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ABSTRACT. We use a natural affine connection with nontrivial torsion on an arbitrary almost-Kähler manifold which respects the almost-Kähler structure to construct a Fedosov-type deformation quantization on this manifold.

1. INTRODUCTION

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Denote by $C^\infty(M)[[\nu]]$ the space of formal series in ν with coefficients from $C^\infty(M)$. Deformation quantization on $(M, \{\cdot, \cdot\})$, as defined in [1], is an associative algebra structure on $C^\infty(M)[[\nu]]$ with the ν-linear and ν-adically continuous product $*$ (named a star-product) given on $f, g \in C^\infty(M)$ by the formula

$$(1.1) \qquad f * g = \sum_{r=0}^{\infty} \nu^r C_r(f, g),$$

where $C_r, r \geq 0$, are bilinear operators on $C^\infty(M)$, $C_0(f, g) = fg$, and $C_1(f, g) - C_1(g, f) = i\{f, g\}$. A star product $*$ is called differential if all the operators $C_r, r \geq 0$, are bidifferential.

Two differential star products $*$ and $*'$ are called equivalent if there exists an isomorphism of algebras $B : (C^\infty(M)[[\nu]], *) \to (C^\infty(M)[[\nu]], *')$, where $B = Id + \nu B_1 + \nu^2 B_2 + \dots$ and $B_r, r \geq 1$, are differential operators on $C^\infty(M)$.

If (M, ω) is a symplectic (and therefore a Poisson) manifold then to each differential star-product $*$ on M one can relate its characteristic class $cl(*) \in (1/i\nu)[\omega] + H^2(M, \mathbb{C})[[\nu]]$ (see [13],[27],[28],[11]). The equivalence classes of differential star-products on (M, ω) are bijectively

parameterized by the elements of the affine vector space $(1/i\nu)[\omega] + H^2(M, \mathbb{C})[[\nu]]$ via the mapping $* \mapsto cl(*)$.

The existence proofs and description of equivalence classes of star-products on symplectic manifolds were given by a number of people (see [10],[29],[14] for existence proofs and [13],[27],[28],[11] for the classification). The questions of existence and classification on general Poisson manifolds were solved by Kontsevich in [24].

Historically the first examples of star products (Moyal-Weyl and Wick star-products) were obtained from asymptotic expansions of related symbol products w.r.t. a numerical parameter $\hbar$ (the 'Planck constant'), as $\hbar \to 0$. Later a number of star-products on Kähler manifolds were obtained from the asymptotic expansion in $\hbar$ of the symbol product of Berezin's covariant symbols (see [7],[8],[9],[25],[26],[21]). These star-products on Kähler manifolds are differential and have the property of 'separation of variables': the corresponding bidifferential operators C_r differentiate their first argument only in antiholomorphic directions and the second argument only in holomorphic ones. The deformation quantizations with separation of variables on an arbitrary Kähler manifold where completely described and parameterized by geometric objects, the formal deformations of the Kähler (1,1)-form (see [19]). The star-products obtained from the product of Berezin's covariant symbols on coadjoint orbits of compact semisimple Lie groups where identified in [21] (that is, the corresponding formal (1,1)-form was calculated).

Another interesting case where star-products are obtained from symbol constructions is the so called Berezin-Toeplitz quantization on arbitrary compact Kähler manifolds [3],[31],[30]. In this quantization scheme Berezin's contravariant symbols are used (see [2]), which, in general, do not have a well defined symbol product. The semiclassical properties of Berezin-Toeplitz quantization in [3] were proved with the use of generalized Toeplitz structures developed by Guillemin and Boutet de Monvel in [6]. With the same technique for compact Kähler manifolds an associated deformation quantization, the Berezin-Toeplitz deformation quantization, was constructed [31],[30]. However, this construction is very implicit and it is a daunting task to calculate with its use the operators C_r even for small values of r. In [23] the Berezin-Toeplitz deformation quantization was shown to be a differentiable deformation quantization with separation of variables and its classifying (1,1)-form was explicitly calculated. Thus the operators C_r of the Berezin-Toeplitz deformation quantization can be calculated recursively by the simple algorithm from [19].

It was shown by Guillemin in [17] that, using generalized Toeplitz structures, it is even possible to construct a star-product on an arbitrary compact symplectic manifold. Later Borthwick and Uribe [5] introduced a natural 'almost-Kähler quantization' on an arbitrary compact almost-Kähler manifold by defining an operator quantization of Berezin-Toeplitz type. From results by Guillemin and Uribe [18] it follows that the generalized Toeplitz structure associated to these operators exists and the correct semi-classical limit in the almost Kähler case follows like in [3]. Therefore there should exist the corresponding natural deformation quantization on an arbitrary compact almost-Kähler manifold. It would be interesting to describe this deformation quantization directly.

The goal of this Letter is to show that one can construct a natural deformation quantization on an arbitrary almost-Kähler manifold using Fedosov's machinery. To this end we use the natural affine connection introduced by Yano [32], which respects the almost-Kähler structure. This connection necessarily has a nontrivial torsion whenever the almost complex structure is non-integrable. We generalize Fedosov's construction to the case of affine connections with torsion and obtain a star-product on an arbitrary almost-Kähler manifold. In the Kähler case this star-product coincides with the star-product with separation of variables (or of Wick type) constructed in [4]. Our considerations were very much motivated by the paper [12], where the calculations concerning the star-product from [4] were made in arbitrary local coordinates (rather than in holomorphic coordinates as in [4]).

It follows from the results obtained in [20] and [22] that the characteristic class of the star-product from [4] is $(1/i\nu)[\omega] - (1/2i)\varepsilon$, where ε is the canonical class of the underlying Kähler manifold. The canonical class is, however, defined for any almost-complex manifold.

We calculate the crucial part of the characteristic class cl of the star-product $*$ which we have constructed on an arbitrary almost-Kähler manifold, it's coefficient c_0 at the zeroth degree of ν. As in the Kähler case, $c_0(*) = -(1/2i)\varepsilon$.

2. A CONNECTION PRESERVING AN ALMOST-KÄHLER STRUCTURE

In this sections we recall some elementary properties of the Nijenhujs tensor and reproduce the construction from [32] of a natural affine connection on an arbitrary almost-Kähler manifold, which respects the almost-Kähler structure.

Let (M, J, ω) be an almost-Kähler manifold, i.e., a manifold M endowed with an almost-complex structure J and a symplectic form ω which are compatible in the following sense, $\omega(JX, JY) = \omega(X, Y)$ for any vector fields X, Y on M, and $g(X, Y) = \omega(JX, Y)$ is a Riemannian metric. Actually, in what follows we only require the metric g to be nondegenerate, but not necessarily positive definite. It is known that on any symplectic manifold (M, ω) one can choose a compatible almost complex structure to make it to an almost-Kähler manifold.

In local coordinates $\{x^k\}$ on a chart $U \subset M$ set

$$\partial_j = \partial/\partial x^j, \ J(\partial_j) = J_j^k \partial_k, \ \omega_{jk} = \omega(\partial_j, \partial_k), \ g_{jk} = g(\partial_j, \partial_k) \ .$$

Then

$$(2.1) \qquad\qquad J_j^k = g_{j\alpha}\omega^{\alpha k} = g^{k\alpha}\omega_{\alpha j},$$

where (g^{jk}) and (ω^{jk}) are inverse matrices for (g_{kl}) and (ω_{kl}) respectively. Here, as well as in the sequel, we use the tensor rule of summation over repeated indices.

The Nijenhuis tensor for the complex structure J is given by the following formula:

$$(2.2) \qquad N(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

where X, Y are vector fields on M. In local coordinates set $N(\partial_j, \partial_k) = N_{jk}^l \partial_l$. Formula (2.2) takes the form

$$(2.3) \qquad N_{jk}^l = (\partial_\alpha J_j^l)J_k^\alpha - (\partial_\alpha J_k^l)J_j^\alpha + (\partial_j J_k^\beta - \partial_k J_j^\beta)J_\beta^l.$$

Let X be a vector field of type $(1,0)$ with respect to the almost-complex structure J. Then it follows from formula (2.2) that

$$N(X, Y) = (1 + iJ)([X, Y] + J[X, JY]) \ .$$

Therefore $N(X, Y)$ is of type $(0,1)$. The latter is also true if Y is of type $(1,0)$ instead. Similarly, if X or Y is of type $(0,1)$ then $N(X, Y)$ is of type $(1,0)$. In particular, if X and Y are of types $(1,0)$ and $(0,1)$ respectively, then $N(X, Y) = 0$. Therefore, $(1 \pm iJ)N((1 \pm iJ)\partial_j, \partial_k) = 0$ and $N((1 \pm iJ)\partial_j, (1 \mp iJ)\partial_k) = 0$, from whence we obtain the two following formulas:

$$(2.4) \qquad\qquad N_{jk}^l = J_j^\alpha N_{\alpha k}^\beta J_\beta^l \text{ and } N_{jk}^l = -J_j^\alpha N_{\alpha\beta}^l J_k^\beta.$$

Let ∇ be an arbitrary affine connection on M. In local coordinates let Γ^l_{jk} be its Christoffel symbols and $T^l_{jk} = \Gamma^l_{jk} - \Gamma^l_{kj}$ the torsion tensor.

In the sequel we shall need the formula

$$(2.5) \qquad \nabla_j \omega_{kl} + \nabla_k \omega_{lj} + \nabla_l \omega_{jk} = -(T^\alpha_{jk}\omega_{\alpha l} + T^\alpha_{kl}\omega_{\alpha j} + T^\alpha_{lj}\omega_{\alpha k}),$$

which easily follows from the closedness of ω. Here $\nabla_i := \nabla_{\partial_i}$.

By a direct calculation we get from formula (2.3) that

$$(2.6) \qquad N^l_{jk} = (\nabla_\alpha J^l_j)J^\alpha_k - (\nabla_\alpha J^l_k)J^\alpha_j + (\nabla_j J^\beta_k - \nabla_k J^\beta_j)J^l_\beta - S^l_{jk},$$

where

$$(2.7) \qquad S^l_{jk} = T^l_{jk} - J^\alpha_j T^l_{\alpha\beta} J^\beta_k + J^\alpha_j T^\beta_{\alpha k} J^l_\beta - J^\alpha_k T^\beta_{\alpha j} J^l_\beta.$$

Assume that the connection ∇ respects the almost-complex structure J. Then formula (2.6) reduces to

$$(2.8) \qquad N^l_{jk} = -S^l_{jk}.$$

Thus (2.8) is a necessary condition for ∇ to respect the almost-complex structure J.

One can check directly with the use of formulas (2.4) and (2.7) that if

$$(2.9) \qquad T^l_{jk} = (-1/4)N^l_{jk}$$

then condition (2.8) is always satisfied.

Proposition 2.1. *Let ∇ be the unique affine connection which respects the metric g and has the torsion given by formula (2.9). Then ∇ also respects the symplectic form ω and therefore the complex structure J.*

This result is due to Yano [32]. For the convenience of the reader we provide here a proof of the statement.

Proof. Denote $Z_{jkl} = T^\alpha_{jk}\omega_{\alpha l} + T^\alpha_{kl}\omega_{\alpha j} + T^\alpha_{lj}\omega_{\alpha k} = (-1/4)(N^\alpha_{jk}\omega_{\alpha l} + N^\alpha_{kl}\omega_{\alpha j} + N^\alpha_{lj}\omega_{\alpha k})$. Clearly, Z_{jkl} is a totally antisymmetric tensor. It follows from (2.5) that

$$(2.10) \qquad \nabla_j\omega_{kl} + \nabla_k\omega_{lj} + \nabla_l\omega_{jk} = -Z_{jkl}.$$

First we show that

$$(2.11) \qquad J^\alpha_j Z_{\alpha kl} = J^\alpha_k Z_{\alpha lj} = J^\alpha_l Z_{\alpha jk}.$$

One has

$$(2.12) \qquad J^\alpha_j Z_{\alpha kl} = (-1/4)(J^\alpha_j N^\beta_{\alpha k}\omega_{\beta l} + J^\alpha_j N^\beta_{l\alpha}\omega_{\beta k} + J^\alpha_j N^\beta_{kl}\omega_{\beta\alpha}).$$

Using (2.4) and (2.1) one obtains $J_j^\alpha N_{\alpha k}^\beta \omega_{\beta l} = -N_{jk}^\alpha J_\alpha^\beta \omega_{\beta l} = -N_{jk}^\alpha g_{\alpha l}$. Similarly, $J_j^\alpha N_{l\alpha}^\beta \omega_{\beta k} = -N_{lj}^\alpha g_{\alpha k}$. Finally, $J_j^\alpha N_{kl}^\beta \omega_{\beta \alpha} = -N_{kl}^\alpha g_{\alpha j}$. Therefore

$$J_j^\alpha Z_{\alpha kl} = (1/4)(N_{jk}^\alpha g_{\alpha l} + N_{lj}^\alpha g_{\alpha k} + N_{kl}^\alpha g_{\alpha j}),$$

from whence (2.11) follows.

Formula (2.6) takes the form

$$(2.13) \qquad (\nabla_\alpha J_j^l)J_k^\alpha - (\nabla_\alpha J_k^l)J_j^\alpha + (\nabla_j J_k^\beta - \nabla_k J_j^\beta)J_\beta^l = 0,$$

due to the fact that (2.8) is now satisfied. Using (2.1) and the condition that ∇ respects the metric g we rewrite (2.13) as follows:

$$(2.14) \quad g^{l\beta}(\nabla_\alpha \omega_{\beta j})J_k^\alpha - g^{l\beta}(\nabla_\alpha \omega_{\beta k})J_j^\alpha + g^{\alpha\beta}(\nabla_j \omega_{\alpha k} - \nabla_k \omega_{\alpha j})J_\beta^l = 0.$$

Using (2.1) once more, we get $g^{\alpha\beta}J_\beta^l = \omega^{\alpha l} = -g^{l\beta}J_\beta^\alpha$ and rewrite (2.14) in the form

$$(2.15) \qquad (\nabla_\alpha \omega_{\beta j})J_k^\alpha - (\nabla_\alpha \omega_{\beta k})J_j^\alpha - (\nabla_j \omega_{\alpha k} - \nabla_k \omega_{\alpha j})J_\beta^\alpha = 0.$$

It follows from formula (2.10) that $\nabla_j \omega_{\alpha k} - \nabla_k \omega_{\alpha j} = \nabla_\alpha \omega_{jk} + Z_{\alpha jk}$. Therefore one gets from (2.15) that

$$(2.16) \qquad (\nabla_\alpha \omega_{\beta j})J_k^\alpha + (\nabla_\alpha \omega_{k\beta})J_j^\alpha - (\nabla_\alpha \omega_{jk})J_\beta^\alpha = J_\beta^\alpha Z_{\alpha jk}.$$

Cyclicly permuting the indices $\beta \to j \to k \to \beta$ in (2.16) and adding the resulting equation to (2.16) one obtains

$$(2.17) \qquad 2(\nabla_\alpha \omega_{\beta j})J_k^\alpha = J_\beta^\alpha Z_{\alpha jk} + J_j^\alpha Z_{\alpha k\beta} = 2J_k^\alpha Z_{\alpha\beta j}.$$

The last equality in (2.17) follows from (2.11). Thus

$$(2.18) \qquad \nabla_\alpha \omega_{\beta j} = Z_{\alpha\beta j}.$$

Summing up (2.18) over all the cyclic permutations of indices α, β, j, one gets from (2.10) that $-Z_{\alpha\beta j} = 3Z_{\alpha\beta j}$. Therefore $Z_{\alpha\beta j} = 0$. Now the statement of the proposition follows from (2.18) and (2.1).   □

Notice that if (M, J, ω) is a Kähler manifold then the connection ∇ from Proposition 2.1 is just the Kähler connection.

3. A MODIFICATION OF FEDOSOV'S CONSTRUCTION

In this section we shall slightly modify Fedosov's construction to obtain a deformation quantization on an almost-Kähler manifold (M, J, ω) endowed with a fixed affine connection ∇ which respects the almost-Kähler structure and has a nontrivial torsion. The existence of such a connection was shown in the previous section.

As in Section 2, we shall work in local coordinates $\{x^k\}$ on a coordinate chart $U \subset M$ and use the same notation. Following Fedosov,

denote by $\{y^k\}$ the fibre coordinates on the tangent bundle w.r.t. the frame $\{\partial_k\}$.

We introduce a tensor $\Lambda^{jk} := \omega^{jk} - ig^{jk}$ on M and define the formal Wick algebra W_x for $x \in M$ associated with the tangent space $T_x M$, whose elements are formal series

$$a(\nu, y) = \sum_{r \geq 0, |\alpha| \geq 0} \nu^r a_{r,\alpha} y^\alpha,$$

where α is a multi-index and the standard multi-index notation is used. The formal Wick product on W_x is given by the formula

$$(3.1) \qquad a \circ b\,(y) := \exp\left(\frac{i\nu}{2} \Lambda^{jk} \frac{\partial^2}{\partial y^j \partial z^k}\right) a(y) b(z)|_{z=y}.$$

Taking a union of algebras W_x we obtain a bundle W of formal Wick algebras. Denote by $\mathcal{W}$ the sheaf of its smooth sections. The fibre product (3.1) can be extended to the space $\mathcal{W} \otimes \Lambda$ of W-valued differential forms by means of the usual exterior product of the scalar forms Λ.

We introduce gradings deg_ν, deg_s, deg_a on $\mathcal{W} \otimes \Lambda$ defined on homogeneous elements ν, y^k, dx^k as follows:

$$deg_\nu(\nu) = 1, \quad deg_s(y^k) = 1, \quad deg_a(dx^k) = 1.$$

All other gradings of the elements ν, y^k, dx^k are set to zero. The grading deg_a is induced from the standard grading on Λ.

The product $\circ$ on $\mathcal{W} \otimes \Lambda$ is bigraded w.r.t. the grading $Deg = 2deg_\nu + deg_s$ and the grading deg_a.

The connection ∇ can be extended to an operator on $\mathcal{W} \otimes \Lambda$ such that for $a \in \mathcal{W}$ and a scalar differential form λ

$$(3.2) \qquad \nabla(a \otimes \lambda) := \left(\frac{\partial a}{\partial x^j} - \Gamma^l_{jk} y^k \frac{\partial a}{\partial y^l}\right) \otimes (dx^j \wedge \lambda) + a \otimes d\lambda.$$

Using formulas (3.1) and (3.2) one can show that ∇ is a deg_a-graded derivation of the algebra $(\mathcal{W} \otimes \Lambda, \circ)$.

We introduce Fedosov's operators δ and δ^{-1} on $\mathcal{W} \otimes \Lambda$ as follows. Assume $a \in \mathcal{W} \otimes \Lambda$ is homogeneous w.r.t. the gradings deg_s and deg_a with $deg_s(a) = p,\ deg_a(a) = q$. Set

$$\delta(a) = dx^j \wedge \frac{\partial a}{\partial y^j} \quad \text{and} \quad \delta^{-1}a = \begin{cases} \frac{1}{p+q} y^j i\left(\frac{\partial}{\partial x^j}\right) a & \text{if } p+q > 0, \\ 0, & \text{if } p = q = 0. \end{cases}$$

Then for $a \in \mathcal{W} \otimes \Lambda$ one has

$$(3.3) \qquad a = \delta\delta^{-1}a + \delta^{-1}\delta a + \sigma(a),$$

where $a \mapsto \sigma(a)$ is the projection on the (deg_s, deg_a)-bihomogeneous part of a of bidegree zero $(deg_s(a) = deg_a(a) = 0)$. It is easy to check that the operator δ is also a deg_a-graded derivation of the algebra $(\mathcal{W} \otimes \Lambda, \circ)$.

Define the elements

$$T := \frac{1}{2} \omega_{s\alpha} T^{\alpha}_{kl} y^s dx^k \wedge dx^l \quad \text{and} \quad R := \frac{1}{4} \omega_{s\alpha} R^{\alpha}_{tkl} y^s y^t dx^k \wedge dx^l$$

of $\mathcal{W} \otimes \Lambda$, where

$$R^s_{tkl} := \frac{\partial \Gamma^s_{lt}}{\partial x^k} - \frac{\partial \Gamma^s_{kt}}{\partial x^l} + \Gamma^s_{k\alpha} \Gamma^{\alpha}_{lt} - \Gamma^s_{l\alpha} \Gamma^{\alpha}_{kt}$$

is the curvature tensor of the connection ∇. Then the formulas

$$(3.4) \qquad [\nabla, \delta] = \frac{i}{\nu} ad_{Wick}(T), \quad \nabla^2 = -\frac{i}{\nu} ad_{Wick}(R)$$

can be obtained by a direct calculation using (3.1), (3.2) and the identity

$$(3.5) \qquad \omega_{s\alpha} R^{\alpha}_{tkl} = \omega_{t\alpha} R^{\alpha}_{skl}$$

proved in [15] (the proof is valid also for connections with torsion). In (3.4) $[\cdot, \cdot]$ is the deg_a-graded commutator of endomorphisms of $\mathcal{W} \otimes \Lambda$ and ad_{Wick} is defined via the deg_a-graded commutator in $(\mathcal{W} \otimes \Lambda, \circ)$.

The following two theorems are minor modifications of the standard statements of Fedosov's theory adapted to the case of affine connections with torsion. We shall denote the Deg-homogeneous component of degree k of an element $a \in \mathcal{W} \otimes \Lambda$ by $a^{(k)}$.

Theorem 3.1. *There exists a unique element $r \in \mathcal{W} \otimes \Lambda$ such that $r^{(0)} = r^{(1)} = 0$, $deg_a(r) = 1$, $\delta^{-1} r = 0$, satisfying the equation*

$$\delta r = T + R + \nabla r - \frac{i}{\nu} r \circ r .$$

It can be calculated recursively with respect to the total degree Deg as follows:

$$r^{(2)} = \delta^{-1} T,$$

$$r^{(3)} = \delta^{-1} \left(R + \nabla r^{(2)} - \frac{i}{\nu} r^{(2)} \circ r^{(2)} \right),$$

$$r^{(k+3)} = \delta^{-1} \left(\nabla r^{(k+2)} - \frac{i}{\nu} \sum_{l=0}^{k} r^{(l+2)} \circ r^{(k-l+2)} \right), k \geq 1.$$

Then the Fedosov connection $D := -\delta + \nabla - \frac{i}{\nu} ad_{Wick}(r)$ is flat, i.e., $D^2 = 0$.

The proof of the theorem is by induction, with the use of the identities

$$\delta T = 0 \quad \text{and} \quad \delta R = \nabla T.$$

The identity $\delta T = 0$ follows from (2.5) and the fact that the connection ∇ respects the form ω. The identity $\delta R = \nabla T$ can be proved by a direct calculation with the use of (3.5).

The Fedosov connection D is a deg_a-graded derivation of the algebra $(\mathcal{W} \otimes \Lambda, \circ)$. Therefore $\mathcal{W}_D := \ker D \cap \mathcal{W}$ is a subalgebra of $(\mathcal{W}, \circ)$.

Theorem 3.2. *The projection $\sigma : \mathcal{W}_D \to C^\infty(M)[[\nu]]$ onto the part of deg_s-degree zero is a bijection. The inverse mapping $\tau : C^\infty(M)[[\nu]] \to \mathcal{W}_D$ for a function $f \in C^\infty(M)$ can be calculated recursively w.r.t. the total degree Deg as follows:*

$$\tau(f)^{(0)} = f,$$

$$\tau(f)^{(k+1)} = \delta^{-1}\left(\nabla \tau(f)^{(k)} - \frac{i}{\nu} \sum_{l=0}^{k} ad_{Wick}\left(r^{(l+2)}\right)\left(\tau(f)^{(k-l)}\right) \right), k \geq 0.$$

The product $*$ on $C^\infty(M)[[\nu]]$ defined by the formula

$$f * g := \sigma(\tau(f) \circ \tau(g)),$$

is a star-product on (M, J, ω).

Assume that the connection ∇ in Theorem 3.1 is as in Proposition 2.1. Then in the case when (M, J, ω) is a Kähler manifold the star-product given by Theorem 3.2 coincides with the star-product of Wick type constructed in [4].

4. CALCULATION OF THE CLASS c_0

It is well known that to each star-product $*$ on a symplectic manifold (M, ω) a formal cohomology class $cl(*) \in (1/i\nu)[\omega] + H^2(M, \mathbb{C})[[\nu]]$ is related (see, e.g., [16]). This class (named the characteristic class of deformation quantization) determines the star-product up to equivalence. Denote by $c_0(*)$ the coefficient of $cl(*)$ at zeroth degree of the formal parameter ν. The class c_0 is, in some sense, the most intriguing part of the characteristic class cl. Only the coefficient $c_0(*)$ of the class $cl(*) = (1/i\nu)[\omega] + c_0(*) + \ldots$ can not be recovered from Deligne's intrinsic class. Also the cohomology class of the formal Kähler form parameterizing a quantization with separation of variables on a Kähler manifold differs from the characteristic class of this quantization only in the coefficient c_0 (see [20]).

In this section we shall calculate the class c_0 of the deformation quantization obtained in Theorem 3.2. First we recall the definition of the class c_0 of a star-product (1.1) (see, e.g., [20]). For a function

$f \in C^\infty(M)$ on a symplectic manifold (M, ω) denote by ξ_f the corresponding Hamiltonian vector field. For a bilinear operator $C = C(f, g)$ denote by C^- its antisymmetric part, $C^-(f, g) := (1/2)(C(f, g) - C(g, f))$. A star-product $*$ given by (1.1) is called *normalized* if $C_1(f, g) = (i/2)\{f, g\}$. For a normalized star-product $*$ the bilinear operator C_2^- is a de Rham – Chevalley 2-cocycle. There exists a unique closed 2-form $\varkappa$ such that for all $f, g \in C^\infty(M)$ one obtains $C_2^-(f, g) = (1/2)\varkappa(\xi_f, \xi_g)$. The class c_0 of a normalized star-product $*$ is defined as $c_0(*) := [\varkappa]$.

It is well known that each star-product on a symplectic manifold is equivalent to a normalized one. One defines the class $c_0(*)$ of a star-product $*$ as the cohomology class $c_0(*')$ of an equivalent normalized star-product $*'$. In order to calculate the class $c_0(*)$ of the star-product $*$ from Theorem 3.2 we shall first construct an equivalent normalized star-product $*'$.

We introduce the fibrewise equivalence operator on $\mathcal{W}$ defined by the formula

(4.1) $$G := \exp\left(-\nu\Delta\right),$$

where Δ is given in local coordinates as follows:

$$\Delta = \frac{1}{4} g^{jk} \frac{\partial^2}{\partial y^j y^k}.$$

It is well known that the fibrewise star-product $\circ'$ defined on $\mathcal{W}$ as follows, $a \circ' b := G(G^{-1} \circ G^{-1}b)$, is the Weyl star-product:

(4.2) $$a \circ' b\,(y) = \exp\left(\frac{i\nu}{2}\omega^{jk}\frac{\partial^2}{\partial y^j \partial z^k}\right) a(y)b(z)|_{z=y}.$$

The following formulas

(4.3) $$[\nabla, \Delta] = [\nabla, G] = 0 \quad \text{and} \quad [\delta, \Delta] = [\delta, G] = 0$$

can be checked directly.

Pushing forward the Fedosov connection D obtained in Theorem 3.1 via G and taking into account formulas (4.3) we obtain a connection

$$D' = GDG^{-1} = -\delta + \nabla - \frac{i}{\nu}\, ad_{Weyl}(r')\,,$$

where $r' = Gr$ and ad_{Weyl} is calculated with respect to the $\circ'$-commutator.

Denote by $\mathcal{W}_{D'} := \ker D' \cap \mathcal{W}$ the Fedosov subalgebra of the algebra $(\mathcal{W}, \circ')$. Clearly, $\mathcal{W}_{D'} = G\mathcal{W}_D$. One can show just as in Theorem 3.2 that the restriction of the projection σ to $\mathcal{W}_{D'}$, $\sigma : \mathcal{W}_{D'} \to C^\infty(M)[[\nu]]$, is a bijection. Denote its inverse by τ'. Then $f *' g := \sigma(\tau'(f) \circ' \tau'(g))$ is a star-product on (M, ω) which is equivalent to the star-product $*$. The

operator $B : (C^\infty(M)[[\nu]], *) \to (C^\infty(M)[[\nu]], *')$ given by the formula $Bf = \sigma(G\tau(f))$ establishes this equivalence.

From now on let $C_r, r \geq 1$, denote the bidifferential operators defining the star-product $*'$. We have to show that $C_1(f, g) = (i/2)\{f, g\}$ and calculate C_2^- in order to determine the class $c_0(*) := c_0(*')$.

For $a \in \mathcal{W}$ we prefer to write $a|_{y=0}$ instead of $\sigma(a)$. For $f \in C^\infty(M)$ set

$$\tau'(f) = t_0(f) + \nu t_1(f) + \nu^2 t_2(f) + \dots .$$

Since $\sigma(\tau'(f)) = \tau'(f)|_{y=0} = f$ we have $t_0(f)|_{y=0} = f$ and $t_r(f)|_{y=0} = 0$ for $r \geq 1$. It follows from (4.2) that for $f, g \in C^\infty(M)$

$$f *' g = (\tau'(f) \circ' \tau'(g))|_{y=0} = \left(t_0(f)t_0(g) + \right.$$

$$\nu \left(t_0(f)t_1(g) + t_1(f)t_0(g) + \frac{i}{2}\omega^{pq}\left(\frac{\partial t_0(f)}{\partial y^p}\frac{\partial t_0(g)}{\partial y^q} \right) \right) + \dots \left. \right)|_{y=0} =$$

$$fg + \frac{i\nu}{2}\omega^{pq}\left(\frac{\partial t_0(f)}{\partial y^p}\frac{\partial t_0(g)}{\partial y^q} \right)|_{y=0} + \dots ,$$

from whence

$$(4.4) \qquad C_1(f, g) = \frac{i}{2}\omega^{pq}\left(\frac{\partial t_0(f)}{\partial y^p}\frac{\partial t_0(g)}{\partial y^q} \right)|_{y=0}.$$

Similarly one obtains that

$$(4.5) \qquad C_2^-(f, g) = \frac{i}{2}\omega^{pq}\left(\frac{\partial t_0(f)}{\partial y^p}\frac{\partial t_1(g)}{\partial y^q} + \frac{\partial t_1(f)}{\partial y^p}\frac{\partial t_0(g)}{\partial y^q} \right)|_{y=0}.$$

For an element $a^{(d)} \in \mathcal{W} \otimes \Lambda$ of Deg-degree d denote by $a_s^{(d)}$ its homogeneous component of deg_s-degree s. We have to calculate $\tau'(f)^{(1)}$ and the component $\tau'(f)_1^{(3)}$ of $\tau'(f)^{(3)}$. It follows from the condition $D'\tau'(f) = 0$ that
(4.6)

$$\tau'(f)^{(k+1)} = \delta^{-1}\left(\nabla\tau'(f)^{(k)} - \frac{i}{\nu}\sum_{l=0}^{k} ad_{Weyl}\left((r')^{(l+2)} \right)\left(\tau'(f)^{(k-l)} \right) \right), k \geq 0.$$

Since $\tau'(f)^{(0)} = f$, we get from (4.6) for $k = 0$ that $\tau'(f)^{(1)} = (\partial f/\partial x^p)y^p$. Therefore $(\partial t_0(f)/\partial y^p)|_{y=0} = \partial f/\partial x^p$. Now (4.4) implies that $C_1(f, g) = (i/2)\{f, g\}$, i.e., that the star-product $*'$ is normalized.

Taking into account that $\tau'(f)^{(2)}$ is of deg_s-degree 2, we obtain from (4.6) for $k = 2$ that

(4.7)
$$\tau'(f)_1^{(3)} = -\frac{i}{\nu}\delta^{-1}\left(ad_{Weyl}\left((r')_1^{(3)}\right)\left(\tau'(f)^{(1)}\right)\right) = \delta^{-1}\left(\omega^{pq}\frac{\partial\left((r')_1^{(3)}\right)}{\partial y^p}\frac{\partial f}{\partial x^q}\right).$$

Before calculating $(r')_1^{(3)}$ one can directly derive from (4.5) and (4.7) the following formula:

(4.8)
$$\varkappa = \frac{i}{\nu}\delta\left((r')_1^{(3)}\right).$$

Denote by $[\cdot,\cdot]_\circ$ the commutator with respect to the Wick multiplication $\circ$. We shall need the following technical lemma which can be proved by a straightforward calculation.

Lemma 4.1. *Let* $a = a_2^{(2)}, b = b_2^{(2)}$ *be two homogeneous elements of* $\mathcal{W}$, $(1/\nu)[a,b]_\circ = c^{(2)} = c_0^{(2)} + c_2^{(2)}$, *then* $c_0^{(2)} = \nu\Delta\left(c_2^{(2)}\right)$.

Using the fact that the operator G respects the total grading Deg and δ lowers both Deg- and deg_s-gradings by 1, one can obtain from formula (4.3), and the formula $r' = Gr$ that

(4.9)
$$\delta\left((r')_1^{(3)}\right) = \delta r_1^{(3)} - \nu\Delta\left(\delta r_3^{(3)}\right).$$

We get from Theorem 3.1 that

(4.10)
$$\delta r^{(3)} = R + \nabla r^{(2)} - \frac{i}{\nu}r^{(2)}\circ r^{(2)},$$

where
$$r^{(2)} = \delta^{-1}T = \frac{1}{3}\omega_{s\alpha}T_{tl}^\alpha y^s y^t dx^l.$$

Since the element $r^{(2)}$ is of deg_a-degree 1, we have

(4.11)
$$\frac{i}{\nu}r^{(2)}\circ r^{(2)} = \frac{i}{2\nu}[r^{(2)},r^{(2)}]_\circ = c^{(2)} = c_0^{(2)} + c_2^{(2)}.$$

We obtain from (4.10) and (4.11) that

(4.12)
$$\delta r_1^{(3)} = -c_0^{(2)} \quad \text{and} \quad \delta r_3^{(3)} = R + \nabla r^{(2)} - c_2^{(2)}.$$

It follows from (4.8), (4.9), (4.12), and Lemma 4.1 that

(4.13)
$$\varkappa = -i\Delta\left(R + \nabla r^{(2)}\right) = -\frac{i}{8}J_s^t R_{tkl}^s dx^k \wedge dx^l - i\lambda,$$

where $\lambda = \Delta\left(\nabla r^{(2)}\right)$. Introduce a global differential one-form $\mu = (1/6)J_s^t T_{tl}^s dx^l$ on M. A direct calculation shows that $\lambda = d\mu$, therefore the form λ is exact.

Recall the definition of the canonical class ε of an almost complex manifold (M, J). The class ε is the first Chern class of the subbundle $T'_{\mathbb{C}}M$ of vectors of type (1,0) of the complexified tangent bundle $T_{\mathbb{C}}M$. To calculate the canonical class of the almost-Kähler manifold (M, J, ω) take the same affine connection ∇ on M as that used in the construction of the star-product $*$. Denote by $\hat{R} = (1/2)R^s_{tkl}dx^k \wedge dx^l$ the curvature matrix of the connection ∇ and by $\Pi = (1/2)(Id - iJ)$ the projection operator onto the (1,0)-subspace. It follows immediately from (3.5) that $R^t_{tkl} = 0$ (see [15]), i.e., $Tr\hat{R} = 0$. The matrix $\Pi\hat{R}\Pi$ is the curvature matrix of the restriction of the connection ∇ to $T'_{\mathbb{C}}M$. The Chern-Weyl form

$$\gamma = (1/i)Tr(\Pi\hat{R}\Pi) = (1/i)Tr(\Pi\hat{R}) = (-1/4)J^t_s R^s_{tkl}dx^k \wedge dx^l$$

is closed. The canonical class is, by definition, $\varepsilon := [\gamma]$. Now it is clear from (4.13) that

$$(4.14) \qquad\qquad c_0(*) = [\varkappa] = -(1/2i)\varepsilon.$$

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