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MATHEMATICAL THEORY OF UNIFORM ELASTIC STRUCTURES

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0.INTRODUCTION

The theory of continuous distributions of material imperfections, dislocations in particular, the origin of which can be traced back to the period of 1950-1967, has been approached from at least two different points of view, i.e., structural dynamics and continuum mechanics. While the pioneering works of Bilby, Eshelby, Kröner, Kondo (see e.g., [B], [Kr]) and others represent a structural point of view the mathematical theory of materially uniform simple elastic bodies of Noll and Wang, [N], [W], [Bl], is firmly based on continuum mechanics notions. Seen as a natural generalization of the structural approach, this theory takes as its fundamental assumption that the presence of imperfections does not modify the general constitutive nature of the elastic material and that the information required to identify and describe smooth distributions of defects can be found in the material response functional of a given uniform body without introducing any extra parameters or a priori geometries. Following this line of thought, imperfections are seen as being responsible for a breakdown of homogeneity of these constitutive functionals. Geometric periodicity of the underlying atomic lattice corresponds, on the other hand, to material uniformity and the form of the material symmetry group. Using the language of modern differential geometry the theory shows that for a materially uniform simple elastic body a linear connection can be defined in a manner consistent with the given constitutive relations but not necessarily in a unique way.

The process of analyzing a given material body is at least two-fold. First, one needs to determine if the given constitutive functional indeed defines a uniform material (see e.g., [EEp1]). Only after this has been established the question of local and global homogeneity can be addressed. It has been shown by Noll [N] and Wang [W] that the existence of locally homogeneous configurations is expressed mathematically by the availability of a locally flat material connection. If the material symmetry group is a continuous group this task proves to be, in general, a very difficult one. Guided by these difficulties, in effort to develop some comprehensive approach to this problem, it was shown by Elżanowski at al. [EEpŚ2] that a definite G-structure can be associated with a materially uniform simple elastic body and that the local homogeneity of such a body is equivalent to the local integrability of the underlying G-structure (material structure).

However, one thing is to determine that the given material structure is locally integrable the other is to explicitly find the corresponding homogeneous configurations. Indeed, in the case of a material having at each point a stress-free uniform reference configuration (e.g. an isotropic elastic solid) one does not know how to arrange a collection of stress-free pieces to fit them together into a global configuration without introducing internal stresses. In the language of the differential geometric theory of linear connections the process described above is equivalent to finding a uniform global reference configuration generating a flat material connection, i.e., a local coordinate system on the body manifold inducing in the corresponding bundle of linear frames, in a manner consistent with the constitutive information, a locally integrable connection possessing a vanishing torsion.

It was shown in [EpEŚ] and [EEpŚ2], that one possible way of resolving this problem is to associate with the given material structure a geometric object (called the characteristic object) capturing the essential geometric features of the structure in question. The analysis of the object's homogeneity (point independence) as a field on the body manifold becomes then the means of analyzing the integrability of the corresponding material structure. On the other hand, looking at the material symmetry group as a gauge group and at the changes of uniform configurations as gauge transformations one is also able to develop, through rather straightforward calculations, a system of quasilinear partial differential equations for the symmetry group controlled configuration changes leading from an arbitrary uniform reference to a uniform configuration possessing the required geometric characteristics, if such a configuration exists, [EP1].

It has been often pointed out, by critics and supporters alike, that the original theory of Noll and Wang does not enjoy the generality often demanded by those propagating the so-called lattice model. This is mainly because in the structural approach to the theory of continuous distribution of defects it has been suggested that although the presence of dislocations shows through the non-vanishing torsion of a material connection, disclinations (rotational defects) are measured by the curvature of such a connection [An], [Lr]. The structural approach suggests also that the bodies with defects, disclinations in particular, are subject to couple and multipolar stresses, [Kr]. Since any constitutive functional associated with a simple elastic material body induces, by definition, a curvature-free material parallelism (field of isomorphisms) it appears that the disclinations, and possibly other defects, are ruled out. Therefore, as it has been suggested by Elżanowski and Epstein [EEp2], it seems only natural to investigate the possibility of describing disclinations in the context of the so-called second grade material. This seems to be also supported by the non-local nature of disclinations, [Lr].

In this paper we present a comprehensive mathematical foundation of the theory of material structures of uniform multipolar hyperelastic bodies. Although based on the original ideas of Noll and Wang the research undertaken here, which grew out of our early works (see e.g., [EEp2], [EP1] and [EP2]), aims at formulating and analyzing the theory of uniform material structures far more complex than simple elasticity. We not only show that such a generalization is mathematically possible but also, in the process of doing so, which often leads through rather unexplored areas of the differential geometry of frame bundles of higher order contact, we show some rather intriguing possibilities of discovering intrinsically higher order defects. Such defects have not yet been, as far as we know, reported in the literature.

The paper is divided into six chapters. In the first chapter we present a covariant constitutive theory of elasticity. Starting from a completely global approach we proceed to study simple hyperelasticity emphasizing different levels of non-locality as well as such primitive concepts as body manifold, ambient space, global and local configurations and constitutive law. The second chapter deals with the notion of symmetry both material and spatial. The concepts of material isomorphism, material uniformity and material transitivity are introduced and discussed in the third chapter. Chapters 4, 5 and 6 constitute the core of this work. The concepts of the modern differential geometry of frame bundles are applied to show that a definite principal bundle, being the reduction of the bundle of k-frames, can be associated with a uniform elastic body. The k-principal material connection, the analog of the material connection of Noll and Wang, is introduced. To analyze the material structure of the uniform body completely we introduce also the concepts of the projected and the induced material connections. These connections provide partial characteristics (lower grade characteristics) of principal material connection and help to identify different stages of inhomogeneities. We analyze in detale the structure of connections on holonomic and semi-holonomic frame to be able in Chapter 6 to derive explicite conditions for the local flatness of such connections. We show that in the case of a curvature-free k-connection the local flatness can be measured by the vanishing of some special tensor which, in the context of continuum mechanics, we call the inhomogeneity tensor. Although we are mostly concerned with the uniform hyperelastic material bodies we also make some comments on material bodies with microstructures.

1.BASIC CONSTITUTIVE THEORY

1.1. Global Model

Let $\mathcal{B}$ denote an oriented smooth n-dimensional compact manifold, possibly with boundary, called the **body**. We assume that the body $\mathcal{B}$ manifests itself through smooth embeddings[1] $\psi : \mathcal{B} \to \mathcal{S}$ into some, in general different, smooth boundaryless m-dimensional manifold $\mathcal{S}$ called the **ambient space**. We also assume that dim $\mathcal{S} \geq$ n. A smooth embedding ψ of $\mathcal{B}$ into $\mathcal{S}$ represents therefore a **configuration** of the continuous body $\mathcal{B}$ while $\psi(\mathcal{B})$ is recognized as its possible **placement** in the ambient space. In fact, as pointed out by Marsden [M2], one should accept as configurations immersions, rather than embeddings. This would allow, for example, a contact at the folding boundary. Classically one assumes that the body is a differentiable manifold admitting a global atlas and that $\mathcal{S} = \mathbb{R}^3$. For the most part we will not limit ourselves to this particular case.

The set $\mathcal{C}_{\mathcal{B}}$ of all smooth embeddings of $\mathcal{B}$ to $\mathcal{S}$, which equipped with Whitney's C^∞-topology is an infinite dimensional Fréchet manifold (see e.g., [BiŚF] or [Mi]) is called the **configuration space** of $\mathcal{B}$. In a more general approach one can regard the space of configurations of a continuous body as the space of sections of some fibre (specially vector) bundle $\pi : \mathbf{E} \to \mathcal{B}$. Such an approach was shown to be particularly useful in the context of the unified Lagrangian field theory of elasticity (see e.g., [MH]). Here, not to cloud the picture, we restrain, for this general part of the exposition, from any unnecessary generalizations. However, later on we will resort briefly to this approach in the context of materials with microstructures. Nevertheless, in our simple case we have $\mathbf{E} \equiv \mathcal{B} \times \mathcal{S}$ where, given a configuration ψ, the corresponding section of $\mathbf{E}$ is a mapping $\mathcal{B} \ni \mathrm{X} \mapsto (\mathrm{X}, \psi(\mathrm{X}))$.

Let $\pi_C : T\mathcal{C}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{B}}$ denote the tangent space of the manifold of all configurations $\mathcal{C}_{\mathcal{B}}$.

Definition 1.1 *An element $\eta_\psi \in T\mathcal{C}_{\mathcal{B}}$ has the physical meaning of the* **virtual displacement** *measured away from the configuration $\psi = \pi_C(\eta_\psi)$.*

[1] An embedding is an open and one-to-one immersion (*cf.*, [K]).

Any element of the tangent space $TC_{\mathcal{B}}$ is uniquely represented by the mapping $\eta_\psi : \mathcal{B} \to T\mathcal{S}$ from the body $\mathcal{B}$ into the tangent space $T\mathcal{S}$ of its ambient space $\mathcal{S}$ such that $\pi_{\mathcal{S}} \circ \eta_\psi = \psi$ where $\pi_{\mathcal{S}}$ denotes the standard projection of $T\mathcal{S}$.[2] In other words, at the placement $\psi(\mathcal{B})$ each material point $\mathrm{X} \in \mathcal{B}$ is assigned a displacement vector $\eta_\psi(\mathrm{X}) \in T_{\psi(\mathrm{X})}\mathcal{S}$ in the ambient space. Although a virtual displacement induces a vector field on the placement $\psi(\mathcal{B}) \subset \mathcal{S}$ the assignment of a vector to a material point X depends, in general, on the whole current configuration.

As pointed out in Epstein at al. [EpEŚ], a force exerted on the body $\mathcal{B}$ is intuitively conceived of as an object which performs work linearly on a virtual displacement. Accepting this point of view we postulate:

Definition 1.2 *A* **force** f *is a 1-form on the configuration space $C_{\mathcal{B}}$, that is, a section of the cotangent bundle $T^*C_{\mathcal{B}}$ of the configuration space.*

Given the force f and the virtual displacement η_ψ, at the same current configuration ψ , the virtual work of f on η_ψ is given by evaluating the 1-form f on the vector η_ψ , i.e., $\mathrm{f}(\eta_\psi) \in \mathbb{R}$. Note, that despite the fact that any tangent vector (virtual displacement) to the configuration space $C_{\mathcal{B}}$ can be represented by a vector field on the placement of the body in the ambient space, there is no natural representation of the force f as a field of 1-forms on such a placement. Such a representation would, however, be possible had we allowed for example some choice of the matric on the configuration space (see e.g., [Bi]).

Definition 1.3 *The* **elastic constitutive law**, *completely defining the mechanical response of the body $\mathcal{B}$, is a smooth field $\mathfrak{c} : C_{\mathcal{B}} \to T^*C_{\mathcal{B}}$.*

Such a constitutive law is *global* not only because it assigns forces to entire configurations but also because the action of those assigned forces involves,

[2] In general, the tangent space to the space of sections of a fibre bundle, e.g. $\mathbf{E} = \mathcal{B} \times \mathcal{S}$, is the space of sections of the bundle the fibre of which is the tangent space to the fibre of the original bundle (see e.g., [EnM] or [Mi]).

as it has been mentioned before, the whole of η_ψ rather than any particular characteristic of it.

Definition 1.4 *We say that the elastic constitutive law* $\mathfrak{c}$ *is of* **local action** *if there exists a linear mapping* $\wp$ *from the space* $T\mathcal{C}_\mathcal{B}$ *of virtual displacements to the space of n-forms on the body* $\mathcal{B}$ *with*

$$supp\, \wp(\eta_\psi) \subset supp\, \eta_\psi \tag{1.1}$$

and such that for any given configuration $\psi \in \mathcal{C}_\mathcal{B}$ *and any compatible virtual displacement* η_ψ *the* **virtual work** *of the force field* $\mathfrak{c}(\psi)$ *on* η_ψ *is given by*

$$\mathcal{C}(\psi)(\eta_\psi) = \int_\mathcal{B} \wp(\eta_\psi). \tag{1.2}$$

Note that we have ignored here a possible contribution from the boundary of the body $\mathcal{B}$. Note also that as the map $\wp$ is supposed to represent a density of work, to ensure that work would not be assigned to a placement of a material point unless there is a non-vanishing virtual displacement on a neighbourhood of it, it is essential to impose the *localization condition* (1.1). The linear mapping $\wp$ of the Definition 1.4 represents a localization of the action of the constitutive law $\mathfrak{c}$ in $\mathcal{C}_\mathcal{B}$ but it does not define the *local* material. Its action at any given material point may still depend on the placement of points away from it.

Definition 1.5 *The material body* $\mathcal{B}$ *is* **jet-local of order k** *or* **k-grade elastic** *if there exists a mapping* $\sigma : J^k(\mathcal{B}, T\mathcal{S}) \longrightarrow \Lambda^n\mathcal{B}$*, called the* **local response functional** *such that for each material point* $\mathrm{X} \in \mathcal{B}$

$$\wp(\eta_\psi)(\mathrm{X}) = \sigma(j^k\eta_\psi(\mathrm{X})). \tag{1.3}$$

Here $j^k\eta_\psi$ is to be understood as the jet extension of the virtual displacement $\eta_\psi \in C^k(\mathcal{B}, T\mathcal{S})$, i.e., a section of the k-jet bundle $J^k(\mathcal{B}, T\mathcal{S})$, while $\Lambda^n\mathcal{B}$ denotes the space of differentiable n-form on $\mathcal{B}$. Due to the localization condition (1.1) it follows immediately from the Local Peetre Theorem (see e.g., [K], Theorem 6.2) that $\wp$ is a linear differential operator and as such is locally of finite order, i.e., it is generated locally by a finite number of derivatives of η_ψ. As $\mathcal{B}$ is assumed to be a compact manifold, the latter implies that $\wp$ is of a finite order. The condition (1.3) is therefore always satisfied for some integer k:

Proposition 1.1 *Any elastic constitutive law $\mathfrak{c}$ of local action $\wp$ represents a jet-local elastic material of some finite order.*

Definition 1.6 *Given the elastic material body $\mathcal{B}$, a smooth real-valued function $\mathcal{W}$ on $C_\mathcal{B}$, such that*

$$\mathfrak{c}(\psi)(\eta_\psi) = \eta_\psi(\mathcal{W}) \tag{1.4}$$

for any configuration ψ and any virtual displacement $\eta_\psi \in \pi_C^{-1}(\psi)$, is called the **elastic potential**. *Any elastic body possessing some elastic potential is called* **hyperelastic**. *The elastic potential $\mathcal{W}$ is said to be* **localizable** *in $\mathcal{B}$ if there exists a smooth real-valued function $\varphi : \mathcal{B} \times C_\mathcal{B} \to I\!R$ such that at any given configuration, say ψ, $\mathcal{W}(\psi) = \int_\mathcal{B} \varphi(\mathrm{X}, \psi)\mu_\mathcal{B}$ where $\mu_\mathcal{B}$ denotes a volume element on $\mathcal{B}$.*

In the case of the hyperelastic body the virtual work is given by the Fréchet derivative (for the definition see e.g., [L]) of the potential $\mathcal{W}$ in the direction of a virtual displacement. Thus, if the hyperelastic material with a localizable elastic potential $\mathcal{W}$ is of local action

$$\wp(\eta_\psi)(\mathrm{X}) = d\varphi(j^k\psi(\mathrm{X}))(j^k\eta_\psi)\mu_\mathcal{B} \tag{1.5}$$

for every virtual displacement η_ψ, any configuration $\psi = \pi_{\mathcal{C}}(\eta_\psi)$ and every material point $\mathrm{X} \in \mathcal{B}$, assuming that one can differentiate under the integral. The virtual work is now given by the first variation of φ [EpEŚ].

The density of the elastic potential φ of the k-grade hyperelastic material becomes, at a given material point and relative to the choice of local charts on the body manifold $\mathcal{B}$ and the ambient space $\mathcal{S}$, a smooth function

$$\varphi : L(\mathbb{R}^n, \mathbb{R}^n) \oplus S^2(\mathbb{R}^n, \mathbb{R}^n) \oplus \ldots \oplus S^k(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R} \qquad (1.6)$$

where $L(\mathbb{R}^n, \mathbb{R}^n)$ denotes the set of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$, $S^l(\mathbb{R}^n, \mathbb{R}^n)$ is the algebra of all symmetric $\mathbb{R}^n$-valued $l-$linear maps from $\mathbb{R}^n$ and where the translational invariance in the ambient space $\mathcal{S}$ was enforced to eliminate the target point dependence. Indeed, suppose that $\mathrm{X} \in \mathcal{B}$, $\psi \in \mathcal{C}_{\mathcal{B}}$, $y = \psi(\mathrm{X})$, $\{\mathrm{U}, \alpha\}$ and $\{\mathrm{V}, \beta\}$ are manifold charts at X and y respectively while α is chosen so that $\alpha(\mathrm{X})$ is the origin of $\mathbb{R}^n$. $\phi \in j^k\psi(\mathrm{X})$ [3] if, and only if, the corresponding k^{th} order Taylor polynomials $\mathcal{T}^k$ are identical at $\alpha(\mathrm{X})$, i.e., $\mathcal{T}^k(\beta \circ \psi \circ \alpha^{-1})(\alpha(\mathrm{X})) = \mathcal{T}^k(\beta \circ \phi \circ \alpha^{-1})(\alpha(\mathrm{X}))$ [BŚF]. This enables us to identify $j^k\psi(\mathrm{X})$ with its representation $j^k(\beta \circ \psi \circ \alpha^{-1})(\alpha(\mathrm{X}))$, the principal part of which is an element of $L(\mathbb{R}^n, \mathbb{R}^n) \oplus_{i=2}^k S^i(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 1.7 *A* **k-local configuration** *of the material point* X *is an element of the space of all invertible k-jets $J^k(\mathcal{B}, \mathcal{S})$. Given two, in general different, configurations ψ and ϕ the* **deformation gradient** *at* X *of the* **deformation** $\chi \equiv \psi \circ \phi^{-1}$ *from the placement $\phi(\mathcal{B})$ to the placement $\psi(\mathcal{B})$* [4] *is the tangent map $\chi_*(\mathrm{X}) : T_{\phi(\mathrm{X})}\mathcal{S} \rightarrow T_{\psi(\mathrm{X})}\mathcal{S}$ the Euclidean representation of which is an element of GL(n,$\mathbb{R}$).*

Higher order deformation gradients can then be thought of as the tangent maps of automorphisms of the bundle of local configurations over corresponding deformations.

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[3] The jet is understood here as an equivalence class of differentiable functions.

[4] If $\mathcal{S} = \mathbb{R}^n$ and $\mathcal{B}$ is an open submanifold of $\mathbb{R}^n$ a deformation is just another name for a configuration.

Given the grade one (simple) hyperelastic body, defined by the density of its elastic potential $\varphi : J^1(\mathcal{B},\mathcal{S}) \to I\!R$, the *first Piola-Kirchhoff stress tensor* is introduced as $\mathbf{P} = D_{\mathbf{F}}\varphi$ where, if χ is a deformation, $\mathbf{F}$ denotes the principal part of its tangent map χ_*, and where D stands for the Fréchet derivative. Note that as the deformation gradient can be looked at as the change of frames or a deformed frame (all the same) $\mathbf{P}$ can be understood as a vector bundle automorphism of $T\mathcal{S}$ over χ^{-1} (see e.g., [MN] and also the next section). Having such a morphism (stress tensor) available one could attempt to express the local action operator $\wp$ in a more classical way as the trace of a composition of linear maps [TN]. To be able to do this, however, one needs to have a splitting (a linear connection) on $T\mathcal{S}$.[5] To this end and to show how to introduce the concept of the stress tensor in the context of a simple, yet not necessarily potential, elasticity we will sketch, following Segev and Epstein [Se], [EpSe], the so-called local (first order) model - the alternative to the localized global model presented above.

1.2 Local Model

In contrast to the global model of a continuous deformable body the local approach considers as its prime object a material point and its neighborhood rather than the body as a whole. By the neighborhood of a material point one can understand, on the one hand, a topological neighborhood, i.e. an open subbody containing the point in question, or on the other hand, in a more abstract sense, the point and an object attached to it which fully characterizes the mechanical properties of the given material point. In the tangent space model of Segev and Epstein [Se] the neighborhood of a material point X $\in \mathcal{B}$ is modeled by $T_X\mathcal{B}$, the tangent space to $\mathcal{B}$ at X. The configuration of that material point is therefore given by an immersion $T_X\mathcal{B} \to T\mathcal{S}$. The *local configuration of the body* $\mathcal{B}$ is a vector bundle morphism (VB-morphism [L]) $\kappa : T\mathcal{B} \to T\mathcal{S}$ [6] where the underlying map $\kappa_o : \mathcal{B} \to \mathcal{S}$, such that $\kappa_o \circ \pi_{\mathcal{B}} = \pi_{\mathcal{S}} \circ \kappa$, is not necessarily an embedding. The set $C^s_k(\mathcal{B}, T\mathcal{S})$ of all

[5] For the discussion of this point see Marsden and Hughes [MH].

[6] Equivalently, a section of $J^1(\mathcal{B},\mathcal{S})$-see Definition 1.6.

VB-morphisms of class C^{s} over C^{k} base maps, where s≤k, is a C^{∞} vector bundle over $C^{k}(\mathcal{B},\mathcal{S})$ [V], [Se]. Therefore, we postulate:

Definition 1.8 *The* **local configuration space** *of the body $\mathcal{B}$ is a submanifold $\hat{\mathcal{C}}$ of $C^{s}_{k}(T\mathcal{B},T\mathcal{S})$.*

In particular, as the set $\mathcal{C}_{\mathcal{B}}$ of all embeddings of $\mathcal{B}$ into $\mathcal{S}$ is open in $C^{k}(\mathcal{B},\mathcal{S})$, one can select as the local configuration space the set of all VB-morphisms $T\mathcal{B}\rightarrow T\mathcal{S}$ over embeddings $\mathcal{B}\rightarrow\mathcal{S}$, as was proposed in [Sv]. The *local virtual displacement* is then a vector $\delta\eta\in T\hat{\mathcal{C}}$ which can be identified with the map $\delta\eta_{\kappa}: T\mathcal{B}\rightarrow T(T\mathcal{S})|_{\kappa(T\mathcal{B})}$. The *local force*, similarly to the global case, is a 1-form on the space of local configurations, i.e., $\delta\mathrm{f}\in T^{*}\hat{\mathcal{C}}$. Suppose now that a connection is given on $T\mathcal{S}$. Thus, every vector $\mathbf{u}\in TT\mathcal{S}$ decomposes uniquely into its horizontal and vertical parts and a VB-morphism $\mathfrak{v}$ which assigns to every tangent vector $\mathbf{u}$ its vertical component $\mathfrak{v}(\mathbf{u})\in T_{\mathbf{u}}(T_{\pi(\mathbf{u})}\mathcal{S})$ can be defined. Moreover, any vertical component of a vector tangent to $T\mathcal{S}$ as a tangent vector to a vector space can be canonically identified with an element of $T\mathcal{S}$. If one now chooses to represent the local virtual displacement $\delta\eta_{\kappa}$ by $\Delta\eta_{\kappa}=i\circ\mathfrak{v}\circ\delta\eta_{\kappa}$, where i represents the above mentioned canonical identifications, the restriction of $\Delta\eta_{\kappa}$ to the tangent space at X becomes a linear transformation from $T_{X}\mathcal{B}$ into $T_{\kappa_{o}(X)}\mathcal{S}$. The corresponding covector $\mathbf{p}_{X}$, known as the *local first Piola-Kirchhoff stress*, is then a restriction of a linear mapping $\mathbf{p}$: $\kappa(T\mathcal{B})\rightarrow T\mathcal{B}$ to $T_{\kappa_{o}(X)}\mathcal{S}$ such that $\kappa_{o}\circ\pi_{\mathcal{B}}\circ\mathbf{p}_{X}(\mathbf{v})=\pi_{\mathcal{S}}(\mathbf{v})$ for every vector $\mathbf{v}\in\kappa(T\mathcal{B})$. The total work of the local forces $\delta\mathrm{f}$ acting on the local virtual displacement $\delta\eta$ can now be given by

$$\delta\mathbf{f}(\delta\eta)=\int_{\mathcal{B}}tr(\mathbf{p}_{X}\circ\Delta\eta)(X)\mu_{\mathcal{B}}. \tag{1.7}$$

The local stress $\mathbf{p}_{X}$ is hence identifiable with the value at X of the first Piola-Kirchhoff stress tensor $\mathbf{P}$, defined in the context of the localized hyperelasticity, provided that both model are made compatible. Hence we say that the local configuration κ is compatible with the global configuration ψ if $\kappa=\psi_{*}$. On the other hand the local virtual displacement $\delta\eta$ is said to be compatible with the global virtual displacement η if $\delta\eta = \omega \circ \eta_*$ where ω is the canonical involution on the double tangent TTS [AM]. Finally, we postulate that the local force $\delta\mathbf{f}$ is compatible with the global force $\mathbf{f}$ if $\delta\mathbf{f}(\delta\eta) = \mathbf{f}(\eta)$ for any pair of compatible virtual displacements $\delta\eta$, η at compatible configurations.[7]

[7] Details can be found in [EpSe] and [Se].

2. MATERIAL SYMMETRIES

By a symmetry of the body $\mathcal{B}$ with the constitutive response function $\mathfrak{c}$ one understands a change of a configuration which leaves the material response unchanged. In the context of the global theory we postulate, as in [EpEŚ], that:

Definition 2.1. *The* **symmetry** *of the material body $\mathcal{B}$ characterized by the constitutive functional $\mathfrak{c}$ is a diffeomorphism γ of its configuration space $C_{\mathcal{B}}$ such that*

$$\gamma^* \mathfrak{c} = \mathfrak{c} \tag{2.1}$$

where the superscript star denotes the pull-back operator.

Thus, if γ is a symmetry of $\mathcal{B}$,

$$\mathfrak{c}(\gamma(\psi))(\gamma_*(\eta_\psi)) = \mathfrak{c}(\psi)(\eta_\psi) \tag{2.2}$$

for every configuration $\psi \in C_{\mathcal{B}}$ and every virtual displacement $\eta_\psi \in T_\psi C_{\mathcal{B}}$. Clearly, the set $\mathcal{G}_{\mathfrak{c}}$ of all the diffeomorphisms of the configuration space of $\mathcal{B}$ satisfying relation (2.2) forms a group under composition. The following two special subgroups are of particular interest. First, let $\beta : \mathcal{B} \to \mathcal{B}$ be a diffeomorphism of the body manifold. It induces, by composition on the right, a unique diffeomorphism $\gamma_\beta \in Diff_{C}$, i.e., $\gamma_\beta(\psi) = \psi \circ \beta$ for any configuration $\psi \in C_{\mathcal{B}}$. $Diff_C$ denotes here the space of all diffeomorphisms of the configuration space C. Similarly, a diffeomorphism s of the ambient space $\mathcal{S}$ induces a diffeomorphism γ_s of the configuration space by composition on the left.

Definition 2.2 *The subgroups $\mathcal{G}_{\mathcal{B}}$ and $\mathcal{G}_{\mathcal{S}}$ generated by the diffeomorphisms of the body manifold and the ambient space, respectively, will be called the* **material** *and the* **spatial global symmetry groups** *of B.*

Note, that if $\mathcal{S} = \mathbb{R}^3$ and the diffeomorphism $s : \mathcal{S} \to \mathcal{S}$ is a global isometry, the relation (2.2) is the expression of the *material frame indifference principle* [TN].

The symmetry group as defined above, whether material or spatial, is both configuration and coordinate chart independent. Often, however, it is convenient to introduce the material symmetry group relative to a particular configuration, say ψ_0, called the *reference configuration.* Namely, the material symmetry of the body $\mathcal{B}$ relative to the reference ψ_0 is an element of $\mathcal{G}_{\psi_0} \equiv \psi_0 \circ \mathcal{G}_{\mathcal{B}} \circ \psi_0^{-1}$. It is then easy to see that given another reference, say ϕ_0,

$$\mathcal{G}_{\phi_0} = \chi_0^{-1} \circ \mathcal{G}_{\psi_0} \circ \chi_0 \tag{2.3}$$

where $\chi_0 = \psi_0 \circ \phi_0^{-1}$ denotes the deformation from one reference configuration to another reference configuration.

We shall look now at some particular classes of materials and the relations between their different but often overlapping symmetry groups. To this end, let us assume that the material body $\mathcal{B}$ is hyperelastic. It follows from the definition of the elastic potential $\mathcal{W}$ (Definition 1.6) that for every configuration $\psi \in \mathcal{C}_{\mathcal{B}}$ and any material symmetry $\gamma \in \mathcal{G}_{\mathcal{B}}$

$$\mathcal{W}(\gamma(\psi)) = \mathcal{W}(\psi). \tag{2.4}$$

Moreover, if $\mathcal{B}$ is a local hyperelastic material body with φ as the density of its elastic potential $\mathcal{W}$, it is elementary to see that if there exists $\beta \in Diff_{\mathcal{B}}$ such that

$$\varphi(\mathrm{X}, \gamma_\beta(\psi))J(\beta_*)(\mathrm{X}) = \varphi(\beta(\mathrm{X}), \psi), \tag{2.5}$$

at every material point X, and for any configuration ψ, the induced diffeomorphism $\gamma_\beta \in \mathcal{G}_{\mathcal{B}}$. β_* denotes here the tangent map and $\mathrm{J}(\beta_*)$ is its Jacobian.

Note that if we consider incompressible elasticity (e.g. rubber) not only will the configuration space $\mathcal{C}_{\mu_{\mathcal{B}}}$ contain only volume preserving embeddings but also, to check for the material symmetries, as well as the spatial symmetries, one can only draw from the respective subgroups of volume preserving diffeomorphisms.[8] The set of all symmetries of a local hyperelastic material (incompressible or not), obeying the relation (2.5), forms a subgroup $\mathcal{U}_{\mathcal{B}}$ of $\mathcal{G}_{\mathcal{B}}$. For reasons which will be clear later we will call it the *uniform subgroup* of the global material symmetry group of the local hyperelastic material body $\mathcal{B}$.

Any local hyperelastic material body is, in fact, k-jet local for some finite grade k (Proposition 1.1.). Consequently, the density of its elastic potential $\mathcal{W}$ at the material point $\mathrm{X} \in \mathcal{B}$ can be affected by a configuration change only if the new configuration has a different k-jet at X.

Definition 2.3 *$\gamma_\alpha \in \mathcal{G}_{\mathcal{B}}$ is the* **local material symmetry** *of a hyperelastic material point* $\mathrm{X} \in \mathcal{B}$ *if the diffeomorphism* $\alpha \in Diff_{\mathcal{B}}$ *preserves the point* X *and*

$$\varphi(j^k\psi(\mathrm{X}) \circ j^k\alpha(\mathrm{X}))J(j^1\alpha(\mathrm{X})) = \varphi(j^k\psi(\mathrm{X})) \tag{2.6}$$

for every k-jet local configuration $j^k\psi(\mathrm{X})$.[9]

Note that if $\gamma_\alpha \in \mathcal{U}_{\mathcal{B}}$, for some diffeomorphism α having the material point X as its fixed point, then according to the relation (2.4) $j^k\alpha(\mathrm{X})$ is a local material symmetry at X. Note also that whether we use the global model or a compatible local model the definition of the local symmetry group as the set

[8] Note that $\mathcal{C}_{\mu_{\mathcal{B}}}$ is a submanifold of $\mathcal{C}_{\mathcal{B}}$, [EbM].

[9] To define the local material symmetry one could invoke all diffeomorphisms γ of the configuration space $\mathcal{C}_{\mathcal{B}}$ satisfying (2.6) and such that for every configuration ψ and any material point X $\gamma(\psi)(\mathrm{X}) = \psi(\mathrm{X})$. The jets of such γ's could be considered local symmetries. This would, however, unnecessarily involve also symmetries of the ambient space $\mathcal{S}$.

of k-jets of local diffeomorphisms of the reference configuration $\mathcal{B}$ preserving, up to the Jacobian, the value of the constitutive functional will always be the same. All despite the fact that we may choose the stress tensor or the density of its elastic potential, if there exists one, to be used as such a constitutive functional. However, although the definitions are the same, the local symmetry group based on the knowledge of the elastic potential is in general different from the symmetry group of its first Piola-Kirchhoff stress tensor. Indeed, adding any material point only dependent function to the density of the elastic potential will not change the mechanical response of the material point, as highlighted by the definition of the stress tensor (Definition 1.7), but it will affect the choice of symmetries.

3. MATERIAL UNIFORMITY

Intuitively speaking, a material body is thought of as materially uniform if all its points are made of the same material. That is, if different material points respond the same way to the compatible changes in their mechanical states. As pointed out by Epstein at al. [EpEŚ] in the context of a completely global theory this way of formulating the idea of uniformity seems to be problematic as it presupposes some kind of locality. For a truly global material body it is impossible to measure the response of any single material point but only the response of the body as a whole. The key idea of checking the uniformity, however, is that of placing one piece of the body in the same configuration as another piece and then checking for the local response.

To make this point more clear and the idea of uniformity more precise let us first introduce the concept of the *non-local symmetry group relative to a material point.* Let $\mathcal{U}$ be an open set in $\mathcal{B}$. Denote by $\mathcal{U}_{\mathrm{X}}$ the family of all open neighbourhoods of the given material point X and let $\eta_{\mathcal{U}}$ be any virtual displacement with a compact support in $\mathcal{U}$.

Definition 3.1

a. $\gamma \in Diff_C$ *is called the* **global symmetry of the subbody** $\mathcal{U}$ *if*

$$\gamma^* \mathfrak{c}(\psi)(\eta_{\mathcal{U}}) = \mathfrak{c}(\psi)(\eta_{\mathcal{U}}) \tag{3.1}$$

for every virtual displacement $\eta_{\mathcal{U}}$ *and every configuration* $\psi = \pi_C(\eta_{\mathcal{U}})$.

b. *The* **global symmetry group of the material point** X *is the union*

$$\mathcal{G}_{\mathfrak{c}}(\mathrm{X}) = \bigcup_{\mathcal{U} \in \mathcal{U}_{\mathrm{X}}} \mathcal{G}_{\mathfrak{c}}(\mathcal{U}) \tag{3.2}$$

where $\mathcal{G}_{\mathfrak{c}}(\mathcal{U})$ *denotes the set of all global symmetries of the subbody* $\mathcal{U} \subset \mathcal{B}$.

Having the group $\mathcal{G}_\mathfrak{c}(\mathrm{X})$ defined we are now in a position to introduce the concept of a *material isomorphism.*[10]

Definition 3.2 *The material point* $\mathrm{Y} \in \mathcal{B}$ *is* **globally materially isomorphic** *to a material point* $\mathrm{X} \in \mathcal{B}$*, if there exists a diffeomorphism* $\alpha \in Diff_\mathcal{B}$ *such that* $\alpha(\mathrm{Y}) = \mathrm{X}$ *and* $\gamma_\alpha \in \mathcal{G}_\mathfrak{c}(\mathrm{Y})$*. The symmetry* γ_α *is then called the* **global material isomorphism** *and the corresponding diffeomorphism* α *the* **material isomorphism generator**.

It is not difficult to see that being materially isomorphic is an equivalence relation as it is both reflexive and transitive. Moreover, if $\beta_1, \beta_2 \in Diff_\mathcal{B}$ are such that the corresponding diffeomorphisms of the configuration space, γ_{β_1} and γ_{β_2} are the global symmetries of the material points X and Y, respectively, then $\gamma_{\beta_1 \circ \alpha \circ \beta_2^{-1}}$ generates another global material isomorphism. Also, if α_1 and α_2 are generators of two material isomorphisms of X and Y then $\gamma_{\alpha_1^{-1} \circ \alpha_2}$ is a global symmetry of the material point Y. A conjugation of a material isomorphism by the material symmetries is again a material isomorphism and a composition of a material isomorphism with the inverse of another material isomorphism is an element of a symmetry group [TW]. Incidentally, any element of the uniform subgroup $\mathcal{U}_\mathcal{B}$ of a local hyperelastic material is a material isomorphism. In fact, for this class of hyperelastic local materials one could alternatively postulate that a diffeomorphism $\alpha \in Diff_\mathcal{B}$ such that $\alpha(\mathrm{Y}) = \mathrm{X}$ and satisfying the relation (2.5) over some open neighbourhood of the point Y makes the material points X and Y materially isomorphic. Imitating the standard definition of uniformity of Noll and Wang (see e.g., [TW]) we say that:

Definition 3.3 *A material body* $\mathcal{B}$ *represented by the constitutive functional* $\mathfrak{c}$ *is* **materially transitive** *if, and only if, all its points are pairwise globally materially isomorphic.*[11]

[10] The concept of the global symmetry group of a material point can also be used to present locality as a symmetry, as shown by Epstein at al. [EpEŚ].

[11] The term transitive is borrowed from Sternberg ([S], p.321) in anticipation of the fact

As noted before and also in [EpEŚ] and [EEpŚ1] the proposed definition of global material uniformity may imply, due to the required compactness of $\mathcal{B}$, some physically unreasonable behaviour at the perimeters of a truly global body. This may be particularly true if the body has a material boundary. To deal with this problem one should probably incorporate into the definition of uniformity some limiting process (similar to the one proposed by Epstein at al. [EpEŚ] in dealing with the concept of locality) to describe the transition of material properties from the interior of the body into its boundary and compatible with some definition of uniformity of a material boundary element. This, however, will not be investigated in this exposition where, to avoid any future confusion, we assume that *as far as the uniformity problem is concerned the manifold $\mathcal{B}$ is boundaryless.*

For a k-grade local material, in addition to the concept the global uniformity, we can also adopt the standard definition of a material isomorphism of Noll [N] and Wang [W] by saying that:

Definition 3.4 *Two material points, say* X *and* Y, *of the local material body $\mathcal{B}$ are* **materially isomorphic** *if, and only if, there exists an isomorphism $\mathcal{P}_{XY} : J^k_Y(\mathcal{B}, T\mathcal{S}) \to J^k_X(\mathcal{B}, T\mathcal{S})$ such that*

$$\sigma(j^k\eta_\psi(Y)) = \sigma(\mathcal{P}_{XY}(j^k\eta_\psi(Y))) \tag{3.3}$$

for every configuration $\psi \in C_{\mathcal{B}}$ and every $\eta_\psi \in TC_{\mathcal{B}}$.[12] *If in addition, any two material points are materially isomorphic and for every material point* Z $\in \mathcal{B}$ *there exists an open neighbourhood $\mathcal{U}$ in $\mathcal{B}$ containing* Z *over which the material isomorphisms $\mathcal{P}_{ZY}$ are distributed smoothly the material body is called* **smoothly materially uniform.**

that the material body which is materially transitive (globally uniform) induces in a natural way a frame transitive G-structure.

[12] Note that for k = 1 the above condition can be realized by a linear isomorphism from $T_Y\mathcal{B}$ to $T_X\mathcal{B}$, as originally postulated by Noll and Wang [N], [W], [CoEp].

For the local material we have now two notions of material uniformity, the global one called transitivity which requires for each pair of material points the existence of a local diffeomorphism generating a configuration space diffeomorphism satisfying (3.1) and the local uniformity of the Definition 3.4. Clearly, for this class of materials, transitivity implies local uniformity since for any global material isomorphism γ_α $T^k\alpha|_{J^k_Y(\mathcal{B},T\mathcal{S})}$ is a local material isomorphism of (3.3). The converse, however, need not be true as even the existence of a smooth collection of material isomorphisms $\mathcal{P}_{ZY}$ does not guarantee the existence of a global material isomorphism in the sense of the Definition 3.2. The discussion of the necessary and sufficient conditions for a materially uniform local material body to be materially transitive consult Elżanowski at al. [EEpŚ1].

We end this section by deriving the global uniformity condition for the one-dimensional localized simple elasticity in terms of the Piola-Kirchhoff stress tensor[13] and by showing a simple example of how to determine in a direct fashion whether or not a given constitutive law describes a uniform material body [EEp1]. To this end let $\mathcal{B} \subset \mathcal{S} = I\!R$ and assume that the local response function

$$\sigma(j^1\eta_\psi)(\mathrm{X}) = \frac{1}{2}\mathbf{p}(j^1\psi(\mathrm{X}))\delta\mathrm{g}\mathrm{dX} \tag{3.4}$$

where $\mathbf{p}$ denotes, as before, the Piola-Kirchhoff stress and where the deformed metric $\mathrm{g} = [\psi'(\mathrm{X})]^2$. The virtual work takes the well known form

$$\mathfrak{c}(\psi)(\eta_\psi) = \int_0^1 \mathbf{p}(\psi'(\mathrm{X},\mathrm{X})\psi'(\mathrm{X}))\delta\psi'(\mathrm{X})\mathrm{dX}. \tag{3.5}$$

where prime denotes the differentiation in $\mathcal{B}$. Suppose now that $\beta \in Diff_{\mathcal{B}}$ is the material isomorphism generator and let γ_β be the global material isomorphism. Then,

[13] This is based on [E1] and some notes made available to me by Marcelo Epstein.

$$\gamma_\beta^* \mathfrak{c}(\psi)(\eta_\psi) = \mathfrak{c}(\psi \circ \beta)(\gamma_{\beta *}(\eta_\psi)) = \int_0^1 \mathbf{p}((\psi \circ \beta)', \mathrm{Z})(\psi \circ \beta)'(\mathrm{Z})\delta(\psi \circ \beta)' \mathrm{dZ}. \quad (3.6)$$

On the other hand, if $\mathrm{X} = \beta(\mathrm{Z})$ then,

$$\mathfrak{c}(\psi)(\eta_\psi) = \int_0^1 \mathbf{p}(\psi'(\beta(\mathrm{Z})), \beta(\mathrm{Z}))\psi'(\beta(\mathrm{Z}))\delta\psi'(\beta(\mathrm{Z}))\beta'(\mathrm{Z})\mathrm{dZ}. \quad (3.7)$$

Using the global uniformity condition (Definition 3.2) one obtaines from the fundamental theorem of calculus of variations that the localized simple elastic material body is globaly uniform only if

$$\mathbf{p}(\psi'(\beta(\mathrm{Z})), \beta(\mathrm{Z})) = \mathbf{p}((\psi \circ \beta)'(\mathrm{Z}), \mathrm{Z})\beta'(\mathrm{Z}) \quad (3.8)$$

for every $\mathrm{Z} \in \mathcal{B}$.

For the second example let us consider a simple hyperelastic material with the density of its elastic potential $\varphi = \varphi(j^1\psi(\mathrm{X}))$, for every configuration ψ and every material point $\mathrm{X} \in \mathcal{B} = \mathbb{R}^3$. As pointed out before the first jet of an embedding ψ at a point X can be identified with a source, a target point and a linear map $\mathbf{F}(\mathrm{X}) = \psi_*(\mathrm{X}) : T_X\mathcal{B} \to T_{\psi(\mathrm{X})}\mathcal{S}$. Consequently, because of the translational invariance in $\mathcal{S}$ the elastic potential becomes a function of a material point and the deformation gradient $\mathbf{F}$. Moreover, if $\mathcal{S}$ is a Riemannian manifold, φ depends on $\mathbf{F}$ only through $\mathbf{C} = \mathbf{F}^*\mathbf{F}$, due to the material frame indifference (see e.g., [M1] or [TN]) where $\mathbf{F}^*$ denotes the dual operator, [L]. Thus, given a smooth field of local configurations $\mathfrak{p}^1 : \mathcal{U} \to J^1(\mathcal{U}, \mathcal{S})$, where $\mathcal{U}$ is an open neighbourhood in $\mathcal{B}$, let us consider the elastic potential density function

$$\varphi(\mathrm{Y}, \mathbf{F}(\mathrm{Y})) = tr(\mathbf{A}(\mathrm{Y})\mathbf{C}(\mathrm{Y})) + \varphi_0(\mathrm{Y}) \quad (3.9)$$

where $\varphi_o(Y)$ is a scalar function of position only and where $\mathbf{A}(Y) \in L(T_Y\mathcal{B}, T_Y\mathcal{B})$ is assumed to be positive definite and symmetric. In the context of a simple hyperelastic material for the body $\mathcal{U}$ to be materially uniform the Definition 3.4 can be realized by assuming that there exists a smooth VB-automorphism $\mathcal{P}$ of the tangent bundle $T\mathcal{U}$ and a scalar-valued function f such that

$$tr(\mathbf{A}(Y)\mathbf{C}(Y)) + f(Y) = tr(\mathbf{A}(X)\mathcal{P}_{XY}\mathbf{C}(Y)\mathcal{P}_{YX}) + \varphi_0(X) \tag{3.10}$$

holds identically for all nonsingular $\mathbf{F}(Y)$ at any X and Y $\in \mathcal{U}$.[14] To show that this is possible we start by setting $f(Y) = \varphi_0(Y)$ and by observing that the condition (3.10) implies that $\mathbf{A}(Y)\mathcal{P}_{YX} = \mathcal{P}_{YX}\mathbf{A}(X)$. Invoking polar decomposition theorem ([L] p.156) for the isomorphism $\mathbf{Q}_{YX} = \mathcal{P}_{YX}\mathbf{A}(X)^{\frac{1}{2}}$ one obtains, in view of the uniqueness of the polar decomposition, $\mathbf{Q}_{YX} = \mathbf{A}(Y)^{\frac{1}{2}}\mathbf{R}_{YX}$ where $\mathbf{R}_{YX} : T_X\mathcal{B} \to T_Y\mathcal{B}$ is an orthogonal isomorphism. It follows that any linear isomorphism

$$\mathcal{P}_{YX} = \mathbf{A}(Y)^{\frac{1}{2}}\mathbf{R}_{YX}\mathbf{A}(X)^{-\frac{1}{2}} \tag{3.11}$$

can serve as a material isomorphism. Incidently, we have just proved:

Proposition 3.1 *The material body $\mathcal{B}$ with the constitutive law* (3.10) *is always smoothly materially uniform provided the map* $X \longmapsto \mathbf{A}(X)$ *is locally smooth.*

This fact is unfortunately by no means a rule but rather an exception, as shown in [EEp1]. Indeed, applying the method presented above to the higher order polynomial analogy of the constitutive law (3.9)

[14] $\mathcal{P}$, when restricted to the fibers at X and Y, becomes the linear isomorphism $\mathcal{P}_{XY}$.

$$\varphi(\mathrm{Y}, \mathbf{F}(\mathrm{Y})) = tr(\mathbf{A}_1(\mathrm{Y})\mathbf{C}(\mathrm{Y})) + tr(\mathbf{A}_2(\mathrm{Y})\mathbf{C}^2(\mathrm{Y})) + \varphi_0(\mathrm{Y}) \tag{3.12}$$

it is easy to see that the uniformity condition (3.10) is, in general, impossible to satisfy unless *material coefficients* $\mathbf{A}_1$ and $\mathbf{A}_2$ are related through the respective fields of orthogonal isomorphisms. By rather straightforward calculations one can show that:

Proposition 3.2 *The material body $\mathcal{B}$ defined by the elastic potential* (3.12) *is uniform only if for any pair of material points* X *and* Y *the material coefficients* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are such that*

$$\mathbf{A}_2(\mathrm{Y})^{\frac{1}{2}} = \mathbf{A}_1(\mathrm{Y})^{\frac{1}{2}}\mathbf{R}_{XY}\mathbf{A}_1(\mathrm{X})^{-\frac{1}{2}}\mathbf{A}_2(\mathrm{X})^{\frac{1}{2}}\mathbf{S}_{YX} \tag{3.13}$$

where $\mathbf{R}$ *and* $\mathbf{S}$ *are arbitrary orthogonal automorphisms of the tangent bundle* $T\mathcal{B}$.

4. UNIFORM MATERIAL STRUCTURES

After introducing in the previous section the concepts of material isomorphism and that of a material uniformity we are now in a position to unveil the intrinsic geometric structure associated with a smoothly uniform local material body of an arbitrary finite grade. For the clarity and also the simplicity of our exposition we shall restrict the class of materials considered here to the finite grade local hyperelasticity.

Hence, suppose that φ denotes the density of an elastic potential of the continuous material body $\mathcal{B}$ with placements in the ambient space $\mathcal{S}$. As we are going to deal only with unconstrained elastic materials [15] we assume that the body manifold $\mathcal{B}$ and the ambient space $\mathcal{S}$ are manifolds of the same dimension, say n. Our first objective is to show that φ as the constitutive functional of a k-grade local hyperelastic material body is in fact a function on the fibre bundle of k- frames of the body $\mathcal{B}$. To this end, select a material point $X \in \mathcal{B}$. We recognize that two embeddings of $\mathcal{B}$ into $\mathcal{S}$ give rise to the same k-jet at X if, and only if, they have at X the same partial derivative up to order k, with respect to some local coordinate systems on $\mathcal{B}$ and $\mathcal{S}$. Note that this definition is independent of the choice of the coordinate systems. Moreover, any k-jet at X of the configuration ψ is an invertible jet (see e.g., Kobayashi [Ko]) as

$$j^k\psi(X) \circ j^k\psi^{-1}(\psi(X)) \equiv j^k(\psi \circ \psi^{-1})(\psi(X)) = j^k id_{\mathcal{S}}. \tag{4.1}$$

where $id_{\mathcal{S}}$ denotes the local identity in $\mathcal{S}$. The collection of all the k-jets of all possible embeddings of $\mathcal{B}$ into $\mathcal{S}$, denoted by $J^k(\mathcal{B},\mathcal{S})$ is a fibre bundle over the manifold $\mathcal{B}$ with the source map $\pi^k(j^k\psi(X)) = X$ being the natural projection onto $\mathcal{B}$, [Sa]. Its fibre at each and every material point X is isomorphic, modulo the translations in $\mathcal{S}$[16], with the set $\mathcal{G}^k$ of all invertible k-jets of the differentiable mappings $g : \mathbb{R}^n \to \mathbb{R}^n$ with the source and the

[15] Some discussion on the interplay of uniformity and constraints was presented in [EEp0].

[16] This, in fact, has been taken care of in the definition of the k-jet.

target at the origin of $\mathbb{R}^n$. Indeed, given an invertible k-jet $j^k\psi(\mathrm{X})$ and selecting, without loss of generality, local coordinate charts α and β on some open neighbourhoods of X and $\psi(\mathrm{X})$ respectively, such that $\alpha(\mathrm{X}) = \beta(\psi(\mathrm{X})) = 0$, $j^k(\alpha \circ \psi \circ \beta^{-1})(0) \in J_0^k(\mathbb{R}^n, \mathbb{R}^n)$ and it is obviously invertible. Evidently, the converse is true as well. Let $\mathrm{H}^k(\mathcal{B})$ denote the bundle of all holonomic k-frames of $\mathcal{B}$, i.e. the set of k-jets at $0 \in \mathbb{R}^n$ of all local diffeomorphisms of $\mathbb{R}^n$ into $\mathcal{B}$, [Sa]. It is now easy to see that the set of k-jets of all configurations of $\mathcal{B}$ can be identified with $\mathrm{H}^k(\mathcal{B})$. Consequently we have:

Proposition 4.1 *Given a k-grade local hyperelastic material its density of the elastic potential φ is a smooth real valued function on the bundle of holonomic k-frames of $\mathcal{B}$.*

This fact is particularly transparent in the case of a simple material body. Indeed, the first jet of a configuration at X can be identify with the pair $(\mathrm{X}, \mathbf{F})$ where $\mathbf{F} : T_{\mathrm{X}}\mathcal{B} \rightarrow T\mathcal{S}$ is a deformation gradient and so a nonsingular linear transformation. Selecting an orthonormal frame at $T_{\mathrm{X}}\mathcal{B}$ $\mathbf{F}$ induces another basis in $T\mathcal{S}$ at $\pi(\mathbf{F}(T_{\mathrm{X}}\mathcal{B}))$.

The set $\mathcal{G}^k$ is a group with the multiplication defined by the composition of jets. It acts on $\mathrm{H}^k(\mathcal{B})$ on the right. Namely, given a k-frame $\mathrm{p}^k = j^k\psi(\mathrm{X})$, for some local diffeomorphism ψ, and $\mathrm{g}^k = j^k\mathrm{g}(0) \in \mathcal{G}^k$, where $\mathrm{g} \in J_0^k(\mathbb{R}^n, \mathbb{R}^n)$, $\mathrm{p}^k\mathrm{g}^k \equiv j^k\phi(\mathrm{X})$ such that $j^k(\phi^{-1}\circ\beta^{-1})(0) = j^k(\psi^{-1}\circ\beta^{-1}\circ\mathrm{g})(0)$ for some local coordinate map β on $\mathcal{S}$. It is then easy to see that locally $\phi = \beta^{-1}\circ\mathrm{g}^{-1}\circ\beta\circ\psi$. It is also straightforward to show that $\mathrm{H}^k(\mathcal{B})$ is a principal bundle over $\mathcal{B}$ with the structure group $\mathcal{G}^k$ (see e.g., [CDL] or [Sa]). Looking closer at the collections of all holonomic frame bundles we first notice that the structure group $\mathcal{G}^1 = \mathrm{GL}(\mathrm{n}, \mathbb{R})$ and that $\mathrm{H}^1(\mathcal{B})$ is the bundle of linear frames of $\mathcal{B}$, [CDL]. In turn, $\mathcal{G}^2$ is the semidirect product of the general linear group $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ and the space of bilinear symmetric $\mathbb{R}^n-$valued forms $\mathrm{S}^2(\mathbb{R}^n, \mathbb{R}^n)$ (see e.g., [CDL] and also [EEp2]). $\mathrm{H}^2(B)$, which in the literature appears under the name of the *holonomic second order frame bundle*[17], is not only a principal bundle over

[17] The term holonomic, which as a matter of fact can be applied to any order frame,

$\mathcal{B}$ with $\mathcal{G}^2$ as its structure group but also an affine bundle over $H^1(\mathcal{B})$ with the standard fibre $\mathcal{N}_1^2(n) = S^2(\mathbb{R}^n, \mathbb{R}^n)$ and the projection $\pi_1^2 : H^2(\mathcal{B}) \to H^1(\mathcal{B})$ such that $\pi_2(p^2) = \pi_1(\pi_1^2(p^2))$ for any $p^2 \in H^2(\mathcal{B})$.

Suppose now that $\varphi : H^k(\mathcal{B}) \to \mathbb{R}$ is the density of the Lagrangian (strain energy function) $\mathcal{W}$ of the k-grade local hyperelastic body $\mathcal{B}$. By the *isotropy group* of φ at X we understand the collection of the elements of $\mathcal{G}^k$ on the orbits of which $\varphi|_{\pi_k^{-1}(X)}$ is constant.

Definition 4.1 *The (local)* **symmetry group of the material point** $X \in \mathcal{B}$ *is the maximum subgroup* $\mathcal{G}_X^k$ *of* $\mathcal{G}^k$ *such that* $\nu_1^k(\mathcal{G}_X^k)$ *is contained in the special linear group* $SL(n, \mathbb{R})$ *and which is also a subgroup of the isotropy group of* φ *at* X.

Note that the Implicit Function Theorem (see e.g., [K]) implies that for every element of $\mathcal{G}_X^k$ there exists a corresponding local material symmetry of Definition 2.3. Note also that the definition of the symmetry group at X depends on how the set of invertible jets of all embeddings of $\mathcal{B}$ in $\mathcal{S}$ is identified with the bundle of holonomic k-frames, i.e. on the choice of an atlas on $\mathcal{S}$ or equivalently the selection of a local reference configuration. Hence, for the rest of this paper, we assume that such an identification is given.

Materials (or rather material points) are classified according to their symmetry group, [TN]. For example, *the elastic fluid* is a material body the points of which have $SL(n, \mathbb{R})$ as their symmetry group. $\mathcal{B}$ is made of an *isotropic solid* if for every material point X there exists a local configuration relative to which $\mathcal{G}_X^k = SO(n, \mathbb{R})$, the special orthogonal group. These and other material structures were analyzed in [WT], [EEpŚ1], [EEp0] and [EP1].

relates to the fact that the elements of $H^k(\mathcal{B})$ are equivalent classes of embeddings rather than jets of sections of bundles of frames of lower order. Only for $k = 1$ there is naturally no difference between a holonomic frame and a non-holonomic frame. For the precise definition of a non-holonomic and a semi-holonomic frame we refer the reader to [EP3], [Sa] and [Y]. Some aspects of these definitions will also be reviewed in Chapter 5.

Even if for two different material points, say X and Y, of the k- grade local hyperelastic material body $\mathcal{B}$ the corresponding symmetry groups are identical one cannot be sure yet that both points are made of the same material. For, the symmetry group of a material point is only a partial characteristic of a material while the ultimate test is that of measuring the response of these material points to the superimposed deformations. As we have argued before, the mathematically correct test is that of the existence of a material isomorphism of the Definition 3.3. Thus, suppose that $\mathrm{X}, \mathrm{Y} \in \mathcal{B}$ are materially isomorphic, i.e. there exists a volume preserving isomorphism $\mathcal{P}_{\mathrm{XY}} : \pi_k^{-1}(\mathrm{Y}) \rightarrow \pi_k^{-1}(\mathrm{X})$ such that

$$\varphi(\mathcal{P}_{\mathrm{XY}}(\mathrm{p}^k)) = \varphi(\mathrm{p}^k) \tag{4.3}$$

for every $\mathrm{p}^k \in \pi_k^{-1}(\mathrm{Y})$. Given $\mathrm{g}^k \in \mathcal{G}_{\mathrm{X}}^k \subset \mathcal{G}^k$, let $\mathfrak{R}_{\mathrm{g}^k} : \mathrm{H}^k(\mathcal{B}) \rightarrow \mathrm{H}^k(\mathcal{B})$ represent the principal bundle automorphism induced by the right action by the element g^k. It is then immediate from the relation (4.3) that

$$\mathfrak{R}_{\mathrm{h}^k}(\mathcal{P}_{\mathrm{XY}}(\mathrm{p}^k) = \mathcal{P}_{\mathrm{XY}}(\mathfrak{R}_{\mathrm{g}^k}(\mathrm{p}^k)) \tag{4.4}$$

for every k-frame p^k over X and any $\mathrm{g}^k \in \mathcal{G}_{\mathrm{X}}^k$ and $\mathrm{h}^k \in \mathcal{G}_{\mathrm{Y}}^k$. The relation (4.4) makes the respective symmetry groups not only homomorphic but also renders $\mathfrak{R}_{\mathrm{g}^k} \circ \mathcal{P}_{\mathrm{XY}} \circ \mathfrak{R}_{\mathrm{h}^k}$ to be a material isomorphism for any $\mathrm{g}^k \in \mathcal{G}_{\mathrm{X}}^k$, any $\mathrm{h}^k \in \mathcal{G}_{\mathrm{Y}}^k$, and any material isomorphism $\mathcal{P}_{\mathrm{XY}}$ (see also [WT]).

Definition 4.2

a. *We say that two k-frames (local configurations)* p_1^k *and* p_2^k *at* X *and* Y, *respectively, are* **materially compatible** *if there exists a material isomorphism* $\mathcal{P}_{\mathrm{XY}}$ *such that* $\mathrm{p}_2^k = \mathcal{P}_{\mathrm{XY}}(\mathrm{p}_1^k)$. *Hence, the* **material reference** *is a smooth local section* $\mathfrak{l}^k : \mathcal{U} \subset \mathcal{B} \rightarrow \mathrm{H}^k(\mathcal{B})$ *such that any two k-frames in its image are materially compatible.*

b. *Any collection* $\mathcal{M}^k(\mathcal{B})$ *of all materially compatible k-frames will be called the* **material structure**.

Obviously, if the frames p_1^k and p_2^k are materially compatible then for any material symmetry g^k and h^k, at $\pi_k(p_1^k)$ and $\pi_k(p_2^k)$ respectively, $p_1^k g^k$ and $p_2^k h^k$ are materially compatible where, $p^k g^k$ is the standard shorthand for the right action $\mathfrak{R}_{g^k}(p^k)$. Also, given the material reference $\mathfrak{l}^k : \mathcal{U} \subset \mathcal{B} \rightarrow H^k(\mathcal{B})$ it induces a local trivialization of the bundle of holonomic k-frames, i.e. an isomorphism $\Psi^k : \pi_k^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times \mathcal{G}^k$ such that $\Psi^k(\mathfrak{l}^k(X)) = (X, e^k)$, for any material point $X \in \mathcal{U}$ where e^k denotes the identity of $\mathcal{G}^k$. By doing so it establishes a homomorphism of the symmetry group of each and every point in $\mathcal{U}$ with a unique (base point independent) subgroup $\mathcal{G}^k_{\mathfrak{l}^k}$ of the structure group $\mathcal{G}^k$ called the *material symmetry group relative to the material reference* $\mathfrak{l}^k$.

Theorem 4.1 *Let φ be the density of the strain energy of the smooth materially uniform k-grade local hyperelastic body $\mathcal{B}$. Then, $\mathcal{M}^k(\mathcal{B})$ is a reduction*[18] *of the bundle of k-frames of $\mathcal{B}$ to some material symmetry groups of $\mathcal{B}$.*

Proof.[19] The statement of the theorem is deliberately generic as there exist many different "collections of materially compatible frames" and many corresponding material symmetry groups all parametrized by different material references. To show that any particular material structure $\mathcal{M}^k(\mathcal{B})$ is a reduction of the principal bundle $H^k(\mathcal{B})$ it is enough to show that there exists a trivialization of $H^k(\mathcal{B})$ whose transition functions take values in the material symmetry group relative to some material reference ([S], Lemma 1.1). This is, however, immediate from the previous discussion. Namely, taking an arbitrary k-frame $p^k \in H^k(\mathcal{B})$ and choosing in its neighbourhood the material reference $\mathfrak{l}^k$, the existence of which is guaranteed by the assumption of smooth local uniformity, will automatically select the material symmetry group $\mathcal{G}^k_{\mathfrak{l}^k}$. It is then apparent from the Definition 4.2 that the only means of collecting all materially compatible frames over $\pi_k(p^k)$ is by the right action of the material symmetry group $\mathcal{G}^k_{\mathfrak{l}^k}$. Extending the given section $\mathfrak{l}^k$ or selecting another

[18] A subbundle of $H^k(\mathcal{B})$ with the structure group being a closed subgroup of $\mathcal{G}^k$. See also Sternberg [S].

[19] See also [EEpŚ2].

section at another materially compatible frame from $\pi_k^{-1}(\mathcal{U})$ will induce local trivialization with transition functions taking values in the given material symmetry groups as implied by (4.4) ♣

It is also evident from the given construction of the particular material structure that if we start the construction from a different frame, say r^k, for which there exists a group element $\mathrm{g}^k \in \mathcal{G}^k/\mathcal{G}^k_{\mathfrak{l}^k}$, the space of right cosets of $\mathcal{G}^k_{\mathfrak{l}^k}$ in $\mathcal{G}^k$, such that $\mathrm{r}^k = \mathrm{p}^k\mathrm{g}^k$ then the corresponding material structures are conjugate, i.e. one is obtained from another by the right action by g^k. The associated material symmetry groups are then conjugate subgroups of the structure group $\mathcal{G}^k$ of $\mathrm{H}^k(\mathcal{B})$. It is worth mentioning at this point that if $\mathrm{H}^k(\mathcal{B})$ can be reduced to $\mathcal{M}^k(\mathcal{B})$ then there exists a global section m^k : $\mathcal{B} \to \mathrm{H}^k(\mathcal{B}) \times_{\mathcal{G}^k} \mathcal{G}^k/\mathcal{G}^k_{\mathfrak{l}^k}$ to the associated bundle of $\mathrm{H}^k(\mathcal{B})$ with the standard fibre $\mathcal{G}^k/\mathcal{G}^k_{\mathfrak{l}^k}$. In our case such a section is easily available by gluing overlapping material references. In fact, the existence of such a global section is both sufficient and necessary for the existence of a reduction [KoNo]. This property is the basis of the analysis of the integrability of G-structures possessing the so-called characteristic object, [EEpŚ2], [F]. Thus we have:

Corollary 4.1 *Any two material structures of the same k-grade local hyperelastic body are conjugate.*

Given a smoothly uniform k-grade hyperelastic body $\mathcal{B}$, a (material) covering $\{\mathcal{U}_{\alpha_i}\}_{i\in I}$ of $\mathcal{B}$ is available such that transition functions of the subordinate trivialization $\{\pi_k \times t_{\alpha_i}\}_{i\in I}$ of $\mathrm{H}^k(\mathcal{B})$ all take values in some material symmetry group. As we know from the proof of Theorem 4.1 such a trivialization is induced by the family of local material references $\mathfrak{l}^k_{\alpha_i} : \mathcal{U}_{\alpha_i} \to \mathrm{H}^k(\mathcal{B})$. Namely, for every $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{U}_{\alpha_i})$

$$\mathrm{p}^k = \mathfrak{l}^k_{\alpha_i}(\pi_k(\mathrm{p}^k))t^k_{\alpha_i}(\mathrm{p}^k) \tag{4.5}$$

where $t^k_{\alpha_i} : \pi_k^{-1}(\mathcal{U}_{\alpha_i}) \to \mathcal{G}^k$ and $t^k_{\alpha_i}t^{-1}_{\alpha_j} \in \mathcal{G}^k_{\mathfrak{l}_{\alpha_i}}$. On the basis of such a *material trivialization* we can now represent, at least locally, the density of the strain energy function φ by a function on the structure group $\mathcal{G}^k$. To this end let us therefore define $\tilde{\mathcal{W}} : \mathcal{G}^k \to I\!R$ such that for every $\mathrm{p}^k \in \pi_k^{-1}(\mathcal{U}_{\alpha_i})$

$$\tilde{\mathcal{W}}(t_{\alpha_i}(\mathrm{p}^k)) = \varphi(\mathrm{p}^k). \tag{4.6}$$

Note that although the definition of the function $\tilde{\mathcal{W}}$ does depend on the choice of a particular material trivialization it is a well defined smooth function on the whole structure group.[20] Note also that its isotropy group is the particular material symmetry group induced by the choice of the material trivialization $\{\pi_k \times t_{\alpha_i}\}_{i\in I}$. Indeed, let $\mathrm{h}^k \in \mathcal{G}^k_{\mathrm{Y}}$ and let $\mathrm{p}^k = \mathfrak{l}^k_{\alpha_i}(\mathrm{X})$ then $\tilde{\mathcal{W}}(t_{\alpha_i}(\mathfrak{l}^k_{\alpha_i}(\mathrm{X})) = \tilde{\mathcal{W}}(t_{\alpha_i}(\mathrm{p}^k)) = \varphi(\mathrm{p}^k) = \varphi(\mathcal{P}_{\mathrm{XY}} \circ \mathfrak{R}_{\mathrm{h}^k} \circ \mathcal{P}_{\mathrm{YX}}(\mathrm{p}^k)) = \varphi(\mathcal{P}_{\mathrm{YX}}(\mathrm{p}^k)\mathrm{h}^k) = \tilde{\mathcal{W}}(t_{\alpha_i}(\mathfrak{l}^k_{\alpha_i}(\mathrm{X}))\mathrm{h}^k)$. As the inducing trivialization has its transition functions taking values in the material symmetry group $\mathcal{G}^k_{\mathfrak{l}^k}$ the relation (4.5) holds for every $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{B})$. Thus we have:

Theorem 4.2 *Given a smoothly uniform k-grade hyperelastic material body $\mathcal{B}$ represented by the density φ of its elastic potential, and selecting a particular material trivialization $\{(\pi_k \times t_{\alpha_i})\}_{i\in I}$, there exists a smooth function $\tilde{\mathcal{W}} : \mathcal{G}^k \to I\!R$ such that the relation* (4.6) *is satisfied for every* $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{B})$.

In fact, the converse is true as well. Namely, given any collection of smooth invariant mappings $t_{\beta_i} : \mathrm{H}^k(\mathcal{U}_{\beta_i}) \to \mathcal{G}^k$ and a smooth function $\tilde{\mathcal{W}} : \mathcal{G}^k \to I\!R$ such that the relation (4.6) is satisfied on $\mathrm{H}^k(\mathcal{B})$ it is easy to see that the material body is smoothly materially uniform. Respective material references $\mathfrak{l}^k$ are then given by $t_{\alpha_i}^{-1}(\mathrm{e}^k)$.

[20] The availability of this relation is not only a reflection of the fact that material isomorphisms are volume preserving but also that the density of the strain energy function at the stress free state, should there exist one, is assumed zero. Other relations where postulated, or derived, in [CoEp] and [EP1].

5. MATERIAL CONNECTIONS

5.1. Principal Material Connections

Suppose that $\mathfrak{l}^k$ is a smooth (local) material reference of the open subbody $\mathcal{U} \subset \mathcal{B}$. Having this available we can lift the tangent space $T\mathcal{U}$ to the bundle of holonomic k-frames $\mathrm{H}^k(\mathcal{B})$ creating a *horizontal distribution*[21] $\mathcal{H}^k$ on $\pi_k^{-1}(\mathcal{U})$ viz:

$$\mathcal{H}^k(\mathrm{p}^k) = T\mathfrak{R}_{t^k(\mathrm{p}^k)}(\mathfrak{l}^k_*(T_{\pi^k(\mathrm{p}^k}\mathcal{B})) \tag{5.1}$$

for every $\mathrm{p}^k \in \pi_k^{-1}(\mathcal{U})$ where T denotes the tangent map and where $t^k : \pi_k^{-1}(\mathcal{U}) \to \mathcal{G}^k$ is defined by the relation (4.5). This distribution is obviously equivariant and such that for every $\mathrm{r}^k \in \pi_k^{-1}(\mathcal{U})$ it splits the tangent space $T\mathrm{H}^k(\mathcal{B})$ i.e., $T_{\mathrm{r}^k}\mathrm{H}^k(\mathcal{B}) = T_{\mathrm{r}^k}\pi_k^{-1}(\mathrm{r}^k)) \oplus \mathcal{H}^k(\mathrm{r}^k)$. Let $\mathfrak{g}^k$ denote the Lie algebra of the structure group $\mathcal{G}^k$, i.e. $\mathfrak{g}^k = T_{\mathrm{e}^k}\mathcal{G}^k$, and let $\omega^k : T\mathrm{H}^k(\mathcal{U}) \to \mathfrak{g}^k$ be the Lie algebra valued 1- form on $\mathrm{H}^k(\mathcal{U})$ such that at any $\mathrm{p}^k \in \pi_k^{-1}(\mathcal{U})$ and for every $\xi \in T_{\mathrm{p}^k}\mathrm{H}^k(\mathcal{U})$

$$\omega^k(\xi) = T\mathcal{L}_{t^k(\mathrm{p}^k)} \circ t^k_*(\xi) \tag{5.2}$$

where $\mathcal{L}_{\mathrm{g}^k} : \mathcal{G}^k \to \mathcal{G}^k$ denotes the left translation by the group element g^k. Using standard arguments (see e.g., [Po], [S]) one can show now that $\mathcal{H}^k(\mathcal{U})$ is exactly the kernel of the 1-form ω^k. It also easy to see from the definition (5.2) that due to the equivariance of the horizontal distribution $\mathcal{H}^k(\mathcal{U})$ the form ω^k is an equivariant 1-form. The extension of the distribution $\mathcal{H}^k(\mathcal{U})$ and the form ω^k to the bundle $\mathrm{H}^k(\mathcal{B})$ is then easily achieved by covering the entire body $\mathcal{B}$ by local material references, generating locally connection forms as per (5.2) and utilizing the partition of unity subordinate to the covering of $\mathcal{B}$ (see [S] and [WT]). As we are able to cover $\mathcal{B}$ by local material references the connection introduced above reduces to a connection on the corresponding material structure $\mathcal{M}^k(\mathcal{B})$. Thus, we postulate:

[21] Being horizontal means that $\pi_{k*}\mathcal{H}^k(\mathrm{p}^k) = T_{\pi_k(\mathrm{p}^k)}\mathcal{B}$.

Definition 5.1 *Any k-connection on the material structure $\mathcal{M}^k(\mathcal{B})$ of $\mathcal{B}$ will be called the* **k-order principal material connection** *of the body $\mathcal{B}$.*

As $\mathcal{M}^k(\mathcal{B})$ is locally trivial, for any $X \in \mathcal{B}$ there exists a principal material connection[22] such that in some neighbourhood of X it is generated by a local material reference. That is, for every material point X there exists an open neighbourhood and a material reference such that the tangent space of its image in $H^k(\mathcal{B})$ coincides with the horizontal distribution of some principal material connection. Consequently the local holonomy group of such a locally integrable principal material connection is trivial and we have the distant material parallelizm [Po]. In the future analysis of material structures we will, in fact, restrict, for most part, our choice of material connections to locally integrable ones only.

Having the principal material connection available we can now restate Theorem 4.2:

Proposition 5.1 *Given the k-grade hyperelastic material body $\mathcal{B}$, represented by the density φ of its strain energy function $\tilde{\mathcal{W}}$, it is smoothly materially uniform if, and only if, for every $p^k \in H^k(\mathcal{B})$ there exists a neighbourhood $\mathcal{U} \ni \pi_k(p^k)$, a k-order connection ω^k and a smooth function $\tilde{\mathcal{W}} : \mathcal{G}^k \to I\!R$ such that the principal material connection $\omega^k|_{\pi_k^{-1}(\mathcal{U})}$ is integrable and that for every r^k in $\pi_k^{-1}(\mathcal{U})$ and every $\xi \in T_{r^k}H^k(\mathcal{B})$*

$$d\varphi(\xi) = d\tilde{\mathcal{W}} \circ \mathfrak{R}_{t^k(p^k)} \circ \omega^k(\xi) \tag{5.3}$$

for some smooth function $t^k : H^k(\mathcal{U}) \to \mathcal{G}^k$ of (4.5), *usually p^k dependent.*

Proof. If $\mathcal{B}$ is smoothly materially uniform one gets the relation (5.3) by differentiating the relation (4.6) and invoking the definition of the connection

[22] The first-order principal material connection is a material connection in the sense of Noll and Wang [WT] (see also Bloom [B]).

1-form (5.2). On the other hand, if there exists a locally integrable connection such that (5.3) holds then the corresponding horizontal distribution $\mathcal{H}^k$ is locally integrable as a differential distribution. Thus, for any $X \in \mathcal{B}$ there exists a local section $\mathfrak{l}^k$ which in turn induces a local trivialization of $H^k(\mathcal{U})$ and the function t^k obeying the relation (4.5). Equivalently, as shown by Poor [Po], there exists a distant parallelism $\mathcal{P}$ which can be taken for the material isomorphisms ♣

The principal material connection we have constructed above is clearly not unique as it strongly depends on the choice of a material section of the bundle of holonomic frames. However, it should be quite obvious from the discussion in this and in the previous chapter that the only two degrees of freedom available to us, as far as choosing another material connection is concerned, are: choosing another material structure or another material reference within the current material structure. As any two material structures are conjugate (Corollary 4.1) the first choice is only apparent, at least for $k = 1$. For the simple material structures translating everything by a constant element of the structure group $\mathcal{G}^k$ is going to change nothing. The connection itself will obviously change but all its essential geometric characteristics will remain the same. For the higher order cases this is not too obvious [EP4]. We will come back to this problem a bit later on once we know more about the higher order connections.

However, it appears that if we change the local material reference of (5.1) from $\mathfrak{l}^k$ to another local material reference the horizontal distribution will change and so will the corresponding connection form. To observe how these changes occur let $\mathfrak{l}_1^k$ and $\mathfrak{l}_2^k$ represent two different local material references but such that the corresponding standard isotropy groups are identical. Thus, $\mathfrak{l}_1^k$ and $\mathfrak{l}_2^k$ are local sections of the same reduction of $H^k(\mathcal{B})$, say $\mathcal{M}^k(\mathcal{B})$. For simplicity, but without any loss of generality, we assume that their respective domains of definition are identical, say $\mathfrak{V}$. Being sections of the same principal bundle $\mathfrak{l}_1^k$ and $\mathfrak{l}_2^k$ differ by a base point dependent deformation by the isotropy group, i.e. there exists a smooth *gauge* $\varrho : \mathfrak{V} \rightarrow \mathcal{G}^k_{\mathfrak{l}_1^k}$ such that

$$\mathfrak{l}_2^k(\mathrm{Y}) = \mathfrak{R}_{\varrho(\mathrm{Y})} \circ \mathfrak{l}_1^k(\mathrm{Y}) \tag{5.4}$$

for any $\mathrm{Y} \in \mathfrak{V}$. Consequently, if ω_1^k and ω_2^k represent the corresponding principal material connection 1-forms then for any $\mathrm{p}^k \in \pi_k^{-1}(\mathfrak{V})$ and any vector $\xi \in T_{\mathrm{p}^k}\mathrm{H}^k(\mathfrak{V})$

$$\omega_2^k(\xi) = \mathfrak{a}d(\varrho(\pi_k(\mathrm{p}^k))^{-1})\omega_1^k(\xi) + \tilde{\varrho}^*(\zeta)(\xi) \tag{5.5}$$

where $\mathfrak{a}d$ denotes the adjoint action of the group on its algebra, ζ is the Maurer-Cartan form on $\mathcal{G}^k$,(see e.g., [Po]), and $\tilde{\varrho} : \pi_k^{-1}(\mathfrak{V}) \to \mathcal{G}^k$ is a constant along fibers function, induced by the gauge ϱ such that $\varrho \circ \pi_k = \tilde{\varrho}$. The same is true even if the connections are not locally integrable. In particular, we may choose to represent locally any material connection by a 1-form on the body $\mathcal{B}$. This is done relative to a trivialization induced by a section, material or not, specially by the coordinate map $\alpha : \mathfrak{V} \to I\!R$. Indeed, such a map induces automatically, through its tangent map α_*, a choice of frames in the tangent space and also higher order frames. The connection forms ω_1^k and ω_2^k are then represented by the $\mathfrak{g}^k$-valued 1- forms $\omega_{\alpha i}^k$ such that

$$\omega_{\alpha i}^k = j^k\alpha^*\omega_i^k \qquad i = 1, 2 \tag{5.6}$$

where $j^k\alpha$ is understood as the local section of $\mathrm{H}^k(\mathcal{B})$ induced by the coordinate map α. Thus, using the standard shorthand, one can write

$$\omega_{\alpha 2}^k(\mathrm{Y}) = \varrho(\mathrm{Y})^{-1}\omega_{\alpha 1}^k(\mathrm{Y})\varrho(\mathrm{Y}) + \varrho(\mathrm{Y})^{-1}\varrho_*(\mathrm{Y}) \tag{5.7}$$

for any $\mathrm{Y} \in \mathfrak{V}$. Generalizing the above relations we have:

Proposition 5.2 *Let $\mathfrak{h}^k$ denote the Lie algebra of a particular isotropy group $\mathcal{G}_{\mathfrak{l}^k}^k$. Given the principal material connection ω^k and a $\mathfrak{h}^k$-valued 1-form*

τ^k *on* $\mathrm{H}^k(\mathcal{B})$, $\omega^k + \tau^k$ *represents another principal material connection if, and only if, for any* $\xi \in T\mathcal{M}^k(\mathcal{B})$

a) $\tau^k(\mathfrak{v}(\xi)) = 0$ *for every vertical vector* $\mathfrak{v}(\xi) \in T\mathcal{M}^k(\mathcal{B})$,

b) τ^k *is* $\mathcal{G}^k$*- equivariant.*

Proof. Clearly, if $\omega^k + \tau^k$ represents a principal material connection on $\mathcal{M}^k(\mathcal{B})$ then conditions a) and b) are satisfied. On the other hand if τ^k is equivariant then $\omega^k + \tau^k$ is equivariant too. Also, for every $\mathrm{p}^k \in \mathcal{M}^k(\mathcal{B})$ and $\xi \in T\mathcal{M}^k(\mathcal{B})$ $\quad d\tilde{\mathcal{W}} \circ \mathfrak{R}_{t^k(\mathrm{p}^k)} \circ (\omega^k + \tau^k)(\mathfrak{v}(\xi)) = d\tilde{\mathcal{W}} \circ \mathfrak{R}_{t^k(\mathrm{p}^k)} \circ \omega^k(\mathfrak{v}(\xi)) = d\varphi(\mathfrak{v}(\xi))$ and $d\tilde{\mathcal{W}} \circ \mathfrak{R}_{t^k(\mathrm{p}^k)} \circ (\omega^k + \tau^k)(hor(\xi)) = d\tilde{\mathcal{W}} \circ \mathfrak{R}_{t^k(\mathrm{p}^k)} \tau^k(hor(\xi)) = 0 = d\varphi(hor(\xi))$ as τ^k is $\mathfrak{h}^k$-valued and $\mathcal{G}^k_{t^k}$ is the isotropy group of $\tilde{\mathcal{W}}$. Thus, the equation (5.2) holds for the connection $\omega^k + \tau^k$ which makes it a principal material connection ♣

5.2. Induced Material Connections

To facilitate our future developments we need first to present the relevant mathematical preliminaries. This is done not only to make this exposition as self contained as possible and not only because the theory of linear connections on frame bundles of order higher than one is not easily available in the mathematical literature but also to present some relevant recent results [EP3].

We start by pointing out that the relation between the second order frame bundle and the bundle of linear frames of $\mathcal{B}$, as presented at the beginning of Chapter 4, is, in fact, typical for the whole chain of frame bundles (holonomic or not). That is, if we consider the following chain of frame bundles:

$$\mathrm{H}^k(\mathcal{B}) \rightarrow \mathrm{H}^{k-1}(\mathcal{B}) \rightarrow \cdots \rightarrow \mathrm{H}^2(\mathcal{B}) \rightarrow \mathrm{H}^1(\mathcal{B}). \tag{5.8}$$

then for any ordered pair of positive integers $s > m > 1$ there is a projection

$$\pi^{\mathrm{s}}_{\mathrm{r}} : \mathrm{H}^{\mathrm{s}}(\mathcal{B}) \rightarrow \mathrm{H}^{\mathrm{r}}(\mathcal{B}) \tag{5.9}$$

making $H^s(\mathcal{B})$ into an affine bundle over $H^r(\mathcal{B})$ with the kernel of the epimorphism $\tilde{\pi}_r^s : \mathcal{G}^s \to \mathcal{G}^r$ being its structure group $\mathcal{N}_r^s(n)$. The group $\mathcal{N}_r^s(n)$ is a normal subgroup of $\mathcal{G}^s$ and for $r = s-1$ is canonically isomorphic to the abelian vector group of all multilinear symmetric $\mathbb{R}^n$-valued $(s-1)$-forms on $\mathbb{R}^n$ [Ko], [Y]. The group $\mathcal{G}^s$ is the semidirect product of $\mathcal{G}^1 = GL(n, R)$ and the vector group $\mathcal{N}_{(k-1)}^k(n)$. The algebra of $\tilde{\pi}_r^s(\mathcal{G}^s)$ is a graded Lie algebra isomorphic to the algebra $\mathfrak{n}_r^s$ of $\mathcal{N}_r^s(n)$. Let us now introduce some technical definitions.

Suppose that $h^{k-1} : H^{k-1}(\mathbb{R}^n) \to H^{k-1}(\mathcal{B})$ denotes a local isomorphism about $(0, e^{k-1})$. We say that h^{k-1} is *admissible* if there exists an embedding $\psi : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{B}$ such that ψ and h^{k-1} commute with the respective projections π_{k-1}, $0 \in \mathcal{U}$ and $h^{k-1}(e^{k-1}) = j^{k-1}\psi(0)$. Thus, given a k-frame p^k there exists an admissible isomorphism h^{k-1} such that $p^k = j^1 h^{k-1}(e^{k-1})$. To show this we point out that for any k-frame p^k there exists an embedding f of a neighbourhood of the origin of $\mathbb{R}^n$ into $\mathcal{B}$ such that $p^k = j^k f(0)$. The corresponding admissible isomorphism h^{k-1} is then defined by the condition that $j^{k-1}f \circ f = h^{k-1} \circ j^{k-1}id$ where $j^{k-1}f$ denotes the jet extension of f. The admissible isomorphism h^{k-1} induces a linear isomorphism $\tilde{h}^{k-1}$: $T_{e^{k-1}}H^{k-1}(\mathbb{R}^n) \to T_{\pi_{k-1}^k(p^k)}H^{k-1}(\mathcal{B})$. Since $H^{k-1}(\mathbb{R}) = \mathbb{R}^n \times \mathcal{G}^{k-1}$ we have that $T_{e^{k-1}}H(\mathbb{R}^n) = \mathbb{R}^n \oplus \mathfrak{g}^{k-1}$.

Definition 5.2 *Let* $p^k \in H^k(\mathcal{B})$ *and let* h^{k-1} *denote the corresponding admissible isomorphism. The* **standard horizontal space of the frame** p^k *is the n-dimensional vector space* $\mathcal{SH}(p^k) \equiv \tilde{h}^{k-1}(\mathbb{R}^n, 0)$.

Generalizing the concept of the solder form the following is the standard definition of the *fundamental form* on a frame bundle.

Definition 5.3 *The* **fundamental form** *on* $H^k(\mathcal{B})$ *is the* $\mathbb{R}^n \oplus \mathfrak{g}^{k-1}$*-valued 1-form* θ^k *such that given a k-frame* p^k*, the corresponding admissible isomorphism* h^{k-1}*, and the tangent vector* $\xi \in T_{p^k}H^k(\mathcal{B})$

$$\tilde{h}^{k-1}(\theta^k(\xi)) = T\pi_{k-1}^k(\xi). \tag{5.10}$$

The form θ^k is equivariant with respect to the right action of $\mathcal{G}^k$ on $\mathrm{H}^k(\mathcal{B})$ and the action ρ^k of $\mathcal{G}^k$ on the tangent space $T\mathrm{H}^k(\mathcal{B})$. The latter being just an extension of the natural action of GL(n,$I\!R$) on $I\!R^n$. Namely,

$$\theta^k(T\mathfrak{R}^k_{\mathrm{g}^k}(\xi)) = \rho^k((\mathrm{g}^k)^{-1})\theta^k(\xi) \tag{5.11}$$

for any $\mathrm{g}^k \in \mathcal{G}^k$ and any tangent vector $\xi \in \mathrm{TH}^k(\mathcal{B})$. The adjoint action ρ^k of the structure group $\mathcal{G}^k$ on $I\!R^n \oplus \mathfrak{g}^{k-1}$ is such that for any vector $\mathfrak{X}^{k-1} \in \mathfrak{g}^{k-1}$ and any $\mathrm{g}^k \in \mathcal{G}^k$

$$\rho^k(\mathrm{g}^k)\mathfrak{X}^{k-1} = \mathfrak{a}d^k(\tilde{\pi}^k_{k-1}(\mathrm{g}^k))\mathfrak{X}^{k-1}. \tag{5.12}$$

On the other hand, for any $v \in I\!R^n$

$$\rho^k(\mathrm{g}^k)(v,0) = (\tilde{\pi}^k_1(\mathrm{g}^k)v, \lambda^k(\mathrm{g}^k, v)) \tag{5.13}$$

for some mapping $\lambda^k : \mathcal{G}^k \times I\!R^n \to \mathfrak{g}^{k-1}$ such that $T\tilde{\pi}^{k-1}_{k-2} \circ \lambda^k \equiv \lambda^{k-1} \circ \{\tilde{\pi}^k_{k-1} \times id_{I\!R^n}\}$. $\tilde{\pi}^l_k$ denotes here the projection $\mathcal{G}^l \to \mathcal{G}^k$, $l \geq k$. For a fixed $\mathrm{g}^k \in \mathcal{G}^k$ $\lambda^k(\mathrm{g}^k, \cdot) : I\!R^n \to \mathfrak{g}^{k-1}$ is linear and it is identically zero if, and only if, $\mathrm{g}^k \in \mathcal{G}^1$ [Y]. Moreover,

$$\lambda^k(\mathrm{g}^k_2\mathrm{g}^k_1, v) = \lambda^k(\mathrm{g}^k_2, \tilde{\pi}^k_1(\mathrm{g}^k_1)v) + \mathfrak{a}d^k(T\tilde{\pi}^k_{k-1}(\mathrm{g}^k_2))\lambda^k(\mathrm{g}^k_1, v) \tag{5.14}$$

for any $\mathrm{g}^k_1, \mathrm{g}^k_2 \in \mathcal{G}^k$.

The fundamental form θ^k decomposes canonically into the sum of 1-forms with values in the subalgebras of $I\!R^n \oplus \mathfrak{g}^{k-1}$. In particular, $\theta^k = \theta^k_1 + \theta_k$ where θ^k_1 is just a projection onto $I\!R^n$ while θ_k takes values in $\{0\} \oplus \mathfrak{g}^{k-1}$. Furthermore, as for any $r < k$ the group $\mathcal{G}^k$ can be represented as the semidirect product of $\mathcal{G}^r \equiv \tilde{\pi}_r^k(\mathcal{G}^k)$ and the kernel $\mathcal{N}_r^k(\mathrm{n})$ of the epimorphism $\tilde{\pi}_r^k$, we can write

$$\theta^k = \theta_1^k + \theta_r + \mu_r^k \tag{5.15}$$

where $\pi_r^{k*}\theta_r = \tilde{\pi}_{r*}^k\theta_k$ and where μ_r^k takes values in n_{r-1}^{k-1}, the algebra of the Lie group $\mathcal{N}_{r-1}^{k-1}(\mathrm{n})$. As a result of the equivariance of the fundamental form θ^k, Eqn.(5.11), we get that

$$\theta_1^k(\mathfrak{R}_{\mathrm{g}^k *}(\xi)) = \pi_1^k((\mathrm{g}^k)^{-1})\theta_1^k(\xi) \tag{5.16a}$$

and that

$$\theta_k(\mathfrak{R}_{\mathrm{g}^k *}(\xi)) = \mathrm{ad}^k(\tilde{\pi}_{k-1}^k((\mathrm{g}^k)^{-1}))\theta_k(\xi) + \lambda^k((\mathrm{g}^k)^{-1}, \theta_1^k(\xi)). \tag{5.16b}$$

for any vector $\xi \in \mathrm{TH}^k(\mathcal{B})$.

Suppose now that $q : \mathrm{H}^{k-1}(\mathcal{B}) \to \mathrm{H}^k(\mathcal{B})$ is a local section and let p^k be in the image of q. Given the element of the standard horizontal space at p^k, $\xi \in \mathcal{SH}(\mathrm{p}^k)$, $q^*\theta^k(\xi) = \theta^k(q_*(\xi)) \in I\!R^n \oplus \{0\}$ as $\tilde{h}^{k-1}(\theta^k(q_*(\xi)) = T\pi_{k-1}^k(q_*(\xi)) = \xi$ by the Definition 5.3. Note that this is true irrespective of the section q as long as p^k belong to its image. All the above implies immediately that:

Proposition 5.3 (Elżanowski and Prishepionok [EP2]) *Let* p^k *be a k-frame.* $\xi \in \mathcal{SH}(\mathrm{p}^k)$ *if, and only if, given a section* $q : \mathrm{H}^{k-1}(\mathcal{B}) \to \mathrm{H}^k(\mathcal{B})$ *such that* p^k *is in the image of* q, $q^*\theta_k(\xi) \equiv 0$.

To get some true insight into the structure of connections on the bundle of k-frames we start by recalling the construction of an arbitrary k-connection

ω^k in terms of the so-called $\mathcal{E}$-connection (see e.g., [Ko], [Y]). We adapt this presentation, however, to our particular needs. To do this we need however to broaden a little our picture and to imbedded the bundle of holonomic frames $\mathrm{H}^k(\mathcal{B})$ into the bundle of the *non-holonomic frames* $\hat{\mathrm{H}}^k(\mathcal{B})$ and specially the bundle of *semi-holonomic frames* $\tilde{\mathrm{H}}^k(\mathcal{B})$.[23] We also recall that the local section $q : \mathrm{H}^r(\mathcal{B}) \rightarrow \mathrm{H}^k(\mathcal{B})$ is *invariant* (*$\mathcal{G}^r$-invariant*) if for any $\mathrm{p}^r \in \mathrm{H}^r(\mathcal{B})$ and every $\mathrm{g}^r \in \mathcal{G}^r$

$$\mathrm{q}(\mathfrak{R}_{\mathrm{g}^r}(\mathrm{p}^r)) = \mathfrak{R}_{\nu_r^k(\mathrm{g}^r)}(\mathrm{q}(\mathrm{p}^r)) \tag{5.17}$$

where ν_r^k is a canonical embedding of $\mathcal{G}^r$ into $\mathcal{G}^k$. For the simplicity of our exposition but without any loss of generality, at least for what we intend to do, let us restrict our analysis to the semi-holonomic case only. Therefore, let $\varepsilon^{k+1} : \mathrm{H}^1(\mathcal{B}) \rightarrow \tilde{\mathrm{H}}^{k+1}(\mathcal{B})$ be a $\mathcal{G}^1$-invariant section called the *$\mathcal{E}$-connection of order* k. It defines a $\mathcal{G}^1$ reduction of the bundle $\mathrm{H}^{k+1}(\mathcal{B})$ given by the image $\varepsilon^{k+1}(\mathrm{H}^1(\mathcal{B}))$. We shall denote it by M_{ω^k}. The projection of M_{ω^k} to the bundle $\mathrm{H}^k(\mathcal{B})$, that is $\mathrm{N}_{\omega^k} \equiv \pi_k^{k+1}(\varepsilon^{k+1}(\mathrm{M}_{\omega^k}))$, is also a $\mathcal{G}^1$ reduction. This, in turn, induces the $\mathcal{G}^1$-invariant partial section $q^k : \mathrm{N}_{\omega^k} \rightarrow \mathrm{M}_{\omega^k}$. The

[23] Although, for the precise definitions we refer the reader to Saunders [Sa] and Yuen [Y] we also would like to point out at the way the space of non-holonomic k-frames $\hat{\mathrm{H}}^k(\mathcal{B})$ can be thought of recursively as the space of the first jets of all local sections of the bundle of non-holonomic (k-1)-frames $\hat{\mathrm{H}}^k(\mathcal{B})$. For example, let $\mathrm{f} : \mathcal{U}(0) \rightarrow \mathrm{H}^1(\mathcal{B})$ (for k=1 all frame bundles are the same) be a differentiable map of a neighbourhood of the origin of $I\!R^n$ into $\mathrm{H}^1(\mathcal{B})$ and such that $\pi^1 \circ \mathrm{f} : \mathcal{U}(0) \rightarrow \mathcal{B}$ is a local diffeomorphism where $\pi^1 : \mathrm{H}^1(\mathcal{B}) \rightarrow \mathcal{B}$ is the standard projection. The first jet of f at 0 can be considered a *non-holonomic* 2-frame of $\mathcal{B}$ at $\pi^1(\mathrm{f}(0))$. If, in addition, f is such that the first jet of $\pi^1 \circ \mathrm{f}$ at 0 is equal to $\mathrm{f}(0)$ the corresponding 2-frame is called *semi-holonomic*. Extending this definition recursively to an arbitrary k-order we obtain the set of all non-holonomic and semi-holonomic frames of $\mathcal{B}$. The space $\tilde{\mathrm{H}}^k(\mathcal{B})$ (also $\hat{\mathrm{H}}^k(\mathcal{B})$) is a principal bundle over $\mathcal{B}$. Its structure group $\tilde{\mathcal{G}}^k$ is the fibre at 0 of $\tilde{\mathrm{H}}^k(I\!R^n)$, i.e., the group of first jets at the origin of all local sections of $\tilde{\mathrm{H}}^{k-1}(I\!R^n)$ satisfying the semi-holonomicity condition. It can be easely shown (see e.g., Saunders [Sa]) that $\mathrm{H}^k(\mathcal{B}) \subset \tilde{\mathrm{H}}^k(\mathcal{B}) \subset \hat{\mathrm{H}}^k(\mathcal{B})$.

connection ω^k on $\mathrm{H}^k(\mathcal{B})$ is then defined by selecting as its horizontal space at $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{B})$ $\mathcal{SH}(q^k(\mathrm{p}^k))$ if $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$ and $T\mathfrak{R}_{\mathrm{n}_1^k}\mathcal{SH}(q^k(\mathrm{p}^k))$ for any other k-frame, where n_1^k denotes the appropriate element of the affine group $\mathcal{N}_1^k(\mathrm{n})$. The $\mathcal{G}^1$-invariant submanifold N_{ω^k} of $\mathrm{H}^k(\mathcal{B})$, fundamental for the construction of the connection ω^k, will be called its *characteristic manifold.* We point out here that to define a connection on the holonomic frame bundle $\mathrm{H}^k(\mathcal{B})$, called the *holonomic connection*, the defining $\mathcal{E}$-connection does not need to be a section into the holonomic (k+1)-frame bundle. As a matter of fact, if it is, it has very special properties, as we show later.

We are now in the position to represent the k-connection ω^k through the fundamental form θ^{k+1}:

Theorem 5.1 (Elżanowski and Prishepionok [EP3]) *Let ω^k be a connection of order k on the bundle of holonomic k-frames $\mathrm{H}^k(\mathcal{B})$ and let ε^{k+1} denotes its generating $\mathcal{E}$-connection with N_{ω^k} as its characteristic manifold. Then, for any $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$ and any $\mathrm{g}^k \in \mathcal{G}^k$*

$$\omega^k(\mathfrak{R}_{\mathrm{g}^k}(\mathrm{p}^k))(\mathfrak{R}_{\mathrm{g}^k *}\xi) = \tilde{q}^{k*}\theta_{k+1}(T\mathfrak{R}_{\mathrm{g}^k}\xi) - \lambda^k((\mathrm{g}^k)^{-1}, \tilde{q}^{k*}\theta_1^k(\xi))$$

where $\xi \in T_{\mathrm{p}^k}\mathrm{N}_{\omega^k}$ and $\tilde{q}^k$ denotes the $\mathcal{G}^k$-equivariant extension of the $\mathcal{G}^1$-invariant partial section q^k induced by the $\mathcal{E}$-connection ε^{k+1}.

Proof. As implied by (5.14a) the 1-form on the right hand side of the identity is equivariant. What remains to be shown is that both sides are identical on the characteristic manifold of the connection ω^k. Thus, let $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$ then $\omega^k(\mathrm{p}^k)(\xi) = 0$ if, and only if, $\xi \in \mathcal{SH}(q^k(\mathrm{p}^k))$. On the other hand if $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$ so does pg^k for any $\mathrm{g}^k \in \nu_1^k(\mathcal{G}^1)$. However, $\lambda^k((\mathrm{g}^k)^{-1}, \cdot)$ is identically zero for any $\mathrm{g}^k \in \mathrm{GL}(\mathrm{n}, \mathrm{R}^\mathrm{n}) \oplus \{0\}$. Also, $q^{k*}\theta_{k+1}(T\mathfrak{R}_{\mathrm{g}^k}\xi) = 0$ if, and only if $\xi \in \mathcal{SH}(q^k(\mathrm{p}^k))$ as attested by the Proposition 5.3 ♣

To get an even more detailed description of a k-connection as well as to understand better the role of the mapping λ^k let us compare the standard horizontal spaces corresponding to two different $(k+1)$-frames over the same k-frame. Hence, let us take $\hat{p}^{k+1}, p^{k+1} \in H^{k+1}(\mathcal{B})$ such that p^k is their projection onto $H^k(\mathcal{B})$. This implies that there exists $n_k^{k+1} \in \mathcal{N}_k^{k+1}(n)$ such that $\hat{p}^{k+1} = p^{k+1}n_k^{k+1}$. Moreover, there exists an admissible local isomorphism $\alpha^k : H^k(I\!R^n) \to H^k(I\!R^n)$ preserving the neutral element and such that $n^{k+1} = j^1\alpha^k(e^k)$. Also, there is an admissible local isomorphism $h^k : H^k(I\!R^n) \to H^k(\mathcal{B})$ such that $j^1h^k(e^k) = p^{k+1}$ (see Definition 5.2). The composition $h^k \circ \alpha^k$ is then an admissible local isomorphism the first jet of which at e^k gives the $(k+1)$-frame $\hat{p}^{k+1}$. According to the Definition 5.2 $(\widetilde{h^k \circ \alpha^k})(v,0) \in \mathcal{SH}(\hat{p}^{k+1})$ for any $(v,0) \in I\!R^n \oplus \mathfrak{g}^k$. Recalling the definition of the fundamental form and that of the action ρ^{k+1} of the group $\mathcal{G}^{k+1}$ on the tangent space of $H^k(\mathcal{B})$ we obtain $\widetilde{h^k \circ \alpha^k}(v,0) = \tilde{h}^k \circ \rho^{k+1}((n_k^{k+1})^{-1})(v,0) = \tilde{h}^k(\tilde{\pi}_1^{k+1}(n_k^{k+1})v, \lambda^k((n_k^{k+1})^{-1}, v)) = \tilde{h}^k(v,0) + \tilde{h}^k(0, \lambda^k((n_k^{k+1})^{-1}, v)) = \tilde{h}^k(v,0) + h_*^k(\lambda^k((n_k^{k+1})^{-1}), v) = \tilde{h}^k(v,0) + \widetilde{\lambda^k((n_k^{k+1})^{-1}, v)}$ for every $(v,0) \in I\!R^n \oplus \mathfrak{g}^k$ where, $\widetilde{\lambda^k(\cdot,\cdot)}$ denotes a vertical vector at p^k corresponding to the Lie algebra element $\lambda^k(\cdot,\cdot)$. All of the above shows that:

Lemma 5.1 *Given two, in general different, $(k+1)$-frames $\hat{p}^{k+1}, p^{k+1}$ over the same k-frame p^k, i.e. $\pi_k^{k+1}(\hat{p}^{k+1}) = \pi_k^{k+1}(p^{k+1}) = p^k$, the standard horizontal space of $\hat{p}^{k+1}$ is the $\mathfrak{g}^k$ translate, through λ^k, of the standard horizontal space of p^k.*

Therefore, the statement of the Theorem 5.4 can be made even more precise:

Proposition 5.4 (Elżanowski and Prishepionok [EP3]) *Let ω^k be a k-connection with N_{ω^k} as its characteristic manifold. Let $l_1^k : H^k(\mathcal{B}) \to \mathcal{N}_1^k(n)$ be an equivariant mapping, i.e. $l_1^k(p^k n_1^k) = l_1^k(p^k)n_1^k$ for any k-frame p^k and any $n_1^k \in \mathcal{N}_1^k(n)$ while $l_1^k(p^k g) = g^{-1} l_1^k(p^k) g$ for any $g \in \mathcal{G}^1$. Assume that l_1^k is such that $p^k l_1^k(p^k)^{-1} \in N_{\omega^k}$ for every $p^k \in H^k(\mathcal{B})$. Also, let $q^k : N_{\omega^k} \to \tilde{H}^{k+1}(\mathcal{B})$ be the $\mathcal{G}^1$-equivariant section such that $\omega^k = q^*\theta_{k+1}$ when restricted to N_{ω^k}. Then,*

$$\omega^k(p^k)(\xi) = \tilde{q}^{k*}\theta_{k+1}(\xi) - \lambda^k(l_1^k(p^k)^{-1}, \theta_1^{k+1}(\tilde{q}_*^k\xi)) \tag{5.20}$$

for any $p^k \in H^k(\mathcal{B})$ *and* $\xi \in T_{p^k}H^k(\mathcal{B})$. *Moreover, there is a one-to-one correspondence between linear connections on* $H^k(\mathcal{B})$ *and pairs of mappings* $(\tilde{q}^k, l_1^k)$.

Proof. Given the pair $(\tilde{q}^k, l_1^k)$ where $\tilde{q}^k : H^k(\mathcal{B}) \rightarrow \tilde{H}^{k+1}(\mathcal{B})$ is an equivariant section and where $l_1^k : H^k(\mathcal{B}) \rightarrow \mathcal{N}_1^k(n)$ is an equivariant mapping the k-connection is uniquely defined by Eqn.(5.18). On the other hand, given the connection ω^k the mapping l_1^k is uniquely defined, modulo the $\mathcal{G}^1$ action, from the equation: $\pi_k^{k+1*}\omega^k - \theta_{k+1} = \pi_k^{k+1*}\lambda^k((l_1^k)^{-1}, \theta_1^{k+1})$. Once l_1^k is available the equivariant section $\tilde{q}^k$ can be obtained from the condition that $\omega^k|_{(l_1^k)^{-1}(0)} = \tilde{q}^k|_{(l_1^k)^{-1}(0)}\theta_{k+1}$. We remark here that $\lambda^1 \equiv 0$ and that for $k = 2$ we get the known expression for a 2-connection of Garcia [G]. We also point out that the theorem shows that there exists a one-to-one correspondence between the $\mathcal{E}$-connections of order k and the k-connections, as shown in a different way by Libermann [Li]♣

A k-connection ω^k on $H^k(\mathcal{B})$ induces, through a projection, a (k-1)-connection $proj_1\omega^k$ on $H^{k-1}(\mathcal{B})$. Namely, for any $\xi \in TH^k(\mathcal{B})$

$$\tilde{\pi}_{k-1*}^k\omega^k(\xi) = \pi_{k-1}^{k*}proj_1\omega^k(\xi). \qquad (5.19)$$

If N_{ω^k} is the characteristic manifold of ω^k then the characteristic manifold of $proj_1\omega^k$ is the projection of N_{ω^k}, i.e. $N_{proj_1\omega^k} = \pi_{k-1}^k(N_{\omega^k})$. Indeed, suppose that ε^{k+1} is the $\mathcal{E}$- connection of order k generating ω^k. Then, $N_{\omega^k} = \pi_k^{k+1}(\varepsilon^{k+1}(H^1(\mathcal{B})))$ and there exists a partial section $q^k : N_{\omega^k} \rightarrow \varepsilon^{k+1}(H^1(\mathcal{B}))$ such that for any $p^k \in N_{\omega^k}$ the horizontal space of ω^k at p^k is $\mathcal{SH}(q^k(p^k))$ that is the kernel of $q^{k*}\theta_{k+1}$. Now, let q^{k-1} be a partial section on $\pi_{k-1}^k(N_{\omega^k})$ with the property that $q^{k-1} \circ \pi_{k-1}^k = \pi_k^{k+1} \circ q^k$. Recalling that the projections π_k^{k+1} and π_{k-1}^k, when restricted to the characteristic manifolds, are one-to-one and invoking the definition of a standard horizontal space, as well as Proposition 5.3, we get:

Lemma 5.2 *The standard horizontal space of a projection of a frame is a projection of the standard horizontal space of that frame, i.e. if* $p^{k+1} \in$

$H^{k+1}(\mathcal{B})$ *then* $\pi^k_{k-1*}\mathcal{SH}(p^{k+1}) = \mathcal{SH}(\pi^{k+1}_k(p^{k+1}))$. *Thus, the characteristic manifold of the projected connection* $proj_1\omega^k$ *is the projection of the characteristic manifold* N_{ω^k}.

This is obviously also true for a projection of a k – connection to any r-order frame bundle, where $0 < r < k$.

We are now ready to introduce the concept of the *induced material connection.* But first, let ω^k be some principal material connection of the materially uniform k-grade hyperelastic body $\mathcal{B}$.

Definition 5.4 *The* (k – r)-**material connection** *of the k-grade uniform hyperelastic body* $\mathcal{B}$ *is the r-th projection of the principal material connection* ω^k, *i.e.* $proj_r\omega^k$.

As we have stated before (see also Wang and Truesdel [WT]) for every material point X of the smoothly uniform material body $\mathcal{B}$ there exists a principal material connection ω^k such that in some neighbourhood of X, say $\mathcal{U}$, it is generated by a (local) material section. Let $\mathfrak{l}^k : \mathcal{U} \subset \mathcal{B} \rightarrow H^k(\mathcal{B})$ be such a section. Therefore, there exists the local section $\mathfrak{p}^1 : \mathcal{U} \rightarrow H^1(\mathcal{B})$ and the map $\varepsilon^k_{\mathfrak{l}^k} : \mathfrak{p}^1(\mathcal{U}) \rightarrow H^k(\mathcal{B})$ such that for any $Y \in \mathcal{U}$ $\mathfrak{l}^k(Y) = \varepsilon^k_{\mathfrak{l}^k}(\mathfrak{p}^1(Y))$. We extend the mapping $\varepsilon^k_{\mathfrak{l}^k}$, by the action of $\mathcal{G}^1$ on $H^1(\mathcal{B})$, to the $\mathcal{G}^1$-equivariant section $\tilde{\varepsilon}^k_{\mathfrak{l}^k} : H^1(\mathcal{U}) \rightarrow H^k(\mathcal{B})$. As we have shown before (Theorem 5.1) such an equivariant section defines a local (k - 1)-connection $i_1\omega^k$ where $N_{i_1\omega^k} \equiv \pi^k_{k-1}[\tilde{\varepsilon}^k_{\mathfrak{l}^k}(p^1(\mathcal{U})\mathcal{G}^1)] = \pi^k_{k-1}[\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1]$.

Definition 5.5 *Given the local material section* $\mathfrak{l}^k$ *the* **induced material connection** $i_1\omega^k$ *is the locally defined* $(k-1)$*-connection such that* $\pi^k_{k-1}[\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1]$ *is its characteristic manifold and* $\tilde{q}^{k-1} : \pi^k_{k-1}[\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1] \rightarrow \mathfrak{l}^k(\mathcal{U})\mathcal{G}^1$ *is its generating section.*

In general $N_{i_1\omega^k} \neq N_{proj_1\omega^k}$. However, if the section $\mathfrak{l}^k$ defines locally the principal material connection ω^k the section $\pi^k_{k-1} \circ \mathfrak{l}^k$ defines $proj_1\omega^k$. This, in turn, enables one to define the $\mathcal{G}^1$-invariant section $\tilde{\varepsilon}^{k-1}_{\pi^k_{k-1}\circ\mathfrak{l}^k}$ inducing the (k - 2)-connection $i_1 proj_1 \omega^k$ with $\pi^k_{k-2}[\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1]$ as its characteristic manifold. The space $\pi^{k-1}_{k-2}(\mathrm{N}_{i_1\omega^k})$ is the characteristic manifold of the projection of $i_1\omega^k$ to $\mathrm{H}^{k-2}(\mathcal{B})$ proving:

Proposition 5.5 *Given a material point* X *let* ω^k *be the principal material connection integrable in the neighbourhood* $\mathcal{U}$ *of* X. *Then, for any pair of positive integers* $j < k$

$$i_1 proj_j \omega^k = proj_j i_1 \omega^k$$

in $\mathcal{U}$.

The analysis of the locally induced connections, the projections of connections and the relation between them will be fundamental for resolving the problem of the local flatness of a principal k- material connection and so the integrability of material structures for k-grade uniform hyperelastic material bodies. This will be presented at length in the next chapter. Yet, even at this point, on the bases the definition of the induced material connection (Definition 5.5) and the Proposition 5.4, we can safely claim that the main advantage of having the induced and the projected material connections lays in the fact that the analysis of the k-order principal material connection can be performed on although two, but lover order, connections. Indeed, it is immediate from the definition of the induced connection and the construction of the connection from its $\mathcal{E}$-connection that:

Proposition 5.6 *Given an integrable connection* ω^{k-1} *and another* $k-1$*-connection* $\tilde{\omega}^{k-1}$ *which characteristic manifold* $\mathrm{N}_{\omega^{k-1}}$ *is the integral manifold of the horizontal distribution of* ω^{k-1} *there is only one integrable k-connection* ω^k *such that* $proj_1\omega^k = \omega^{k-1}$ *and* $i_1\omega^k = \tilde{\omega}^{k-1}$.

We end this part by looking in more detail at the second order holonomic frame bundle and the second order connections. We shall follow here Elżanowski and Prishepionok [EP2], [EP4].

Suppose, for the simplicity and the clarity of our presentation, that the body $\mathcal{B}$ can be covered by a single (global) chart and that $\mathcal{S} = I\!R^n$. Thus, we assume the body $\mathcal{B}$ is equipped with the coordinate system $\{x^1, \ldots, x^n\}$. Let us select as the reference placement $\mathcal{U}(0)$, a neighbourhood of the origin of $I\!R^n$. Then, any (local about the origin) diffeomorphism $\chi : \mathcal{U}(0) \in I\!R^n \rightarrow \mathcal{B}$ can be viewed as a deformation of the body $\mathcal{B}$. Consider a linear frame p^1 and a holonomic 2-frame p^2 such that $\pi_2(p^2) = \pi_1(p^1) = Y = (y^1, \ldots y^n) \in \mathcal{B}$. These frames are represented in $H^1(\mathcal{B})$ and respectively in $H^2(\mathcal{B})$ by the sets of local coordinates (y^i, y^i_k) and (y^i, y^i_k, y^i_{kl}) such that $\det(y^i_k) \neq 0$ and $y^i_{kl} = y^i_{lk}$. Let us add here that a non-holonomic frame is respectively characterized by the set of coordinates $(y^i, y^i_j, \bar{y}^i_l, y^i_{kl})$ where y^i_{kl} is not necessarily symmetric. If $y^i_l = \bar{y}^i_l$ the the frame is semi-holonomic.

In the locally induced by the coordinate system $\{x^1 \ldots x^n\}$ bases

$$p^1 = (y^1, \ldots, y^n; y^i_k \frac{\partial}{\partial x^i}), \quad p^2 = (y^1, \ldots y^n; y^i_k \frac{\partial}{\partial x^i}; y^i_{ks} \frac{\partial}{\partial x^i_s}) \tag{5.20}$$

where the summation convention is enforced. One can think of y^i_k as the components of the deformation gradient at $Y \in \mathcal{B}$ of χ while the 2-frame p^2 represents the first and the second deformation gradients. Given an element (g^i_k, n^i_{kl}) of the structure group $\mathcal{G}^2 = GL(n, I\!R) \oplus S^2(n)$ of $H^2(\mathcal{B})$, where $n^i_{kl} = n^i_{lk}$, it acts on the right on the holonomic 2-frame $p^2 = (y^i, y^i_k, y^i_{kl})$ by (see e.g., [CDL] and [EEp2])

$$(y^i, y^i_k, y^i_{kl})(g^k_r, n^k_{rp}) = (y^i, y^i_k g^k_r, y^i_{kl} g^k_r g^l_r + y^i_k n^k_{rp}) \tag{5.21}$$

As we have shown before (see Definition 3.4 and Proposition 5.1) the second-grade hyperelastic material body $\mathcal{B}$ is smoothly uniform if there exists a gauge $(p^i_j, q^i_{jk}) : \mathcal{B} \rightarrow \mathcal{G}^2$ and a smooth function $\tilde{\mathcal{W}} : \mathcal{G}^2 \rightarrow I\!R$ such that

$$\mathcal{W}(y^i, y^i_k, y^i_{kj}) = \tilde{\mathcal{W}}(y^i_k p^k_l, y^i_{kj} p^k_r p^j_p + y^i_k q^k_{rp}) \tag{5.22}$$

for all material points and any pair of the first and second deformation gradients y^i_k, y^i_{kj}.[24] The material section $\mathfrak{l}^2$, being just a collection of local configurations relative to which $\mathcal{W}$ becomes material point independent, is then given as

$$\mathfrak{l}^2(\mathrm{Y}) = (\mathrm{y}^i, \mathrm{a}^i_j(\mathrm{Y}), \mathrm{b}^i_{jk}(\mathrm{Y})), \tag{5.23}$$

where $\mathrm{b}^i_{jk} = \mathrm{b}^i_{kj}, \mathrm{p}^i_j = (\mathrm{a}^{-1})^i_j$ and $\mathrm{q}^i_{jk} = (\mathrm{a}^{-1})^i_l \mathrm{b}^l_{nm} (\mathrm{a}^{-1})^n_j (\mathrm{a}^{-1})^m_k$. This is set up so that, for any $\mathrm{Y} \in \mathcal{B}$, $(\mathrm{p}^i_j, \mathrm{q}^i_{jk})(\mathfrak{l}(\mathrm{Y})) = \mathrm{e}^2 = (\delta^i_j, 0)$, the neutral element of the structure group $\mathcal{G}^2$.

The material reference $\mathfrak{l}^2$ induces, by projection, the section $\mathfrak{p}^1 : \mathcal{B} \to \mathrm{H}^1(\mathcal{B})$, i.e. $\pi^2_1 \circ \mathfrak{l}^2 = \mathfrak{p}^1$ and

$$\mathfrak{p}^1(\mathrm{Y}) = (y^i, \mathrm{a}^i_j(\mathrm{Y})). \tag{5.24}$$

Consequently, there exists the partial section $q^2 : \mathfrak{p}^1(\mathcal{B}) \to \mathrm{H}^2(\mathcal{B})$ such that $q^2 \circ \mathfrak{p}^1 = \mathfrak{l}^2$. As it follows from (5.21) and (5.23) this section, when extended equivariantly by the action of $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ to the entire $\mathrm{H}^1(\mathcal{B})$, gives the $\mathcal{G}^1$−invariant section $\tilde{q}^2 : \mathrm{H}^1(\mathcal{B}) \to \mathrm{H}^2(\mathcal{B})$ such that

$$\tilde{q}^2(y^i, y^i_k) = (y^i, y^i_k, \mathrm{b}^i_{mn}(\mathrm{a}^{-1})^m_s(\mathrm{a}^{-1})^n_r y^s_k y^r_j). \tag{5.25}$$

Choosing a basis in the Lie algebra $\mathfrak{g}^1 = \mathfrak{gl}(\mathrm{n}, \mathbb{R})$ a linear connection on the bundle of linear frames $\mathrm{H}^1(\mathcal{B})$ is given locally by a collection of real-valued $\mathcal{G}^1$−equivariant 1-forms

$$\omega^i_j = (\mathrm{x}^i_k)^{-1}(\mathrm{dx}^k_j + \Gamma^k_{ln}\mathrm{x}^l_j \mathrm{dx}^n) \tag{5.26}$$

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24. We deliberately ignore here the fact that, in general, the body $\mathcal{B}$ has some non-trivial symmetry group.

while the corresponding horizontal distribution is spanned by

$$\mathcal{D}_i = \frac{\partial}{\partial \mathrm{x}^i} - \Gamma^k_{ij}\mathrm{x}^j_r \frac{\partial}{\partial \mathrm{x}^k_r} \tag{5.27}$$

where Γ^k_{ij} are the Christoffel symbols. If , as it happens in the case of the 1-material connection, the horizontal space is a lift of the tangent space $T\mathcal{B}$ by the local section $\mathfrak{p}^1$ to the bundle of linear frames

$$\mathcal{D}_i = \mathfrak{p}^1_*(\frac{\partial}{\partial \mathrm{x}^i}) = \frac{\partial}{\partial \mathrm{x}^i} + \frac{\partial \mathrm{a}^k_j}{\partial \mathrm{x}^i}(\mathrm{a}^{-1})^j_l \mathrm{x}^l_r \frac{\partial}{\partial \mathrm{x}^k_r}. \tag{5.28}$$

Indeed, the horizontal distribution at $\mathfrak{p}^1(\mathcal{B})$ is spanned by $\mathfrak{p}^1_*(\frac{\partial}{\partial \mathrm{x}^s}) = (\frac{\partial}{\partial \mathrm{x}^s})_{\mathfrak{p}^1(\mathcal{B})}$ $+\frac{\partial \mathrm{a}^i_j}{\partial \mathrm{x}^s}(\frac{\partial}{\partial \mathrm{x}^i_j})_{\mathfrak{p}^1(\mathcal{B})}$. On the other hand, any invariant vector field on $\mathrm{H}^1(\mathcal{B})$ has the form $\alpha^s \frac{\partial}{\partial \mathrm{x}^s} + \beta_{sj}\mathrm{x}^j_i \frac{\partial}{\partial \mathrm{x}^s_i}$. Comparing these two expressions yields (5.28). The section $\mathfrak{p}^1$ induces on $\mathrm{H}^1(\mathcal{B})$ the integrable connection ω^1 (the 1-material connection) the Christoffel symbols of which take the form

$$\Gamma^k_{nl} = -\frac{\partial \mathrm{a}^k_i}{\partial y^n}(\mathrm{a}^{-1})^i_l. \tag{5.29}$$

The fundamental form on the bundle of 2-frames is represented by a collection of the following forms (see e.g., [CDL]):

$$\theta^i = (\mathrm{x}^i_k)^{-1}\mathrm{dx}^k \tag{5.30a}$$

and

$$\theta^i_j = (\mathrm{x}^i_k)^{-1}(\mathrm{dx}^k_j - \mathrm{x}^k_{rj}(\mathrm{x}^r_l)^{-1}\mathrm{dx}^l). \tag{5.30b}$$

Invoking the Proposition 5.4 and the Eqn. (5.25) this implies through straightforward calculations, that the Christoffel symbols of the induced material connection $i_1\omega^2$, i.e. the connection having as its characteristic manifold $\pi_1^2[\mathfrak{l}^2\mathcal{G}^1] = \mathrm{H}^1(\mathcal{B})$, are given by

$$\tilde{\Gamma}^i_{mn} = -\mathrm{b}^i_{pr}(\mathrm{a}^{-1})^p_m(\mathrm{a}^{-1})^r_n. \tag{5.31}$$

Note, that this fact suggests that the $\mathcal{E}$- connection of order 1 generating a linear connection on $\mathrm{H}^1(\mathcal{B})$ with the Christoffel symbols Γ^i_{nm} is given as $\mathcal{E}^1(\mathrm{z}^i, \mathrm{z}^i_j) = (\mathrm{z}^i, \mathrm{z}^i_j, -\Gamma^i_{nm}\mathrm{z}^n_j\mathrm{z}^m_k)$. Note also, as we have mentioned before (footnote 24), that although the $\mathcal{E}$-connection generating a holonomic connection does not need to be a section of a holonomic frame bundle, as evident from its form, if it is the connection it induces has the Christoffel symbols are symmetric. This fact will later be proved for an arbitrary order connection (see Collorary 6.2).

Finally, given the material reference $\mathfrak{l}^2$ it generates the horizontal distribution on $\mathfrak{l}^2(\mathcal{B}) \subset \mathrm{H}^2(\mathcal{B})$ spanned by

$$\mathfrak{l}^2_*(\frac{\partial}{\partial \mathrm{x}^i}) = \frac{\partial}{\partial \mathrm{x}^i} + \frac{\partial \mathrm{a}^j_k}{\partial \mathrm{x}^i}\frac{\partial}{\partial \mathrm{x}^j_k} + \frac{\partial \mathrm{b}^l_{nm}}{\partial \mathrm{x}^i}\frac{\partial}{\partial \mathrm{x}^l_{nm}}. \tag{5.32}$$

On the other hand, as shown by Cordero at al. [CDL], any invariant horizontal vector field on $\mathrm{H}^2(\mathcal{B})$ is of the form

$$\frac{\partial}{\partial \mathrm{x}^i} - \Gamma^k_{il}\mathrm{x}^l_r\frac{\partial}{\partial \mathrm{x}^k_r} - (\Gamma^s_{im}\mathrm{x}^m_{rk} + \Gamma^s_{iml}\mathrm{x}^m_r\mathrm{x}^l_k)\frac{\partial}{\partial \mathrm{x}^s_{rk}}. \tag{5.33}$$

where Γ's are functions of position. Consequently, the generalized Christoffel symbols of the principal material connection of the second-grade hyperelastic material induced by the material reference $\mathfrak{l}^2$ are given by

$$\Gamma^k_{is} = -\frac{\partial \mathrm{a}^k_r}{\partial \mathrm{x}^i}(\mathrm{x}^{-1})^r_s, \tag{5.34}$$

$$\Gamma^l_{ipq} = \frac{\partial \mathrm{a}^l_r}{\partial \mathrm{x}^i}(\mathrm{x}^{-1})^r_s(\mathrm{x}^{-1})^n_p(\mathrm{x}^{-1})^k_q \mathrm{x}^s_{nk} - \frac{\partial \mathrm{b}^l_{rk}}{\partial \mathrm{x}^i}(\mathrm{x}^{-1})^r_p(\mathrm{x}^{-1})^k_q. \qquad (5.35)$$

6. INTEGRABLE MATERIAL STRUCTURES: HOMOGENEITY

We have shown so far that if the k-grade hyperelastic body $\mathcal{B}$ is locally smoothly uniform then there exists the corresponding material stucture $\mathcal{M}^k(\mathcal{B})$ being a reduction of the bundle of holonomic k-frames to the symmetry group of $\mathcal{B}$. This istructure is defined uniquely up to a conjugation by the elements of $\mathcal{G}^k$, the structure group of $\mathrm{H}^k(\mathcal{B})$. We have determined also that the uniformity of the material body $\mathcal{B}$ is equivalent to the existance of the so-called k-order principal material connection being a k-connection on the subbundle $\mathcal{M}^k(\mathcal{B})$ locally induced by the material sections. As the Proposition 5.6 shows every such a connection is uniquely characterized by its own 1-projection and the induced material connection (Definition 5.5). What remains to be shown is under what condition the arrangement of local configurations of a truly uniform material body into a local material reference can possibly be chosen such a way that it is locally generated by a (global) configuration. The afforded degree of freedom of choice comes naturally from the symmetry group of the body $\mathcal{B}$. This problem will be investigated in this chapter.

Definition 6.1 *The materially uniform k-grade hyperelastic body $\mathcal{B}$ is said to be* **locally homogeneous** *if for every material point $\mathrm{X} \in \mathcal{B}$ there exist an open neighborhood $\mathcal{U}(\mathrm{X})$ and an integrable (local) material reference $\mathfrak{l}^k : \mathcal{U}(\mathrm{X}) \rightarrow \mathrm{H}^k(\mathcal{B})$, i.e. there exists a local (about the origin) diffeomorphism $\chi : \mathcal{U}(0) \subset \mathbb{R}^n \rightarrow \mathcal{B}$ such that $\chi(0) = \mathrm{X}$, $\chi(\mathcal{U}(0)) \subset \mathcal{U}(\mathrm{X})$ and $\mathfrak{l}^k(\mathcal{U}(\mathrm{X})) = j^k\chi(\mathcal{U}(0))$. Such an integrable material reference at X will be called the* **homogeneous material reference**.

Suppose then that $\mathfrak{l}^k : \mathcal{U}(\mathrm{X}) \rightarrow \mathrm{H}^k(\mathcal{B})$ is a homogeneous material reference at $\mathrm{X} \in \mathcal{B}$. Given some chart $\alpha : \mathcal{U} \subset \mathcal{S} \rightarrow \mathbb{R}^n$ such that $\alpha(\mathcal{U}) \subset \mathcal{U}(0)$ there obviously exist at X a local embedding (configuration) $\psi : \mathcal{V}(\mathrm{X}) \subset \mathcal{B} \rightarrow \mathcal{S}$ such that $j^k(\alpha \circ \psi)^{-1} = j^k\chi$ on some neighborhood of the origin of $\mathbb{R}^n$. We have agreed in Chapter 4 on how to identify $J^k(\mathcal{B}, \mathcal{S})$ with the bundle of holonomic k-frames and so the above argument proves that:

Proposition 6.1 *If the materially uniform k-grade hyperelastic body $\mathcal{B}$ is locally homogeneous at X then there exists a subbody $\mathcal{V}(\mathrm{X}) \subset \mathcal{B}$ and a* *configuration* $\psi : \mathcal{V}(\mathrm{X}) \to \mathcal{S}$ *such that the k-jet extension* $j^k\psi$ *is a material reference at* X.

Intuitively speaking, in the case of the material having at each point a stress-free uniform reference, the homogeneity means that in a vicinity of X one can arrange the stress-free pieces into a global configuration in such a way that no internal stress is introduced. The equilibrium of a finite sample with the free boundary can be mentained with no internal stress.

As we know from our previous considerations, Theorem 4.1 in particular, if the k-grade hyperelastic body $\mathcal{B}$ is smoothly materially uniform then there exists the corresponding material structure $\mathcal{M}^k(\mathcal{B}) \subset \mathrm{H}^k(\mathcal{B})$. In fact, as stated by the Corollary 4.1, if the symmetry group of $\mathcal{B}$ is a continuous closed subgroup of $\mathcal{G}^k$ there exist a whole conjugate class of material structures. Furthermore, if the material body is locally homogeneous and so at every material point there is an integrable material reference, say $\mathfrak{l}^k$, one can find the material structure such that the material reference $\mathfrak{l}^k$ is its local section. Consequently, as stated in the definition of local homogeneity, given a material point $\mathrm{X} \in \mathcal{B}$ there exists at X a coordinate chart $\beta : \mathcal{U} \in \mathcal{B} \to \mathbb{R}^n$ such that the k- jet extension of $\beta^{-1}|_{\beta(\mathcal{U})}$ is identical, at some neighborhood of X, with the material reference $\mathfrak{l}^k$.

Let us recall that two k-order $\mathcal{G}$-structures $\mathcal{M}^k(\mathcal{B})$ and $\mathcal{M}^k(\tilde{\mathcal{B}})$ on $\mathcal{B}$ and $\tilde{\mathcal{B}}$, respectively, where $\mathcal{G}$ is a subgroup of the structure group $\mathcal{G}^k$, are said to be *equivalent* if there exist a diffeomorphism $f : \mathcal{B} \to \tilde{\mathcal{B}}$ such that $f^{\natural} : \mathcal{M}^k(\mathcal{B}) \to \mathcal{M}^k(\tilde{\mathcal{B}})$ given by the usuall composition of jets is the principal bundle isomorphism over f. In particular, the structure is called *locally flat* if, and only if, it is locally equivalent to the flat $\mathcal{G}$-structure, i.e. the trivial bundle $\mathbb{R}^n \times \mathcal{G}$. It is not hard to show (Sternberg [S], for $k = 1$ and Saunders [Sa] for $k > 1$) that the $\mathcal{G}$-structure $\mathcal{M}^k(\mathcal{B})$ is locally flat if near every point on the manifold $\mathcal{B}$ there is a coordinate system $\{\mathrm{x}^i, \cdots, \mathrm{x}^n\}$ the k-jet extension of which is a local section of the $\mathcal{G}$-structure in question. Invoking the Definition 4.2 and the discussion thereafter, as well as the Corollary 4.1, one immediately gets that:

Theorem 6.1 (Elżanowski at al. [EEpŚ2] for $k = 1$) *If the k-grade hyperelastic body $\mathcal{B}$ is locally homogeneous then there exists a material structure $\mathcal{M}^k(\mathcal{B})$ which is a locally flat $\mathcal{G}^k_{\mathfrak{h}^k}$-structure over $\mathcal{B}$ where $\mathcal{G}^k_{\mathfrak{h}^k}$ denotes the symmetry group of $\mathcal{B}$ relative to some homogeneous reference $\mathfrak{h}^k$.*[25]

Let ω^k (resp. $\tilde{\omega}^k$) be a k-order $\mathcal{G}$- connection on $\mathcal{M}^k(\mathcal{B})$ (resp. $\mathcal{M}^k(\tilde{\mathcal{B}})$). We say that these two connections are equivalent if there exists a principal bundle isomorphism $f^\natural : \mathcal{M}^k(\mathcal{B}) \rightarrow \mathcal{M}^k(\tilde{\mathcal{B}})$ such that $f^{\natural *}\tilde{\omega}^k = \omega^k$. We say that ω^k is a *locally flat k-connection* if it is locally equivalent to the canonical flat connection on the trivial bundle $\mathbb{R}^n \times \mathcal{G}$. It is then immediate that a k-order $\mathcal{G}$-structure is locally flat if, and only if, it admits a locally flat k-order $\mathcal{G}$-connection.

Thus, having a locally homogeneous k-grade hyperelastic body $\mathcal{B}$ there exists the material structure $\mathcal{M}^k(\mathcal{B})$ which is locally flat. There exist, therefore, a locally flat connection on $\mathcal{M}^k(\mathcal{B})$. As every locally flat $\mathcal{G}-$valued connection is locally generated by a section into the subbundle $\mathcal{M}^k(\mathcal{B}) \subset \mathrm{H}^k(\mathcal{B})$ and as any local section of a material structure is a material reference, $\mathcal{M}^k(\mathcal{B})$ admits a locally flat principal material connection. Such a connection as locally equivalent to the canonical connection on the corresponding trivial bundle is locally induced by a coordinate system on the body manifold $\mathcal{B}$. The above discussion yields therefore that:

Theorem 6.2[26] *A k-grade hyperelastic body $\mathcal{B}$ is locally homogeneous if, and only if, there exists a locally flat principal material connection.*

Indeed, given the locally homogeneous material body $\mathcal{B}$ there exists a locally flat principal material connection generated by the corresponding ho-

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25 Recall that although not every material reference of the given material structure $\mathcal{M}^k(\mathcal{B})$ is a homogeneous reference (if there is any at all) the symmetry groups relative to any material reference, homogeneous or not, of the particular structure are always identical.

26 This theorem was originally proved by Noll [N] and Wang [W] for k=1 (see also Elżanowski at al. [EEpŚ2]). For the second-grade hyperelastic material the same was shown by Elżanowski and Prishepionok [EP2] and independently by de Leon and Epstein [LE].

mogeneous material reference, say $\mathfrak{h}^k$. Any other principal material connection generated by some other material reference does not need to be locally flat as the gauging by the symmetry group $\mathcal{G}^k_{\mathfrak{h}^k}$ (see the relation (5.4)) takes the homogeneous material reference into, in general, arbitrary local material reference unless, the symmetry group $\mathcal{G}^k_{\mathfrak{h}^k}$ is a discrete subgroup of $\mathcal{G}^k$ or the corresponding gauge is induced by the coordinate change on the body manifold $\mathcal{B}$. In the discrete case, due to the smoothness of any material reference, if there is a homogeneous material reference then there is only one. On the other hand, if the gauge is generated by the coordinate change on $\mathcal{B}$ it is only natural as evident from the Definition 6.1 that a homogeneous material reference is taken into another homogeneous material reference.

Given some principal material connection, to determine that it is locally flat, is to show that its horizontal distribution is locally induced by some homogeneous material reference. In the linear case (k=1, simple elasticity) when the vanishing of the torsion form (see e.g., Sternberg [S]) guarantees the flatness this amounts, as shown by Noll [N] and Wang [W], to finding, through gauging by the symmetry group, the (principal) material connection with the zero torsion. In the case of the second and the higher grade materials the vanishing of the torsion is only, as we show below, a necessary but certainly not a sufficient condition for the principal material connection to be locally flat. However, we will be able to invoke some other coordinate change invariant objects which in the way simillar to the torsion measure the local flatness of a principal material connection and so characterize the local homogeneity. To be able to do this we need first to introduce the notion of the prolongation of a k-connection and the concept of a simple connection.

Definition 6.2 *Given the k-connection ω^k let ε^{k+1} be its generating $\mathcal{E}$-connection, $\tilde{q}^k : \mathrm{H}^k(\mathcal{B}) \to \tilde{\mathrm{H}}^{k+1}(\mathcal{B})$ the corresponding $\mathcal{G}^k$-equivariant section and N_{ω^k} its characteristic manifold. The* **prolongation** *of ω^k is the $(k+1)$-connection $\mathcal{P}(\omega^k)$ such that its horizontal space at any $\mathrm{p}^{k+1} \in q^k(\mathrm{N}_{\omega^k})$ is the q^k-lift of the horizontal space of ω^k, i.e. for any $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$*

$$hor_{q^k(\mathrm{p}^k)}\mathcal{P}(\omega^k) = q^k_*(hor_{\mathrm{p}^k}\omega^k).$$

The following facts are easy consequences of the definition of prolongation.

Proposition 6.2

a. *Given the k-connection ω^k there is only one prolongation $\mathcal{P}(\omega^k)$.*
b. $proj_1\mathcal{P}(\omega^k) = \omega^k$.
c. *The connection ω^{k+1} is the prolongation of its projection $proj_1\omega^{k+1}$ if. and only if,* $\mathrm{N}_{\omega^{k+1}} = q^k(\mathrm{N}_{proj_1\omega^{k+1}})$.

Definition 6.3 (Yuen [Y]) *The k-connection ω^k is called* **simple**, *and we write $\omega^k = \mathcal{P}^{k-1}(\omega^1)$, if it is the $(k-1)$-prolongation of some linear connection ω^1.*

It appears that any simple k-connection can be characterized by the "position" of its horizontal distribution relative to its characteristic manifold. Indeed, we have:

Proposition 6.3 *If ω^k is a simple connection then its horizontal distribution is tangent to its characteristic manifold at all points.*

Proof. It is enough to point out that if the 2-connection ω^2 is the prolongation (simple) of some linear connection ω^1 then, by the definition of a simple connection, $hor_{q^1(\mathrm{p}^1)}\omega^2 = q^1_*(hor_{\mathrm{p}^1}\omega^1)$ for any $\mathrm{p}^1 \in \mathrm{N}_{\omega^1}$. However, according to Proposition 6.2(c) $q^1_*(\mathrm{H}^1(\mathcal{B})) = q^1_*(\mathrm{N}_{\omega^1}) = \mathrm{M}_{\omega^1} = \mathrm{N}_{\omega^2}$. Therefore, the definition of the prolongation implies immediately that $hor\mathcal{P}^1(\omega^1)|_{\mathrm{N}_{\omega^2}} \subset T\mathrm{N}_{\omega^2}$. Applying this argument recursively proves the original claim ♠

In fact, somewhat more general statement can be made.

Theorem 6.3 *The connection ω^k on the bundle of holonomic k-frames $\mathrm{H}^k(\mathcal{B})$ is the $(k-s)$-prolongation of its projection $proj_{k-s}\omega^k$ if, and only if, its horizontal distribution is tangent to the induced by the characteristic manifold N_{ω^k} $\mathcal{G}^s$-reduction of the bundle $\mathrm{H}^k(\mathcal{B})$, i.e. if it is tangent to $\mathrm{N}_{\omega^k}\mathcal{N}^s_{s-1}(\mathrm{n})$. In particular, ω^k is simple if, and only if, its horizontal distribution is tangent to its characteristic manifold.*[27]

[27] In fact, the same is true in the semi-holonomic case.

Proof. The condition is obviously necessary as easily attested by the definition of the prolongation of connection and Proposition 6.3. Also, as the projection of the characteristic manifold of a connection is the characteristic manifold of the projected connection $\mathrm{N}_{proj_{k-s}\omega^k} = \pi^k_s(\mathrm{N}_{\omega^k})$. Therefore, the horizontal distribution of $proj_{k-s}\omega^k$ is tangent to $\mathrm{N}_{proj_{k-s}\omega^k}\mathcal{N}^s_{s-1} = \mathrm{H}^s(\mathcal{B})$. Consequently, the sequence of invariant sections $\{\tilde{q}^l\}_{l=s,\dots,k-1}$, corresponding to the sequence of prolongations of $proj_{k-s}\omega^k$ to $\mathrm{H}^k(\mathcal{B})$, maps the horizontal distribution of the $(k-s)$-projection of ω^k onto the horizontal distribution of ω^k satisfying conditions of Definition 6.2♠

If the horizontal distribution of ω^k is locally integrable Theorem 6.3 has particularly far reaching consequences.

Collorary 6.1 *A locally integrable k-connection ω^k is simple, i.e., $\omega^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$, if and only if, $i_1\omega^k = proj_1\omega^k$.*

Proof. If the connection ω^k is simple then, by Theorem 6.2, $hor_{\mathrm{p}^k}\omega^k \subset T_{\mathrm{p}^k}\mathrm{N}_{\omega^k}$ for every $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$. On the other hand, as ω^k is locally integrable, for any $\pi^k(\mathrm{p}^k)$ there exists a local section $\mathfrak{l}^k : \mathcal{U} \subset \mathcal{B} \rightarrow \mathrm{H}^k(\mathcal{B})$ such that $hor_{\mathrm{p}^k}\omega^k = T_{\mathrm{p}^k}\mathfrak{l}^k(\mathcal{U})$. This implies that $T_{\mathrm{p}^k}\mathfrak{l}^k(\mathcal{U}) \subset T\mathrm{p}^k\mathrm{N}_{\omega^k}$ for any $\mathrm{p}^k \in \mathrm{N}_{\omega^k}$. Moreover, as N_{ω^k} is a $\mathcal{G}^1$-reduction of $\mathrm{H}^k(\mathcal{B})$, $\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1 = \mathrm{N}_{\omega^k}|_{\mathcal{U}}$ and $\mathrm{N}_{proj_1\omega^k} = \pi^k_{k-1}(\mathrm{N}_{\omega^k}) = \pi^k_{k-1}(\mathfrak{l}^k\mathcal{G}^1) = \mathrm{N}_{i_1\omega^k}$ by the definition of the induced connection (Definition 5.4). Therefore, the induced connection $i_1\omega^k$ has the same characteristic manifold as the 1-projection of ω^k. Having the same characteristic manifold the connections do not need to be the same however, $i_1\omega^k$ and $proj_1\omega^k$ not only have the same characteristic manifolds but also have the same generating q-sections as $\mathrm{M}_{proj_1\omega^k} = \mathrm{N}_{\omega^k} = \mathfrak{l}^k(\mathcal{U})\mathcal{G}^1 = \mathrm{M}_{i_1\omega^k}$. Conversely, if for some integrable connection ω^k, $i_1\omega^k = proj_1\omega^k$ then $\pi^k_{k-1}(\mathrm{N}_{\omega^k}) = \mathrm{N}_{1\omega^k} = \mathrm{N}_{i_1\omega^k} = \pi^k_{k-1}(\mathfrak{l}^k(\mathcal{U})\mathcal{G}^1)$. This, in general, may yet not guarantee that the horizontal distribution of ω^k is tangent to its characteristic manifolds but as the corresponding generating q-sections are identical it indeed does conclude the proof ♣

Applying the above argument recursively one can easily conclude the following:

Collorary 6.2 *Let the k-connection ω^k be a simple connection, i.e. $\omega^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$. Then, the horizontal distribution of ω^k is locally integrable if, and only if, the horizontal distribution of $proj_{k-1}\omega^k$ is locally integrable.*

We are ready now to determine under what conditions a k-order holonomic connection is locally equivalent to the standard flat connection on $\mathbb{R}^n \times \mathcal{G}^k$. To this end, let us recall first that it was shown by Yuen [Y] and in the context of continuum mechanics by Elżanowski and Prishepionok [EP2], and independently by de Leon and Epstein [LE1], that:

Theorem 6.4 *The k-connection ω^k is locally flat if, and only if, it is simple and its curvature and torsion vanish, i.e. $\omega^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$ and $\Omega_{\omega^k} = 0$, and $\Theta_{\omega^k} = 0$ where the* **curvature** *Ω_{ω^k} of the k-connection ω^k is the $\mathfrak{g}^k$-valued 2-form $d\omega^k|_{hor\omega^k}$ while the* **torsion** *Θ_{ω^k} is the $\mathbb{R}^n \oplus \mathfrak{g}^{k-1}$-valued 2-form $d\theta^k|_{hor\omega^k}$.*

Note that the curvature and torsion of the j$^{\text{th}}$-projection of ω^k are respectively defined by the following identities (*cf.* [CDL]):

$$\pi^{k*}_{k-j}\Theta_{proj_j\omega^k} = id_{\mathbb{R}^n} \times \tilde{\pi}^{k-1}_{k-j-1*}\Theta_{\omega^k}, \tag{6.1}$$

$$\pi^{k*}_{k-j}\Omega_{proj_j\omega^k} = \tilde{\pi}^{k}_{k-j*}\Omega_{\omega^k}. \tag{6.2}$$

Thus, if the connection ω^k has a vanishing torsion and/or curvature then its projections $proj_j\omega^k$ have the same properties.

Although Theorem 6.4 sets the explicit sufficient and necessary conditions for the k-connection to be locally flat, we shall try to determine if these conditions could not be weaken, in particular, in the locally integrable case, i.e., $\Omega_{\omega^k} = 0$, that is particular in the case of the principal material connection. To this end let us recall that it was proved by Garcia [G] and Yuen [Y] that:

Lemma 6.1 *Let the (holonomic) connection* ω^k *be induced by the* $\mathcal{E}$*- connection* $\varepsilon^{k+1} : \mathrm{H}^1(\mathcal{B}) \to \mathrm{H}^{k+1}(\mathcal{B})$ *into the holonomic frame bundle. Then,* ω^k *has a vanishing torsion.*[28]

This simple fact enables us to show that:

Collorary 6.3 *If the k-connection* ω^k *is holonomic and has the vanishing curvature then the induced connection* $i_1\omega^k$ *has vanishing torsion.*

Proof. Let $\mathfrak{l}^k : \mathcal{U} \subset \mathcal{B} \to \mathrm{H}^k(\mathcal{B})$ define locally the horizontal distribution of ω^k. The corresponding $\mathcal{E}$-connection of $i_1\omega^k$ is a section into the holonomic k-frame bundle (see Definition 5.5). This, according to Lemma 6.1, guarantees the vanishing of the torsion of $i_1\omega^k$ ♣

Moreover,

Proposition 6.4 *A k-connection (locally integrable or not) cannot be prolonged (see Definition 6.2) into the holonomic frame bundle* $\mathrm{H}^{k+1}(\mathcal{B})$ *unless it has the vanishing torsion.*

Proof. Suppose that ω^k has non-vanishing torsion and let $\mathcal{P}^1(\omega^k)$ be its prolongation into the holonomic frame bundle $\mathrm{H}^{k+1}(\mathcal{B})$. As the prolongation is holonomic $\mathrm{M}_{\omega^k} = \mathrm{N}_{\mathcal{P}^1(\omega^k)} \subset \mathrm{H}^{k+1}(\mathcal{B})$. This, however, means that the $\mathcal{E}$-connection inducing ω^k is a section of the holonomic frame bundle which in turn, due to the Lemma 6.1, implyes that ω^k has vanishing torsion ♠

Finally, we have come to the point when we can conclude our analysis by proving two important statements about locally flat connections. Some other interesting intermediate cases will be presented elsewhere as they require somewhat deeper look at the form of k-connections (Theorem 5.1 and Proposition 5.4) and the properties of their curvature and torsion forms.

Proposition 6.5 *A simple holonomic k-connection* ω^k *is locally flat if, and only if,* $\omega^1 \equiv proj_{k-1}\omega^k$ *is locally flat.*

[28] The same can be show directly from the Proposition 5.4.

Proof. If the $(k-1)$-prolongation $\mathcal{P}^{k-1}(\omega^1)$ is locally flat then obviously ω^1 is locally flat as $\omega^1 = proj_{k-1}\mathcal{P}^{k-1}(\omega^1)$. We also know, from Collorary 6.1, that ω^1 is curvature free if, and only if, its prolongations are curvature free. What remains to be shown is that if the torsion of ω^1 vanishes then any of its prolongations has vanishing torsion. This is, however, immediate by Collorary 6.1, Proposition 6.4 and the uniqueness of the prolongation ♠

Proposition 6.6 *Let the holonomic k-connection ω^k be simple and curvature free. Then, it is locally flat.*

Proof. If a holonomic connection ω^k is simple and curvature-free then by Collorary 6.1 $i_1\omega^k = proj_1\omega^k$. Moreover, because $\omega^k = \mathcal{P}(i_1\omega^k)$, the induced connection has vanishing torsion as otherwise, according to Proposition 6.4, it could not be prolonged into the holonomic frame bundle. This proves that $proj_1\omega^k$ is locally flat as it simple (is a projection of a simple connection), locally integrable (Collorary 6.2) and has no torsion as it is identical to $i_1\omega^k$. This, in fact, concludes the proof as the prolongation of a locally flat connection is a locally flat connection as attested by Proposition 6.5 ♣

The message of the Proposition 6.6 is that for a locally integrable holonomic k-connections to be locally flat is equivalent to being simple. Combining this with Collorary 6.1 enables one to state that:

Theorem 6.5 *A curvature-free holonomic k-connection ω^k is locally flat if, and only if, its projection $proj_1\omega^k$ is identical to its induced connection $i_1\omega^k$.*

For a curvature-free linear connection to be locally flat is to be symmetric, i.e., to have a vanishing torsion. Similarly, for a curvature-free holonomic k-connection, $k \geq 2$, the local flatness is equivalent to the vanishing of the tensor ($\mathfrak{g}^k$-valued tensorial 1-form) $\mathfrak{D}_{\omega^k} \equiv proj_1\omega^k - i_1\omega^k$. We therefore have:

Proposition 6.7 *Let ω^k be a curvature-free holonomic connection and let $k \geq 2$. Then,*

(1) ω^k *is locally flat if, and only if,* $\mathfrak{D}_{\omega^k} \equiv 0$,

(2) *if* $\mathfrak{D}_{\omega^k} = 0$ *then* $\mathfrak{D}_{proj_1\omega^k} = 0$,

(3) *if* $\mathfrak{D}_{proj_1\omega^k} = 0$ *then* $proj_1\omega^k$ *is a simple connection but, in general,* $\mathfrak{D}_{\omega^k} \neq 0$.

Proof.

(1) This statement is equivalent to the statement of Theorem 6.5 and is a straitforward consequence of Proposition 6.6 and Collorary 6.1.[29]

(2) $\mathfrak{D}_{proj_1\omega^k} \equiv proj_1(proj_1\omega^k) - i_1(proj_1\omega^k) = proj_2\omega^k - proj_1(i_1\omega^k) = proj_1(\mathfrak{D}_{\omega^k})$ by Proposition 5.5. Therefore, if $\mathfrak{D}_{\omega^k}$ vanishes so does its projection $\mathfrak{D}_{proj_1\omega^k}$. Note that the tensor $\mathfrak{D}_{proj_1\omega^k}$ is indeed well defined as if ω^k is curvature-free so is its projection guaranteeing the existence of the induced connection $i_1(proj_1\omega^k)$.

(3) If $\mathfrak{D}_{proj_1\omega^k} = 0$ then $proj_1\omega^k$ is a simple connection as stated in (1). However, even if $\mathfrak{D}_{proj_1\omega^k}$ vanishes ω^k may not be simple. Indeed, it is enough to choose as ω^k a curvature-free holonomic connection which projection is simple but which has an arbitrary $\mathfrak{n}^k_{k-1}$-component (see Eqs. (5.15) and (5.20))♣

We are now in a position to go back the main topic of this presentation and with the general results we have obtained above continue the analysis of the

[29] We would like to add that somewhat similar, but not identical, statement can be made in case ω^k is a semi-holonomic connection. The similarity comes from the fact that in order to secure the local flatness of a curvature-free semi-holonomic connection one must require, like in the holonomic case, that the tensor $\mathfrak{D}_{\omega^k}$ vanishes. To make the condition sufficient one must also demand that the vanishing of the torsion of $proj_{k-1}\omega^k$. The difference between the semi-holonomic case and the holonomic case comes from the fact that, in general, the semi-holonomicity of ω^k does not guarantee the vanishing of the torsion of the induced connection $i_1\omega^k$. Consequently, the vanishing of $\mathfrak{D}_{\omega^k}$ although makes $proj_1\omega^k = i_1\omega^k$ it does not force it to have a zero torsion. If however $\Theta_{proj_{k-1}\omega^k} \equiv 0$ and $\mathfrak{D}_{\omega^k} \equiv 0$ then the linear connection $proj_{k-1}\omega^k$ is locally flat making, by virtue of (2), the 2-connection $proj_{k-2}\omega^k$ holonomic and simple. Iterating this upwards will imply that ω^k is simple and holonomic and so locally flat.

problem of the local homogeneity of smoothly uniform hyperelastic material bodies. To this end let us recall once again that every principal material connection of a k-grade hyperelastic material body $\mathcal{B}$ is by definition holonomic and curvature-free as it is locally induced by a material reference being a local section into the holonomic frame bundle $\mathrm{H}^k(\mathcal{B})$. It always generates locally the induced material connection as well as its projections. As we have argued before (Theorem 6.2), the local homogeneity of $\mathcal{B}$ is equivalent to the existance of a locally flat principal material connection, say ω^k. The local flatness of the principal material connection of a simple uniform elastic body is guaranteed by the vanishing of its torsion while for the second-grade and higher materials it corresponds to the vanishing of the appropriate tensor $\mathfrak{D}_{\omega^k}$, as shown by Propsition 6.7. In the context of continuum mechanics we shall call the tensor $\mathfrak{D}_{\omega^k}$ the *inhomogeneity tensor*.

The discussion above can now be summarized in the following form:

Theorem 6.5 *A smoothly uniform k-grade hyperelstic body $\mathcal{B}$ is locally homogeneous if, and only if, there exists a principal material connection, say ω^k, such that:*

(1) *if $k = 1$ its torsion $\Theta_{\omega^k} \equiv 0$,*

(2) *if $k > 1$ its inhomogeneity tensor $\mathfrak{D}_{\omega^k} \equiv 0$.*

We can now go back to our second order holonomic example from the end of Chapter 5. We point out that, as stated above, the principal material connection ω^2 induced by the section $\mathfrak{l}^2(\mathrm{y}^i) = (\mathrm{y}^i, \mathrm{a}^i_j(\mathrm{y}^i), \mathrm{b}^i_{jk}(\mathrm{y}^i))$ is simple if, and only if,

$$(\mathfrak{D}_{\omega^2})^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk} \equiv 0 \tag{6.3}$$

where the Christoffel symbols Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ are defined by Eqs. (5.29) and (5.31). The vanishing of the inhomogeneity tensor implies that

$$\frac{\partial \mathrm{a}^i_j}{\partial \mathrm{x}^k}\mathrm{a}^k_l = \mathrm{b}^i_{jl}. \tag{6.4}$$

As b^i_{jl} is always symmetric the above relation is, in fact, the integrability conditon for a^k_l. Thus, there exist smooth functions $\zeta^i(x^k)$ such that the gauge $p^i_j = \frac{\partial \zeta^i}{\partial x^j}$ and $q^i_{jk} = \frac{\partial^2 \zeta^i}{\partial x^j \partial x^k}$ proving that if the inhomogeneity tensor vanishes the body is locally homogeneous.

The importance of the simplicity condition for determining the local flatness of the principal material connection can be illustrated by the following example. Let us assume that our second-grade hyperelastic material body $\mathcal{B}$ is not locally homogeneous (there is no locally flat principal material connection) but there exists a principal material connection ω^2_o such that its projected material connection $proj_1\omega^2_o$ as well as the induced connection $i_1\omega^2_o$ are both locally flat but different. Therefore, there is no coordinate system in which the corresponding Christoffel symbols ${}_o\Gamma^i_{jk}$ and ${}_o\tilde{\Gamma}^i_{jk}$ vanish simultanieusly. The inhomogeneity tensor $\mathfrak{D}_{\omega^2_o}$ does not vanish, it only becomes symmetric. The principal material connection ω^2_o has a vanishing torsion but it is not a prolongation of the locally flat linear connection $proj_1\omega^2_o$. Despite the fact that ω^2_o is curvature-free and has no torsion it is not locally induced by a single coordinate system.

In the case of a simple elastic material the torsion of the material connection is in some way a measure of the density of the distribution of dislocations [Kr], [W]. Following this line of interpreting the geometric quantities appearing in the theory one might say that the curvature of the induced connection measures the distribution of disclinations while the non-vanishing of the symmetric inhomogeneity tensor (like in the example above) can possibly be regarded as the indecation the presence of some intrinsically *second order defects*, as suggested in [EEp2]. Note also that in order to be able to detect the presence of these second order defects one must have no first order once. Otherwise, the non-vanishing of the inhomogeneity tensor indecates only that there are all kinds of defects present.[30]

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[30] We would like to point out here that the theory of non-holonomic frame bundles can also be utilized to model the uniformity of material bodies w microstructure. For example, the uniformity of a first-grade material body consisting of a rigid matrix and a smoothly distributed micro-inclusions described by the deformable triades of vectors could be modeled

We end this chapter by reiterating once again that to determine that the material body, possesing a continuous symmetry group, is locally homogeneous one must find a locally flat principal material connection. Normally there are many principal material connections available (compare Eqs. (5.5) and (5.7) as well as Proposition 5.2) and only through gauging them by the symmetry group one can possibly determine if there exists any which is locally flat. One must find such a principal material connection which is a prolongation of a locally flat linear connection. It must be stressed here that gauging does not, in general, preserve the differential lifting (prolongation) as evident from Propsition 6.3. The non-vanishing of the inhomogeneity tensor for some choice of the principal material connection does not prejudice its vanishing for some other principal material connection as $\mathfrak{D}_{\omega^k}$ is not invariant under the action of the symmetry group.

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by the analogous theory on semi-holonomic frame bundles. Indeed, the deformation of the triad can be presented as a 2×2 matrix while its deformation gradient is not symmetric due to the fact that the distribution of these bases is, in general, non-integrable. In such a case the local homogeneity is guaranteed by the existence of the principal material connection such that not only its inhomogeneity tensor vanishes but also the projected to the first level material connection is symmetric (see footnote 29 and also [LE2]).

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