SINGULAR PROJECTIVE VARIETIES AND QUANTIZATION

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Abstract. By the quantization condition compact quantizable Kähler manifolds can be embedded into projective space. In this way they become projective varieties. The quantum Hilbert space of the Berezin-Toeplitz quantization (and of the geometric quantization) is the projective coordinate ring of the embedded manifold. This allows for generalization to the case of singular varieties. The set-up is explained in the first part of the contribution. The second part of the contribution is of tutorial nature. Necessary notions, concepts, and results of algebraic geometry appearing in this approach to quantization are explained. In particular, the notions of projective varieties, embeddings, singularities, and quotients appearing in geometric invariant theory are recalled.

CONTENTS

Introduction

Compact Kähler manifolds which are quantizable, i.e. which admit a holomorphic line bundle with curvature form equal to the Kähler form (a so called quantum line bundle) are projective algebraic manifolds. This means that with the help of the global holomorphic sections of a suitable tensor power of the quantum line bundle they can be embedded into a projective space of certain dimension. Submanifolds of

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the projective space are always projective varieties, i.e. can be given as zero sets of finitely many homogeneous polynomials. As will be explained in this contribution the basic objects in the set-up of Berezin-Toeplitz (or equivalently geometric) quantization of quantizable compact Kähler manifolds can be completely described inside this algebraic-geometric context. For example the quantum Hilbert space will be essentially the projective coordinate ring of the variety.

By definition Kähler manifolds are nonsingular hence the varieties obtained are nonsingular. But from the point of view of varieties the singular ones are on equal footing. Hence one might expect that it is possible to find a direct way towards quantization of singular spaces by exploiting the theory of varieties.

In this contribution I do not present a solution for the quantization of singular spaces. I will only explain the above mentioned path from compact quantizable Kähler manifolds to projective varieties. The quantization procedure I am considering is the Berezin-Toeplitz quantization, resp. the Berezin-Toeplitz deformation quantization. This quantization procedure is adapted to the complex structure which is a requirement for the fact that it can be formulated in terms of complex algebraic geometry. I recall the results on this quantization scheme in Section 1.

The rest of the contribution is considered to be tutorial. There is nothing new there, and everything is well-known for researchers working in algebraic geometry. But I hope that the collection of concepts and results will be useful for researcher in quantization. Some concepts used elsewhere in this volume are explained. In Section 2 and in Section 3 basic concepts of algebraic geometry are introduced. First projective varieties are defined. Compactified moduli spaces are candidates for projective varieties. The projective (homogeneous) coordinate ring is discussed. It will turn out to be the quantum Hilbert space of the theory. It incorporates the vector space of global holomorphic sections of all tensor power of the quantum line bundle at once. On this quantum Hilbert space the total Berezin-Toeplitz quantization operator operates. It is used to show that the quantization scheme has the correct semi-classical limit and to prove the existence of an associated deformation quantization. As already pointed out, quantizable Kähler manifolds can be embedded with the help of the very ample quantum line bundle into projective space (as complex manifolds, not necessarily as Kähler manifolds). Such embeddings are discussed in detail in Section 2.

Projective varieties are not necessarily smooth, they can have singularities. After giving some examples of singular varieties in Section 2 (e.g. the singular cubic curves) singularities are treated in more detail in Section 3. Beside the definition of a singular point using the rank of the Jacobi matrix of the defining equations for the variety, a more intrinsic definition in terms of the local ring $\mathcal{O}_{V,\alpha}$ of a point α on the variety V and the Zariski tangent space at the point α is given. In terms of algebraic properties of the local ring a hierarchy of types of singularities can be introduced. As special examples normal singularities are discussed. Whereas on an arbitrary singular variety the set of singular points can have codimension one, on a normal variety (i.e. on a variety where all local rings are normal rings, see the definition below) this subset has codimension at least two. The singularities of moduli spaces are very often normal singularities.

Typically, moduli spaces are obtained by dividing out a group action on a nonsingular variety. The main question is whether it is possible to define a geometric structure on the orbit space, i.e. whether there exists some algebraic geometric quotients. The "Geometric Invariant Theory (GIT)" as developed by Mumford [13] gives a powerful tool how to deal with such quotients. If one considers only certain suitable subsets of points of the variety the group is acting on (i.e. the subset of semi-stable, or stable points) one obtains a good quotient (which is also a categorical quotient), resp. a geometric quotient. They will carry a compatible structure of a projective variety, resp. of an open subset of a projective variety. This will be explained in Section 4. Also there the results on the relation with the symplectic quotients obtained via moment maps and symplectic reduction due to Kirwan, Kempf and Ness will be explained. These results are taken from the appendix to [13], written by Kirwan. Roughly speaking, the geometric quotient and the symplectic quotient coincides on the regular points of the symplectic reduction (see Theorem 4.5 for a precise statement). But in general the singularity structure will differ.

1. From quantizable compact Kähler manifolds to projective varieties

Let (M,ω) be a Kähler manifold, i.e. M a complex manifold and ω a Kähler form on M. In this contribution I will only consider compact Kähler manifolds. If nothing else is said compactness is assumed. A further data we need is the triple (L,h,∇), with a holomorphic line bundle L on M, a hermitian metric h on L (with the convention that it is conjugate linear in the first argument) and a connection ∇ compatible with the metric on L and the complex structure. With respect to local holomorphic coordinates of the manifold and with respect to a local holomorphic frame for the bundle the metric h can be given as

$$h(s_1,s_2)(x) = \hat{h}(x)\overline{\hat{s}_1}(x)\hat{s}_2(x), \tag{1}$$

where $\hat{s}_i$ is a local representing function for the section s_i $(i=1,2)$ and $\hat{h}$ is a locally defined real-valued function on M. The compatible connection is uniquely defined and is given in the local coordinates as $\nabla = \partial + (\partial \log \hat{h}) + \overline{\partial}$. The curvature of L is defined as the two-form

$$curv_{L,\nabla}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\ , \tag{2}$$

where X and Y are vector fields on M. In the local coordinates the curvature can be expressed as $curv_{L,\nabla} = \overline{\partial}\partial \log \hat{h} = -\partial\overline{\partial}\log \hat{h}$.

A Kähler manifold (M,ω) is called *quantizable* if there exists such a triple (L,h,∇) which obeys

$$curv_{L,\nabla}(X,Y) = -\,\mathrm{i}\,\omega(X,Y)\ . \tag{3}$$

The condition (3) is called the (pre)quantum condition. The bundle (L, h, ∇) is called a (pre)quantum line bundle. Usually we will drop ∇ and sometimes also h in the notation.

For the following we assume (M, ω) to be a quantizable Kähler manifold with quantum line bundle (L, h, ∇).

There is an important observation. If M is a compact Kähler manifold which is quantizable then from the prequantum condition (3) we obtain for the Chern form of the line bundle the relation

$$c(L) := \frac{\mathrm{i}}{2\pi} curv_{L,\nabla} = \frac{\omega}{2\pi} \ . \tag{4}$$

This implies that L is a positive line bundle. In the terminology of algebraic geometry it is an ample line bundle, see Definition 2.5 for the definition of ampleness. By the Kodaira embedding theorem M can be embedded (as algebraic submanifold) into projective space $\mathbb{P}^N(\mathbb{C})$ using a basis of the global holomorphic sections s_i of a suitable tensor power L^{m_0} of the bundle L

$$z \ \mapsto \ (s_0(z) : s_1(z) : \ldots : s_N(z)) \in \mathbb{P}^N(\mathbb{C}) \ . \tag{5}$$

These algebraic submanifolds can be described as zero sets of homogeneous polynomials, i.e. they are projective varieties. Note that the dimension of the space $\Gamma_{hol}(M, L^{m_0})$ consisting of the global holomorphic sections of L^{m_0}, can be determined by the Theorem of Grothendieck-Hirzebruch-Riemann-Roch, see [7], [15]. By passing to the Kähler form $m_0\omega$ and to the associated quantum line bundle L^{m_0} we might assume that the sections of our quantum line bundle do already the embedding (i.e that it is already very ample).

So even if we start with an arbitrary Kähler manifold the quantization condition will force the manifold to be an algebraic manifold and we are in the realm of algebraic geometry. This should be compared with the fact that there are "considerable more" Kähler manifolds than algebraic manifolds.

In Section 2 I will explain what projective varieties are. But first I like to introduce the quantum operator we are dealing with. We take $\Omega = \frac{1}{n!}\omega^n$ as volume form on M. On the space of C^∞ sections of the bundle L we have the scalar product

$$\langle \varphi, \psi \rangle := \int_M h(\varphi, \psi)\, \Omega \ , \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle} \ . \tag{6}$$

Let $\mathrm{L}^2(M, L)$ be the L^2-completion of the space of C^∞-sections of the bundle L and $\Gamma_{hol}(M, L)$ be its (due to compactness of M) finite-dimensional closed subspace of holomorphic sections. Let $\Pi : \mathrm{L}^2(M, L) \to \Gamma_{hol}(M, L)$ be the projection.

Definition 1.1. *For $f \in C^\infty(M)$ the Toeplitz operator T_f is defined to be*

$$T_f := \Pi\,(f\cdot) : \quad \Gamma_{hol}(M, L) \to \Gamma_{hol}(M, L) \ .$$

In words: We multiply the holomorphic section with the differentiable function f. This yields only a differentiable section. To obtain a holomorphic section again, we project it back to the subspace of global holomorphic sections. From the point of view of Berezin's approach [2], T_f is the operator with contravariant symbol f.

The linear map

$$T : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M,L)\big), \qquad f \to T_f \ ,$$

is called the Berezin-Toeplitz quantization. Recall that $(C^\infty(M), \cdot, \{.,.\})$ is a Poisson algebra. To define the Poisson bracket (i.e. a compatible Lie algebra structure) on $C^\infty(M)$ we use the Kähler form ω as symplectic form and define $\{f,g\} := \omega(X_f, X_g)$ where X_f is the Hamiltonian vector field assigned to $f \in C^\infty(M)$ given by $\omega(X_f, .) = df(.)$. The Berezin-Toeplitz quantization map is neither a Lie algebra homomorphism nor an associative algebra homomorphism, because in general

$$T_f\,T_g = \Pi\,(f\cdot)\,\Pi\,(g\cdot)\,\Pi \neq \Pi\,(fg\cdot)\,\Pi \ .$$

Due to the compactness of M this defines a map from the commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. A lot of information will get lost. To recover this information one should consider not just the bundle L alone but all its tensor powers L^m for $m \in \mathbb{N}_0$ and apply all the above constructions for every m. In this way one obtains a family of matrix algebras and maps

$$T_f^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M,L^m)\big), \qquad f \to T_f^{(m)} \ .$$

This infinite family should in some sense "approximate" the algebra $C^\infty(M)$.(See [3] for a definition of such an approximation.)

If we group all $T_f^{(m)}$ together we obtain a map

$$C^\infty(M) \to \prod_{m\in\mathbb{N}_0} \mathrm{End}(\Gamma_{hol}(M,L^m)) \subseteq \mathrm{End}(\prod_{m\in\mathbb{N}_0} \Gamma_{hol}(M,L^m)), \tag{7}$$

$$f \quad \mapsto \quad T_f^{(*)} := (T_f^{(m)})_{m\in\mathbb{N}_0} \ . \tag{8}$$

We will see later on that $\prod_{m\in\mathbb{N}} \Gamma_{hol}(M,L^m)$ with a slight modification (i.e. $\prod$ is replaced by $\bigoplus$) is the projective coordinate ring of the embedded M. The operator $T_f^{(*)}$ is called the total Berezin-Toeplitz operator. It operates on the projective coordinate ring.

It was shown by Bordemann, Meinrenken and Schlichenmaier [4] that this quantization scheme has the correct semi-classical behavior and yields an associated star product (a deformation quantization). Denote by $||f||_\infty$ the sup-norm of f on M and by $||T_f^{(m)}|| = \sup_{s\in\Gamma_{hol}(M,L^m),s\neq 0} \frac{||T_f^{(m)}s||}{||s||}$ the operator norm on $\Gamma_{hol}(M,L^m)$.

Theorem 1.2. *[Bordemann, Meinrenken, Schlichenmaier]* [4]
(a) For every $f \in C^\infty(M)$ *there exists* $C > 0$ *such that*

$$||f||_\infty - \frac{C}{m} \leq ||T_f^{(m)}|| \leq ||f||_\infty \ . \tag{9}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = ||f||_\infty$.

(b) For every $f, g \in C^\infty(M)$

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m}) \,. \tag{10}$$

(c) For every $f, g \in C^\infty(M)$

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m}) \,. \tag{11}$$

Let me recall the definition of a star product. Let $\mathcal{A} = C^\infty(M)[[\nu]]$ be the algebra of formal power series in the variable ν over the algebra $C^\infty(M)$. A product $\star$ on $\mathcal{A}$ is called a (formal) star product if it is an associative $\mathbb{C}[[\nu]]$-linear product such that

1. $\mathcal{A}/\nu\mathcal{A} \cong C^\infty(M)$, i.e. $f \star g \bmod \nu = f \cdot g$,
2. $\frac{1}{\nu}(f \star g - g \star f) \bmod \nu = -\,\mathrm{i}\,\{f, g\}$,

where $f, g \in C^\infty(M)$. We can also write

$$f \star g = \sum_{j=0}^{\infty} \nu^j C_j(f, g) \,, \tag{12}$$

with $C_j(f,g) \in C^\infty(M)$. The C_j should be $\mathbb{C}$-bilinear in f and g. The conditions 1. and 2. can be reformulated as

$$C_0(f,g) = f \cdot g, \qquad \text{and} \qquad C_1(f,g) - C_1(g,f) = -\,\mathrm{i}\,\{f,g\} \,. \tag{13}$$

Theorem 1.3. *There exists a unique (formal) star product on $C^\infty(M)$*

$$f \star g := \sum_{j=0}^{\infty} \nu^j C_j(f,g), \quad C_j(f,g) \in C^\infty(M), \tag{14}$$

in such a way that for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| \le K_N(f,g) \left(\frac{1}{m}\right)^N \,. \tag{15}$$

See [16], [17] and [18].

It has a couple of nice properties, i.e. (i) $1 \star f = f \star 1 = f$, (ii) the selfadjointness $\overline{f \star g} = \overline{g} \star \overline{f}$, and (iii) it admits a naturally defined trace (see [18]).

As is shown in [10] the star product is a differential star product, i.e. the C_j are bidifferential operators and it has the property of "separation of variables" [11] (resp. it is of Wick type [5]). This says that it respects the holomorphic structure. In more precise terms: if the star product is restricted to open subsets the star multiplication from the right with local holomorphic functions is pointwise multiplication, and the star multiplication from the left with local anti-holomorphic functions is pointwise multiplication.

Let me close this section with two remarks.

Remark. More traditionally one considers the operator Q of geometric quantization (with Kähler polarization) defined as $Q = \Pi \circ P$ with

$$P : C^\infty(M) \to \mathrm{End}(\Gamma_\infty(M, L)), \quad f \mapsto P_f := -\nabla_{X_f} + \mathrm{i}\, f \cdot id \,,$$

where $\Gamma_\infty(M, L)$ is the space of C^∞ sections of the bundle L and Π is the projection onto the space of global holomorphic sections. Now $Q_f \in \mathrm{End}(\Gamma_{hol}(M, L))$. Again one should consider $Q_f^{(m)}$ for all $m \in \mathbb{N}_0$. For compact Kähler manifolds both quantization procedures are related via the Tuynman relation. It reads as

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f} = \mathrm{i}\left(T_f^{(m)} - \frac{1}{2m} T^{(m)}_{\Delta f}\right) . \tag{16}$$

Hence, the $T_f^{(m)}$ and the $Q_f^{(m)}$ have the same asymptotic behavior.

Remark. There is another kind of embedding of the manifold M into projective space. It is the embedding using the coherent states of Berezin-Rawnsley. This embedding turns out to be a special case of the embedding considered at the beginning of this section where one uses a orthogonal basis of the sections, resp. (depending on the conventions) the conjugate of it, see [1] for details.

2. Projective varieties

2.1. The definition of a projective variety

Let $\mathbb{K}$ be an algebraically closed field and let us assume for simplicity that its characteristic is zero. Without any harm the reader might even assume $\mathbb{K} = \mathbb{C}$. The projective space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{K})$ is given as the space of lines through the origin in $\mathbb{K}^{n+1}$, i.e. as the equivalence classes of points in $\mathbb{K}^{n+1} \setminus \{0\}$ where two points α and β are equivalent if $\exists \lambda \in \mathbb{K} \setminus \{0\}$ with $\beta = \lambda \cdot \alpha$. The point $[\alpha]$ in projective space defined by the point $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, $\alpha \neq 0$ can be given by its (non-unique) homogeneous coordinates $[\alpha] = (\alpha_0 : \alpha_1 : \cdots : \alpha_n)$.

Let $f \in \mathbb{K}[X_0, X_1, \ldots, X_n]$ be a homogeneous polynomial of degree k. As usual we obtain an associated $\mathbb{K}$-valued function on $\mathbb{K}^{n+1}$ by assigning to the point $\alpha \in \mathbb{K}^{n+1}$ the value $f(\alpha)$ obtained by "setting" X_i to be α_i. If $\beta = \lambda\alpha$ with $\lambda \in \mathbb{K}$, $\lambda \neq 0$, is another point in the same equivalence class as α, then we obtain $f(\lambda\alpha) = \lambda^k f(\alpha)$. In particular, the induced function is only well-defined on the whole projective space if $k = 0$, i.e. f is a constant. But we also see that if α is a zero of f then any other element $\beta = \lambda\alpha$ will be a zero too. Hence the zero-set

$$\mathcal{Z}(f) := \{[\alpha] \in \mathbb{P}^n \mid f(\alpha) = 0\} \tag{17}$$

is a well-defined subset of $\mathbb{P}^n$. The Zariski topology is the coarsest topology in which the sets $\mathcal{Z}(f)$ are closed subsets for all polynomials f, or equivalently for which the complements $D_f = \mathbb{P}^n \setminus \mathcal{Z}(f)$ are open sets. Because the zero-sets of polynomials are also closed in the "usual" topology if the base field is $\mathbb{C}$ the sets which are closed (open) in the Zariski topology are closed(open) in the "usual" topology. The Zariski topology has a number of quite unusual properties. For example, it is not separated, i.e. two distinct points do not necessarily have disjoint open neighborhoods. Even more is true: every non-empty Zariski open set U is automatically dense in $\mathbb{P}^n$.

Definition 2.1. *(a) A subset W of $\mathbb{P}^n$ is called a (projective) variety if it is the set of common zeros of finitely many homogeneous polynomials $f_1, f_2, \ldots, f_m$ (which are not necessarily of the same degree)*

$$W = \mathcal{Z}(f_1, f_2, \ldots, f_m) := \{[\alpha] \in \mathbb{P}^n \mid f_i(\alpha) = 0,\ i = 1, \ldots, m\}. \tag{18}$$

(b) A variety is called a linear variety if it can be given as the zero-set of linear polynomials.
(c) A variety is called irreducible if every decomposition

$$W = V_1 \cup V_2 \tag{19}$$

with varieties V_1 and V_2 implies that

$$V_1 \subseteq V_2 \quad \text{or} \quad V_2 \subseteq V_1 . \tag{20}$$

A variety which is not irreducible is called reducible.
(d) A Zariski open set of a projective variety is called a quasiprojective (or sometimes just algebraic) variety.

Note that some authors reserve the term variety for irreducible ones.

Definition 2.2. *Let V be an irreducible variety, then its dimension* $\dim V$ *is defined as the maximal length n of chains of strict subvarieties which are irreducible*

$$\emptyset \subsetneqq V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_{n-1} \subsetneqq V_n = V . \tag{21}$$

For arbitrary varieties the dimension is defined to be the maximum of the dimensions of its irreducible subvarieties.

Subvarieties of dimension 0 are called points, subvarieties of dimension 1 curves, etc.

Let V be a projective variety i.e. $V = \mathcal{Z}(f_1, f_2, \ldots, f_m)$. Take $I = (f_1, f_2, \ldots f_m)$ to be the ideal generated by the polynomials $f_1, f_2, \ldots f_m$, i.e.

$$I = \{\sum_{i=1}^{m} g_i f_i \mid g_i \in \mathbb{K}[X_0, X_1, \ldots X_n], i = 1, \ldots, k\} . \tag{22}$$

Obviously $V = \mathcal{Z}(I)$. Ideals which can be generated by homogeneous elements are called homogeneous ideals. Hence, projective varieties can always be given as zero-sets of homogeneous ideals. The converse is also true. Clearly

$$\mathcal{Z}(I) := \{x \in \mathbb{P}^n \mid f(x) = 0, \forall f \in I\} . \tag{23}$$

is by the homogeneity of the generators a well-defined subset of $\mathbb{P}^n$. Because the polynomial ring is a Noetherian ring, i.e. every ideal I can be generated (as ideal in the sense of (22)) by finitely many elements, e.g. $I = (g_1, g_2, \ldots, g_s)$, we get $\mathcal{Z}(I) = \mathcal{Z}(g_1, g_2, \ldots, g_s)$ and hence $\mathcal{Z}(I)$ is a projective variety in the sense of

Definition 2.1. Any other set of generators of the ideal will define the same zero-set. Even the ideal is not fixed uniquely by V. As a simple example one might consider the hyperplane $H = \mathcal{Z}(I)$ with $I = (X_0)$. The same variety might be defined as $H = \mathcal{Z}(I'))$ with $I' = (X_0^2)$. But note that $I' \subset I$. One might expect that for a given variety V there is a largest ideal which still defines V. This is indeed true.

Definition 2.3. *Let V be a projective variety, i.e. $V = \mathcal{Z}(I)$ for some ideal I. The vanishing ideal $\mathcal{I}(V)$ is defined to be*

$$\mathcal{I}(V) := \{f \in \mathbb{K}[X_0, \dots, X_n] \mid f(x) = 0, \forall x \in V\} . \tag{24}$$

The subset $\mathcal{I}(V)$ is a homogeneous ideal and contains any other defining ideal I for V. It can completely be described in algebraic terms. For this we define for any ideal I its radical ideal

$$Rad(I) := \{f \in \mathbb{K}[X_0, \dots, X_n] \mid \exists n \in \mathbb{N} : f^n \in I\} . \tag{25}$$

If I is homogeneous $Rad(I)$ will again be homogeneous. We obtain

$$\mathcal{I}(\mathcal{Z}(I)) = Rad(I) , \tag{26}$$

with only one exception in the case when $\mathcal{Z}(I) = \emptyset$. Note that $\emptyset$ corresponds to two homogeneous radical ideal, the full ring $\mathbb{K}[X_0, \dots, X_n]$ and the ideal $I_0 := (X_0, X_1, \dots, X_n)$. Note that the only possible zero of I_0 is the point 0 which is not an element of projective space.

There is another warning necessary. One might think that the dimension r of a variety is exactly $n - k$ if k is the minimal number of necessary polynomials to generate its vanishing ideal I. Unfortunately this is not true. The only information one has is that $r \geq n - k$, with equality if $k = 1$. A variety is called a complete intersection if indeed $r = n - k$.

Projective varieties are not always manifolds (of course not all manifolds are projective varieties either). Varieties have not necessarily to be smooth. They might have singularities. In Section 3 I will deal with singularities in more detail. Here I would like to show some non-trivial examples of singular varieties. For this I give a first definition of a singular point. Further definitions will follow in Section 3.

Definition 2.4. *Let $V = \mathcal{Z}(f_1, f_2, \dots, f_m)$ be a projective variety of dimension r in $\mathbb{P}^n$ with vanishing ideal $\mathcal{I}(V)$ generated by the polynomials $f_1, f_2, \dots, f_m$. Consider the $m \times (n+1)$-matrix (the Jacobi matrix)*

$$J(X) = \begin{pmatrix} \frac{\partial f_1}{\partial X_0} & \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \frac{\partial f_2}{\partial X_0} & \frac{\partial f_2}{\partial X_1} & \cdots & \frac{\partial f_2}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial X_0} & \frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_n} \end{pmatrix} . \tag{27}$$

A point x on V with

$$\operatorname{rank} J(x) < n - r \tag{28}$$

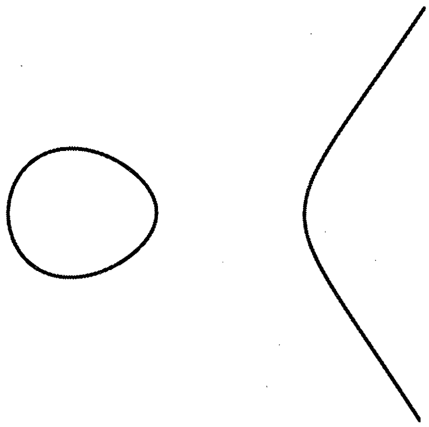

FIGURE 1. A nondegenerate cubic curve

is called a singular point of the variety. A point on a variety which is not a singular point is called a regular point. If V has no singular points it is called a non-singular (or smooth, or regular) variety.

For a variety which is the union of two different subvarieties the points where the subvarieties meet are always singular points. As a typical example one might take the variety $V = \mathcal{Z}(X_0X_1)$ in $\mathbb{P}^2$. Then $V = \mathcal{Z}(X_0) \cup \mathcal{Z}(X_1)$ and the singular point is the point $(0:0:1) = \mathcal{Z}(X_0) \cap \mathcal{Z}(X_1)$. But even irreducible varieties can have singularities. As an example let me consider the varieties Y in $\mathbb{P}^2$ defined by irreducible cubic polynomials. These polynomials can be written (after a suitable change of coordinates) as

$$f(X,Y,Z) = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 \ , \tag{29}$$

with certain elements $g_2, g_3 \in \mathbb{K}$. The variety $\mathcal{Z}(f)$ is non-singular if and only if the coefficients g_2 and g_3 are such that $g_2^3 - 27g_3^2 \neq 0$. One obtains in this way the *elliptic curves*. See the Figure 1. Here the curve is defined over $\mathbb{C}$ and the real-valued points are plotted. The elliptic curves correspond to 1-dimensional complex tori, see Section 2.2 below. In the singular cases we obtain two different types of curves. The first one is the *nodal cubic* $\mathcal{Z}(Y^2Z - 4X^2(X+Z))$, see Figure 2. The only singular point is the point $(0:0:1)$. Moving along the curve we pass through the singular point twice, each time with a different tangent direction. The second one is the *cuspidal cubic* $\mathcal{Z}(Y^2Z - 4X^3)$, see Figure 3. Again the point $(0:0:1)$ is a singular point. But now there is only one tangent direction at this point.

2.2. Embeddings into Projective Space

Let us now take the complex numbers $\mathbb{C}$ as base field $\mathbb{K}$. With the complex topology $\mathbb{P}^n$ is a compact, n-dimensional complex manifold. A coordinate covering is given

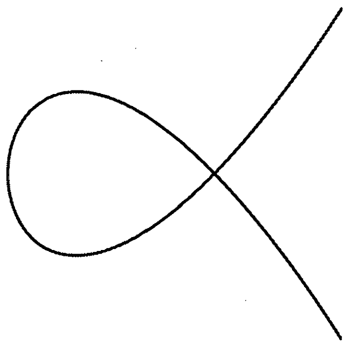

FIGURE 2. Nodal cubic

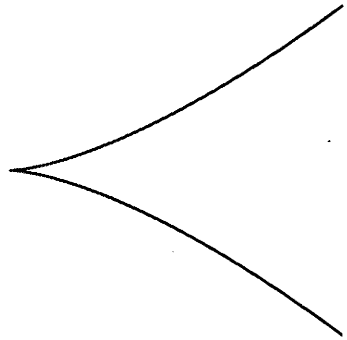

FIGURE
3. Cuspidal cubic

by the "affine" sets

$$D_{X_i} := \mathcal{Z}(X_i) = \{(\alpha_0 : \alpha_1 : \cdots : \alpha_n) \mid \alpha_i \neq 0\} \;\cong\; \mathbb{C}^n \tag{30}$$

for $i = 0, 1, \ldots, n$. Every nonsingular projective variety is closed in the Zariski topology and hence also in the complex topology and hence a compact submanifold of $\mathbb{P}^n$. An abstract compact complex manifold M is called a projective algebraic manifold if there exists an injective holomorphic embedding

$$\Phi : M \to \mathbb{P}^n \ , \tag{31}$$

such that $\Phi(M) \cong M$ as complex manifolds. The Theorem of Chow [7, p.166] says that in this case $\Phi(M)$ is a nonsingular projective variety, i.e. it can be given as the zero-set of finitely many homogeneous polynomials. This is even true in the strong sense that every meromorphic function on $\Phi(M)$ is a rational function (i.e. it can be expressed as quotient of homogeneous polynomials of the same degree in $(n+1)$ variables), every meromorphic differential is a rational differential, and every holomorphic map between two embedded complex manifolds is an algebraic map, i.e. can be given locally as a set of rational functions without poles.

Let me illustrate this in the case of the above introduced elliptic curves. Let $T = \mathbb{C}/\Gamma$ be the one-dimensional complex torus defined as the quotient of $\mathbb{C}$ by the lattice $\Gamma := \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{Z}\}$ for fixed $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. The associated Weierstraß $\wp$-function and its derivative $\wp'$ are doubly-periodic meromorphic functions with respect to the lattice Γ, i.e.

$$\wp(z+\omega) = \wp(z), \quad \wp'(z+\omega) = \wp'(z), \quad \text{for all } \omega \in \Gamma,\ z \in \mathbb{C}\ .$$

Hence they are meromorphic functions on T. The function $\wp$ fulfills the famous differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{32}$$

with the Eisenstein series

$$g_2 := 60 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4}, \qquad g_3 := 140 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6} .$$

An embedding of the torus into the projective plane is given by

$$\Psi : T \to \mathbb{P}^2, \qquad [z] \mapsto \begin{cases} (\wp(z) : \wp'(z) : 1), & [z] \neq 0 \\ (0:1:0), & [z] = 0. \end{cases} \tag{33}$$

Here $[z] = z \mod \Gamma$ denotes the point on the torus represented by $z \in \mathbb{C}$. If one compares the differential equation (33) with the polynomial (29) one sees that $\Psi(T)$ is a cubic curve hence a projective variety (indeed it is nonsingular). Via Ψ the meromorphic function $\wp$ corresponds to the rational function X/Z and $\wp'$ corresponds to Y/Z. Note that the field of meromorphic functions on the torus consists of rational expressions in $\wp$ and $\wp'$. For more details see, [15], p.34 and p.62 ff.

After this excursion let me return to the situation discussed in the Section 1. Let M be a compact complex manifold and $\pi : L \to M$ a holomorphic line bundle (not necessarily a quantum line bundle). Choose a basis of the global holomorphic sections $s_0, s_1, \dots, s_n \in \Gamma_{hol}(M, L)$. For every point $x \in M$ there exists an open neighborhood U of x such that L can be locally trivialized over U, i.e. that there is an (holomorphic) bundle map $\rho : L_U := \pi^{-1}(U) \cong U \times \mathbb{C}$. With respect to this trivialization the section s_i can be given by a local holomorphic function $\hat{s}_i : U \to \mathbb{C}$ defined by $\rho(s_i(x)) = (x, \hat{s}_i(x))$. The map

$$U \to \mathbb{C}^{n+1}, \qquad y \mapsto \tilde{\Phi}(y) := (\hat{s}_0(y), \hat{s}_1(y), \dots, \hat{s}_n(y)) \tag{34}$$

is a holomorphic map. It depends not only on the basis chosen, but also on the trivialization. If ρ' is a different trivialization defined over the open set U where ρ is defined (or a subset of it) then $\rho' \circ \rho^{-1}(x, \lambda) = (x, g(x)\lambda)$ with a holomorphic function $g : U \to \mathbb{C}$ nowhere vanishing on U. The map $\tilde{\Phi}' : U \to \mathbb{C}^{n+1}$ corresponding to ρ' fulfills $\tilde{\Phi}'(y) = g(y) \cdot \tilde{\Phi}(y)$. Hence, $[\Phi(y)] := \tilde{\Phi}(y)$ will be well-defined, i.e. not depend on the trivialization chosen if we assure that $\tilde{\Phi}(y) \neq 0$. But $\tilde{\Phi}(y) = 0$ if and only if $s(y) = 0$ for all sections $s \in \Gamma_{hol}(M, L)$. Hence we obtain a well-defined holomorphic map

$$\Phi : M \setminus \{y \in M \mid s(y) = 0, \forall s \in \Gamma_{hol}(M,L)\} \ \to \ \mathbb{P}^n , \tag{35}$$

obtained by glueing together the local maps $\tilde{\Phi}$. A change of basis of $\Gamma_{hol}(M, L)$ is given by an element of $\mathrm{GL}(n+1, \mathbb{C})$. The images of the two mappings obtained by the two set of basis elements are related by the corresponding $\mathrm{PGL}(n+1, \mathbb{C})$ action. Note that if there exists a nontrivial section s (i.e. $s \not\equiv 0$) then the map (35) is defined on a dense open subset of M.

Definition 2.5. *(a) A line bundle L is called very ample if the map Φ (with respect to one and hence to all set of basis elements) is an embedding.*
(b) A line bundle L is called ample if there exists $m \in \mathbb{N}$ such that $L^{\otimes m}$ is very ample.

It follows that a compact complex manifold is projective algebraic if it admits an ample line bundle. The converse is also true. To see this we first study $\mathbb{P}^n$. Here we have the tautological line bundle whose fiber over the point $[z]$ is the complex line trough 0 and the point $z \in \mathbb{C}^{n+1}$. The hyperplane section bundle H is the dual of the tautological line bundle. Its space of global sections is generated by the coordinate functions $X_0, X_1, \ldots, X_n$, i.e. it can be identified with the space of linear polynomials in $(n+1)$ variables. All line bundles over $\mathbb{P}^n$ are given as H^m where this denotes for $m > 0$ the m-th tensor power of H, for $m < 0$ the $|m|$-th tensor power of the dual bundle of H, and for $m = 0$ the trivial bundle $\mathcal{O}$. The space of global holomorphic sections of H^m can canonically be identified with the space of homogeneous polynomials of degree m in $(n+1)$ variables. In particular, there exists no nontrivial sections for $m < 0$. If $\Phi : M \to \mathbb{P}^n$ is a holomorphic map then the pullback $\Phi^* H$ is a holomorphic line bundle on M. The space of global sections of $\Phi^* H$ is generated by the pullback $\Phi^*(X_i) = X_i \circ \Phi$ of the global sections X_i, $i = 0, 1, \ldots, n+1$. If Φ is a holomorphic embedding than $\Phi^* H$ is a very ample line bundle. If the pull-backs of the $(n+1)$ sections X_i stay linearly independent then Φ is exactly given by the embedding defined via the bundle $\Phi^* H$. If not, then the embedding defined via $\Phi^* H$ goes into a linear subvariety of $\mathbb{P}^n$ of lower dimension, hence in a $\mathbb{P}^k$ for $k < n$.

Altogether we see that the embeddings of M into projective space correspond to very ample line bundles over M. The pair (M, L) where M is a compact complex manifold and L is a very ample line bundle is called a *polarized projective algebraic manifold.*

Note that the same manifold considered with different L may "look" quite differently. As a simple example take $M = \mathbb{P}^1$ and $L = H$ then $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ is the identity. Now consider $L = H^2$, which gives an embedding into $\mathbb{P}^2$. Let X_0, X_1 be the basis of the sections of H then $X_0^2,\ X_0 X_1,\ X_1^2$ is a basis of H^2. If $(\alpha_0 : \alpha_1)$ are homogeneous coordinates on $\mathbb{P}^1$ the image of $\mathbb{P}^1$ in $\mathbb{P}^2$ is given as

$$\Phi(\mathbb{P}^1) = \{(\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2) \in \mathbb{P}^2 \mid \alpha_0, \alpha_1 \in \mathbb{C}\} = \mathcal{Z}(X_1^2 - X_0 X_3)\ .$$

The obtained subvariety is not linear anymore. Nevertheless it is algebraically isomorph to the linear variety $\mathbb{P}^1$.

Let us come back to the quantization condition. Recall that the quantization condition says that the Chern form of the quantum line bundle L is essentially the Kähler form. But the Kähler form is a positive form, hence L is a positive line bundle. Kodairas embedding theorem says that a certain positive tensor power of L will give an embedding into projective space. Hence L is an ample line bundle. This implies that quantizable compact Kähler manifolds are always projective algebraic. In Section 2.3 we will see that the converse is also true.

2.3. The projective coordinate ring

Let V be a projective variety in $\mathbb{P}^n$ and $I = \mathcal{I}(V)$ its vanishing ideal (24). Recall that it is a homogeneous ideal.

Definition 2.6. *The projective (or homogeneous) coordinate ring is the graded ring*

$$\mathbb{K}[V] := \mathbb{K}[X_0, X_1, \dots X_n]/\mathcal{I}(V). \tag{36}$$

For $V = \mathbb{P}^n$, we have $\mathcal{I}(\mathbb{P}^n) = (0)$, hence

$$\mathbb{K}[\mathbb{P}^n] := \mathbb{K}[X_0, X_1, \dots X_n] = \bigoplus_{m\geq 0} H^0(\mathbb{P}^n, H^m)$$

is the full polynomial ring.

Inside $\mathbb{K}[V]$ the whole geometry of the variety V is encoded. For example the points correspond to maximal homogeneous ideals $M \subseteq I$ which are not identical to the ideal $(X_0, X_1, \dots X_n)$. Note that the only element of $\mathbb{K}^{n+1}$ which is a zero of all X_i is 0, which is not an element of projective space.

Definition 2.7. *The Krull dimension* $\dim R$ *of a ring* R *is defined to be the maximal length* k *of strict chains of prime ideals* P_i

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_k \subseteqq R\,. \tag{37}$$

Recall that an ideal P is called a prime ideal if from $f \cdot g \in P$ it follows that $f \in P$ or $g \in P$. Clearly, for prime ideals P we have $Rad(P) = P$, hence $\mathcal{I}(\mathcal{Z}(P)) = P$. Moreover, the variety $\mathcal{Z}(P)$ is always irreducible. Any chain (21) of irreducible subvarieties of an irreducible variety gives a chain of homogeneous prime ideals

$$\mathbb{K}[X_0, \dots, X_n] \supsetneqq \mathcal{I}(V_0) \supsetneqq \mathcal{I}(V_1) \supsetneqq \cdots \supsetneqq \mathcal{I}(V_n) = \mathcal{I}(V)\,. \tag{38}$$

lying between the vanishing ideal of V and the whole ring. Passing to the quotient, i.e. to the coordinate ring one obtains a chain of prime ideals of the coordinate ring $\mathbb{K}[V]$. This works also in the opposite direction with the one exception that to both the whole ring $\mathbb{K}[V]$ and to the ideal $(X_0, X_1, \dots, X_n) \mod I(V)$ corresponds the empty set. This implies [1]

$$\dim V = \dim \mathbb{K}[V] - 1\,.$$

Now let $\Phi : M \to \mathbb{P}^n$ be the embedding obtained via the quantum line bundle L, which we assume already to be very ample. Let $I := \mathcal{I}(\mathcal{Z}(\Phi(M))$ be the vanishing ideal of $\Phi(M)$. We obtain $\Phi^* H = L$, $i^* X_i = s_i$ for $i = 0, 1, \dots, n$ for the sections s_i used for the embedding, and $\Phi^*(H^m) = (\Phi^* H)^m = L^m$. In particular, the pull-backs of the global sections of H^m generate the space of global sections of L^m. But in general they will not be a basis. The algebraic relations between them are exactly given by the elements of the ideal I. The projective coordinate ring $\mathbb{C}[V]$ can be identified with $\bigoplus_{m\geq 0} H^0(M, L^m)$.

In Section 1 we have defined the Berezin-Toeplitz quantization map

$$C^\infty(M) \to \operatorname{End}\left(\prod_{m\in\mathbb{N}_0} H^0(M, L^m)\right), \quad f \mapsto T_f^{(*)} = (T_f^{(m)})_{m\in\mathbb{N}_0}\,. \tag{39}$$

[1] Note that for homogeneous coordinate rings to determine the Krull dimension it is enough to consider chains of homogeneous prime ideals.

Due to the fact that $T_f^{(*)}$ respects the grading given by m, it can also be considered as an element of

$$\operatorname{End}\left(\bigoplus_{m\in\mathbb{N}_0} H^0(M, L^m)\right)$$

and is fixed by this restriction. Hence, $T_f^{(*)}$ is an algebraic object operating on an algebraic vector space which coincides with the coordinate ring. The coordinate ring should be considered as the quantum Hilbert space. Note that this set-up makes perfect sense also for singular projective varieties.

Clearly, there is also a metric aspect in the theory. Our line bundle comes with a hermitian metric. On $\mathbb{P}^n$ we have the Fubini-Study Kähler form ω_{FS} induced by the standard metric in $\mathbb{C}^{n+1}$. This defines a metric on the tautological bundle and by taking the inverse metric a hermitian metric h_{FS} on the hyperplane section bundle H. Suitable normalized it turns out that H with h_{FS} is the quantum line bundle of the Kähler manifold $(\mathbb{P}^n, \omega_{FS})$. If N is a closed submanifold of $\mathbb{P}^n$, i.e. a nonsingular projective variety and $i : N \to \mathbb{P}^n$ is the embedding then the pair $(N, i^*\omega_{FS})$ is a Kähler manifold with associated quantum line bundle (i^*H, i^*h_{FS}). In particular, nonsingular projective varieties are always quantizable. But note that if we start with a fixed Kähler manifold (M, ω_M) with very ample quantum line bundle (L, h) and induced embedding $\Phi : M \to \mathbb{P}^n$ then $(M, \Phi^*\omega_{FS})$ is again a quantizable Kähler manifold with quantum line bundle $(L \cong \Phi^*H, \Phi^*h_{FS})$. But in general we have for the two Kähler forms defined on the same complex manifold $\Phi^*\omega_{FS} \neq \omega_M$. We only know that they are cohomologous because they are representatives of the Chern class of the same bundle L. The question whether they coincide as forms has to do with the question whether the embedding is a Kähler embedding. This is related to Calabi's diastatic function, respectively to Rawnsley's epsilon function. I will not discuss this matter here, but see [1] for a discussion and references to further results.

Via the metric the projective coordinate ring $\bigoplus_{m\geq 0} H^0(M, L^m)$ carries also a metric structure. To have a full description of the quantization also in the singular case the metric should be studied in more detail.

3. Singularities

In the last section a point on a projective variety was called a singular point if the rank of the matrix (27) is less than expected (see Definition 2.4). In this section I will give a different characterization of singular points. In particular, it will turn out, that there exist singularities which are better than others.

Clearly, the definition of a singular point as given in Definition 2.4 is a local one. Hence it is enough to study the local situation. For the local situation it is more convenient to consider affine varieties instead of projective varieties. If the projective space is replaced by an affine space the definitions work accordingly.

After choosing coordinates the n-dimensional affine space is given as $\mathbb{K}^n$. A subset V of $\mathbb{K}^n$ is called an affine variety if there exists finitely many polynomials $f_1, f_2, \ldots, f_m \in \mathbb{K}[X_1, X_2, \ldots, X_n]$ such that

$$V = \mathcal{Z}(f_1, f_2, \ldots, f_m) = \{\alpha \in \mathbb{K}^n \mid f_i(\alpha) = 0,\ i = 1, \ldots, m\} .$$

If one replaces homogeneous polynomials, homogeneous ideals, etc. by arbitrary polynomials, arbitrary ideals, etc, the whole theory develops like in the projective case. Again, let $I = (f_1, \ldots, f_m)$ be the ideal generated by the above polynomials then $V = \mathcal{Z}(I)$. Vice versa, given a variety V in $\mathbb{K}^n$ we can define its vanishing ideal

$$\mathcal{I}(V) := \{f \in \mathbb{K}[X_1, X_2, \ldots, X_n] \mid f(\alpha) = 0, \forall \alpha \in V\} . \tag{40}$$

With the same definition (25) of the radical ideal we obtain $\mathcal{I}(\mathcal{Z}(I)) = Rad(I)$. The affine coordinate ring of the variety V is defined to be

$$\mathbb{K}[V] := \mathbb{K}[X_1, X_2, \ldots, X_n]/\mathcal{I}(V) . \tag{41}$$

The subset

$$U^{(i)} := \mathbb{P}^n \setminus \mathcal{Z}(X_i) = \{(\alpha_0 : \alpha_1 : \ldots : \alpha_n) \mid \alpha_i \neq 0\} \tag{42}$$

of $\mathbb{P}^n$ is a Zariski open (and hence dense) subset of $\mathbb{P}^n$. It can be identified with the affine space $\mathbb{K}^n$ via the map

$$\Phi_i((\alpha_0 : \alpha_1 : \cdot : \alpha_n)) \mapsto (\frac{\alpha_0}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}) . \tag{43}$$

In this way $\mathbb{P}^n$ is covered by $(n+1)$ copies of affine n-space, i.e. $\mathbb{P}^n = \cup_{i=0}^n U^{(i)}$. Every projective variety can be covered by affine varieties. Let $f_l(X_0, X_1, \ldots, X_n)$ for $l = 1, \ldots, m$ be defining homogeneous polynomials for the projective variety V. Fix i with $0 \le i \le n$ and let $f_l^{(i)}$ be the polynomials in n variables obtained from the f_l by setting the variable X_i to 1. Then $V^{(i)} = \mathcal{Z}(f_1^{(i)}, \ldots, f_m^{(i)})$ defines an affine variety. Via the map (43) we can identify $V^{(i)} = V \cap U^{(i)}$. Again $V = \cup_{i=0}^n V^{(i)}$. In particular every point of the projective variety lies at least in one of these affine varieties $V^{(i)}$.

In the following let V be an affine variety. Again the dimension $\dim V$ can be defined by Definition 2.2. This coincides with the Krull dimension of the coordinate ring $\mathbb{K}[V]$, i.e. $\dim V = \dim \mathbb{K}[V]$. In the affine case there is no subtraction of 1 necessary, because in this case there is a complete 1:1 correspondence between prime ideals and irreducible subvarieties. Note that if Y is a irreducible projective variety all covering affine varieties $Y^{(i)}$ will be irreducible affine varieties and vice versa. Additionally we have $\dim Y = \dim Y^{(i)}$ for non-empty $Y^{(i)}$.

Singular points of affine varieties can be defined according to Definition 2.4 (of course now only n variables will appear, hence we get an $m \times n$ matrix) using generators $f_1, f_2, \ldots, f_m$ of the vanishing ideal of the variety. If the affine variety $V^{(i)}$ comes from a projective variety V as described above then $x \in V$ corresponding to $\Phi(x)$ will be a singular point of V if and only if $\Phi(x)$ is a singular point of $V^{(i)}$.

There are some problems with this definition of a singular point. First, it is not a priori clear that it does not depend on the chosen generators of the ideal $\mathcal{I}(V)$. Second, the starting point of the definition is a variety lying in some affine space. But the singularity should be something intrinsic to the variety and not depend on the affine space the variety is lying in. It can be shown that indeed the notion does not depend on these choices. Nevertheless, a more intrinsic definition of a singularity would be desirable.

There is such a definition which deals with the local ring $\mathcal{O}_{V,\alpha}$ of the point α on V. This local ring is defined as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$ be a point on V. The vanishing ideal of α in the polynomial ring is the ideal $M_\alpha = (X_1 - \alpha_1, X_2 - \alpha_2, \ldots, X_n - \alpha_n)$. It is a maximal ideal. This says that every ideal which is strictly bigger than M_α is the whole polynomial ring. The condition $\alpha \in V$ is equivalent to $M_\alpha \supseteq I = \mathcal{I}(V)$. From this it follows that $M_\alpha \bmod I$ is a maximal ideal of $\mathbb{K}[V]$. The local ring $\mathcal{O}_{V,\alpha}$ of the variety V at the point α is defined as the *localization* of the ring $\mathbb{K}[V]$ with respect to the maximal ideal $M_\alpha \bmod I$ (for simplicity we will denote it also by M_α)

$$\mathcal{O}_{V,\alpha} = \mathbb{K}[V]_{M_\alpha} \ . \tag{44}$$

The localization is the ring of fractions where the denominators are elements from the multiplicative set $\mathbb{K}[V] \setminus M_\alpha$. It is a Noetherian local ring. Noetherian means that every ascending chain of ideals becomes stationary (or terminates). Local means that the ring has only one maximal ideal. Here the unique maximal ideal is $M_\alpha \bmod I/1$ (again simply denoted by M_α).

Definition 3.1. *A local ring R with maximal ideal M is called a regular local ring if*

$$\dim_{R/M} M/M^2 = \dim R \ . \tag{45}$$

Due to the fact that M is a maximal ideal of R the quotient R/M is a field, and M/M^2 is a vector space over R/M. On the left hand side of (45) the vector space dimension is meant, on the right hand side the Krull dimension is meant.

In our case where R is the local ring coming from the coordinate ring of a variety over an algebraically closed field K we have $R/M \cong \mathbb{K}$.

Definition 3.2. *A point $\alpha \in V$ is called a non-singular (or regular) point of V if its local ring $\mathcal{O}_{V,\alpha}$ is regular. If not it is called a singular point. A variety is called non-singular (or smooth, or regular) if all points are regular. The subset of singular points of V is denoted by $Sing(V)$.*

$Sing(V)$ is always an algebraic subvariety of codimension $\operatorname{codim}_V Sing(V) \geq 1$. In particular the non-singular locus is a non-empty Zariski open subset of V. If V is irreducible then $Sing(V)$ is dense. For irreducible V the dimensions of all local rings $\mathcal{O}_{V,\alpha}$ are constant and equal to the dimension of V.

The $\mathbb{K}$-vector space M_α/M_α^2 is also called the *Zariski cotangent space*, resp. its dual $(M_\alpha/M_\alpha^2)^*$ the *Zariski tangent space*. In general

$$\dim_{\mathbb{K}} M_\alpha/M_\alpha^2 \geq \dim \mathcal{O}_{V,\alpha} = dim V, \tag{46}$$

where we assume for the last equality V to be irreducible. Hence we can also define the singular points to be the points where the dimension of the Zariski tangent space is bigger than the dimension of the (irreducible) variety.

Let me illustrate this in the case of cubic curves. In an affine chart using the ideal

$$I = \big(Y^2 - 4X(X-a)(X-b)\big) \tag{47}$$

with $a, b \in \mathbb{K}$ we obtain the cubic curves as $\mathcal{Z}(I)$. In this normalisation $\alpha = (0,0)$ lies on the cubic. The cotangent space at α is given as

$$M_\alpha / M_\alpha^2 = (X, Y) \bmod I / (X^2, Y^2, XY) \bmod I\ . \tag{48}$$

From the relations given by I we calculate

$$Y^2 = 4abX - 8(a+b)X^2 + 4X^3 \bmod I\ . \tag{49}$$

Hence

$$4abX \in (X^2, Y^2, XY) \bmod I\ . \tag{50}$$

If $a \cdot b \neq 0$ the element Y will be enough to generate the quotient (48). The tangent space will be one-dimensional and $(0,0)$ will be a nonsingular point. If either a or b equals 0 the element X will also be necessary to generate the tangent space. Hence $(0,0)$ will be a singular point.

Given an arbitrary irreducible (projective or affine) variety V then there exists a stratification of the singularity set $Sing(V)$ obtained in the following manner. Let $U = V \setminus Sing(V)$ be the Zariski open set of regular points then $V \setminus U$ is a closed subvariety. It can be decomposed into finitely many irreducible varieties of dimension less than $\dim V$

$$V \setminus U = V_1^{(1)} \cup V_2^{(1)} \cup \cdots \cup V_l^{(1)}\ .$$

Again from this complement $Sing(V \setminus U)$ can be determined. It is a subvariety of the variety V from higher codimension. This process can be repeated as long as there are singularities. Because the codimension strictly increases it has to stop after finitely many steps.

Guided by the algebraic properties of the local rings we have an hierarchy for the types of singularities. If R is a ring without zero divisors then $Quot(R)$ is the ring whose elements are the fractions of elements in R with denominator $\neq 0$.

Definition 3.3. *A ring R (without zero divisors) is called a normal ring if the elements of $Quot(R)$ which are solutions of algebraic equations with coefficients from R and highest coefficient 1 lie already in R.*

It is a classical result (Gauß Lemma) that $\mathbb{Z}$ is normal and also that polynomial rings over fields are normal.

Definition 3.4. *Let V be an irreducible variety.*
(a) A variety is normal at a point $\alpha \in V$ if the local ring $\mathcal{O}_{V,\alpha}$ is normal.
(b) A variety is called normal if it is normal at every point $\alpha \in V$.
(c) A singular point is called a normal singular point, and the singularity is called a normal singularity, if the variety is normal at this point.

A regular local ring is always normal. Hence, regular points are always normal. If V is a normal variety it follows

$$\operatorname{codim}_V Sing(V) \geq 2 \ . \tag{51}$$

This says that normal singular varieties are "less singular" than generic singular varieties. A lot of the singular varieties which appear as moduli spaces are normal. Normal varieties behave from the point of functions defined on them similar to nonsingular varieties. For example, if V is a variety of dimension ≥ 2 and $x \in V$ a normal point then every regular function in $V \setminus \{x\}$ can be extended to a regular function on V. Additionally normality is necessary to have a well-behaved theory of (Weil-)divisors based on codimension 1 irreducible subvarieties.

For every irreducible affine variety V with singularity set $Sing(V)$ there exists a normal affine variety $\widetilde{V}$ and an algebraic morphism $\pi : \widetilde{V} \to V$ such that

$$\pi^{-1}(V \setminus Sing(V)) \cong V \setminus Sing(V) \ . \tag{52}$$

The variety $\widetilde{V}$ is called the normalization of V. It is obtained by a purely algebraic process, i.e. by taking the normal closure of the coordinate ring in its quotient field. This can be extended to the projective case too.

For algebraic curves normal points are always regular (there is no space for codimension two subvarieties). Hence the normalization gives already a desingularization. In the case of the above discussed singular cubic curves the normalization is given by the line (affine, resp. projective).

The question arises whether it might be even possible to find for every projective variety V with singularity set $Sing(V)$ a nonsingular projective variety Y which coincides with V outside $Sing(V)$, and is minimal in a certain sense. Such a Y is called a desingularization and the whole process is called a resolution of singularities. It was shown by Hironaka [8] (see also [12]) that there exists for projective varieties over fields of characteristic zero (and this is the case we are dealing with) a resolution of singularities. More precisely, for every projective variety V there exists a nonsingular projective variety Y and a proper [2] algebraic map $f : Y \to V$ such that f is an isomorphism over an open non-empty subset $U \subseteq V$, i.e. $f^{-1}(U) \cong U$.

4. Quotients

In this section let us assume $\mathbb{K} = \mathbb{C}$ for the base field.

4.1. Quotients in algebraic geometry

Moduli spaces of geometric objects are very often varieties with singularities. Typically, they are obtained starting from a smooth variety classifying the objects with respect to a certain "presentation". To obtain the moduli space one has to "divide out" the different presentations. Usually, one has a group operating on the presentations and a candidate of the moduli space is given by the quotient set

[2] the algebraic equivalent of a compact map

under the group action, the orbit space. Unfortunately, it is not always possible to endow the quotient set with a compatible structure of a variety again. Even if we allow the quotient to be an algebraic scheme it will not be possible. In our context schemes will appear as "varieties with multiplicities". It is quite reasonable that one should at least incorporate such objects in the theory. E.g. if we have two lines in the plane meeting at a point and we move one line with the intersection point fixed, we will nearly always have two lines. There is only one exception, if the moving line coincides with the fixed one. In this case the configuration consists of one line. But from the deformation point of view we should better count this special line twice, i.e. we should consider it as double line. The language of schemes deals with such objects and even with much more general ones. Nevertheless to avoid giving additional definitions I will still work on algebraic varieties (affine, projective, quasiprojective) in the following. But the reader should keep in mind that the language of algebraic schemes would be more appropriate for moduli problems. See [6] for an introduction to this field.

Let X be an algebraic variety and G a reductive algebraic group acting algebraically on X. This means that G is the complexification of a maximal compact subgroup K of G. Of special importance (and this are the examples that the reader should keep in mind) are the groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, and $\mathrm{PGL}(n)$. As indicated above it is important to study "quotients" of X under actions of the group G. Mumford has given with his geometric invariant theory (GIT) [13] the principal tool to deal with such quotients.

Definition 4.1. *A morphism of algebraic varieties $f : X \to Y$ is called a good quotient if*

(1) *f is surjective and G-invariant, i.e. $f(gx) = f(x)$, for all $g \in G$ and $x \in X$,*
(2) *$(f_*(\mathcal{O}_X))^G = \mathcal{O}_Y$,*
(3) *if V is a G-invariant closed subset of X then $f(V)$ is closed in Y, and if V_1 and V_2 are G-invariant closed subsets of X then*

$$V_1 \cap V_2 = \emptyset \implies f(V_1) \cap f(V_2) = \emptyset\ .$$

In Condition (2) $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the structure sheaves of the varieties X and Y. They are essentially nothing else as the sheaves of local regular functions on X and Y respectively. Condition (2) states that the local regular functions on Y can be given as those local regular functions on X which are constant along the fiber and invariant under G.

A good quotient is a *categorical quotient* in the sense that

(1) f is constant on the orbits of the action,
(2) for every algebraic variety Z with a morphism $g : X \to Z$ which is constant on the orbits of the G-action on X there exist a unique morphism $\bar{g} : Y \to Z$ with $g = \bar{g} \circ f$.

Definition 4.2. *A morphism of algebraic varieties $f : X \to Y$ is called a geometric quotient if*

(1) *$f : X \to Y$ is a good quotient,*
(2) *for every $y \in Y$ the fiber $f^{-1}(y)$ consists exactly of one orbit under the group action.*

If any of these quotients exists then they are unique.

For a good quotient there might exists fibers consisting of several orbits under the group action and these orbits are not necessarily closed (for a geometric quotient the orbits are always closed).

If we have a geometric quotient than the orbit space carries a structure of an algebraic variety. But this condition is very often to strong to be fulfilled. We have sometimes to assign several orbits to one geometric point to obtain a geometric structure and to end up (hopefully) with a good quotient. Mumford's concept of stability will help to decide what to do. Let $X \subseteq \mathbb{P}^n$ be a projective algebraic variety and G a reductive algebraic group embedded into $\mathrm{GL}(n+1)$ with an action of G on X given by the standard linear action of $\mathrm{GL}(n+1)$ on the points in $\mathbb{P}^n$.

Definition 4.3. *(1) A point $x \in X$ is called semi-stable if and only if there exists a non-constant G-invariant homogeneous polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$ with $F(x) \neq 0$.*
(2) A point $x \in X$ is called stable if and only if

(a) *the dimension of the Orbit $O(x)$ under the G-action equals the dimension of the group and*
(b) *there exists a non-constant G-invariant homogeneous polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$ with $F(x) \neq 0$, and the action of G on the zero set $X_F := \{y \in X \mid F(y) = 0\}$ is closed, i.e. if for every $y_0 \in X_F$ the orbit $O(y_0)$ is closed.*

The set of stable [3] points of X under the above group action G is denoted by X^s, the set of semi-stable points is denoted by X^{ss}. Both are open subset. Clearly, $X^s \subseteq X^{ss} \subseteq X$.

Let me point out that the notion of stability might depend on the embedding of the projective variety X into projective space and a corresponding linearization of the action of G. Recall from Section 2.2 that for an abstract projective variety X such an embedding is defined by the choice of a very ample line bundle L on X and a choice of basis of its global sections.

Theorem 4.4. *Assume that X^s is non-empty then there exists a projective algebraic variety Y and a morphism $f_{ss} : X^{ss} \to Y$ such that*

(1) *f_{ss} is a good quotient of X^{ss} by G,*
(1) *there exists an open subset $U \subseteq Y$ such that $f_{ss}^{-1}(U) = X^s$ and $f_s := f_{ss|X^s} : X^s \to U$ is a geometric quotient of X^s by G.*

[3] "Stable" in the above introduced sense corresponds to "properly stable" in the definition of Mumford. Stability in his sense does not require the condition on the dimension of the orbit.

The good quotient is projective, but the geometric quotient is as an open subset of something projective in general only quasi-projective. If we interpret this in the opposite way, we see that we will need in general also non-stable (but still semi-stable) points to obtain projective (this means compact in the complex topology) moduli spaces. Clearly, even if the projective variety we started with was smooth there is no reason to expect that the quotient will be smooth.

For more details one might consult [13], or for a more leisurely reading [14].

4.2. The relation with the symplectic quotient

In this subsection I want to quote results on the relation between the quotients in algebraic geometry and the symplectic quotients. The results are taken from Francis Kirwan's appendix to the third edition of Mumford's book on GIT [13] and are due to Kirwan, Kempf and Ness. More details and references can be found there.

Let X be a nonsingular projective complex variety in $\mathbb{P}^n$, and G a reductive group acting linearly on X via $\rho : G \to \mathrm{GL}(n+1)$. If K is any fixed maximal compact subgroup of G then after a suitable choice of coordinates the subgroup K acts unitarily on X, i.e. $\rho_{|K} : K \to \mathrm{U}(n+1)$. Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{k}^*$ its dual, and $\mu : X \to \mathfrak{k}^*$ the standard moment map defined for all $a \in \mathfrak{k}$ by

$$\mu(x)(a) := \frac{{}^t\overline{\hat{x}} \cdot \rho_*(a)\hat{x}}{2\pi \mathrm{i}\,||\hat{x}||}. \tag{53}$$

Here $\hat{x} \in \mathbb{C}^{n+1} \setminus \{0\}$ is any vector of homogeneous coordinates representing the element $x \in X \subseteq \mathbb{P}^n$ and $\rho_* : \mathfrak{k} \to \mathfrak{u}(n+1)$ is the tangent map of $\rho_{|K}$.

In this situation the *symplectic quotient* (or *Marsden-Weinstein reduction*) is defined as $\mu^{-1}(0)/K$. On the other hand we can define the good quotient (which is also a categorical quotient) of the semi-stable points X^{ss} by G. By Theorem 4.4 it is a projective variety which is commonly also denoted by $X//G$. It contains as open subset the geometric quotient X^s/G. Immediately the following question arrises: How are these quotients related?

Theorem 4.5. *(a) The point $x \in X$ is semi-stable if and only if $\overline{O_G(x)} \cap \mu^{-1}(0) \neq \emptyset$, i.e. the closure of the orbit of x under G meets $\mu^{-1}(0)$.*
(b) $\mu^{-1}(0) \subseteq X^{ss}$.
(c) The inclusion under (b) induces a homeomorphism

$$\mu^{-1}(0)/K \to X//G\ . \tag{54}$$

(d) If we denote by $\mu^{-1}(0)_{reg}$ the set of the $x \in \mu^{-1}(0)$ for which the tangent map $d\mu_x$ of the moment map is surjective, then the homeomorphism under (c) restricts to a homeomorphism

$$\mu^{-1}(0)_{reg}/K \to X^s/G\ . \tag{55}$$

Hence we see that as topological spaces the symplectic quotient is isomorphic to the good quotient and the subspace of the regular points $\mu^{-1}(0)_{reg}/K$ is isomorphic to the geometric quotient.

Things get slightly more complicated if we consider also the complex structure. If X is a compact Kähler manifold then the symplectic quotient $\mu^{-1}(0)_{reg}/K$ carries a structure of a complex Kähler manifold away from the singularities. But X^s/G carries also a complex structure from the geometric quotient construction. These two structures coincide on the subset where the symplectic quotient has no singularities. Hence if $\mu^{-1}(0)_{reg}/K$ is a Kähler manifold the complex structure coincides with the complex structure of the geometric quotient. But there exists examples (e.g. given by Kirwan [13, p.159]) where the geometric quotient has no singularities but the symplectic quotient has singularities in the sense that the Kähler structure coming from the reduction process is singular at certain points. Further examples of the relation between the structure of the singularities of the two type of quotients are given in the contribution of J. Huebschmann [9] to this volume.

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