thus obtained algebra $(W, \circ)$ is called a *formal Wick algebra*.

A *formal Fock space* V on $\mathbf{C}^n$ is the subspace of W of formal series in ν and ζ^k, i.e., of the formal series $v = \sum_{r \geq r_0, \alpha} \nu^r v_{r,\alpha} \zeta^\alpha$. Denote by $\bar{V}$ the subspace of W of formal series in ν and $\bar{\zeta}^l$.

Consider the following projection operators in W, $\Pi' w = w|_{\bar\zeta=0}$, $\Pi'' w = w|_{\zeta=0}$ and $\Pi w = w|_{\zeta=\bar\zeta=0}$, $w \in W$. Then $\Pi' W = V, \Pi'' W = \bar V$ and $\Pi W = \mathbf{K}$.

The kernels of the projections Π' and Π'' consist of the formal series $w \in W$ with all the terms containing at least one antiholomorphic variable $\bar\zeta^l$ or a holomorphic variable ζ^k respectively. It is easy to check that $Ker\ \Pi'$ and $Ker\ \Pi''$ are a left and a right ideals in the Wick algebra $(W, \circ)$ respectively. It follows, in particular, that $Ran\ \Pi' = V \cong W/Ker\ \Pi'$ is a left W-module. An element $w \in W$ acts on V by a formal holomorphic differential operator T_w on $\mathbf{C}^n$ given by the formula $T_w v = \Pi'(w \circ v),\ v \in V$. One can show that if $w \in \mathbf{K}[\zeta, \bar\zeta]$ then $T_w = \hat w$, i.e., T_w is the differential operator with the Wick symbol w. We shall say for general $w \in W$ that w is the Wick symbol of T_w and denote $T_w = \hat w$. It is easy to check that the mapping $W \ni w \mapsto \hat w$ is an injective homomorphism of the algebra $(W, \circ)$ to the algebra of formal differential operators on $\mathbf{C}^n$.

Lemma 1. *For $w \in W$ $\Pi' w = 0$ iff the operator $\hat w$ annihilates the subspace of formal constants $\mathbf{K} \subset V$, and $\Pi'' w = 0$ iff $Ran\ \hat w \subset Ker\ \Pi$.*

The proof of the lemma follows from elementary properties of Wick symbols.

Given a Kähler manifold (M, ω_{-1}) of the complex dimension $dim_{\mathbf{C}} M = n$, consider the unions of the formal Wick algebras and of the formal Fock spaces associated to each tangent space to M. Thus we obtain the bundles of formal Wick algebras $\mathbf{W}$ and of formal Fock spaces $\mathbf{V}$ on M. For an open subset $U \subset M$ denote by $\mathcal{W}(U)$ and $\mathcal{V}(U)$ the spaces of local sections of $\mathbf{W}$ and $\mathbf{V}$ on U respectively. Set $\mathcal{W} = \mathcal{W}(M),\ \mathcal{V} = \mathcal{V}(M)$.

On a coordinate chart $(U, \{z^k\})$ on M introduce the following gradings on 1-forms $dz^k, d\bar z^l$: $deg'_a(dz) = deg''_a(d\bar z) = 1$, $deg'_a(d\bar z) = deg''_a(dz) = 0$; $deg_a = deg'a + deg''_a$. Denote $\Lambda = \oplus_r \Lambda^r$ the deg_a-graded algebra of differential forms on M.

There exist natural inclusions of the spaces $\mathcal{F} \otimes \Lambda \subset \mathcal{V} \otimes \Lambda \subset \mathcal{W} \otimes \Lambda$ of the (formal) scalar, $\mathbf{V}$- and $\mathbf{W}$-valued differential forms on M respectively (the tensor product is taken over $C^\infty(M)$, $\otimes = \otimes_{C^\infty(M)}$).

The fibrewise Wick product and the action of W on V in the first factor of the tensor product together with the wedge-product of differential forms in the second factor define the structures of deg_a-graded algebra on $\mathcal{W} \otimes \Lambda$ and of its deg_a-graded module on $\mathcal{V} \otimes \Lambda$. The product in $\mathcal{W} \otimes \Lambda$ will be denoted also $\circ$. The projections Π, Π' and Π'' define fibrewise projections in $\mathcal{W} \otimes \Lambda$ denoted by the same symbols. The action of an element $w \in \mathcal{W} \otimes \Lambda$

on the space $\mathcal{V} \otimes \Lambda$ is given by the operator $\hat{w}$ defined, as above, by the expression $\hat{w}v = \Pi'(w \circ v)$, where $v \in \mathcal{V} \otimes \Lambda$. We have $\Pi'(\mathcal{W} \otimes \Lambda) = \mathcal{V} \otimes \Lambda$ and $\Pi(\mathcal{W} \otimes \Lambda) = \mathcal{F} \otimes \Lambda$.

In the sequel we shall always denote by $\zeta^k, \bar{\zeta}^l$ the fiber coordinates on the tangent bundle TM in the frame $\{\partial/\partial z^k, \partial/\partial \bar{z}^l\}$ on a coordinate chart $(U, \{z^k\})$ on M.

Notice that for a local section $w(z, \bar{z}) = \sum_{r \geq r_0, \alpha, \beta} \nu^r w_{r,\alpha,\beta}(z, \bar{z}) \zeta^\alpha \bar{\zeta}^\beta \in \mathcal{W}(U)$ the coefficients $w_{r,\alpha,\beta}(z, \bar{z})$ are symmetric covariant tensor fields on M.

3. Fedosov star-product of Wick type

Recall the construction by Bordemann and Waldmann of the Fedosov star-product of Wick type on a Kähler manifold (M, ω_{-1}) from [3]. (We use, however, different conventions and notations.)

Let ∇ denote the standard Kähler connection on M. It can be naturally extended to symmetric covariant tensors, and thus to the bundles $\mathbf{W}$ and $\mathbf{V}$. For technical reasons it will be convenient to denote its extension to $\mathbf{W}$ also by ∇, and its extension to $\mathbf{V}$ by $\hat{\nabla}$.

Express the Kähler form ω_{-1} on M and the Kähler connection ∇ on $\mathcal{W} \otimes \Lambda$ in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\}$: $\omega_{-1} = i g_{kl} dz^k \wedge d\bar{z}^l$, $\nabla = d - \Gamma^s_{ki} \zeta^i (\partial/\partial \zeta^s) dz^k - \Gamma^t_{lj} \bar{\zeta}^j (\partial/\partial \bar{\zeta}^t) d\bar{z}^l$, where $\Gamma^s_{ki} = g^{ls} \partial g_{kl}/\partial z^i$ and $\Gamma^t_{lj} = g^{kt} \partial g_{kl}/\partial \bar{z}^j$ are the Kristoffel symbols and (g^{lk}) is the matrix inverse to (g_{kl}). Then $\hat{\nabla} = d - \Gamma^s_{ki} \zeta^i (\partial/\partial \zeta^s) dz^k$.

Introduce an element $R \in \mathcal{W} \otimes \Lambda^2$ such that it is given in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\}$ by the formula $R = (-g^{ts} \partial g_{kt} \wedge \bar{\partial} g_{sl} + \partial \bar{\partial} g_{kl}) \zeta^k \bar{\zeta}^l$.

The curvature of the connection ∇ on the bundle $\mathbf{W}$ was calculated in [3]: $\nabla^2 = (1/\nu) ad_{Wick}(R)$. A straightforward calculation leads to the following

Lemma 2. *The curvature of the connection* $\hat{\nabla}$ *on the bundle* $\mathbf{V}$ *is expressed via* R *as follows,* $\hat{\nabla}^2 = (1/\nu)\hat{R}$.

Introduce Fedosov's operators δ and δ^{-1} on $\mathcal{W} \otimes \Lambda$. In local coordinates $\delta = (\partial/\partial \zeta^k) dz^k + (\partial/\partial \bar{\zeta}^l) d\bar{z}^l$ and the operator δ^{-1} is defined as follows. For an element $a \in \mathcal{W} \otimes \Lambda^q$ such that $deg_s = p$ set $\delta^{-1} a = 0$ if $p + q = 0$ and $\delta^{-1} a = (p+q)^{-1} (\zeta^k i(\partial/\partial z^k) + \bar{\zeta}^l i(\partial/\partial \bar{z}^l)) a$ if $p + q > 0$.

Then $\delta = (1/\nu) ad_{Wick}(\vartheta)$, where $\vartheta = g_{kl} \bar{\zeta}^l dz^k - g_{kl} \zeta^k d\bar{z}^l$ (see [3]).

It was shown in [3] that there exists a unique element $r \in \mathcal{W} \otimes \Lambda^1$ which satisfies the equations $\delta^{-1}\, r = 0$ and $\delta r = R + \nabla r + (1/\nu) r \circ r$, and contains only non-negative powers of ν.

In [3] a flat Fedosov's connection D on $\mathbf{W}$ is defined as follows, $D = -\delta +$

$\nabla + (1/\nu)ad_{Wick}(r)$. It is a deg_a-graded derivation in the algebra $(\mathcal{W}\otimes\Lambda, \circ)$. Therefore $\mathcal{W}_D = Ker\ D \cap \mathcal{W}$ is closed under Wick multiplication.

It was proved in [3] that the mapping $\Pi : \mathcal{W}_D \to \mathcal{F}$ is, in fact, a bijection. Transferring the product from the Fedosov algebra $(\mathcal{W}_D, \circ)$ to $\mathcal{F}$ via this bijection, one obtains a star-product $*$ on (M, ω_{-1}). Moreover, it was proved in [3] that $*$ is a star-product with separation of variables. The proof was based on the following important statement (Lemma 4.5 in [3]): $r \in Ker\ \Pi' \cap Ker\ \Pi''$, i.e., in any local expression of r each term contains variables ζ^k and $\bar\zeta^l$ for some indices k, l. We reformulate this statement using Lemma 1.

Lemma 3. *The operator $\hat r$ in $\mathcal{V}$ annihilates the subspace $\mathcal{F}\otimes\Lambda \subset \mathcal{V}\otimes\Lambda$. In particular, $\hat r 1 = 0$. Moreover, Ran $\hat r \subset Ker\ \Pi$.*

We are going to show that the star-product with separation of variables $*$ constructed in [3] corresponds to the trivial deformation $\omega = (1/\nu)\omega_{-1}$ of the Kähler form $(1/\nu)\omega_{-1}$.

4. The Fock algebra

Using the fact that $\delta = (1/\nu)ad_{Wick}(\vartheta)$, one can express D as follows, $D = \nabla + (1/\nu)ad_{Wick}(\gamma)$, where $\gamma = -\vartheta + r$.

Introduce a connection $\hat D$ on $\mathbf{V}$ by the formula $\hat D = \hat\nabla + (1/\nu)\hat\gamma$.

One can split the connections $\nabla, D, \hat D$, the operator δ and the element r into the sums of their (1,0)- and (0,1)-components, $\nabla = \nabla' + \nabla'', D = D' + D'', \hat D = \hat D' + \hat D'', \delta = \delta' + \delta'', r = r' + r''$. In local coordinates denote $\nabla_k = \nabla_{\partial/\partial z^k},\ \nabla_l = \nabla_{\partial/\partial \bar z^l}$, so that $\nabla' = \nabla_k dz^k,\ \nabla'' = \nabla_l d\bar z^l$. Introduce similarly $D_k, D_l, \hat D_k, \hat D_l$. Let $r = r_k dz^k + r_l d\bar z^l$ be a local expression of the element r. Then $r' = r_k dz^k, r'' = r_l d\bar z^l$.

A simple calculation shows that $(1/\nu)\hat\vartheta = \partial/\partial\zeta^k dz^k - \eta_l d\bar z^l$, where $\eta_l = (1/\nu)g_{kl}\zeta^k$. Therefore,

$$\hat D_k = \partial/\partial z^k - \partial/\partial\zeta^k + (1/\nu)\hat r_k;\ \hat D_l = \partial/\partial\bar z^l + \eta_l + (1/\nu)\hat r_l. \qquad (2)$$

Lemma 4. *Let $f \in \mathcal{F}(U)$, where $(U, \{z^k\})$ is a coordinate chart on M. Then $\hat D_k f = \partial f/\partial z^k$. In particular, $\hat D_k 1 = 0$.*

The lemma trivially follows from Lemma 3 and formula (2).

Lemma 5. *For $w \in \mathcal{W}\otimes\Lambda$ one has $[\hat\nabla, \hat w] = \widehat{\nabla w}$.*

Here, as well as below, the commutator is the deg_a-graded commutator in the graded algebra of endomorphisms of $\mathcal{V}\otimes\Lambda$.

The lemma is an easy consequence of the fact that ∇ is a deg_a-graded derivation of the algebra $(\mathcal{W}\otimes\Lambda, \circ)$. It implies the following

Proposition 1. *For $w \in \mathcal{W} \otimes \Lambda$ the formula $[\hat{D}, \hat{w}] = \widehat{Dw}$ holds.*

Denote $\omega = (1/\nu)\omega_{-1}$.

Lemma 6.

(i) $[\hat{\nabla}, \hat{\vartheta}] = 0$*;*

(ii) $(1/\nu)[\hat{\vartheta}, \hat{r}] = \widehat{\delta r}$*;*

(iii) $\hat{\vartheta}^2 = i\nu^2\omega$.

Lemma is proved by straightforward calculations. It implies the following

Proposition 2. *The connection $\hat{D}$ on $\mathbf{V}$ has a scalar curvature, $\hat{D}^2 = i\omega$.*

The subspace $\mathcal{W}_{D''} = Ker\ D'' \cap \mathcal{W}$ of the algebra $(\mathcal{W}, \circ)$ is closed under the Wick product. We shall use the algebra $(\mathcal{W}_{D''}, \circ)$ to define a product on the space $\mathcal{V}$.

Introduce Fedosov's operator δ''^{-1} on $\mathcal{W} \otimes \Lambda$ defining it in the local coordinates on a chart $(U, \{z^k\})$ as follows. Let $w \in (\mathcal{W} \otimes \Lambda)(U)$ be such that $deg''_s(w) = p$, $deg''_a(w) = q$. Set $\delta''^{-1}a = 0$ if $p + q = 0$ and $\delta''^{-1}a = (p + q)^{-1}\bar{\zeta}^l i(\partial/\partial \bar{z}^l)a$ if $p+q > 0$. Then for $w \in \mathcal{W} \otimes \Lambda$ one has $(\delta''\delta''^{-1} + \delta''^{-1}\delta'')w = w - w_0$, where w_0 is the $(deg''_s + deg''_a)$-homogeneous component of w of the degree 0. For an element $w \in \mathcal{W} \otimes \Lambda$ denote by $w^{(q)}$ its Deg''-homogeneous component of the degree q.

The following proposition can be proved by Fedosov's technique developed in [6].

Proposition 3. *The mapping $\Pi' : \mathcal{W}_{D''} \to \mathcal{V}$ is a bijection. For an element $v \in \mathcal{V}$ such that $deg_\nu(v) = 0$ (i.e., which does not depend on the formal parameter ν) the unique element $w \in \mathcal{W}_{D''}$ such that $v = \Pi' w$ can be calculated recursively with respect to the degree Deg'' by*

$$w^{(0)} = v;$$

$$w^{(q+1)} = \delta''^{-1}(\nabla'' w^{(q)} + (1/\nu) \sum_{p=0}^{q} ad_{Wick}(r''^{(p+1)}) w^{(q-p)}).$$

Denote by $\bullet$ the product in $\mathcal{V}$ obtained by pushing forward the product in the algebra $(\mathcal{W}_{D''}, \circ)$ by the mapping Π'. Thus we obtain a *Fock algebra* $(\mathcal{V}, \bullet)$. For $v \in \mathcal{V}$ denote by $L^\bullet_v, R^\bullet_v$ the operators of left and right multiplication by v in the algebra $(\mathcal{V}, \bullet)$ respectively. Set $\mathcal{L}^\bullet = \{L^\bullet_v | v \in \mathcal{V}\}$, $\mathcal{R}^\bullet = \{R^\bullet_v | v \in \mathcal{V}\}$.

Lemma 7. *For $w \in \mathcal{W}_{D''}$ the operator $\hat{w}$ coincides with the left multiplication operator by the element $v = \Pi' w$ in the Fock algebra $(\mathcal{V}, \bullet)$, $\hat{w} = L^\bullet_v$.*

Proof. For $w_1, w_2 \in \mathcal{W}_{D''}$ set $v_1 = \Pi' w_1,\ v_2 = \Pi' w_2$. Then, by definition, $v_1 \bullet v_2 = \Pi'(w_1 \circ w_2)$. Since Π' is a projection, $w_2 - v_2 \in Ker\ \Pi'$. Taking into account that $Ker\ \Pi'$ is a left ideal in the algebra $(\mathcal{W}, \circ)$, we get $w_1 \circ (w_2 - v_2) \in Ker\ \Pi'$. Therefore $\Pi'(w_1 \circ w_2) = \Pi'(w_1 \circ v_2) = \hat{w}_1 v_2$, whence the Lemma follows. □

Since the action of the operators $\hat{w},\ w \in \mathcal{W}$, on $\mathcal{V}$ is fibrewise, it follows from Lemma 7 that the operator of point-wise multiplication by $f \in \mathcal{F}$ (also denoted by f) commutes with all operators from $\mathcal{L}^\bullet$. Therefore, $f \in \mathcal{R}^\bullet$, namely, $R^\bullet_f = f$.

Fix a coordinate chart $(U, \{z^k\})$ on M.

Lemma 8. $R^\bullet_{\eta_l} = \hat{D}_l$.

Proof. Let $w \in \mathcal{W}_{D''}(U),\ v = \Pi' w \in \mathcal{V}(U)$. It follows from Lemma 7 and Proposition 1 that $[\hat{D}_l, L^\bullet_v] = [\hat{D}_l, \hat{w}] = \widehat{D_l w} = 0$, therefore $\hat{D}_l \in \mathcal{R}^\bullet$. Using formula (2) and Lemma 3 we get $\hat{D}_l 1 = \eta_l$, whence $\hat{D}_l = R^\bullet_{\eta_l}$. □

Denote $\mathcal{U} = \Pi'(\mathcal{W}_D) \subset \mathcal{V}$. Since $\mathcal{W}_D \subset \mathcal{W}_{D''}$, and the projection Π' establishes an isomorphism of the algebras $(\mathcal{W}_{D''}, \circ)$ and $(\mathcal{V}, \bullet)$, the subspace $\mathcal{U} \subset \mathcal{V}$ is closed under multiplication $\bullet$ and the projection Π' maps the Fedosov algebra $(\mathcal{W}_D, \circ)$ isomorphically onto the subalgebra $(\mathcal{U}, \bullet)$ of the Fock algebra $(\mathcal{V}, \bullet)$.

Lemma 9. *For $w \in \mathcal{W}_{D''}(U)$ and $v = \Pi' w \in \mathcal{V}(U)$ one has $D_k w \in \mathcal{W}_{D''}(U)$ and $[\hat{D}_k, L^\bullet_v] = \widehat{D_k w} = L^\bullet_{\hat{D}_k v}$.*

Proof. Using Lemma 7 and Proposition 1 we obtain $[\hat{D}_k, L^\bullet_v] = [\hat{D}_k, \hat{w}] = \widehat{D_k w}$. Since Fedosov's connection D is flat, $D^2 = 0$, we have $[D_k, D_l] = 0$, whence $D_l D_k w = D_k D_l w = 0$, i.e., $D_k w \in \mathcal{W}_{D''}(U)$ and therefore $\widehat{D_k w} \in \mathcal{L}^\bullet(U)$. Using Lemma 4 we get $[\hat{D}_k, L^\bullet_v] 1 = \hat{D}_k v - L^\bullet_v \hat{D}_k 1 = \hat{D}_k v$ and thus $\widehat{D_k w} = L^\bullet_{\hat{D}_k v}$, which concludes the proof. □

Denote $\mathcal{V}_{\hat{D}'}(U) = Ker\ \hat{D}' \cap \mathcal{V}(U)$ the space of local sections of the Fock bundle $\mathbf{V}$ on an open subset $U \subset M$, annihilated by $\hat{D}'$. Set $\mathcal{V}_{\hat{D}'} = \mathcal{V}_{\hat{D}'}(M)$.

Proposition 4. $\mathcal{U} = \mathcal{V}_{\hat{D}'}$.

Proof. We have to show that on any coordinate chart $(U, \{z^k\})$ on M, $w \in \mathcal{W}_{D''}(U)$ and $v = \Pi' w \in \mathcal{V}(U)$ the condition $D_k w = 0$ holds iff $\hat{D}_k v = 0$. The assertion follows immediately from the equality $L^\bullet_{\hat{D}_k v} = \widehat{D_k w}$ proved in Lemma 9 and the fact that the mapping $\mathcal{W} \ni w \mapsto \hat{w}$ is injective. □

We can obtain the star-product $*$ on M from the algebra $(\mathcal{V}_{\hat{D}'}, \bullet) = (\mathcal{U}, \bullet)$. Let $v_1, v_2 \in \mathcal{V}_{\hat{D}'},\ f_1 = \Pi v_1, f_2 = \Pi v_2 \in \mathcal{F}$. Then $f_1 * f_2 = \Pi(v_1 \bullet v_2)$.

Let Φ_{-1} be a local potential of the form $\omega_{-1} = i g_{kl} dz^k \wedge d\bar{z}^l$ on a coordinate chart $(U, \{z^k\})$ on M, so that $\partial^2\Phi_{-1}/\partial z^k\partial\bar{z}^l = g_{kl}$. Then $\Phi = (1/\nu)\Phi_{-1}$ is a local potential of the form $\omega = (1/\nu)\omega_{-1}$. Set $Q_l = \partial\Phi/\partial\bar{z}^l + \eta_l$.

Proposition 5. $Q_l \in \mathcal{V}_{\hat{D}'}(U)$.

Proof. Using Lemma 4 we get $\hat{D}_k\partial\Phi/\partial\bar{z}^l = \partial^2\Phi/\partial z^k\partial\bar{z}^l = (1/\nu)g_{kl}$. It follows from Proposition 2 that $[\hat{D}_l, \hat{D}_k] = (1/\nu)g_{kl}$. Now, $\hat{D}_k\eta_l = \hat{D}_k\hat{D}_l 1 = \hat{D}_l\hat{D}_k 1 - (1/\nu)g_{kl} = -(1/\nu)g_{kl}$ and therefore $\hat{D}'Q_l = 0$. □

Since $*$ is known to be a star-product with separation of variables, then $R^*_{\bar{z}^l} = \bar{z}^l$ holds. This can be checked also directly. It follows from Lemma 4 that $\hat{D}_k\bar{z}^l = 0$, i.e., $\bar{z}^l \in \mathcal{V}_{\hat{D}'}(U)$. Let $v \in \mathcal{V}_{\hat{D}'}(U)$ and $f = \Pi v \in \mathcal{F}(U)$. Now $f * \bar{z}^l = \Pi(v \bullet \bar{z}^l) = \Pi(v\bar{z}^l) = f\bar{z}^l$, which proves the assertion.

In order to identify the star-product with separation of variables $*$ it remains to calculate $R^*_{\partial\Phi/\partial\bar{z}^l}$. Let $v \in \mathcal{V}_{\hat{D}'}(U)$ and $f = \Pi v$ as above. Calculate first $\Pi\hat{D}_l v$. Using formula (2) we get $\Pi\hat{D}_l v = \Pi(\partial v/\partial\bar{z}^l + \eta_l v + (1/\nu)\hat{r}_l v)$. Since $\Pi\eta_l = 0$, we have $\Pi(\eta_l v) = 0$. Lemma 3 implies that $\Pi(\hat{r}_l v) = 0$. Finally we obtain that $\Pi\hat{D}_l v = \partial f/\partial\bar{z}^l$.

Since $\Pi Q_l = \partial\Phi/\partial\bar{z}^l$, we get $f * \partial\Phi/\partial\bar{z}^l = \Pi(v \bullet Q_l) = \Pi(R^\bullet_{Q_l} v) = \Pi((\partial\Phi/\partial\bar{z}^l + \hat{D}_l)v) = (\partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l)f$. Therefore $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$. Thus we have proved the desired

Theorem. *The Fedosov star-product of Wick type $*$ on a Kähler manifold (M, ω_{-1}) is the star-product with separation of variables corresponding to the trivial deformation of the form $(1/\nu)\omega_{-1}$.*

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