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FRED E. BENTH[1], THOMAS DECK[2], JÜRGEN POTTHOFF[3]

1 Introduction

This paper is inspired by articles of Chow [Ch] and Nualart–Zakai [NZ], in which certain (linear) stochastic heat equations are treated within the framework of generalized Brownian functionals. In particular, in [Ch] a stochastic heat equation with a gradient–coupled noise (namely, the noise associated with a Wiener integral with respect to Brownian motion) is proposed as a model for the transport of a substance in a turbulent medium. The present article extends the work in [DP,P2] (and also [CLP]) in several ways: most notably, to the non–linear case and to very general noise terms which may depend on space and time.

We consider a class of Cauchy problems of the type:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= A\phi + F(\phi) + \nabla G(\phi) \diamond N, \\ \phi(0) &= \phi_0, \end{aligned}$$

where A is a second order differential operator on $\mathbb{R}^d$, N is a noise and F, G are (possibly) non–linear functions of the solution ϕ (F may contain noise terms as well), and $\diamond$ denotes the Wick product; – for a more precise formulation of the problem we refer to Section 2.2. We prove existence and uniqueness results for these Cauchy problems under various conditions of (global and local) Lipschitz type on the non–linearities F, G, and for various types of noise. (This is for example motivated by the works [LØU1,2] in which positive noise of exponential type has been employed.)

Our method is an extension of the ideas in [DP, P1, P2] (cf. also [CLP, CP]), namely of the combination of the S–transform with classical fixed point theorems and so–called characterization theorems (e.g., [HKPS, KLPSW, KLS, Ou, PS] and references quoted there), which serve to reverse the S–transform. The basic reason for this procedure is the fact that the S–transform turns the Itô integral and its generalizations (i.e., the Skorokhod and Hitsuda–Skorokhod integrals, and even more generally, integrals of Wick products of random variables

[1]Matematisk Institutt, Universitet i Oslo, Norway, and Lehrstuhl für Mathematik V, Universität Mannheim, Germany. Supported by the Norwegian Research Council (NFR) under grant NAVF: 100549/410

[2]Lehrstuhl für Mathematik V, Universität Mannheim, Germany. Supported by the Deutsche Forschungsgemeinschaft (DFG).

[3]Lehrstuhl für Mathematik V, Universität Mannheim, Germany. Partially supported by the Deutsche Forschungsgemeinschaft (DFG).

and noise terms) into a usual Lebesgue integral. Hence, after the S–transform is carried out on the above Cauchy problem, it is amenable to a more or less standard contraction method. On a more technical level: due to the non-linearities one has to work with Banach's contraction mapping theorem on various spaces of so-called U–functionals (which are images of generalized random variables under the S–transform). In order to cover a large class of Cauchy problems we make use as well of the Hida triple of random variables as of the Kondratiev triples (cf. Section 2.1). We also apply our methods to anticipating stochastic differential equations and stochastic Volterra equations.

We combine our machinery with the Wick calculus (e.g., [KLS]) in order to produce a number of examples as illustrations of our results. Among others, we consider: stochastic reaction–diffusion equations, stochastic Burgers equations and an equation modelling population growth in a random medium (all in Wick form). In this context we also want to mention the works [HØUZ, HLØUZ2, HLØUZ3, LØU1, LØU2]. In the linear case (and for space–time white noise), one finds in [PW] an approach via the Feynman–Kac and Cameron–Martin–Girsanov–Maruyama formulae.

Our article is organized as follows. Section 2 provides some mathematical background from white noise analysis, a precise formulation of the Cauchy problem and the notion of solution to be used. In Section 3 the necessary Banach spaces of U–functionals are introduced. In Sections 4 and 5 we prove existence and uniqueness of solutions of the Cauchy problem under certain global (Section 4), resp. local (Section 5) Lipschitz conditions on the coefficients F and G. Moreover, a number of examples are considered there. Finally, as a by-product of our method we treat in Section 6 non-linear anticipating stochastic differential equations and stochastic Volterra equations, and give again several examples.

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2 Mathematical Preliminaries

2.1 Some Elements From White Noise Analysis

For $k \in \mathbb{N}$, denote the space of Schwartz functions by $\mathcal{S}(\mathbb{R}^k)$ and its dual by $\mathcal{S}'(\mathbb{R}^k)$. We have the well-known family of semi-norms on $\mathcal{S}(\mathbb{R}^k)$

$$\|\xi\|_{(\alpha,\gamma)} := \sup_{x\in\mathbb{R}^k} |\partial^\alpha \xi(x)|(1+|x|)^\gamma.$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i, \gamma \in \mathbb{N}_0$. We are going to make use of the norms

$$|\xi|_{2,p} := |(H^{\otimes k})^p \xi|_{2,0}$$

where $p \in \mathbb{N}_0$ and $|\cdot|_{2,0}$ is the $L^2(\mathbb{R}^k)$–norm. Here, H is the harmonic oscillator on $\mathbb{R}$

$$H = -\frac{d^2}{dx^2} + (1+x^2).$$

The complexification of the Schwartz space and its dual will be denoted by $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$ and $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^k)$, respectively. The notation for the corresponding families of semi–norms and norms stays unchanged.

We shall make use of the probability space $\left(\mathcal{S}'(\mathbb{R}^k), \mathcal{B}, \mu\right)$, where $\mathcal{B}$ is the Borel σ–algebra induced by the weak topology and μ is the measure with characteristic functional $\exp(-\frac{1}{2}|\cdot|_{2,0}^2)$. For each $\xi \in \mathcal{S}(\mathbb{R}^k)$ the dual pairing with $\omega \in \mathcal{S}'(\mathbb{R}^k)$ defines the random variable $\omega \rightarrow <\omega, \xi>$. To $X \in (L^2) := L^2(\mu)$ there is associated a function on $\mathcal{S}(\mathbb{R}^k)$, called the $\mathcal{S}$–transform of X:

$$\mathcal{S}X(\xi) = \int_{\mathcal{S}'(\mathbb{R}^k)} X(\omega) e^{<\omega,\xi> - \frac{1}{2}|\xi|_2^2} d\mu(\omega). \tag{1}$$

It is known that elements $X \in (L^2)$ admit the Wiener–Itô–Segal decomposition

$$X = \sum_{n=0}^{\infty} I_n(X^{(n)}),$$

where the $X^{(n)}$ are symmetric elements in $L^2(\mathbb{R}^{kn})$, and I_n denotes the n–fold Wiener integral. The norm of X is

$$||X||_{(L^2)}^2 = \sum_{n=0}^{\infty} n! \, |X^{(n)}|_{L^2(\mathbb{R}^{kn})}^2.$$

For $\beta \in [0,1]$, $p \in \mathbb{N}_0$, introduce the Hilbert spaces $(\mathcal{S})_p^\beta$ consisting of elements

$$\phi = \sum_{n=0}^{\infty} I_n(\phi^{(n)})$$

such that

$$||\phi||_{2,\beta,p}^2 := \sum_{n=0}^{\infty} (n!)^{1+\beta} \, |(H^{\otimes kn})^p \phi^{(n)}|_{L^2(\mathbb{R}^{kn})}^2 < \infty.$$

Denote by $(\mathcal{S})_{-p}^{-\beta}$ the dual. The space $(\mathcal{S})^\beta$ is defined as the projective limit (w.r.t. p) of the spaces $(\mathcal{S})_p^\beta$. Its dual is the inductive limit of $(\mathcal{S})_{-p}^{-\beta}$, and is denoted $(\mathcal{S})^{-\beta}$. For $\beta = 0$, the spaces $(\mathcal{S})^0$ and $(\mathcal{S})^{-0}$ will frequently be denoted $(\mathcal{S})$ and $(\mathcal{S})^*$ respectively. The following chain of inclusions holds: $0 < \beta < 1$,

$$(\mathcal{S})^1 \subset (\mathcal{S})^\beta \subset (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* \subset (\mathcal{S})^{-\beta} \subset (\mathcal{S})^{-1}.$$

Remark $(\mathcal{S})$ and $(\mathcal{S})^*$ are known as the spaces of Hida test functions and of Hida distributions, respectively. For $\beta \in (0,1]$, $(\mathcal{S})^\beta$ and $(\mathcal{S})^{-\beta}$ are called the spaces of Kondratiev test functions and of Kondratiev distributions, respectively. The $\mathcal{S}$–transform (1) can be extended to elements $X \in (\mathcal{S})^{-1}$.

To characterize $(\mathcal{S})^{-\beta}$ we need the notion of U–functionals. First let $\beta \in [0,1)$: Consider functions $U : \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) \to \mathbb{C}$, with the following two properties:

1. For every $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, the mapping $z \mapsto U(\xi + z\eta)$ is entire.

2. There exist $K_1, K_2 > 0$ and $p \in \mathbb{Z}$ such that for all $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$,

$$|U(\xi)| \leq K_1 \exp(K_2 |\xi|_{2,p}^{\frac{2}{1-\beta}}).$$

The space of all such functions U will be denoted $\mathcal{U}^\beta$. Consider complex valued functions U defined on an open neighborhood $\mathcal{O}$ around zero in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, with the following two properties:

1′. For every $\xi, \eta \in \mathcal{O}$ there exists an open set $V_{\xi,\eta}$ around zero in $\mathbb{C}$ such that the following mapping is analytic:

$$z \mapsto U(\xi + z\eta) : V_{\xi,\eta} \to \mathbb{C}.$$

2′. U is locally bounded on $\mathcal{O}$, i.e. every $\xi \in \mathcal{O}$ has a neighborhood $\mathcal{N} \subset \mathcal{O}$ such that $U(\mathcal{N})$ is bounded.

We identify two functions when they are equal on a neighborhood of zero in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$. Hence, the space $\mathcal{U}^1$ consists of germs of functions of above type. We state the characterization theorem for $(\mathcal{S})^{-\beta}$:

Theorem 1 *Let $\beta \in [0,1]$. If $\phi \in (\mathcal{S})^{-\beta}$, then $\mathcal{S}\phi \in \mathcal{U}^\beta$. Conversely, if $U \in \mathcal{U}^\beta$ there exists a unique element $\phi \in (\mathcal{S})^{-\beta}$ such that $\mathcal{S}\phi = U$.*

A proof of this theorem for $\beta = 0$ can be found in [HKPS], [KLPSW], [Ou] and [PS]. The general case $\beta \in [0,1]$ is treated in [KLS].

The spaces $\mathcal{U}^\beta$ are all closed under the pointwise product of functions. Hence, we can define the Wick product of two elements $\phi, \psi \in (\mathcal{S})^{-\beta}$:

$$\phi \diamond \psi := \mathcal{S}^{-1}(\mathcal{S}\phi \cdot \mathcal{S}\psi). \tag{2}$$

In the following we shall be concerned with elements of $(\mathcal{S})^{-\beta}$ parametrized by $(t,x) \in D_T$, where $D_t := [0,t] \times \mathbb{R}^d$ for $0 < t < \infty$, and $d \geq 1$. I.e. mappings

$$(t,x) \mapsto \phi(t,x) : D_T \to (\mathcal{S})^{-\beta}.$$

For $\beta \in [0,1]$ and $n \in \mathbb{N}_0$, we say that $\phi \in C_b^{0,n}(D_T; (\mathcal{S})^{-\beta})$ if and only if $\langle \phi(\cdot,\cdot), \psi \rangle \in C_b^{0,n}(D_T)$ for every $\psi \in (\mathcal{S})^\beta$. Here $C_b^{m,n}(D_T)$ denotes the space of complex valued functions which are m times resp. n times continuously differentiable in $t \in [0,T]$ resp. in $x \in I\!R^d$, and all the derivatives are bounded. We set $C_b^n(I\!R^d) := C_b^{n,n}(I\!R^d)$ and $C_b(I\!R^d) := C_b^0(I\!R^d)$. Consider mappings

$$(t,x) \mapsto f(t,x;\cdot) : D_T \to \mathcal{U}^\beta .$$

For $\beta \in [0,1)$, $f \in C_b^{0,n}(D_T;\mathcal{U}^\beta)$ if and only if $f(\cdot,\cdot;\xi) \in C_b^{0,n}(D_T)$ for every $\xi \in \mathcal{S}_{\mathbb{C}}(I\!R^k)$ and the constants K_1, K_2 in property 2 above are independent of t and x. In case $\beta = 1$, $f \in C_b^{0,n}(D_T;\mathcal{U}^1)$ if and only if $f(\cdot,\cdot;\xi) \in C_b^{0,n}(D_T)$ for every $\xi \in \mathcal{O}$ and property $2'$ holds uniformly in t and x. The proof of the following result is an adoption of the proof given for case $\beta = 0$ in [P2]:

Lemma 2 *Let $\beta \in [0,1]$ and $n \in I\!N_0$. If $f \in C_b^{0,n}(D_T;\mathcal{U}^\beta)$ then $\mathcal{S}^{-1}f \in C_b^{0,n}(D_T;(\mathcal{S})^{-\beta})$.*

2.2 Formulation Of The Problem

We assume $(t,x) \in D_T$, for $d \geq 1$ and $0 < T < \infty$. Let A be a second order uniformly elliptic differential operator,

$$A = \sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x),$$

where, for given $\lambda_2 \geq \lambda_1 > 0$,

$$\lambda_1 |y|^2 \leq \sum_{i,j=1}^d a_{ij}(t,x) y_i y_j \leq \lambda_2 |y|^2$$

for all $y \in I\!R^d$, and $(t,x) \in D_T$. We furthermore assume that the coefficient functions are in $C_b^1(D_T)$ so that the heat equation $\frac{\partial u}{\partial t} - Au = 0$ has a fundamental solution $q(t,x;s,y)$ with bounds

$$\begin{aligned} |q(t,x;s,y)| &\leq K_q |t-s|^{-d/2} \exp(-\lambda \frac{|x-y|^2}{|t-s|}), \\ |\nabla q(t,x;s,y)| &\leq K_q |t-s|^{-(d+1)/2} \exp(-\lambda \frac{|x-y|^2}{|t-s|}), \end{aligned}$$

where λ, $K_q > 0$ are suitable constants, and the gradient is taken either w.r.t. the x or the y variable (see, e.g. [Fr], [LSU]). These bounds imply the following estimates

$$\int_{I\!R^d} |q(t,x;s,y)|\, dy \leq K_q(\frac{\pi}{\lambda})^{d/2} =: C_q, \tag{3}$$

$$\int_{I\!R^d} |\nabla_y q(t,x;s,y)|\, dy \leq C_q (t-s)^{-\frac{1}{2}}. \tag{4}$$

Consider $\beta \in [0,1]$ and $N(t,x) = (N_1(t,x), \dots, N_d(t,x))$, where the components $N_i(t,x)$ belong to $(\mathcal{S})^{-\beta}$, for $i = 1, \dots, d$. Let F and G be two functions on $C_b(D_T; (\mathcal{S})^{-\beta})$ such that for every $\phi \in C_b(D_T; (\mathcal{S})^{-\beta})$

$$F(\phi), G(\phi) : D_T \to (\mathcal{S})^{-\beta}.$$

We consider the stochastic Cauchy problem

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t,x) - A\phi(t,x) &= F(\phi)(t,x) + \nabla G(\phi)(t,x) \diamond N(t,x) \\ \phi(0,x) &= \phi_0(x), \end{aligned} \tag{5}$$

with

$$\nabla G(\phi)(t,x) \diamond N(t,x) = \sum_{i=1}^{d} \frac{\partial G}{\partial x_i}(t,x) \diamond N_i(t,x).$$

The corresponding integral formulation of (5) leads (after an integration by parts) to:

$$\begin{aligned} \phi(t,x) = & \int_{\mathbb{R}^d} \phi_0(y) q(t,x;0,y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^d} q(t,x;s,y) F(\phi)(s,y)\, dy\, ds \\ & - \int_0^t \int_{\mathbb{R}^d} q(t,x;s,y) G(\phi)(s,y) \diamond \sum_{i=1}^{d} \frac{\partial N_i}{\partial y_i}(s,y)\, dy\, ds \\ & - \int_0^t \int_{\mathbb{R}^d} \nabla_y q(t,x;s,y) \cdot N(s,y) \diamond G(\phi)(s,y)\, dy\, ds. \end{aligned} \tag{6}$$

The integrals are understood in the sense of Bochner. By a solution of (5), we shall understand an element $\phi \in C_b(D_T; (\mathcal{S})^{-\beta})$ which satisfies (6). To prove existence and uniqueness results for (5) we are going to study its $\mathcal{S}$-transform: Let $u(t,x;\xi) := \mathcal{S}\phi(t,x,\cdot)(\xi)$ and define the functions

$$\begin{aligned} f(u)(t,x;\xi) &:= \mathcal{S}F(\mathcal{S}^{-1}u(t,x))(\xi), \\ g(u)(t,x;\xi) &:= \mathcal{S}G(\mathcal{S}^{-1}u(t,x))(\xi). \end{aligned}$$

Informal $\mathcal{S}$-transformation of (6) with $n(t,x;\xi) := \mathcal{S}N(t,x)(\xi)$ gives

$$\begin{aligned} u(t,x;\xi) = & \int_{\mathbb{R}^d} u_0(y;\xi) q(t,x;0,y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^d} q(t,x;s,y) f(u)(s,y;\xi)\, dy\, ds \\ & - \int_0^t \int_{\mathbb{R}^d} q(t,x;s,y) g(u)(s,y;\xi) \sum_{i=1}^{d} \frac{\partial n_i}{\partial y_i}(s,y;\xi)\, dy\, ds \\ & - \int_0^t \int_{\mathbb{R}^d} \nabla_y q(t,x;s,y) n(s,y;\xi) g(u)(s,y;\xi)\, dy\, ds. \end{aligned} \tag{7}$$

In this paper, we study problem (6) for f and g satisfying a uniform or local Lipschitz condition. We refer to Section 4 and 5 for a precision of the conditions on f and g. In case that f and g are uniform Lipschitz, we will distinguish between two types of noise $N(t,x)$: If $n = \mathcal{S}N$ grows like a polynomial in $|\xi|$, i.e, for a $\alpha > 0$

$$|n(t,x;\xi)| \leq K_n(1+|\xi|^{\alpha}_{2,p})$$

we call N *polynomial noise*, otherwise *nonpolynomial noise*.

3 Banach Spaces of U–Functionals

In this section we introduce function spaces which will become useful for our treatment of (5). We remark that similar spaces can be found in [CLP, CP].

For $d, k \in \mathbb{N}$ and $T > 0$ fixed, consider functions $u : D_T \times \mathcal{V} \to \mathbb{C}$, where $\mathcal{V}$ is an open subset of $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$. We make the following assumptions:

(i) (*Continuity*) For every fixed element $\xi \in \mathcal{V}$, $u(\cdot,\cdot;\xi) \in C_b(D_T)$.

(ii) (*Measurability*) For every $\xi, \eta \in \mathcal{V}$, there exists an open set $V_{\xi,\eta} \subset \mathbb{C}$ such that the following mapping is Borel–measurable:

$$(t,x,z) \mapsto u(t,x;\xi+z\eta) : D_T \times V_{\xi,\eta} \to \mathbb{C}.$$

(iii) (*Analyticity*) For every $(t,x) \in D_T$ an analytic mapping is defined by

$$z \mapsto u(t,x;\xi+z\eta) : V_{\xi,\eta} \to \mathbb{C}.$$

Denote by $\tilde{\mathcal{U}}$ the space of all such functions u. Introduce the norm

$$\|u\| := \sup\{|u(t,x;\xi)|w(t;\xi);\ (t,x,\xi) \in D_T \times \mathcal{V}\}, \tag{8}$$

where the weight function w is given by

$$w(t;\xi) = \frac{\exp(-tL(\xi))}{1+R(\xi)}.$$

The functions $R, L : \mathcal{V} \to \mathbb{R}_+$ are such that $z \mapsto L(\xi+z\eta)$ and $z \mapsto R(\xi+z\eta)$ are measurable and bounded on every compact subset $\mathcal{K} \subset V_{\xi,\eta}$ for fixed $\xi, \eta \in \mathcal{V}$. Let $\mathcal{U} := \{u \in \tilde{\mathcal{U}} : \|u\| < \infty\}$. Obviously, $(\mathcal{U}, \|\cdot\|)$ is a normed vector space.

Proposition 3 *$(\mathcal{U}, \|\cdot\|)$ is a Banach space.*

Proof The proof follows the arguments given in [CLP]: To show that $\mathcal{U}$ is complete, let $(u_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{U}$. For $(t,x,\xi) \in D_T \times \mathcal{V}$:

$$|u_n(t,x;\xi) - u_m(t,x;\xi)| \leq \|u_n - u_m\|(1+R(\xi))\exp(tL(\xi)).$$

Denoting by $|\cdot|_\infty$ the supremum-norm on D_T we obtain

$$|u_n(\cdot,\cdot;\xi) - u_m(\cdot,\cdot;\xi)|_\infty \leq ||u_n - u_m||(1 + R(\xi))\exp(TL(\xi)).$$

By completeness of the space $(C_b(D_T), |\cdot|_\infty)$, there exists $u(\cdot,\cdot;\xi) \in C_b(D_T)$ such that $u_n(\cdot,\cdot;\xi) \to u(\cdot,\cdot;\xi)$ in $|\cdot|_\infty$–norm. I.e., for a given $\epsilon > 0$

$$\sup\{|u_n(t,x;\xi) - u(t,x;\xi)| : (t,x) \in D_T\} < \epsilon,$$

for an appropriate $n \geq N(\xi)$. This inequality together with $0 < w(t;\xi) \leq 1$ implies

$$\begin{aligned} |u(t,x;\xi)|w(t;\xi) &\leq |u(t,x;\xi) - u_n(t,x;\xi)|w(t;\xi) + |u_n(t,x;\xi)|w(t;\xi) \\ &\leq |u(\cdot,\cdot;\xi) - u_n(\cdot,\cdot;\xi)|_\infty + ||u_n|| \leq \epsilon + ||u_n||, \end{aligned}$$

for $n \geq N(\xi)$. Because $||u_n|| <$ const. for all $n \in \mathbb{N}$ we find

$$||u|| \leq \epsilon + ||u_n|| < \infty.$$

Since $(u_n)_{n=1}^\infty$ is Cauchy in $\mathcal{U}$, there exists $N > 0$ such that

$$||u_n - u_m|| < \epsilon, \quad \forall n, m \geq N.$$

By the triangle inequality we get

$$\begin{aligned} &|u(t,x;\xi) - u_n(t,x;\xi)|w(t;\xi) \\ &\leq |u_n(t,x;\xi) - u_m(t,x;\xi)|w(t;\xi) + |u_m(t,x;\xi) - u(t,x;\xi)|w(t;\xi) \\ &\leq \epsilon + |u_m(\cdot,\cdot;\xi) - u(\cdot,\cdot;\xi)|_\infty. \end{aligned}$$

Since $u_m(\cdot,\cdot;\xi) \to u(\cdot,\cdot;\xi)$ in $|\cdot|_\infty$–norm, we let m tend to ∞ and obtain:

$$|u(t,x;\xi) - u_n(t,x;\xi)|w(t;\xi) \leq \epsilon, \quad \forall n \geq N.$$

This yields $||u - u_n|| \leq \epsilon$ for all $n \geq N$, i.e., $u_n \to u$ w.r.t. the $||\cdot||$–norm.

It remains to show conditions (ii) and (iii): Since by assumption

$$(t,x,z) \to u_n(t,x;\xi + z\eta) : D_T \times V_{\xi,\eta} \to \mathbb{C}$$

is measurable for every n, the same holds for the limit $(t,x,z) \to u(t,x;\xi + z\eta)$. We show analyticity (iii): Fix $\xi, \eta \in \mathcal{V}$ and $(t,x) \in D_T$: Consider the analytic function $\tilde{u}_n(z) := u_n(t,x;\xi + z\eta)$ on $V_{\xi,\eta}$. By direct estimation

$$\begin{aligned} |\tilde{u}_n(z)| = |u_n(t,x;\xi + z\eta)| &\leq ||u_n||(1 + |K(\xi + z\eta)|)\exp(tL(\xi + z\eta)) \\ &\leq (\sup_{n\in\mathbb{N}} ||u_n||)(1 + |K(\xi + z\eta)|)\exp(tL(\xi + z\eta)). \end{aligned}$$

Hence, $\tilde{u}_n$ is bounded uniformly on every compact $K \subset V_{\xi,\eta}$, since so are $L(\xi + z\eta)$ and $K(\xi + z\eta)$. Moreover, since $\tilde{u}_n(z)$ converges pointwise to $\tilde{u}(z) := u(t,x;\xi + z\eta)$, it follows by the theorem of Vitali that $\tilde{u}$ is analytic on $V_{\xi,\eta}$. I.e.,

$$z \mapsto u(t,x;\xi + z\eta) : V_{\xi,\eta} \to \mathbb{C}$$

is analytic, and therefore u satisfies condition (iii), too. ∎

In what follows some explicitly given functions $L(\xi)$, $R(\xi)$ and subsets $\mathcal{V}$ will be used. We fix notation for later purposes:

The case $\beta < 1$: Choose $\mathcal{V} = \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$ and $V_{\xi,\eta} = \mathbb{C}$ in (ii). Condition (iii) now says that the map $z \mapsto u(t,x;\xi + z\eta)$ is entire. In this case, denote $\tilde{\mathcal{U}}$ by $\tilde{\mathcal{U}}_e$, where e indicates entireness. For $p \in \mathbb{N}_0$ and constants $C, \tilde{C} \geq 0$ consider

$$L(\xi) := C(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}}), \quad R(\xi) := \tilde{C}\exp(\tilde{C}|\xi|_{2,p}^{\frac{2}{1-\beta}}). \tag{9}$$

For these particular choices of $L(\xi)$, $R(\xi)$ and $\mathcal{V}$, we shall denote the corresponding space by $(\mathcal{U}_{\beta,p}, \|\cdot\|_{\beta,p})$. Notice that $\beta < \beta'$ implies $\mathcal{U}_{\beta,p} \subset \mathcal{U}_{\beta',p}$.

The case $\beta = 1$: The space $(\mathcal{U}_{1,p}, \|\cdot\|_{1,p})$, for $p \in \mathbb{N}_0$, is defined by setting

$$L(\xi) := C, \quad R(\xi) := \tilde{C} \tag{10}$$

where $C, \tilde{C} > 0$. $\mathcal{V}$ and $V_{\xi,\eta}$ in (ii) (depending on p and $\delta > 0$) are given by

$$\begin{aligned} \mathcal{V}_p(\delta) &:= \left\{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) : |\xi|_{2,p} < \delta\right\}, \\ V_{\xi,\eta} &:= \{z \in \mathbb{C} : |z||\eta|_{2,p} < \delta - |\xi|_{2,p}\}. \end{aligned} \tag{11}$$

Thus $z \mapsto u(t,x;\xi + z\eta)$ is locally analytic. $\tilde{\mathcal{U}}$ will in this case be denoted $\tilde{\mathcal{U}}_l$.

Remark Let $u \in \mathcal{U}_{\beta,p}$ for $\beta \in [0,1)$. Then $u(t,x;\xi)$ satisfies the bound

$$|u(t,x;\xi)| \leq \|u\|_{\beta,p} \exp(TC(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})).$$

Moreover, since $u(\cdot,\cdot;\xi) \in C_b(D_T)$ for every $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, we see by Lemma 2 that there exists $\phi \in C_b(D_T;(\mathcal{S})^{-\beta})$ such that

$$S\phi(t,x)(\xi) = u(t,x;\xi).$$

For $u \in \mathcal{U}_{1,p}$ we correspondingly obtain $\phi \in C_b(D_T;(\mathcal{S})^{-1})$ such that

$$\mathcal{S}\phi(t,x)(\xi) = u(t,x;\xi), \quad \xi \in \mathcal{V}_p(\delta).$$

4 The Uniform Lipschitz Case

In this section we are going to work with the spaces $\tilde{\mathcal{U}}_e$, $\tilde{\mathcal{U}}_l$ and $(\mathcal{U}_{\beta,p}, \|\cdot\|_{\beta,p})$. The treatment of polynomial and non–polynomial noise is separated into two sections. In Section 4.1 the case of polynomial noise is considered, for $\beta \in [0,1)$. The non–polynomial case is treated for $\beta = 1$ in Section 4.2.

4.1 The Case of Polynomial Noise

Define the operator Γ on $\mathcal{U}_{\beta,p}$, for $\beta \in [0,1)$ and $p \in \mathbb{N}$ by

$$\begin{aligned}\Gamma u(t,x;\xi) \;&:=\; \int_{\mathbb{R}^d} u_0(y;\xi)q(t,x;0,y)\,dy \\ &+\; \int_0^t\int_{\mathbb{R}^d} q(t,x;s,y)\{f(u)-g(u)m\}(s,y;\xi)\,dy\,ds \\ &-\; \int_0^t\int_{\mathbb{R}^d} \nabla_y q(t,x;s,y)\cdot n(s,y;\xi)g(u)(s,y;\xi)\,dy\,ds, \qquad (12)\end{aligned}$$

where

$$m(t,x;\xi) := \sum_{i=1}^d \frac{\partial n_i}{\partial x_i}(t,x;\xi).$$

We impose the following technical assumptions on the operator Γ:

(A) The initial function $u_0 : \mathbb{R}^d \times \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k) \to \mathbb{C}$ is such that

$$|u_0(x;\xi)| \le K_0 \exp(K_0|\xi|_{2,p}^{\frac{2}{1-\beta}}),$$

for some $K_0 \ge 0$. Moreover, for every $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, the mapping

$$(y,z) \mapsto u_0(y;\xi + z\eta) : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$$

is Borel measurable, and $z \mapsto u_0(y;\xi+z\eta)$ is entire.

(B) f and g are functions from $\tilde{\mathcal{U}}_e$ into itself. For $u, v \in \tilde{\mathcal{U}}_e$ they satisfy a uniform Lipschitz bound with some constant $K_{f,g} \ge 0$, and $\sigma \in [0,1]$:

$$\begin{aligned}|f(u)(t,x;\xi) - f(v)(t,x;\xi)| \;&\le\; K_{f,g}(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})|u(t,x;\xi)-v(t,x;\xi)|,\\ |g(u)(t,x;\xi) - g(v)(t,x;\xi)| \;&\le\; K_{f,g}(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1-\sigma}{2}}|u(t,x;\xi)-v(t,x;\xi)|.\end{aligned}$$

Also, f and g are of polynomial growth

$$\begin{aligned}|f(u)(t,x;\xi)| \;&\le\; K_{f,g}(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})(1+|u(t,x;\xi)|),\\ |g(u)(t,x;\xi)| \;&\le\; K_{f,g}(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1-\sigma}{2}}(1+|u(t,x;\xi)|).\end{aligned}$$

(C) For $i = 1, \dots d$, n_i and $\frac{\partial n_i}{\partial x_i}$ are elements of $\tilde{\mathcal{U}}_e$. In addition, there exists a positive constant K_n such that for all $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$

$$|n(t,x;\xi)| \le K_n(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{\sigma}{2}}, \quad |m(t,x;\xi)| \le K_n(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1+\sigma}{2}},$$

where σ is the constant in (B).

We remark that a–priori we do not need to have the same p in conditions (A)–(C). However, since the norms $|\cdot|_{2,q}$ are increasing for increasing q, we can always find *one* p for which all the above conditions hold. Define the constant $\tilde{C}$ appearing in (9) to be equal to the constant K_0 in condition (A). To get a more compact notation, introduce the function

$$\tilde{f}(u)(t,x;\xi) := f(u)(t,x;\xi) - g(u)(t,x;\xi)m(t,x;\xi).$$

Note that $\tilde{f}$ fullfills assumption (B), but now with $K_{\tilde{f}} := K_{f,g} + K_{f,g}K_n$.

Proposition 4 *Under conditions (A)–(C), the operator Γ maps $\mathcal{U}_{\beta,p}$ into itself.*

Proof In the first step we verify the properties (i)–(iii) from Section 3:
(i) Let $u \in \mathcal{U}_{\beta,p}$ and fix $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$. Conditions (A)–(C) imply that $u_0(\cdot;\xi)$, $\tilde{f}(u)(\cdot,\cdot;\xi)$ and $g(u)(\cdot,\cdot;\xi)n_i(\cdot,\cdot;\xi)$ for $i = 1,\ldots,d$ are bounded and measurable functions. ¿From Theorem A.3 in [DP] we conclude $\Gamma u(\cdot,\cdot;\xi) \in C_b(D_T)$.
(ii) Fix $\xi,\eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, and let $u \in \mathcal{U}_{\beta,p}$. By conditions (A)–(C), the functions

$$\begin{aligned} (x,z) &\mapsto u_0(x;\xi + z\eta) \\ (t,x,z) &\mapsto \tilde{f}(u)(t,x;\xi + z\eta) \\ (t,x,z) &\mapsto g(u)(t,x;\xi + z\eta)n_i(t,x;\xi + z\eta), \quad i = 1,\ldots,d, \end{aligned}$$

are all Borel measurable. The function $(t,x,z) \mapsto \Gamma u(t,x,\xi + z\eta)$ is therefore Borel measurable, since it is the integral of products of such functions.
(iii) Fix $\xi,\eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^k)$, and consider $u \in \mathcal{U}_{\beta,p}$. We will show by an application of Morera's theorem that $z \mapsto \Gamma u(t,x;\xi + z\eta)$ is entire, for every $(t,x) \in D_T$. Define

$$a(y,z) := q(t,x;0,y)u_0(y;\xi + z\eta) \tag{13}$$

$$\begin{aligned} b(s,y,z) := {} & q(t,x;s,y)\tilde{f}(u)(s,y;\xi + z\eta) \\ & -\nabla_y q(t,x;s,y)\cdot n(s,y;\xi + z\eta)g(u)(s,y;\xi + z\eta). \end{aligned} \tag{14}$$

From (A)–(C) we know that, for every fixed $(s,y) \in D_T$, the functions $a(y,\cdot)$ and $b(s,y,\cdot)$ are entire. Moreover, a and b are Borel measurable. We estimate

$$|a(y,z)| = |q(t,x;0,y)|\cdot|u_0(y;\xi + z\eta)| \le |q(t,x;0,y)|\cdot R(\xi + z\eta). \tag{15}$$

Since $R(\xi + z\eta)$ is bounded on every compact $\mathcal{K} \subset \mathbb{C}$, it follows that a is integrable on $\mathbb{R}^d \times \mathcal{K}$. Let γ be the boundary of a closed rectangle in $\mathbb{C}$. By Fubini's theorem and entireness of $a(y,\cdot)$,

$$\int_{\gamma\times\mathbb{R}^d} a(y,z)\,dy\,dz = \int_{\mathbb{R}^d}\int_\gamma a(y,z)\,dz\,dy = 0.$$

Since this holds for every closed rectangle in $\mathbb{C}$, Morera's theorem implies that

$$z \mapsto \int_{\mathbb{R}^d} a(y,z)\,dy$$

is entire. We argue similarly for b, and estimate in the following way:

$$\begin{aligned}
|b(s,y,z)| &\leq |q(t,x;s,y)| \cdot |\tilde{f}(u)(s,y;\xi+z\eta)| \\
&\quad +|\nabla_y q(t,x;s,y)| \cdot |g(u)(s,y;\xi+z\eta)| \cdot |n(s,y;\xi+z\eta)| \\
&\leq |q(t,x;s,y)| K_{\tilde{f}}(1+|\xi+z\eta|_{2,p}^{\frac{2}{1-\beta}})(1+|u(s,y;\xi+z\eta)|) \\
&\quad +|\nabla_y q(t,x;s,y)| K_{\tilde{f}}(1+|\xi+z\eta|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}(1+|u(s,y;\xi+z\eta)|) \\
&\leq |q(t,x;s,y)| K_{\tilde{f}}(1+|\xi+z\eta|_{2,p}^{\frac{2}{1-\beta}})(1+\|u\|_{\beta,p}) w(t;\xi+z\eta)^{-1} \\
&\quad +|\nabla_y q(t,x;s,y)| K_{\tilde{f}}(1+|\xi+z\eta|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}(1+\|u\|_{\beta,p}) w(t;\xi+z\eta)^{-1}.
\end{aligned}$$

By the assumptions on w, it follows that b is integrable on $D_T \times \mathcal{K}$ for every compact $\mathcal{K} \subset \mathbb{C}$. By Fubini's theorem and the entireness of $b(s,y,\cdot)$, we have

$$\int_{\gamma\times[0,t]\times\mathbb{R}^d} b(s,y,z)\,dz\,ds\,dy = \int_0^t \int_{\mathbb{R}^d}\int_\gamma b(s,y,z)\,dz\,dy\,ds = 0.$$

Again, by Morera's theorem we have that

$$z \mapsto \int_0^t \int_{\mathbb{R}^d} b(s,y,z)\,dy\,ds$$

is entire. This proves (iii).

The second step is to show that Γu is bounded in the $\|\cdot\|_{\beta,p}$-norm, for $u \in \mathcal{U}_{\beta,p}$. We will use the abbreviation

$$K := \max\left(K_{\tilde{f}}, C_q\right), \tag{16}$$

where C_q is defined in (3). Using (B) for f and g we find

$$\begin{aligned}
&|\Gamma u(t,x;\xi)| \\
&\leq Kw(t;\xi)^{-1} + K^2(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})(1+\|u\|_{\beta,p})(1+R(\xi))\int_0^t \exp(sL(\xi))ds \\
&\quad +K^2(1+|\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}(1+\|u\|_{\beta,p})(1+R(\xi))\int_0^t (t-s)^{-\frac{1}{2}}\exp(sL(\xi))ds.
\end{aligned}$$

We estimate $\int_0^t \exp(sL(\xi))ds \leq (L(\xi))^{-1}\exp(tL(\xi))$. In [CLP] we find the following estimate:

$$\int_0^t (t-s)^{-\frac{1}{2}}\exp(sL(\xi))ds \leq \left(\frac{\pi}{L(\xi)}\right)^{\frac{1}{2}}\exp(tL(\xi)).$$

Taking into account $L(\xi) = C(1 + |\xi|^{\frac{2}{1-\beta}})$ we finally obtain

$$|\Gamma u(t,x;\xi)| \leq \left(K + (\frac{K^2}{C} + \frac{\pi^{\frac{1}{2}} K^2}{C^{\frac{1}{2}}})(1 + \|u\|_{\beta,p}) \right) w(t;\xi)^{-1}. \tag{17}$$

This implies $\|\Gamma u\|_{\beta,p} \leq \text{const.}(1 + \|u\|_{\beta,p})$. ■

Proposition 5 *Let K be given by (16). For $C = 4K^2 + 16\pi K^4$ in (9), Γ defined in (12) is a strict contraction and therefore has a unique fixed point $u \in \mathcal{U}_{\beta,p}$.*

Proof Let $u, v \in \mathcal{U}_{\beta,p}$. Using the Lipschitz condition on $\tilde{f}$ and g in (B), the polynomial bound on $n(t,x;\xi)$ in (C) and the definition of $\|\cdot\|_{\beta,p}$, gives

$$|\Gamma u(t,x;\xi) - \Gamma v(t,x;\xi)| \leq \Bigg(K^2(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})(1 + R(\xi)) \int_0^t \exp(sL(\xi))ds$$

$$+K^2(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}(1 + R(\xi)) \int_0^t (t-s)^{-\frac{1}{2}} \exp(sL(\xi))ds \Bigg) \|u - v\|_{\beta,p}.$$

The estimations of the integrals in the proof of Proposition 4 yield

$$\begin{aligned} &|\Gamma u(t,x;\xi) - \Gamma v(t,x;\xi)| \\ \leq\ & \left(\frac{K^2(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})}{L(\xi)} + \frac{\pi^{\frac{1}{2}} K^2 (1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}}{L(\xi)^{\frac{1}{2}}} \right) \|u - v\|_{\beta,p} w(t;\xi)^{-1}. \end{aligned}$$

With the choice $C = 4K^2 + 16\pi K^4$ we obtain

$$|\Gamma u(t,x;\xi) - \Gamma v(t,x;\xi)| \leq \frac{1}{2} \|u - v\|_{\beta,p} w(t;\xi)^{-1},$$

and hence $\|\Gamma u - \Gamma v\|_{\beta,p} \leq \frac{1}{2}\|u - v\|_{\beta,p}$. Banach's fixed point theorem now implies the second claim. ■

By the remark at the end of Section 3, there exists a unique $\phi \in C_b(D_T; (\mathcal{S})^{-\beta})$ such that for the fixed point u we have $\mathcal{S}\phi(t,x)(\xi) = u(t,x;\xi)$. To show that ϕ is a solution of (6), we must prove that the inverse $\mathcal{S}$-transform commutes with the integrals in (7): For the fixed point $u \in \mathcal{U}_{\beta,p}$ and fixed $(t,x) \in D_T$, define

$$\begin{aligned} \phi(s,y;\xi) \ :=\ & q(t,x;s,y)\tilde{f}(u)(s,y;\xi) \\ & +\nabla_y q(t,x;s,y) \cdot n(s,y;\xi) g(u)(s,y;\xi). \end{aligned} \tag{18}$$

Then we have for a suitable constant $\tilde{K}$:

$$\begin{aligned} |\phi(s,y;\xi)| \ \leq\ & |q(t,x;s,y)| K(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})(1 + \|u\|_{\beta,p}) \exp(TL(\xi)) \\ & +|\nabla_y q(t,x;s,y)| K(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})^{\frac{1}{2}}(1 + \|u\|_{\beta,p}) \exp(TL(\xi)) \\ \leq\ & (|q(t,x;s,y)| + |\nabla_y q(t,x;s,y)|) K(1 + \|u\|_{\beta,p}) \exp\big(\tilde{K}(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})\big). \end{aligned}$$

Since $|q(t,x;\cdot,\cdot)|$ and $|\nabla_y q(t,x;\cdot,\cdot)|$ are integrable on D_t, it follows as in the proof of Theorem 4.51 in [HKPS] that

$$\mathcal{S}\left(\int_0^t \int_{\mathbb{R}^d} \mathcal{S}^{-1}\phi(s,y;\cdot)\,dy\,ds\right)(\xi) = \int_0^t \int_{\mathbb{R}^d} \phi(s,y;\xi)\,dy\,ds.$$

Hence, ϕ is a solution of (6). We summarize:

Theorem 6 *Under conditions (A)-(C) and for $\beta \in [0,1)$, there exists a unique solution $\phi \in C_b(D_T;(\mathcal{S})^{-\beta})$ of (6).*

4.2 The Case Of Nonpolynomial Noise

Consider again Γ introduced in (12). In this section Γ will be studied on $\mathcal{U}_{1,p}$ for fixed $p \in \mathbb{N}_0$ and $\delta > 0$. Recall that $V_{\xi,\eta}$ and $\mathcal{V}$ depend on p and δ, and that u_0 is the initial function. We make the following assumptions:

(A1) $u_0 : \mathbb{R}^d \times \mathcal{V}_p(\delta) \to \mathbb{C}$ is bounded and such that for every $\xi, \eta \in \mathcal{V}_p(\delta)$,

$$(y,z) \mapsto u_0(y;\xi + z\eta) : \mathbb{R}^d \times V_{\xi,\eta} \to \mathbb{C}$$

is Borel measurable. Also $z \mapsto u_0(y;\xi + z\eta)$ is analytic on $V_{\xi,\eta}$.

(B1) f and g are functions on $\tilde{\mathcal{U}}_l$ into itself. For $u, v \in \tilde{\mathcal{U}}_l$ they satisfy a uniform Lipschitz bound (h stands for f or g)

$$|h(u)(t,x;\xi) - h(v)(t,x;\xi)| \le K_{f,g}|u(t,x;\xi) - v(t,x;\xi)|,$$

and are of polynomial growth ($K_{f,g}$ is a positive constant)

$$|h(u)(t,x;\xi)| \le K_{f,g}(1 + |u(t,x;\xi)|).$$

(C1) For $i = 1,\dots d$, n_i and $\frac{\partial n_i}{\partial x_i}$ are elements of $\tilde{\mathcal{U}}_l$. In addition, there exists a positive constant K_n such that for all $(t,x,\xi) \in D_T \times \mathcal{V}_p(\delta)$

$$|n(t,x;\xi)| \le K_n, \quad |\sum_{i=1}^d \frac{\partial n_i}{\partial x_i}(t,x;\xi)| \le K_n.$$

Choose the constant $\tilde{C}$ in (10) as $\tilde{C} := \sup\{|u_0(x;\xi)|; (x,\xi) \in \mathbb{R}^d \times \mathcal{V}_p(\delta)\}$, which is finite by (A1). The constant C in (10) is chosen like in Proposition 5.

Proposition 7 *Under conditions (A1)-(C1), the operator Γ defined by (12) is a strict contraction on $\mathcal{U}_{1,p}$ and has a unique fixed point $u \in \mathcal{U}_{1,p}$.*

Proof Similarly as in Section 4.1, but now with conditions (A1)–(C1), we argue as follows: For fixed $\xi, \eta \in \mathcal{V}_p(\delta)$, $\Gamma u(\cdot, \cdot; \xi) \in C_b(D_T)$ and

$$(t, x, z) \mapsto \Gamma u(t, x; \xi + z\eta) : D_T \times V_{\xi,\eta} \to \mathbb{C}$$

is Borel measurable. Let $\mathcal{K}$ be a compact subset of $V_{\xi,\eta}$ and $u \in \mathcal{U}_{1,p}$. Since $L(\xi) = C, R(\xi) = \tilde{C}$, an argument analogous to the one in Section 4.1 shows that the two functions a and b defined in (13) and (14), are integrable on $\mathbb{R}^d \times \mathcal{K}$ and $D_T \times \mathcal{K}$, respectively. Let γ be the boundary to a closed rectangle in $V_{\xi,\eta}$. Fubini's theorem together with analyticity of a and b on $V_{\xi,\eta}$ yield

$$\int_{\gamma \times \mathbb{R}^d} a(y, z)\, dy\, dz = \int_{\mathbb{R}^d} \int_\gamma a(y, z)\, dz\, dy = 0,$$

and

$$\int_{\gamma \times [0,t] \times \mathbb{R}^d} b(s, y, z)\, dz\, ds\, dy = \int_0^t \int_{\mathbb{R}^d} \int_\gamma b(s, y, z)\, dz\, dy\, ds = 0.$$

Hence, by Morera's theorem, the function $z \mapsto \Gamma u(t, x; \xi + z\eta)$ is analytic on $V_{\xi,\eta}$. Γu is bounded in the $\|\cdot\|_{1,p}$–norm, whenever $u \in \mathcal{U}_{1,p}$, and from the choice of C, Γ is strictly contractive in the norm $\|\cdot\|_{1,p}$. By Banach's fixed point theorem the second claim follows. ■

From the remark at the end of Section 3, there exists $\phi \in C_b(D_T; (\mathcal{S})^{-1})$ so that

$$\mathcal{S}\phi(t, x)(\xi) = u(t, x; \xi), \quad \xi \in \mathcal{V}_p(\delta).$$

By estimating the function $\phi(s, y; \xi)$ defined in (18) like in the preceding section, we get from Theorem 6 in [KLS] that the inverse $\mathcal{S}$–transform commutes with integration. Hence ϕ is a solution of (6).

Theorem 8 *Under conditions (A1)–(C1), there exists a unique solution* $\phi \in C_b(D_T; (\mathcal{S})^{-1})$ *of (6).*

4.3 Examples

Assume $G(\phi)(t, x) = \phi(t, x)$ and $F(\phi)(t, x) = \phi(t, x) \diamond \tilde{N}(t, x)$, with $\tilde{N} \in C(D_T; (\mathcal{S})^{-\beta})$. Hence, the equation under consideration is

$$\frac{\partial \phi}{\partial t}(t, x) - A\phi(t, x) = \phi(t, x) \diamond \tilde{N}(t, x) + \nabla\phi(t, x) \diamond N(t, x). \qquad (19)$$

Example 1: Polynomial noise. Let $k = d + 1$, and define N_i and $\tilde{N}$ to be equal to time–space white noise. I.e., informally

$$N_i(t, x, \omega) = \tilde{N}(t, x, \omega) := \langle \omega, \delta_{t,x} \rangle,$$

where $\delta_{t,x}$ is Dirac's δ–function. This space–time white noise will also be denoted as $W_{t,x}$, and is the time–space derivative of the Brownian sheet considered as a generalized random variable in $(\mathcal{S})^*$. Its $\mathcal{S}$–transform is given by

$$\mathcal{S}W_{t,x}(\xi) = \xi(t,x).$$

The hypotheses (B) and (C) are obviously satisfied for $\beta = 0$. Hence, if $\mathcal{S}\phi_0 = u_0$ satisfies (A), we have a solution $\phi \in C_b(D_T;(\mathcal{S})^*)$ of (19).

From a physical point of view, it might be interesting to smear out the noise in the space variable. Given $d+1$ functions $\rho, \psi_i \in C_b^{0,1}(D_T)$, $i = 1,\ldots,d$, define N_i and $\tilde{N}$ to be

$$\begin{aligned} N_i(t,x,\omega) &:= \int_{\mathbb{R}^d} \psi_i(t,x-y)W_{t,y}(\omega)\,dy, \quad i=1,\ldots,d, \\ \tilde{N}(t,x,\omega) &:= \int_{\mathbb{R}^d} \rho(t,x-y)W_{t,y}(\omega)\,dy. \end{aligned}$$

The corresponding $\mathcal{S}$–transforms will be

$$\begin{aligned} \mathcal{S}N_i(t,x,\cdot)(\xi) &= \int_{\mathbb{R}^d} \psi_i(t,x-y)\xi(t,y)\,dy, \quad i=1,\ldots,d, \\ \mathcal{S}\tilde{N}(t,x,\cdot)(\xi) &= \int_{\mathbb{R}^d} \rho(t,x-y)\xi(t,y)\,dy. \end{aligned}$$

Since ρ is bounded, we get for large enough γ

$$\begin{aligned} |\mathcal{S}\tilde{N}(t,x,\cdot)(\xi)| &\le \int_{\mathbb{R}^d} |\rho(t,x-y)||\xi(t,y)|\,dy \le K_\rho \int_{\mathbb{R}^d} |\xi(t,y)|\,dy \\ &\le K_\rho \int_{\mathbb{R}^d} (1+|y|)^{-\gamma}\,dy\;\|\xi\|_{(0,\gamma)} \le C\|\xi\|_{(0,\gamma)}. \end{aligned}$$

We argue in the same way for $\mathcal{S}N_i$. Moreover, since $\rho \in C_b^{0,1}(D_T)$,

$$\frac{\partial}{\partial x_i}\mathcal{S}N(t,x,\cdot)(\xi) = \int_{\mathbb{R}^d} \frac{\partial \rho}{\partial x_i}(t,x-y)\xi(t,y)\,dy, \quad i=1,\ldots,d.$$

We find $|\frac{\partial}{\partial x_i}\mathcal{S}N_i(t,x,\cdot)(\xi)| \le \tilde{C}\|\xi\|_{(0,\gamma)}$. For p large enough, $\|\xi\|_{(0,\gamma)} \le \hat{C}|\xi|_{2,p}$, and thus our hypotheses (B) and (C) are satisfied with $\beta = 0$. We obtain a solution $\phi \in C_b(D_T;(\mathcal{S})^*)$ of (19), whenever $\mathcal{S}\phi_0 = u_0$ satisfies (A).

In [GHØUZ] a noise is considered which is also smeared out in the time variable. I.e.,

$$\begin{aligned} N_i(t,x,\omega) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \psi_i(t-s,x-y)W_{t,y}(\omega)\,dy\,ds, \quad i=1,\ldots,d, \\ \tilde{N}(t,x,\omega) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \rho(t-s,x-y)W_{t,y}(\omega)\,dy\,ds, \end{aligned}$$

for $\rho, \psi_i \in C_b^{0,1}(I\!R \times I\!R^d)$. ρ, ψ_i and their first order space derivatives are elements in $\mathcal{S}_{-p}(I\!R^{d+1})$ for some $p > 0$, so that we can estimate the $\mathcal{S}$–transform with the Cauchy–Schwarz inequality, and obtain

$$|\mathcal{S}\tilde{N}(t,x,\cdot)(\xi)| \leq |\rho|_{2,p}|\xi|_{2,-p}.$$

$\mathcal{S}N_i$ is treated similarly. If $\mathcal{S}\phi_0 = u_0$ satisfies hypothesis (A) for this p and $\beta = 0$, we see that there exists a solution $\phi \in C_b(D_T; (\mathcal{S})^*)$. For $\rho, \psi_i \in \mathcal{S}_p(I\!R^{d+1})$ and p large enough one can even show that

$$\phi(t,x,\cdot) \in (L^2), \text{ for all } (t,x) \in D_T.$$

For example, if we have deterministic initial conditions in $C_b(I\!R^d)$, and $\rho, \psi_i \in \mathcal{S}(I\!R^{d+1})$, such a p obviously exists. We remark that in [GHØUZ] a Feynman–Kac formula for the solution is worked out.

Example 2: Nonpolynomial noise. Examples of noises which cannot be bounded by a polynomial, are easy to produce. For instance, let N_i and $\tilde{N}$ be equal to the Wick exponential of singular white noise $W_{t,x}(\omega)$. I.e., informally

$$N_i(t,x,\omega) = \tilde{N}(t,x,\omega) := :\exp W_{t,x}(\omega):, \quad i = 1,\ldots,d.$$

The Wick exponential of white noise has been considered as a model for *positive* noise (e.g. [HLØUZ1]). Its $\mathcal{S}$-transform (which could be taken as a rigorous definiton in view of Theorem 1) is

$$\mathcal{S}\left(:\exp W_{t,x}(\cdot):\right)(\xi) = \exp(\xi(t,x)),$$

which shows that this noise can not be bounded like in (C). Choose $\delta > 0$, and let $\xi \in \mathcal{V}_p(\delta)$ where p is such that $\sup_{(t,x)\in D_T}|\xi(t,x)| \leq c|\xi|_{2,p}$. Then

$$|\mathcal{S}\left(:\exp W_{t,x}(\cdot):\right)(\xi)| \leq \exp(c\delta), \text{ for all } \xi \in \mathcal{V}_p(\delta).$$

Similarly, by changing p if necessary, we can bound the space derivative of $\mathcal{S}N_i$. It is easily seen that conditions (B1), (C1) hold, so we have a solution $\phi \in C_b(D_T; (\mathcal{S})^{-1})$ for (19), whenever $\mathcal{S}\phi = u_0$ satisfies (A1). The same argument shows that also the normalized exponential of regularized noise (smeared in space or in time–space) gives a solution in $C_b(D_T; (\mathcal{S})^{-1})$.

5 The Local Lipschitz Case

The spaces of U–functionals under consideration in this section will be $\mathcal{U}_{1,p}$ with $L(\xi)$ and $R(\xi)$ both identical to zero. Moreover, we will see that for the non-linear heat equation, only local (in time) solutions exist. Hence, we are going to consider the space $\mathcal{U}_{1,p}$ where the time parameter is in an interval $[0;\tau]$, for a $\tau \leq T$. We denote this space by $\mathcal{U}_{1,p}(\tau)$, and its norm will be

$$||u||_\tau := \sup\{|u(t,x;\xi)|;\ (t,x,\xi) \in D_\tau \times \mathcal{V}_p(\delta)\}.$$

Of course, $(\mathcal{U}_{1,p}(\tau), ||\cdot||_\tau)$ is a Banach space. We shall see that there exists a $\tau = t_0$ for which the problem under investigation has a solution. This t_0 will be explicitly given in Section 5.1 below (see (20)). We remark that the idea to our approach is taken from [Sm].

5.1 Existence And Uniqueness Of Solution

For the operator Γ in (12), we assume (A1) and (C1) of Section 4.2, $p \in \mathbb{N}_0$ fixed and $\delta > 0$. In addition f and g will satisfy a modification of (B1):

(B1′) f and g are functions of $\tilde{\mathcal{U}}_l$ into itself, obeying $f(0) = g(0) = 0$. Consider $u, v \in \tilde{\mathcal{U}}_l$ with $|u(t,x;\xi)|, |v(t,x;\xi)| \leq M$ for a constant $M > 0$. Then there exists $\bar{K}_{f,g}(M)$ such that (h stands for f and g)

$$|h(u)(t,x;\xi) - h(v)(t,x;\xi)| \leq \bar{K}_{f,g}|u(t,x;\xi) - v(t,x;\xi)|.$$

Let $|u_0|_\infty := \sup\{|u_0(x;\xi)| : (x,\xi) \in \mathbb{R}^d \times \mathcal{V}_p(\delta)\}$, and consider the closed ball with radius $C_q|u_0|_\infty$ (t_0 will be specified below):

$$\mathcal{B}_{1,p}(t_0) := \left\{ u \in \mathcal{U}_{1,p}(t_0) :\ ||u - \int_{\mathbb{R}^d} u_0(y;\cdot)q(\cdot,\cdot;0,y)\,dy||_{t_0} \leq C_q|u_0|_\infty \right\}.$$

Obviously $0 \in \mathcal{B}_{1,p}(t_0)$, so $u \in \mathcal{B}_{1,p}(t_0)$ satisfies $||u||_{t_0} \leq 2C_q|u_0|_\infty$. For

$$K_{f,g} := \bar{K}_{f,g}(2C_q|u_0|_\infty)$$

the functions f and g are uniform Lipschitz on $\mathcal{B}_{1,p}(t_0)$ for any t_0, with Lipschitz constant $K_{f,g}$. I.e., for $u, v \in \mathcal{B}_{1,p}(t_0)$, ($h$ stands for f or g)

$$|h(u)(t,x;\xi) - h(v)(t,x;\xi)| \leq K_{f,g}|u(t,x;\xi) - v(t,x;\xi)|.$$

The function $\tilde{f} := f - g \cdot m$ (m is as in Section 4) satisfies (B1′) and is uniform Lipschitz on $\mathcal{B}_{1,p}(t_0)$ with Lipschitz constant

$$K_{\tilde{f}} := K_{f,g} + K_{f,g}K_n.$$

Proposition 9 *Assume (A1), (B1′) and (C1). Then there exists $t_0 > 0$ so that $\Gamma : \mathcal{B}_{1,p}(t_0) \to \mathcal{B}_{1,p}(t_0)$ is a contraction and has a unique fixed point $u \in \mathcal{B}_{1,p}(t_0)$.*

Proof Let $u \in \mathcal{B}_{1,p}(t_0)$ and $t \le t_0$. We estimate, using (A1), (B1′) and (C1),

$$
\begin{aligned}
&|\Gamma u(t,x;\xi) - \int_{\mathbb{R}^d} u_0(y;\xi) q(t,x;0,y)\, dy| \\
\le\ & \int_0^t \int_{\mathbb{R}^d} |q(t,x;s,y)| \tilde{f}(u)(s,y;\xi)|\, dy\, ds \\
&+ \int_0^t \int_{\mathbb{R}^d} |\nabla_y q(t,x;s,y)| \cdot |n(s,y;\xi)| \cdot |g(u)(s,y;\xi)|\, dy\, ds \\
\le\ & K_{\tilde{f}} \|u\|_{t_0} \int_0^t \int_{R^d} |q(t,x;s,y)|\, dy\, ds \\
&+ K_{f,g} K_n \|u\|_{t_0} \int_0^t \int_{\mathbb{R}^d} |\nabla_y q(t,x;s,y)|\, dy\, ds \\
\le\ & \left(C_q K_{\tilde{f}} \cdot t + 2 C_q K_{f,g} K_n \cdot t^{\frac{1}{2}} \right) 2 C_q |u_0|_\infty.
\end{aligned}
$$

Choose t_0 as

$$
t_0 := \left(4 C_q K_{\tilde{f}} + 64 C_q^2 K_{f,g}^2 K_n^2 \right)^{-1}. \tag{20}
$$

From this it is easy to see that for $t \le t_0$

$$
2 C_q K_{\tilde{f}} \cdot t \le \frac{1}{2}, \quad 4 C_q K_{f,g} K_n \cdot t^{\frac{1}{2}} \le \frac{1}{2}. \tag{21}
$$

We therefore obtain that

$$
\|\Gamma u - \int_{\mathbb{R}^d} u_0(y;\cdot) q(\cdot,\cdot;0,y)\, dy\|_{t_0} \le C_q |u_0|_\infty,
$$

thus $\Gamma u \in \mathcal{B}_{1,p}(t_0)$. For $u \in \mathcal{B}_{1,p}(t_0)$, the arguments in Section 4.1 with the modifications made in Section 4.2 show that $\Gamma u(t,x;\xi)$ possesses the correct continuity, measurability and analyticity properties.

If $u, v \in \mathcal{B}_{1,p}(t_0)$, we show that Γ is strictly contractive in the $\|\cdot\|_{t_0}$–norm:

$$
\begin{aligned}
&|\Gamma u(t,x;\xi) - \Gamma v(t,x;\xi)| \\
\le\ & \int_0^t \int_{\mathbb{R}^d} |q(t,x;s,y)| \cdot |\tilde{f}(u)(s,y;\xi) - \tilde{f}(v)(s,y;\xi)|\, dy\, ds \\
&+ \int_0^t \int_{\mathbb{R}^d} |\nabla_y q(t,x;s,y)| \cdot |n(s,y;\xi)| \cdot |g(u)(s,y;\xi) - g(v)(s,y;\xi)|\, dy\, ds \\
\le\ & \left(C_q K_{\tilde{f}} \cdot t + 2 C_q K_{f,g} K_n \cdot t^{\frac{1}{2}} \right) \|u - v\|_{t_0} \le \frac{1}{2} \|u - v\|_{t_0},
\end{aligned}
$$

so $\|\Gamma u - \Gamma v\|_{t_0} \le \frac{1}{2} \|u - v\|_{t_0}$. Since $\mathcal{B}_{1,p}(t_0) \subset \mathcal{U}_{1,p}(t_0)$, we obtain the second claim from Banach's fixed point theorem. ∎

There exists $\phi \in C_b(D_{t_0}; (\mathcal{S})^{-1})$ so that $\mathcal{S}\phi(t,x,\cdot)(\xi) = u(t,x;\xi)$, for $\xi \in \mathcal{V}_p(\delta)$. Moreover, by arguments similar to those in Sections 4.1 and 4.2, ϕ solves (6).

Theorem 10 *Under conditions (A1), (B1') and (C1) there exists $t_0 > 0$ such that problem (6) has a solution $\phi \in C_b(D_{t_0}; (\mathcal{S})^{-1})$.*

Remark The solution ϕ above is unique in the sense that it is the only solution of (6) for which the $\mathcal{S}$–transform is an element of $\mathcal{B}_{1,p}(t_0)$.

5.2 Examples

We shall concentrate on examples produced by the so-called Wick calculus (see [KLS]): Let $h(z)$ be an entire function with Taylor expansion

$$h(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Theorem 12 in [KLS] states that for every $\Psi \in (\mathcal{S})^{-1}$ we have

$$h^{\diamond}(\Psi) := \sum_{n=1}^{\infty} a_n \Psi^{\diamond n} \in (\mathcal{S})^{-1} \quad \text{and} \quad \mathcal{S}\left(h^{\diamond}(\Psi)\right)(\xi) = h\left(\mathcal{S}\Psi(\xi)\right).$$

Example 1: Stochastic Reaction–Diffusion Equations of Wick Type. Let $k = d+1$, $N = 0$, and assume that the noise $\tilde{N}(t,x,\omega)$ satisfies $\tilde{n} := \mathcal{S}\tilde{N} \in \mathcal{U}_{1,p}(T)$, e.g. $\tilde{N} = W_{t,x}$. Then (C1) trivially holds and (B1′) is easily verified for the following two types of reaction–diffusion problems:

$$\begin{aligned}
\frac{\partial \phi}{\partial t}(t,x) - A\phi(t,x) &= h^{\diamond}(\phi(t,x)) + \phi(t,x) \diamond \tilde{N}(t,x) \\
\frac{\partial \phi}{\partial t}(t,x) - A\phi(t,x) &= h^{\diamond}(\phi(t,x)) \diamond \tilde{N}(t,x).
\end{aligned}$$

In view of Theorem 10 we obtain a local solution $\phi \in C_b(D_{t_0}; (\mathcal{S})^{-1})$, whenever $\mathcal{S}\phi_0 = u_0$ satisfies (A1). We remark that the second equation possesses a *global* solution, i.e. $t_0 = T$, if u_0 is just a function of x, and if $\|\tilde{n}\|_T \leq c|\xi|_{2,p}$, for some $p, c > 0$. This can be seen as follows. For $|u(t,x,\xi)|, |v(t,x,\xi)| \leq 2C_q|u_0|_\infty$ and $\xi \in \mathcal{V}_p(\delta)$, i.e. $|\xi|_{2,p} < \delta$, we have

$$\begin{aligned}
|f(u) - f(v)| &= |\tilde{n}| \cdot |h(u) - h(v)| \\
&\leq \|\tilde{n}\|_T \sup\{|h'(z)|;\ |z| \leq 2C_q|u_0|_\infty\}|u - v| \\
&\leq c\delta|h'|_\infty|u - v|,
\end{aligned}$$

where $|h'|_\infty := \sup\{|h'(z)|;\ |z| \leq 2C_q|u_0|_\infty\}$. Hence we choose $K_{f,g} = c\delta|h'|_\infty$. Since $N = 0$ we have $K_n = 0$, thus $K_{\bar{f}} = K_{f,g}$. The definition of t_0 in (20) gives

$$t_0 = \frac{1}{4C_q K_{f,g}} = \frac{1}{4C_q c\delta|h'|_\infty}.$$

We set $\delta = (4C_q cT|h'|_\infty)^{-1}$ and obtain $t_0 = T$, i.e. the solution ϕ is in $C_b(D_T; (\mathcal{S})^{-1})$. Notice that the conditions on $\tilde{N}$ are satisfied for $\tilde{N} = W_{t,x}$ or one of the smeared out versions of $W_{t,x}$ considered in Section 4.3.

Example 2: The Wick Burgers Equation. Let $d = 1$ and $k = 2$. We consider

$$\frac{\partial \phi}{\partial t}(t,x) - A\phi(t,x) = -\phi(t,x) \diamond \frac{\partial \phi}{\partial x}(t,x) + \phi(t,x) \diamond W_{t,x},$$

i.e., we let $N(t,x) = 1$, $F(\phi)(t,x) = -\phi(t,x) \diamond W_{t,x}$ and $G(\phi)(t,x) = -\frac{1}{2}\phi^{\diamond 2}(t,x)$. Of course, we can also let $G(\phi) = h^\diamond(\phi)$ for any entire function h. In any case, we obtain local solutions for this type of equations by Theorem 10. We remark that the following Burgers equation with gradient coupled noise can also be studied in our framework:

$$\frac{\partial \phi}{\partial t}(t,x) - A\phi(t,x) = -\phi(t,x) \diamond \frac{\partial \phi}{\partial x}(t,x) + \frac{\partial \phi}{\partial x}(t,x) \diamond W_{t,x}.$$

Although this equation is not representable by (5), it is not hard to see that with minor modifications of our approach a local solution can be established.

The Burgers equation with non-linearity in Wick form (and additive noise) has also been considered in [HLØUZ2, HLØUZ3].

6 Other Applications

In this last section we shall look at ordinary (anticipating) stochastic differential equations and stochastic Volterra equations within our framework. For earlier work in this direction, we mention [KP], [LØU1+2], [CLP], [ØZ] and [B2]. The equation under consideration reads

$$X_t = X_0 + \int_0^t b(t,s,X_s)\,ds + \int_0^t \sigma(t,s,X_s)\;\delta B_s.$$

X_0, b and σ can all be anticipating. The last integral above is the Skorokhod integral (see e.g. [NZ]), which is a generalization of the Itô integral. It is well-known that the Skorokhod integral coincides with the Hitsuda–Skorokhod integral (cf. [HKPS], [LØU2], [B1]) in the case of Skorokhod integrable processes F_s, i.e.

$$\int_0^t F_s\;\delta B_s = \int_0^t F_s \diamond W_s\,ds.$$

Remark In case of ordinary equations the variable $x \in \mathbb{R}^d$ which appears in the definition of $\tilde{U}$, $\tilde{U}_e$ etc. is absent. We will express a function $f \in \tilde{U}$ which is constant with respect to x simply by $f(t;\xi)$ instead of $f(t,x;\xi)$.

6.1 Anticipating Stochastic Differential Equations

Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) \diamond W_s\, ds \tag{22}$$

The conditions on X_0, b and σ will be imposed below. Take the $\mathcal{S}$–transform of (22): Put $v_t(\xi) := \mathcal{S}X_t(\xi)$, and define the function

$$f(v)(t;\xi) := \mathcal{S}b(t, \mathcal{S}^{-1}v_t)(\xi) + \xi(t)\mathcal{S}\sigma(t, \mathcal{S}^{-1}v_t)(\xi). \tag{23}$$

Then we obtain the fixed point problem

$$v_t(\xi) = v_0(\xi) + \int_0^t f(v)(s;\xi)\, ds. \tag{24}$$

The Uniform Lipschitz Case: Let $\beta \in [0,1)$, $p \in \mathbb{N}_0$, and assume

(a1) The mapping $v_0 : \mathcal{S}_{\mathbb{C}}(\mathbb{R}) \to \mathbb{C}$ is such that for every $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ the function $z \mapsto v_0(\xi + z\eta)$ is entire, and $|v_0(\xi)| \le K_0 \exp(K_0 |\xi|_{2,p}^{\frac{2}{1-\beta}})$.

(b1) f is a function on $\tilde{\mathcal{U}}_e$ into itself. Moreover, for $u, v \in \tilde{\mathcal{U}}_e$:

$$\begin{aligned} |f(u)(t;\xi) - f(v)(t;\xi)| &\le K_f(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})|u(t;\xi) - v(t;\xi)|, \\ |f(u)(t;\xi)| &\le K_f(1 + |\xi|_{2,p}^{\frac{2}{1-\beta}})(1 + |u(t;\xi)|). \end{aligned}$$

Proposition 11 *Under conditions (a1) and (b1) there exist unique solutions $v \in \mathcal{U}_{\beta,p}$ to problem (24), and $X_t \in C_b([0,T]; (\mathcal{S})^{-\beta})$ to problem (22).*

Proof Define the operator Γ on $\mathcal{U}_{\beta,p}$ by

$$\Gamma u(t;\xi) = v_0(\xi) + \int_0^t f(u)(s;\xi)\, ds. \tag{25}$$

Informally speaking, we obtain this operator Γ from the one in Section 4.1 if we set $N = 0$, $p(t,x,s,y) = 1$, and drop the $\mathbb{R}^d$–integration. Also conditions (A) and (B) lead to (a1) and (b1) if there is no dependence on $x \in \mathbb{R}^d$. Simple modifications in the proof of Proposition 4 thus show that Γ is a contraction on $\mathcal{U}_{\beta,p}$. The proposition follows by Banach's fixed point theorem and inverse $\mathcal{S}$–transformation (Theorem 1). ■

The Local Lipschitz Case: Fix $p \in \mathbb{N}_0$ and $\delta > 0$. Consider (24) with the function f given as in (23) on $\mathcal{U}_{1,p}$. We impose the following conditions:

(a2) The function $v_0 : \mathcal{V}_p(\delta) \to \mathbb{C}$ is bounded, and for every $\xi, \eta \in \mathcal{V}_p(\delta)$, $z \mapsto v_0(\xi + z\eta)$ is analytic on $V_{\xi,\eta}$.

(b2) $f : \tilde{\mathcal{U}}_l \to \tilde{\mathcal{U}}_l$ satisfies $f(0) = 0$. If $u, v \in \tilde{\mathcal{U}}_l$ obey $|u(t;\xi)|, |v(t;\xi)| \leq M$ for a constant M, there exists $\bar{K}_f(M)$ such that

$$|f(u)(t;\xi) - f(v)(t;\xi)| \leq \bar{K}_f |u(t;\xi) - v(t;\xi)|.$$

We follow the arguments in Section 5: Let $|v_0|_\infty := \sup\{|v_0(\xi)| : \xi \in \mathcal{V}_p(\delta)\}$ and consider the closed ball (for $0 < t_0 \leq T$ to be defined below)

$$\mathcal{B}_{1,p}(t_0) := \{u \in \mathcal{U}_{1,p}(t_0);\ \|u - v_0\|_{t_0} \leq |v_0|_\infty\}.$$

We see that $u \in \mathcal{B}_{1,p}(t_0)$ implies $\|u\|_{t_0} \leq 2|v_0|_\infty$. With $K_f := \bar{K}_f(2|v_0|_\infty)$ the function f is uniform Lipschitz on $\mathcal{B}_{1,p}(t_0)$ with Lipschitz constant K_f.

Proposition 12 *Under conditions (a2), (b2) there exists $t_0 > 0$ such that (24) has a unique solution $u \in \mathcal{B}_{1,p}(t_0)$, and equation (22) has a solution X_t in $C_b([0,t_0];(\mathcal{S})^{-1})$.*

Proof Let Γ on $\mathcal{U}_{1,p}(t_0)$ be defined by (25). We calculate for $u \in \mathcal{B}_{1,p}(t_0)$:

$$|\Gamma u(t;\xi) - v_0(\xi)| \leq \int_0^t |f(u)(s;\xi)|\, ds \leq K_f \int_0^t |u(s;\xi)|\, ds \leq 2K_f t_0 |v_0|_\infty.$$

From the arguments in Section 5 and with the choice $t_0 := (2K_f)^{-1}$, we find that Γ is a strict contraction on $\mathcal{B}_{1,p}(t_0)$. By Banach's fixed point theorem and inverse $\mathcal{S}$–transformation the proposition follows. ■

Remark The solution X_t is unique in the sense that it is the only solution of (22) for which $\mathcal{S}X_t(\xi) \in \mathcal{B}_{1,p}(t_0)$.

Example 1: We start with an application of the uniform Lipschitz case: Define the functions b and σ in (22) to be

$$b(s, X_s) = B_{T-s}^{\diamond\alpha} \diamond X_s, \quad \sigma(s, X_s) = B_{T-s}^{\diamond\alpha-1} \diamond X_s,$$

where $\alpha \in \mathbb{N}$. It is easy to see that the function f in (23) in this case is

$$f(v)(s;\xi) = \left((\int_0^{T-s} \xi(\tau)\, d\tau)^\alpha + \xi(s)(\int_0^{T-s} \xi(\tau)\, d\tau)^{\alpha-1} \right) v(s;\xi).$$

Choose $p \in \mathbb{N}$ such that $|\xi|_\infty \leq K_p |\xi|_{2,p}$. A straightforward estimation yields

$$\begin{aligned} |(\int_0^{T-s} \xi(\tau)\, d\tau)^\alpha + \xi(s)(\int_0^{T-s} \xi(\tau)\, d\tau)^{\alpha-1}| &\leq (T^\alpha K_p^\alpha + T^{\alpha-1} K_p^\alpha)\, |\xi|_{2,p}^\alpha \\ &\leq K_f(1 + |\xi|_{2,p}^\alpha), \end{aligned}$$

for a constant K_f. We see that f is uniform Lipschitz and of polynomial growth in the sense of condition (b1). Let β be such that $\alpha = \frac{2}{1-\beta}$, i. e. $\beta = \frac{\alpha-2}{\alpha}$. If $\alpha \in \{0,1\}$ then $\beta \leq 0$. In such a case (a1) and (a2) also hold for $\beta = 0$. Hence, (22) has a unique solution $X_t \in C_b([0,T];(\mathcal{S})^*)$. If $\alpha > 2$, then $\beta \in (0,1)$, and the unique solution X_t will be an element of $C_b([0,T];(\mathcal{S})^{-\beta})$.

Example 2: In [LØU2] and [B2], the following nonlinear stochastic equation has been considered as a model for population growth in a random medium:

$$X_t = X_0 + \int_0^t X_s \diamond (1 - X_s)\, ds + \int_0^t X_s \diamond (1 - X_s)\ \delta B_s.$$

The function f will be $f(v)(s;\xi) = (1+\xi(s))v(s;\xi)(1 - v(s;\xi))$ which is a local Lipschitz function. If $|u(s;\xi)|, |v(s;\xi)| \leq 2|\mathcal{S}X_0|_\infty$, we see that

$$|f(u)(s;\xi) - f(v)(s;\xi)| \leq (1 + K_p\delta)(1 + 4|\mathcal{S}X_0|_\infty)|u(s;\xi) - v(s;\xi)|.$$

Hence, from Proposition 12 we obtain a solution $X_t \in C_b([0,t_0];(\mathcal{S})^{-1})$, where $t_0^{-1} = 2(1+K_p\delta)(1+4|\mathcal{S}X_0|_\infty)$. We remark that in [LØU2] and [B2] an explicit solution is found for the above equation.

Example 3: Let h_1, h_2 be any two entire functions for which $h_1(0) = h_2(0) = 0$. From Theorem 12 in [KLS] and our results there exists $t_0 > 0$ such that

$$X_t = X_0 + \int_0^t h_1^\diamond(X_s)\, ds + \int_0^t h_2^\diamond(X_s)\ \delta B_s \tag{26}$$

has a solution $X_t \in C_b([0,t_0];(\mathcal{S})^{-1})$ for any X_0 where $\mathcal{S}X_0$ satisfies condition (a2). With $|h_i'|_\infty := \sup\{|h_i'(z)|;\ |z| \leq |\mathcal{S}X_0|_\infty\}$ the time t_0 is given by

$$t_0 = (2|h_1'|_\infty + 2K_p\delta|h_2'|_\infty)^{-1}.$$

¿From this expression we see that for the sub–class of equations (26) satisfying $h_1 = 0$ we can obtain global solutions: Choose (in advance) $\delta = (2K_pT|h_2'|\infty)^{-1}$. Then $t_0 = T$, i.e. there exists $X_t \in C_b([0,T];(\mathcal{S})^{-1})$ solving

$$X_t = X_0 + \int_0^t h_2^\diamond(X_s)\ \delta B_s.$$

6.2 Nonlinear Stochastic Volterra Equations

We shall concentrate our discussion on Volterra equations of the form

$$X_t = Y_t + \int_0^t \sigma(t,s,X_s)\ \delta B_s. \tag{27}$$

Informal $\mathcal{S}$–transformation yields the equation

$$v(t;\xi) = y(t;\xi) + \int_0^t f(v)(t,s;\xi)\xi(s)\,ds, \tag{28}$$

where we abbreviated $v(t;\xi) := \mathcal{S}X_t(\xi)$, $f(v)(t,s;\xi) := \mathcal{S}\sigma(t,s,\mathcal{S}^{-1}v(s,\cdot))(\xi)$ and $y(t;\xi) := \mathcal{S}Y_t(\xi)$. Choose $p \in \mathbb{N}$ such that $|\xi|_\infty \le K_p|\xi|_{2,p}$, and denote $\Delta_T := \{(t,s) : \ 0 \le t \le T, \ 0 \le s \le t\}$.

The uniform Lipschitz case: For $0 \le \alpha < 1$ and $\lambda \ge 0$, we assume:

(a3) $y \in \tilde{U}_e$, and for a positive constant K_y, $t \in [0,T]$ and $\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$:

$$|y(t;\xi)| \le K_y \exp(tK_y|\xi|_{2,p}^{\frac{\lambda+1}{1-\alpha}}).$$

(b3) $f : \tilde{\mathcal{U}}_e \to \tilde{\mathcal{U}}_e$ is such that $f(u) : \Delta_T \times \mathcal{S}_{\mathbb{C}}(\mathbb{R}) \to \mathbb{C}$. For $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ the mapping $(t,s,z) \mapsto f(u)(t,s;\xi + z\eta)$ is Borel measurable, and $z \mapsto f(u)(t,s;\xi + z\eta)$ is entire. Also, $s \mapsto f(u)(t,s;\xi)$ is integrable on $[0,t]$ for $0 \le t \le T$. There exists a constant K_f such that for $u, v \in \tilde{\mathcal{U}}_e$ and $s < t$,

$$\begin{aligned} |f(u)(t,s;\xi) - f(v)(t,s;\xi)| &\le \frac{K_f|\xi|_{2,p}^{\lambda}}{(t-s)^{\alpha}}|u(s;\xi) - v(s;\xi)|, \\ |f(u)(t,s;\xi)| &\le \frac{K_f|\xi|_{2,p}^{\lambda}}{(t-s)^{\alpha}}(1 + |u(s;\xi)|). \end{aligned}$$

With the definition

$$\beta := \frac{\lambda + 2\alpha - 1}{\lambda + 1}. \tag{29}$$

the exponential growth bound on $y(t;\xi)$ in (a3) says

$$|y(t;\xi)| \le K_y \exp(tK_y|\xi|_{2,p}^{\frac{2}{1-\beta}}).$$

Observe that $\beta \in [-1,1)$ when $\alpha \in [0,1)$ and $\lambda \ge 0$. We are going to work with $(\mathcal{U}_{\beta,p}, \|\cdot\|_{\beta,p})$, with the modification of $L(\xi) := C|\xi|_{2,p}^{\frac{2}{1-\beta}}$ in (9), and choose $\tilde{C}$ in (9) to be equal to K_y in (a3). Notice that $u \in \mathcal{U}_{\beta,p}$ for $\beta \le 0$ implies $\mathcal{S}^{-1}u \in C_b([0,T];(\mathcal{S})^*)$. If the constant C in $L(\xi)$ above is chosen to be

$$C := (2K_pK_f\Gamma(1-\alpha))^{\frac{1}{1-\alpha}},$$

where $\Gamma(x)$ is the Γ–function, we have the following result:

Proposition 13 *Under conditions (a3) and (b3) there exists a unique solution $u \in \mathcal{U}_{\beta,p}$ of (28), with β given as in (29). Also (27) has a unique solution $X_t \in C_b([0,T];(\mathcal{S})^*)$ for $\beta \le 0$ and $X_t \in C_b([0,T];(\mathcal{S})^{-\beta})$, for $0 < \beta < 1$.*

Proof Define the mapping Γ on $\mathcal{U}_{\beta,p}$ by

$$\Gamma u(t;\xi) = y(t;\xi) + \int_0^t f(u)(t,s;\xi)\xi(s)\,ds. \tag{30}$$

By the integrability condition in (b3), Γ is a well–defined operator on $\mathcal{U}_{\beta,p}$. The proof of the proposition follows the arguments from Section 4.1, with the obvious modifications due to the different form of $L(\xi)$. We here just show that Γ is strictly contractive in the $\|\cdot\|_{\beta,p}$–norm with the above choice of C:

$$\begin{aligned}
|\Gamma u(t;\xi) \quad - \quad \Gamma v(t;\xi)| &\le K_p|\xi|_{2,p}\int_0^t |f(u)(t,s;\xi) - f(v)(t,s;\xi)|\,ds \\
&\le K_pK_f|\xi|_{2,p}^{\lambda+1}\int_0^t (t-s)^{-\alpha}|u(s;\xi)-v(s;\xi)|\,ds \\
&\le K_pK_f|\xi|_{2,p}^{\lambda+1}(1+R(\xi))\int_0^t (t-s)^{-\alpha}\exp(sC|\xi|_{2,p}^{\frac{2}{1-\beta}})\,ds\|u-v\|_{\beta,p} \\
&\le \frac{K_pK_f\Gamma(1-\alpha)|\xi|_{2,p}^{\lambda+1}}{(C|\xi|_{2,p}^{\frac{2}{1-\beta}})^{1-\alpha}}\|u-v\|_{\beta,p}(1+R(\xi))\exp(tC|\xi|_{2,p}^{\frac{2}{1-\beta}}).
\end{aligned}$$

In the last inequality we used Lemma 2.2 in [CLP]. By our choice of β, we see that $\frac{2}{1-\beta}(1-\alpha) = \lambda + 1$. Moreover, $C^{1-\alpha} = 2K_pK_f\Gamma(1-\alpha)$. We finally obtain $\|\Gamma u - \Gamma v\|_{\beta,p} \le \frac{1}{2}\|u-v\|_{\beta,p}$. The last claim follows by Banach's fixed point theorem and inverse $\mathcal{S}$–transformation. ■

The local Lipschitz case: Let p be as above, and fix $\delta > 0$. Consider (28) with the following assumptions:

(a4) $y \in \tilde{U}_l$, and for a positive constant K_y,

$$|y(t;\xi)| \le K_y, \quad \text{for all } (t,\xi) \in [0,T] \times \mathcal{V}_p(\delta).$$

(b4) $f : \tilde{\mathcal{U}}_e \to \tilde{\mathcal{U}}_e$ is such that $f(u) : \Delta_T \times \mathcal{S}_{\mathbb{C}}(\mathbb{R}) \to \mathbb{C}$ and $f(0) = 0$. For $\xi, \eta \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$, $(t,s,z) \mapsto f(u)(t,s;\xi + z\eta)$ is Borel measurable and $z \mapsto f(u)(t,s;\xi+z\eta)$ is entire. Also, $s \to f(u)(t,s;\xi)$ is integrable on $[0,t]$ for $0 \le t \le T$. For $0 \le \alpha < 1$ and $u, v \in \tilde{U}_l$ obeying $|u(t;\xi)|, |v(t;\xi)| \le M$ for a constant M, there exists $\bar{K}_f(M)$ such that

$$|f(u)(t,s;\xi) - f(v)(t,s;\xi)| \le \frac{\bar{K}_f(M)}{(t-s)^{\alpha}}|u(s;\xi) - v(s;\xi)|.$$

We follow the arguments in Section 5. Let $|y|_\infty := \sup\{|y(t;\xi)|;\,(t,\xi) \in [0,T] \times \mathcal{V}_p(\delta)\}$, and define for $0 < t_0 \le T$, the closed ball in $\mathcal{U}_{1,p}(t_0)$:

$$\mathcal{B}_{1,p}(t_0) := \{u \in \mathcal{U}_{1,p}(t_0);\ \|u-y\|_{t_0} \le |y|_\infty\}.$$

Obviously $u \in \mathcal{U}_{1,p}(t_0)$ implies $\|u\|_{t_0} \leq 2|y|_\infty$. Let $K_f := \bar{K}_f(2|y|_\infty)$. Then f is uniform Lipschitz on $\mathcal{B}_{1,p}(t_0)$ in the sense that for $u, v \in \mathcal{B}_{1,p}(t_0)$ one has

$$|f(u)(t,s;\xi) - f(v)(t,s;\xi)| \leq \frac{K_f}{(t-s)^\alpha}|u(s;\xi) - v(s;\xi)|.$$

Proposition 14 *Under conditions (a4), (b4) there exists $t_0 > 0$ such that (28) has a unique solution $u \in \mathcal{B}_{1,p}(t_0)$ and (27) has a solution $X_t \in C_b([0,T];(\mathcal{S})^{-1})$.*

Proof Introduce the operator Γ on $\mathcal{U}_{1,p}(t_0)$ defined by (30). Then

$$\begin{aligned} |\Gamma u(t;\xi) - y(t;\xi)| &\leq K_f K_p \delta \int_0^t (t-s)^{-\alpha}|u(s;\xi)|\, ds \\ &\leq 2K_f K_p \delta |y|_\infty \int_0^t (t-s)^{-\alpha}\, ds \leq \frac{2K_f K_p \delta}{1-\alpha} t^{1-\alpha}|y|_\infty. \end{aligned}$$

With the choice

$$t_0 = \left(\frac{1-\alpha}{2K_f K_p \delta}\right)^{\frac{1}{1-\alpha}}$$

Γ is a strict contraction on $\mathcal{B}_{1,p}(t_0)$. When δ decreases then also K_f decreases. Thus, choosing δ small enough we obtain $t_0 = T$. Theorem 1 gives the second statement. ■

Example 1: Let $\mathcal{S}Y_t$ satisfy (a3) and choose σ in (27) to be

$$\sigma(t,s,X_s) := \frac{B_s}{(t-s)^\alpha} \diamond X_s,$$

for $0 \leq \alpha < 1$. Applying the $\mathcal{S}$–transform immediately gives

$$f(v)(t,s;\xi) = \frac{\int_0^s \xi(s')\, ds'}{(t-s)^\alpha} v(s;\xi).$$

f obviously satisfies (b3) with $\lambda = 1$. Consequently there exists a unique solution $X_t \in C_b([0,T];(\mathcal{S})^{-\alpha})$ of (27).

Example 2: If the function σ is given by

$$\sigma(t,s,X_s) := (t-s)^{-\alpha} X_s$$

we find that $f(v)(t,s;\xi) = (t-s)^{-\alpha} v(s;\xi)$ satisfies (b3) with $\lambda = 0$. In view of (29) we will have a unique solution $X_t \in C_b([0,T];(\mathcal{S})^*)$ when $0 \leq \alpha \leq \frac{1}{2}$, and we will have $X_t \in C_b([0,T];(\mathcal{S})^{-(2\alpha-1)})$ when $\frac{1}{2} < \alpha < 1$.

Example 3: Let $\mathcal{S}Y_t$ satisfy (a4) and h be an entire function with $h(0) = 0$. Put

$$\sigma(t,s,X_s) := \frac{h^\diamond(X_s)}{(t-s)^\alpha},$$

for $0 \leq \alpha < 1$. Then

$$f(u)(t,s;\xi) = \frac{h(u(s;\xi))}{(t-s)^\alpha}.$$

One verifies that f obeys (b4) with $\bar{K}_f(M) := \sup\{|h'(z)| : |z| \leq M\}$. We thus obtain a solution $X_t \in C_b([0,T],(S)^{-1})$ of (26).

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