Some Analytic Aspects Concerning the Collatz Problem

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Abstract. A series of relatively simple equivalences to the Collatz conjecture, concerning the Collatz mapping

$$\tau(n) = \begin{cases} \frac{n}{2} & \text{for even n,} \\ \frac{3n+1}{2} & \text{for odd n} \end{cases}$$

are presented. The conjecture reads as follows: To every $n \in \mathbb{N}$ there exists a number $m \in \mathbb{N}$ such that the m^{th} iterate of τ, applied to n, has the value 1. The main topic of this paper consists in investigating a certain linear equation in the space of special Dirichlet series. The conjecture that this equation possesses a null space of dimension 1, generated by the Riemann zeta function, is equivalent to the Collatz conjecture. A number of analytic properties of the operator, which defines the linear equation, is given, some of them concern problems of analytic continuation in the complex domain. A few remarks with respect to generalizations of those problems conclude the paper.

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1.Introduction. Let $\mathbb{N}$ be the set of natural numbers. We consider the so-called Collatz map $\tau : \mathbb{N} \to \mathbb{N}$, defined by

$$\tau(n) = \begin{cases} \frac{n}{2} & \text{for even } n \\ \frac{3n+1}{2} & \text{for odd } n \end{cases} \tag{1}$$

Already in the first half of the last century, it has been conjectured by L. Collatz [3],[4] that to every number $n \in \mathbb{N}$ then exists a number $m \in \mathbb{N}$ such that for the iterates

$$\tau^1 = \tau, \qquad \tau^{k+1} = \tau \circ \tau^k, \qquad k \in \mathbb{N}$$

the assertion

$$\tau^m(n) = 1 \tag{2}$$

is valid. Until now it seems that this conjecture has neither been proved nor disproved. There is a huge literature on this topic, concerning relations to many fields in mathematics. We here refer to the Lecture Notes in Mathematics, no. 1681 by G. J. Wirsching [10],[11] as a main source.

In this paper we will contribute to such investigations by transforming the problem to a functional equation for special Dirichlet series.

2.Elementary Equivalences. Let A denote the set of all bounded sequences

$$a = \{a_\nu\}_{\nu=1}^{\infty}, \qquad a_\nu \in \mathbb{R} \qquad \text{for } \nu \in \mathbb{N}.$$

The subset $B \subset A$ is given by such sequences, for which $a_\nu \in \{0, 1\}$.

If $a = \{a_\nu\}_{\nu=1}^{\infty}$ we denote by a_τ the sequence

$$a_\tau = \{a_{\tau(\nu)}\}_{\nu=1}^{\infty}.$$

Theorem 1. *The following assertions are equivalent:*

(i) The Collatz conjecture is valid;

(ii) The only solutions of the equation

$$a = a_\tau \tag{3}$$

in A are given by the constant sequences

$$a = \{\lambda\}_{\nu=1}^{\infty} \quad \textit{with} \quad \lambda \in \mathbb{R}; \tag{4}$$

(iii) The equation

$$a = a_\tau$$

possesses only the two solutions in B,

$$a = \{0\}_{\nu=1}^{\infty} \qquad \textit{and} \qquad a = \{1\}_{\nu=1}^{\infty}.$$

Proof. We first assume the Collatz conjecture to be true. Then we choose any number $n \in \mathbb{N}$. According to the assumption there is a number $m \in \mathbb{N}$ such that $\tau^m(n) = 1$. We now consider any solution a of equation (3). It follows

$$a_n = a_{\tau(n)} = \cdots = a_{\tau^m(n)} = a_1.$$

Since n was arbitrary chosen, we get $a_n = a_1 = \lambda = \text{const}$ for all $n \in \mathbb{N}$. Hence there are no other solutions of equation (3).
Assuming the Collatz conjecture to be false, then there exists a subset $M \subset \mathbb{N}$, $M \neq \emptyset$, consisting of all $\nu \in \mathbb{N}$ for which $\tau^k(\nu) \neq 1$ for all $k \in \mathbb{N}$. We define the sequence

$$a = \{a_\nu\}_{\nu=1}^{\infty}$$

by

$$a_\nu = \begin{cases} 1 & \text{for } \nu \in M \\ 0 & \text{for } \nu \notin M \end{cases}$$

Since $\nu \in M$ implies $\tau(\nu) \in M$, we get also

$$a_{\tau(\nu)} = \begin{cases} 1 & \text{for } \nu \in M \\ 0 & \text{for } \nu \notin M \end{cases}$$

Hence this sequence is a solution of equation (3). Furthermore it is different from all the solutions (4), because of $1 \notin M$.
The equivalence of 2.) and 3.) in Theorem 1 is obvious. □

Instead of sequences one may consider suitable series, e.g.

$$\varphi(z) = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu(z),$$

in some region of the variable z with some convergence properties. Equivalence theorems concerning the Collatz conjecture can be started by considering functional equations, e.g.

$$\varphi(z) = \varphi_\tau(z),$$

where

$$\varphi_\tau(z) = \sum_{\nu=1}^{\infty} a_{\tau(\nu)} \varphi_\nu(z).$$

This has been performed in several papers [1], [2], [7], provided $\varphi(z)$ stands for a power series around the origin in the complex z-plane.

We turn over now to the special Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

which, for bounded coefficients a_n, converges for all complex variables s with $R(s) > 1$. The vector space of all such Dirichlet series will be denote by Â. The associated series

$$D_\tau(s) = \sum_{n=1}^{\infty} a_{\tau(n)} n^{-s}$$

takes, according to (1), the form

$$D_\tau(s) = \sum_{n=1}^{\infty} a_n (2n)^{-s} + \sum_{\nu=0}^{\infty} a_{3\nu+2}(2\nu+1)^{-s} = 2^{-s} D(s) + (TD)(s)$$

with the operator $T : \hat{A} \to \hat{A}$, defined by

$$(TD)(s) = \sum_{\nu=0}^{\infty} a_{3\nu+2}(2\nu+1)^{-s}. \tag{5}$$

So we get the

Theorem 2. *Let $\hat{B}$ denote the set of Dirichlet series*

$$D(s) = \sum a_n n^{-s}$$

with coefficients $a_\nu \in \{0,1\}$ for $\nu \in N$, and let $R(s) > 1$.
Then the following assertions are equivalent:

(i) The Collatz conjecture is valid;

(ii) The equation

$$D(s) = \frac{1}{1-2^{-s}} \ (TD)(s) \tag{6}$$

possesses in $\hat{B}$ the only non-trivial solution $D(s) = \zeta(s)$, where $\zeta(s)$ denotes the Riemann zeta function.

Proof. We proceed analogously to the proof of Theorem 1.

(i) If the Collatz conjecture is valid, then all the coefficients a_ν of a non-trivial solution $D \in \hat{B}$ are equal to 1. This shows $D(s) = \zeta(s)$.

(ii) Using the same notations as in the proof of Theorem 1, we see that for the Dirichlet series

$$D(s) = \sum_{\nu \in M} \nu^{-s}$$

we get

$$D_\tau(s) = \sum_{\nu \in M} a_{\tau(\nu)} \nu^{-s} = \sum_{\nu \in M} \nu^{-s} = D(s).$$

Therefore

$$D(s) = D_\tau(s) = 2^{-s} D(s) + (TD)(s),$$

i.e. the Dirichlet series $D(s)$ yields a solution of equation (6). Since $1 \notin M$ this series is different from $\zeta(s)$ and, according to the assumption $M \neq \emptyset$, not the trivial solution.

This proves Theorem 2. □

In the following section we will give special representations of the operator T.

3. A Complex representation of T. For real γ and a complex function $f : \{z \in \mathbb{C}, Re(z) = \gamma\} \to \mathbb{C}$ we will use the abbreviation

$$\int_{(\gamma)} f(z) \; dz$$

for the

$$\lim_{\alpha \to \infty} \int_{\gamma - i\alpha}^{\gamma + i\alpha} f(z) \; dz,$$

provided that this limit exists.

Theorem 3. *Let γ be a real number, $\gamma > 1$. Then, for $D \in \hat{A}$,*

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

the operator T is represented by the integral

$$(TD)(s) = \frac{1}{2\pi i}\int\limits_{(\gamma)} D(v)F(v,s)dv, \qquad R(s) > \gamma, \tag{7}$$

with

$$F(v,s) = \int\limits_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(3n+2+\omega)^{v-1}}{(2n+1)^s}\, d\omega, \qquad R(s) > R(v). \tag{8}$$

Remark. The formula (7) is based essentially on the Kronecker-Cahen-Perron Theorem (cp.[5],[9]). We will, however, present a detailed proof. To do this we need some lemmata.

Lemma 1. *Let $\gamma \in \mathbb{R}$, $\gamma > 0$. Then, for $x \in \mathbb{R}$, $x > 0$, we claim the validity of the formula*

$$\frac{1}{2\pi i}\int\limits_{(\gamma)} \frac{x^v}{v}\, dv = \begin{cases} 0, & \text{if } 0 < x < 1 \quad (a) \\ \frac{1}{2}, & \text{if } x = 1 \quad (b) \\ 1, & \text{if } x > 1 \quad (c) \end{cases} \tag{9}$$

This is Dirichlet's discontinuous factor.

Proof of Lemma 1. For demonstration we draw the figures 1a,b

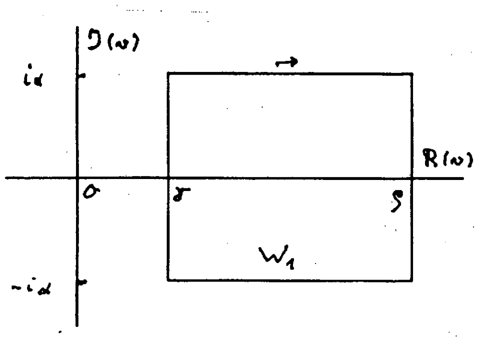

Fig.1a

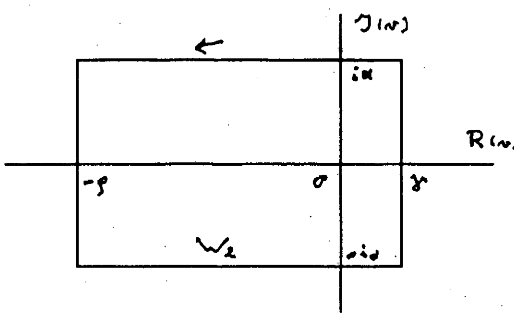

Fig.1b

First we assume $0 < x < 1$. We consider the rectangle with the corners $\gamma - i\alpha$, $\gamma + i\alpha$, $\rho + i\alpha$, $\rho - i\alpha$ (cp fig. 1a), where α and ρ are arbitrary positive numbers and $\rho > \gamma$. Since there are no singularities of the integrand $\frac{x^v}{v}$ in the interior of this rectangle, the value of the corresponding integral equals zero, i.e.

$$\frac{1}{2\pi i}\int\limits_{W_1} \frac{x^v}{v}\, dv = 0.$$

Here W_1 denotes the circumference of the rectangle. So we have

$$\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv=\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\rho-i\alpha}\frac{x^v}{v}\,dv+\frac{1}{2\pi i}\int_{\rho-i\alpha}^{\rho+i\alpha}\frac{x^v}{v}\,dv+\frac{1}{2\pi i}\int_{\rho+i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv.$$

Therefore we may estimate as follows:

$$\left|\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\rho-i\alpha}\frac{x^v}{v}\,dv\right|\leq\frac{x^\gamma}{\alpha\pi\log(\frac{1}{x})}+\frac{\alpha x^\rho}{\pi\rho}.$$

For $\rho\to\infty$ we get the inequality

$$\left|\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv\right|\leq\frac{x^v}{\alpha\pi\log(\frac{1}{x})}$$

for every $\alpha>0$. It follows for $\alpha\to\infty$,

$$\frac{1}{2\pi i}\int_{(\gamma)}\frac{x^v}{v}\,dv=0,\qquad\text{if } x\in(0,1),$$

i.e. eq.(9a).

Now let $x=1$. Using the parameter representation

$$v=\gamma+it,\qquad -\alpha\leq t\leq\alpha,$$

yields

$$\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{dv}{v}=\frac{1}{2\pi}\int_0^\alpha\left\{\frac{1}{\gamma+it}+\frac{1}{\gamma-it}\right\}dt=\frac{1}{\pi\gamma}\int_0^\alpha\frac{dt}{1+(\frac{t}{\gamma})^2}=\frac{1}{\pi}\int_0^{\frac{\alpha}{\gamma}}\frac{du}{1+u^2},$$

which tends to $\frac{1}{2}$ for $\alpha\to\infty$. This proves (9b).

Let us consider the third case $x>1$. This time the rectangle is defined by its corners $\gamma-i\alpha$, $\gamma+i\alpha$, $-\rho+i\alpha$, $-\rho-i\alpha$, (cp. fig.1b). The pole at $s=0$ of the integrand belongs to the interior of our rectangle. The residue equals 1. So we get

$$\frac{1}{2\pi i}\int_{W_2}\frac{x^v}{v}\,dv=1,$$

where W_2 denotes the circumference of the rectangle. Hence

$$\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv=1+\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{-\rho-i\alpha}\frac{x^v}{v}\,dv+\frac{1}{2\pi i}\int_{-\rho-i\alpha}^{-\rho+i\alpha}\frac{x^v}{v}\,dv+\frac{1}{2\pi i}\int_{-\rho+i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv$$

and we gain the estimation

$$\left|\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv-1\right|\leq\frac{x^\gamma}{\alpha\pi\log(x)}+\frac{\alpha x^{-\rho}}{\pi\rho}.$$

For $\rho\to\infty$ we arrive at the inequality

$$\left|\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha}\frac{x^v}{v}\,dv-1\right|\leq\frac{x^\gamma}{\alpha\pi\log(x)}$$

for every $\alpha>0$. It follows for $\alpha\to\infty$:

$$\frac{1}{2\pi i}\int_{(\gamma)}\frac{x^v}{v}\,dv=1,\qquad\text{if } x>1,$$

which is eq. (9c). Thus the Lemma 1 is proved. □

Remark. Using the well known formula (cf. [8], eqs. 3.523 resp. 3.527)

$$I_1(\mu,a):=\int_0^\infty\frac{\cos(\mu t)}{a^2+t^2}\,dt=\frac{\pi}{2a}e^{-\mu a}\qquad\text{for }\mu>0,a>0$$

and

$$I_2(\mu,a):=\int_0^\infty\frac{t\sin(\mu t)}{a^2+t^2}\,dt=\frac{\pi}{2}e^{-\mu a}\qquad\text{for }\mu>0,a>0,$$

the assertions of Lemma 1 follow easily from the identity

$$\frac{1}{2\pi i}\int_{(\gamma)}\frac{x^v}{v}\,dv=\frac{x^v}{\pi}\{\gamma I_1(|\log x|,\gamma)+I_2(|\log x|,\gamma)\},$$

provided $x>0$ and $x\neq 1$.

Lemma 2. *Let the Dirichlet series*

$$D(s):=\sum_{m=1}^\infty a_m m^{-s}$$

be absolutely convergent for all $s\in\mathbb{C}$ with $Re(s)\geq\gamma>1$. Then, for every $x\in\mathbb{R},x>0$ we get the formula

$$\frac{1}{2\pi i}\int_{(\gamma)}D(v)\frac{x^v}{v}\,dv=\sum_{m\leq x}^{*}a_m, \tag{10}$$

where

$$\sum_{m\le x}^{*} a_m := \sum_{m<x} a_m + \begin{cases} \frac{1}{2}\, a_x & \textit{if } x \textit{ is an integer} \\ 0 & \textit{if } x \textit{ is not an integer} \end{cases}$$

Proof. According to Lemma 1 we have, for every natural number m,

$$\frac{1}{2\pi i}\int_{(\gamma)} \left(\frac{x}{m}\right)^v \frac{dv}{v} = \begin{cases} 1 & \text{if } m < x, \\ \frac{1}{2} & \text{if } m = x, \\ 0 & \text{if } m > x, \end{cases} \tag{11}$$

Since $D(v)$ is uniformly convergent in compact set, contained in the half plane $R(v) \ge \gamma > 1$, we get

$$\frac{1}{2\pi i}\int_{(\gamma)} D(v)\frac{x^v}{v}\, dv = \sum_{m=1}^{\infty} a_m \frac{1}{2\pi i}\int_{(\gamma)} \left(\frac{x}{m}\right)^v \frac{dv}{v} = \sum_{m\le x}^{*} a_m$$

□

Proof of Theorem 3. Let $m \in \mathbb{N}_0$. We choose first $x = 3n+\frac{5}{2}$ and then $x = 3n+\frac{3}{2}$ and apply Lemma 2. It follows

$$\frac{1}{2\pi i}\int_{(\gamma)} D(v)\left\{\left(3m+\frac{5}{2}\right)^v - \left(3m+\frac{3}{2}\right)^v\right\}\frac{dv}{v}$$

$$= \frac{1}{2\pi i}\int_{(\gamma)} D(v)\int_{-\frac{1}{2}}^{+\frac{1}{2}} (3m+2+\omega)^{v-1}\, d\omega dv$$

$$= a_{3m+2}.$$

Because of the uniform convergence of the series

$$\sum_{m=0}^{\infty} \frac{(3m+2+\omega)^{v-1}}{(2m+1)^s}$$

for $R(s) > R(v)$, we get the following representation of T :

$$(TD)(s) = \sum_{m=0}^{\infty} \frac{a_{3m+2}}{(2m+1)^s} = \frac{1}{2\pi i}\int_{(\gamma)} D(v)F(v,s)dv,$$

where F is defined in (8). This proves Theorem 3. □

4. The kernel function F. In investigating the properties of the map T one has to study the kernel function F. Here we consider again v as a complex parameter and s as the essential complex variable.

The representation of F in eq.(8) by the infinite series shows that F is a holomorphic function with respect to the variable s in the half plane $R(s) > R(v)$. We consider first some analytic continuations of F.

Theorem 4. *The kernel function F with fixed parameter v possesses an analytic continuation with respect to the variable s, to a meromorphic function in the half plane $R(s) > R(v)-1$. The only singularity of the continuation in this region consists in a first order pole at $s = v$ with residue $3^{v-1}/2^v$.*

Proof. A short consideration shows that it suffices to prove the assertion for the function

$$\varphi(v,s) := \sum_{m=0}^{\infty} \frac{(3m+2+\omega)^{v-1}}{(2m+1)^s}, \qquad R(s) > R(v),$$

where $\omega \in \mathbb{R}$ belongs to the interval $\left[-\frac{1}{2}, +\frac{1}{2}\right]$. Obviously φ represents a holomorphic function of s in the half plane $R(s) > R(v)$. Next we consider, for $m \geq 1$, the difference

$$\begin{aligned} d_m(v,s) : &= \frac{(3m+2+\omega)^{v-1}}{(2m+1)^s} - \int_{m-1}^{m} \frac{(3t+2+\omega)^{v-1}}{(2m+1)^s}\, dt \\ &= \frac{(3m+2+\omega)^{v-1}}{(2m+1)^s} - \int_0^1 \frac{(3m-1+3t+\omega)^{v-1}}{(2m-1+2t)^s}\, dt \\ &= -\int_0^1 t\frac{(3(m+t)-1+\omega)^{v-1}}{(2(m+t)-1)^{s+1}}\, h(m+t)\ dt \end{aligned}$$

with

$$h(t) = (6(v-1)-6s)t - 3(v-1) + 2s - 2st\omega.$$

Here we have used integration by parts. It follows:

(i) The series

$$\sum_{m=1}^{\infty} d_m(v,s)$$

converges uniformly in every compact subset of the half plane $R(s) > R(v)-1$ and hence represents a holomorphic function in that region.

(ii) It is

$$\varphi(v,s) = \sum_{m=1}^{\infty} d_m(v,s) + (2+\omega)^{v-1} + \int_0^{\infty} \frac{(3t+2+\omega)^{v-1}}{(2t+1)^s}\ dt. \qquad (12)$$

We have to investigate the last integral in eq.(12).

One gets

$$\begin{aligned}
\int_0^{\infty} \frac{(3t+2+\omega)^{v-1}}{(2t+1)^s}\ dt &= \int_0^{\infty} \frac{(3(1+t)-1+\omega)^{v-1}}{(2(1+t)-1)^s}\ dt \\
&= \frac{3^{v-1}}{2^s} \int_0^{\infty} (1+t)^{-s+v-1}\ \frac{(1-\frac{1-\omega}{3(1+t)})^{v-1}}{(1-\frac{1}{2(1+t)})^s}\ dt \\
&= \frac{3^{v-1}}{2^s} \int_0^{\infty} \frac{dt}{(1+t)^{s-v+1}}\ dt + R(v,s) \\
&= \frac{3^{v-1}}{2^s}\ \frac{1}{(s-v)} + \tilde{R}(v,s).
\end{aligned}$$

An expansion of the expression

$$\frac{(1-\frac{1-\omega}{3(1+t)})^{v-1}}{(1-\frac{1}{2(1+t)})^s}$$

into a power series

$$\sum_{\nu=0}^{\infty} \frac{c_\nu}{(1+t)^\nu}$$

leads to the result that $\tilde{R}(v,s)$ represents a holomorphic function in the region $R(s) > R(v) - 1$.
The assertion concerning the first order pole at $s = v$ is evident. □

Remark. It may be of interest to mention the recursion formula

$$\varphi(v,s) = \frac{2}{3}\ \varphi(v-1, s-1) + (\omega + \frac{1}{2})\ \varphi(v-1, s) \qquad (13)$$

for $R(s) > R(v)$, which can easily be verified.

Using classical tools we investigate the function φ, defined in (12), in particular

$$\varphi\ (-v,s) = \sum_{m=0}^{\infty} (3m+2+\omega)^{-v-1}\ (2m+1)^{-s} \qquad (14)$$

in the region

$$G = \{(v,s)|\ Rv > 0,\ \ Rs > 0\}. \tag{15}$$

Eulers integral for the Γ - function,

$$\Gamma(z) = \int_0^\infty t^{z-1}\ e^{-t}\ dt, \qquad Rz > 0,$$

yields

$$(3m+2+\omega)^{-v-1}(2m+1)^{-s} = \frac{1}{\Gamma(v+1)\Gamma(s)} \int_0^\infty \int_0^\infty x^v y^{s-1} e^{-m(3x+2y)-(2+\omega)x-y} dx\ \ dy. \tag{16}$$

So we get

Theorem 5. *The representation*

$$\varphi(-v,s) = \frac{1}{\Gamma(v+1)\ \Gamma(s)} \int_0^\infty \int_0^\infty x^v\ \ y^{s-1} \frac{e^{-(2+\omega)x-y}}{1-e^{-3x-2y}}\ dx\ \ dy \tag{17}$$

holds true for all

$$(v,s) \in \hat{G},$$

where

$$\hat{G} = \{(v,s) \in \mathbb{C}^2|\ Rv > 1,\ \ Rs > 2\}.$$

Proof. We have just to sum in (16) the geometric series

$$\sum_{m=0}^\infty e^{-m(3x+2y)}.$$

□

The integral representation (17) of $\varphi(-v,s)$ can be used to gain analytic continuation with respect to s, where v is fixed and, conversely, with respect to v, where s is fixed.
We will not go into details here. It may be possible, using classical methods due to Riemann, to get useful estimations of the kernel function $F(v,s)$, using the representation (17) of $\varphi(-v,s)$.

5. Generalizations. The mapping (cp. [10],p.13)

$$\hat{\tau} : \mathbb{N} \to \mathbb{N}, \qquad \hat{\tau}(n) = \begin{cases} \frac{n}{2} & \text{for even n,} \\ \frac{3n-1}{2} & \text{for odd n,} \end{cases}$$

for $n \in \mathbb{N}$ is closely connected with the Collatz mapping τ, defined in (1).The corresponding kernel function $\hat{F}(v,s)$ (cp.eq.(8)) reads

$$\hat{F}(v,s) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(3n+1+\omega)^{v-1}}{(2n+1)^s} \, d\omega, \quad \text{for } R(s) > R(v).$$

This gives not a big difference to the kernel $F(v,s)$. On the other hand the full information on the mapping $\hat{\tau}$ is contained in the kernel $\hat{F}$. But: There are three known cycles of $\hat{\tau}$,

$$(1), (5,7,10), (17,25,37,55,82,41,61,91,136,68,34),$$

which gives the reason to conjecture that the dimension of the null space of the equation

$$D(s) = \frac{1}{1-2^{-s}} \cdot \frac{1}{2\pi i} \int_{(\gamma)} D(v)\hat{F}(v,s)dv$$

is 3.

A slightly more general mapping is given by

$$\tilde{\tau} : \mathbb{N} \to \mathbb{N}, \qquad \tilde{\tau}(n) = \begin{cases} \frac{n}{2} & \text{for even n,} \\ \frac{an+b}{2} & \text{for odd n,} \end{cases}$$

where $a \in \mathbb{N}, b \in \mathbb{Z}$ and $a+b \equiv 0 \bmod 2$ and $a+b > 0$ holds. Here the corresponding kernel function $\tilde{F}$ is given by

$$\tilde{F}(v,s) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(an+c+\omega)^{v-1}}{(2n+1)^s} \, d\omega,$$

where

$$c = \frac{a+b}{2}.$$

What can be said an the dimension of the null space of the corresponding linear equation for the Dirichlet series ? Under which assumptions is this dimension finite ?

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