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GAUGE SYMMETRIES OF AN EXTENDED PHASE SPACE FOR YANG-MILLS AND DIRAC FIELDS

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Abstract

We identify an extended phase space **P** for minimally interacting Yang-Mills and Dirac fields in the Minkowski space. It is a Sobolev space of Cauchy data for which we prove the finite time existence and uniqueness of the evolution equations.

We prove that the Lie algebra $gs(\mathbf{P})$ of all infinitesimal gauge symmetries of **P** is a Hilbert-Lie algebra, carrying a Beppo Levi topology. The connected group $GS(\mathbf{P})$ of the gauge symmetries generated by $gs(\mathbf{P})$ is proved to be a Hilbert-Lie group acting properly in **P**.

The Lie algebra $gs(\mathbf{P})$ has a maximal ideal $gs(\mathbf{P})_0$. We prove that the action in **P** of the connected group $GS(\mathbf{P})_0$ generated by $gs(\mathbf{P})_0$ is proper and free. The constraint set is shown to be the zero level of the equivariant momentum map corresponding to the action of $GS(\mathbf{P})_0$ in **P**.

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1. Introduction.

Gauge invariance is the most fundamental feature of Yang-Mills theories. The gauge transformations preserving the extended phase space, called here gauge symmetries, give rise to the conservation laws and the constraints. Other gauge transformations intertwine equivalent Hamiltonian descriptions of the theory.

An extended phase space is a space of Cauchy data which admit the existence and the uniqueness of (finite time) solutions of the evolution equations of the theory. For such a space of Cauchy data the space of solutions of the field equations is completely determined by the constraint equation.

The aim of this paper is to study the gauge symmetry group for minimally interacting Yang-Mills and Dirac fields. We study the Yang-Mills-Dirac system over the Minkowski space $M^4 = \mathbb{R} \times \mathbb{R}^3$ with a compact structure group G, which is embedded in the space M_k^k of $k \times k$ matrices. Let $\{T^a\}$ be a basis of the structure algebra $\mathfrak{g}$, and $[T^a, T^b] = f_c^{ab} T^c$ the Lie bracket. The usual $(3+1)$ splitting of space-time yields a splitting of the Yang-Mills field $A_\mu = (\Phi, A)$ into the scalar potential Φ and the vector potential $A = A_i dx^i$. It leads to a representation of the field strength $F_{\mu\nu}$ in terms of the "electric" field E and the "magnetic" field

$$B = \operatorname{curl} A + [A \times, A] . \tag{1.1}$$

We use the Euclidean metric in $\mathbb{R}^3$ to identify vector fields and forms, and $\times$ to denote the cross product. The field equations split into the evolution equations

$$\partial_t A = E + \operatorname{grad} \Phi - [\Phi, A] , \tag{1.2}$$

$$\partial_t E = -\operatorname{curl} B - [A \times, B] - [\Phi, E] + J , \tag{1.3}$$

$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^j A_j + \gamma^0 \Phi)\Psi , \tag{1.4}$$

and the constraint equation

$$\operatorname{div} E + [A; E] = J^0 . \tag{1.5}$$

Here A, E, and B are treated as time dependent $\mathfrak{g}$-valued vector fields on $\mathbb{R}^3$, and Ψ is a time dependent spinor field with values in the space V_G of the fundamental representation of G. Moreover $[A; E]$ means the Lie bracket contracted over the vector indices, and

$$J^0 = \Psi^\dagger (I \otimes T^a) \Psi T_a \quad , \quad J^k = \Psi^\dagger (\gamma^0 \gamma^k \otimes T^a) \Psi T_a . \tag{1.6}$$

The scalar potential Φ does not appear as an independent degree of freedom in (1.2) through (1.4). It can be fixed for all times by the choice of an appropriate gauge transformation. The most common gauge fixing for studies of Yang-Mills fields as a dynamical system is the temporal gauge $A_0 = 0$, cf. [1] and [2]. This leads to difficulties with the linearized equations, discussed in detail by Eardley and Moncrief, who had to modify the dynamical system off the constraint set, [3]. In order to be able deal with the evolution equations also off the constraint set we fix the gauge differently by demanding

$$\Delta \Phi = -\operatorname{div} E \quad \text{and} \quad \int_{\mathbb{R}^3} \rho^{-4} \Phi \, d_3x = 0 \quad \text{where} \quad \rho = \sqrt{1 + |x|^2} . \tag{1.7}$$

We choose the extended phase space

$$\mathbf{P} = \{(A, E, \Psi) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g}) \times H^2(\mathbb{R}^3, V_G)\} , \tag{1.8}$$

where $H^k(\mathbb{R}^3, \mathfrak{g})$ and $H^k(\mathbb{R}^3, V)$ are Sobolev spaces of the $\mathfrak{g}$-valued forms and V_G-valued spinors, respectively, which are square integrable over $\mathbb{R}^3$ together with their partial derivatives up to order k, [6]. For the Cauchy data in $\mathbf{P}$ we prove the finite time existence and the uniqueness of classical solutions of the evolution equations. The mathematically important problem of the infinite time existence of solutions, cf. [3-5], is beyond the scope of this paper. We prove also the following regularity result. If the initial data for the Yang-Mills-Dirac system are in

$$\mathbf{P}^k = \{(A, E, \Psi) | A \in H^{k+1}(\mathbb{R}^3, \mathfrak{g}),\ E \in H^k(\mathbb{R}^3, \mathfrak{g}),\ \Psi \in H^{k+1}(\mathbb{R}^3, V_G)\} \tag{1.9}$$

then the solution curve is in $\mathbf{P}^k$.

Having established that $\mathbf{P}$ is an admissible extended phase space, we can turn to the main objective of this paper, that is to the study of the group of time independent gauge transformations which preserve $\mathbf{P}$. We denote by $gs(\mathbf{P})$ the Lie algebra of all time independent infinitesimal gauge transformations which preserve $\mathbf{P}$. We prove that it is a Banach-Lie algebra which carries a Beppo Levi topology [7] with the norm

$$||\xi||_{\mathcal{B}^3} := \int_{D_1} |\xi| d_3 x + ||\mathrm{grad}\, \xi||_{H^2} , \tag{1.10}$$

where D_1 denotes the unite ball in $\mathbb{R}^3$ centered at 0. We show that this topology is induced by a scalar product, so that $gs(\mathbf{P})$ is a Hilbert-Lie algebra. This algebra admits a splitting

$$gs(\mathbf{P}) = gs(\mathbf{P})_0 \oplus \mathfrak{g} . \tag{1.11}$$

where the subalgebra $gs(\mathbf{P})_0$ is the completion of the space of smooth compactly supported maps $\xi : \mathbb{R}^3 \to \mathfrak{g}$ in the topology given by (1.10), cf. [8].

We construct a connected Hilbert-Lie group $GS(\mathbf{P})$ of gauge symmetries for the Yang-Mills and Dirac fields in the phase space $\mathbf{P}$ with Lie algebra $gs(\mathbf{P})$. It carries the uniform topology induced by topology of $gs(\mathbf{P})$. We prove that the action of $GS(\mathbf{P})$ in $\mathbf{P}$ is continuous and proper.

The extended phase space $\mathbf{P}$ is weakly symplectic. The action of $GS(\mathbf{P})$ on $\mathbf{P}$ is Hamiltonian with an equivariant momentum map $\mathcal{J}$. The vanishing of the restriction of $\mathcal{J}$ to the subalgebra $gs(\mathbf{P})_0$ gives rise to the constraints of the theory. More precisely, if $\mathbf{C}$ is the constraint set of the theory, i.e. the set of all $(A, E, \Psi) \in \mathbf{P}$ satisfying Eq. (1.5), then subalgebra $gs(\mathbf{P})_0$ can be given a geometric interpretation as

$$gs(\mathbf{P})_0 = \{\xi \in gs(\mathbf{P}) \,|\, \langle \mathcal{J}(\mathbf{A}, \mathbf{E}, \Psi) | \xi \rangle = 0 \quad \forall (\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{C}\} . \tag{1.12}$$

We construct a connected Banach-Lie subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$ with Lie algebra $gs(\mathbf{P})_0$ and prove that it acts freely and properly in $\mathbf{P}$. Conversely, the constraint set is shown to be the zero level of the momentum map $\mathcal{J}_0$ for the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$,

$$\mathbf{C} = \mathcal{J}_0^{-1}(0) . \tag{1.13}$$

It follows from Eq. (1.13) that the natural choice for the reduced phase space is the space $\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0$ of $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$. If $\mathbf{C}$ were a submanifold of $\mathbf{P}$, the reduced phase space $\check{\mathbf{P}}$ would be a symplectic (Hausdorff) manifold with an exact symplectic form. The structure of the constraint set and of the reduced phase space will be studied elsewhere.

This paper is organized as follows. In Section 2 we prove the (finite time) existence and uniqueness of solutions of the evolution equations in $\mathbf{P}$ under our gauge condition. Section 3 is devoted to the study of the gauge symmetry group. In Appendix A we consider some decomposition results and estimates for Beppo Levi spaces.

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2. Existence and uniqueness results.

For any vector space V, let $H^k(\mathbb{R}^3, V)$ be the Sobolev space of V-valued vector fields on $\mathbb{R}^3$. Each each $X \in H^k(\mathbb{R}^3, V)$ allows for a Helmholtz decomposition

$$X = X^L + X^T \quad \text{such that} \quad \operatorname{curl} X^L = 0 \quad \text{and} \quad \operatorname{div} X^T = 0 . \tag{2.1}$$

The components $X^L \in H^k(\mathbb{R}^3, V)$ and $X^T \in H^k(\mathbb{R}^3, V)$ are uniquely determined by $\operatorname{div} X$ and $\operatorname{curl} X$, and called the longitudinal and transverse components of X, respectively. For details see the Appendix. Splitting the gauge fields in this way we obtain

$$\begin{aligned} \partial_t A^L &= E^L + \operatorname{grad} \Phi - [\Phi, A]^L , \\ \partial_t E^L &= -[A\times, B]^L - [\Phi, E]^L + J^L , \\ \partial_t A^T &= E^T - [\Phi, A]^T , \\ \partial_t E^T &= -\operatorname{curl} B - [A\times, B]^T - [\Phi, E]^T + J^T . \end{aligned} \tag{2.2}$$

In order to prove that the gauge condition (1.7) can be satisfied for each field E^L we need the Beppo Levi spaces $BL_m(L^2(\mathbb{R}^3, \mathfrak{g}))$, which are defined as the spaces of $\mathfrak{g}$-valued distributions on $\mathbb{R}^3$ with square integrable partial derivatives of order m, cf. [7]. For the intersection of k Beppo Levi spaces we write

$$\mathcal{B}^k(\mathbb{R}^3, \mathfrak{g}) := \bigcap_{m=1}^{k} BL_m(L^2(\mathbb{R}^3, \mathfrak{g})) . \tag{2.3}$$

These spaces are topologized by the norm

$$\|\xi\|_{\mathcal{B}^k} = \int_{D_1} |\xi| d_3x + \|\operatorname{grad} \xi\|_{H^{k-1}} . \tag{2.4}$$

Proposition 2.1
For each $E \in H^1(\mathbb{R}^3, \mathfrak{g})$ there exists a unique scalar potential $\Phi \in \mathcal{B}^2(\mathbb{R}^3, \mathfrak{g})$ obeying the gauge condition

$$\operatorname{grad}\Phi = -E^L \quad \text{and} \quad \int_{\mathbb{R}^3} \rho^{-4}\,\Phi\, d_3x = 0 \quad \text{where} \quad \rho = \sqrt{1+|x|^2}\ . \tag{2.5}$$

If $E \in H^k(\mathbb{R}^3, \mathfrak{g})$, one has $\Phi \in \mathcal{B}^{k+1}(\mathbb{R}^3, \mathfrak{g})$ and

$$\|\Phi\|^2_{\mathcal{B}^{k+1}} \le C\|E\|^2_{H^k}\ . \tag{2.6}$$

Proof.
Let $H^2_{-2}(\mathbb{R}^3, \mathfrak{g})$ and $H^1_{-1}(\mathbb{R}^3, \mathfrak{g})$ denote the weighted Sobolev spaces with respect to the weight function ρ, cf. (A.6). It is shown in [10], that the Laplace operator $\Delta : H^2_{-2}(\mathbb{R}^3, \mathfrak{g}) \to L^2(\mathbb{R}^3, \mathfrak{g})$ is Fredholm, onto and has kernel $\ker(\Delta) = \mathfrak{g}$. Therefore, for each $\chi \in L^2(\mathbb{R}^3, \mathfrak{g})$, there exists a unique $\Phi_\chi \in H^2_{-2}(\mathbb{R}^3, \mathfrak{g})$ such that

$$\Delta\Phi_\chi = \chi \qquad \text{and} \qquad \int_{\mathbb{R}^3} \rho^{-4}(\Phi_\chi \cdot \xi) d_3x = 0 \quad \forall \xi \in \mathfrak{g}\ . \tag{2.7}$$

By Fredholmness of Δ there exists a constant C independent of χ such that

$$\|\Phi_\chi\|^2_{H^2_{-2}} \le C\|\chi\|^2_{L^2}\ . \tag{2.8}$$

Given $E \in H^1(\mathbb{R}^3, \mathfrak{g})$ we consider $\chi_E = -\operatorname{div} E \in L^2(\mathbb{R}^3, \mathfrak{g})$. The corresponding solution of (2.7) we denote by Φ_E. Then the vector field $Y_E := \operatorname{grad}\Phi_E + E^L$ is harmonic, that is $\operatorname{curl} Y_E = 0$ and $\operatorname{div} Y_E = 0$. By the estimates (A.4) and (A.5),

$$\sum_{j=1}^{3} \|\partial_j Y_E\|^2_{L^2} \le \|\operatorname{curl} Y_E\|^2_{L^2} + \|\operatorname{div} Y_E\|^2_{L^2}\ , \tag{2.9}$$

see also [11]. Therefore Y_E is constant. Since, by construction, Y_E has a finite norm in $H^1_{-1}(\mathbb{R}^3, \mathfrak{g})$, this implies that $Y_E = 0$, cf. (A.7). This proves that Φ_E is the unique solution of the problem (2.5). From the a-priori estimate (2.8) and (A.6) we conclude that

$$\frac{1}{4}\Big(\int_{D_1} |\Phi_E|\, d_3x\Big)^2 \le \Big(\int_{\mathbb{R}^3} \rho^{-2}\,|\Phi_E|\, d_3x\Big)^2 \le \|\Phi_E\|^2_{H^2_{-2}} \le C\|\operatorname{div} E\|^2_{L^2}\ . \tag{2.10}$$

Moreover $\|\operatorname{grad}\Phi_E\|^2_{H^k} = \|E^L\|^2_{H^k} \le \|E\|^2_{H^k}$. This proves the estimate (2.6). Q.E.D.

On the constraint set $\mathbf{C}$ this special gauge fixing can be achieved by a gauge transformation

$$\widetilde{\Phi} \longmapsto \Phi = \varphi\widetilde{\Phi}\varphi^{-1} + \varphi\partial_t\varphi^{-1}\ . \tag{2.11}$$

More precisely, let $(\widetilde{\Phi}(t), A(t), E(t), \Psi(t))$ be a C^1 curve in $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \times \mathbf{P}$, such that $(A(t), E(t), \Psi(t))$ satisfy the constraint equation (1.5). Then, for t small enough, there exists a C^1 curve $\varphi(t) : \mathbb{R}^3 \to G$ of gauge transformations such that the transformed scalar potential Φ satisfies the gauge condition (2.5). It is of class $\mathcal{B}^3(\mathbb{R}^3, M^k_k)$. This is shown in [12] for bounded domains. The proof literally generalises to $\mathbb{R}^3$ if one takes the estimates of Lemma A.3 and (2.6) into account.

Using the gauge fixing (2.5) and linearizing the system given by (2.2) and (1.4) we obtain

$$\frac{d}{dt}\begin{bmatrix} A^L \\ E^L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{2.12}$$

$$\frac{d}{dt}\begin{bmatrix} A^T \\ E^T \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} A^T \\ E^T \end{bmatrix} =: \mathcal{T}(A^T, E^T) , \tag{2.13}$$

$$\frac{d}{dt}\Psi = -\gamma^0(\gamma^j \partial_j + im)\Psi =: \mathcal{D}\Psi . \tag{2.14}$$

We shall study these linear equations in the Hilbert spaces

$$\mathbf{H}_L = \{(A^L, E^L) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g})\} , \tag{2.15}$$

$$\mathbf{H}_T = \{(A^T, E^T) \in H^1(\mathbb{R}^3, \mathfrak{g}) \times L^2(\mathbb{R}^3, \mathfrak{g})\} , \tag{2.16}$$

$$\mathbf{H}_D = \{\Psi \in L^2(\mathbb{R}^3, V_G)\} . \tag{2.17}$$

Proposition 2.2

The operator $\mathcal{T}$, defined by (2.13), with domain

$$\mathbf{D}_T = \{(A^T, E^T) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g})\} \tag{2.18}$$

is the generator of a continuous group $\exp(t\mathcal{T})$ of transformations in $\mathbf{H}_T$.

Proof.

By standard arguments, the operator

$$\widetilde{\mathcal{T}} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \quad \text{with domain } H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g}) \tag{2.19}$$

is dissipative, and satisfies

$$\operatorname{range}(\widetilde{\mathcal{T}} - \lambda I) = H^1(\mathbb{R}^3, \mathfrak{g}) \times L^2(\mathbb{R}^3, \mathfrak{g}) \quad \text{and} \quad \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\} \tag{2.20}$$

for $\lambda > 0$. In fact, $\widetilde{\mathcal{T}}$ is the infinitesimal generator corresponding to the wave equation, [13]. We have to show that $\exp(t\widetilde{\mathcal{T}})$ preserves the Hilbert space $\mathbf{H}_T$ of transverse fields. Given $(X^T, Y^T) \in \mathbf{H}_T$ we consider $(\widetilde{A}, \widetilde{E})$, satisfying the equation

$$(\widetilde{\mathcal{T}} - \lambda I)(\widetilde{A}, \widetilde{E}) = (X^T, Y^T) . \tag{2.21}$$

Since Δ maintains the Helmholtz decomposition $\widetilde{A} = \widetilde{A}^T + \widetilde{A}^L$, this implies that

$$(\widetilde{A}^L, \widetilde{E}^L) \in \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\} . \tag{2.22}$$

Therefore, since $\mathcal{T} = \widetilde{\mathcal{T}}|_{\mathbf{D}_T}$, we have

$$\text{range}\big(\mathcal{T} - \lambda I\big) = \text{range}\big(\widetilde{\mathcal{T}} - \lambda I\big)|_{\mathbf{D}_T} = \mathbf{H}_T \ . \tag{2.23}$$

The Lumer-Phillips theorem implies that $\mathcal{T}$ generates a one parameter group of continuous transformations $\exp(t\mathcal{T})$ in $\mathbf{H}_T$. Q.E.D.

Proposition 2.3
(i) The operator $\mathcal{D}$, with domain

$$\mathbf{D}_D = \{\Psi \in H^1(\mathbb{R}^3, V_G)\} \tag{2.24}$$

is the generator of a continuous group of (unitary) transformations $\exp(t\mathcal{D})$ in $\mathbf{H}_D$.
(ii) $\exp(t\mathcal{D})$ restricts to a group of continuous transformations in $H^2(\mathbb{R}^3, V_G)$.

Proof.
(i) It is known, [14], that the operator $\mathcal{D}$ with domain $\mathbf{D}_D$ is skew-adjoint in $\mathbf{H}_D$. Thus, $\mathcal{D}$ generates a group $\exp(t\mathcal{D})$ of unitary transformations in $\mathbf{H}_D$.
(ii) The operator $\mathcal{D} : H^1(\mathbb{R}^3, V_G) \to L^2(\mathbb{R}^3, V_G)$ is continuous, and its square

$$\mathcal{D}^2 = \Delta - m^2 : H^2(\mathbb{R}^3, V_G) \longrightarrow L^2(\mathbb{R}^3, V_G) \tag{2.25}$$

is continuous and elliptic. With the elliptic a-priori estimate this implies that

$$C_1\|\mathcal{D}^2\Psi\|_{L^2} \leq \|\Psi\|_{H^2} \leq C_2\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\Psi\|_{H^1}\big) \ . \tag{2.26}$$

Moreover, from the identity $\gamma^i\gamma^k = -\delta^{ik} + \frac{1}{2}[\gamma^i, \gamma^k]$, we obtain

$$\|\mathcal{D}\Psi\|^2_{L^2} = \sum_{j=1}^{3} \|\partial_j\Psi\|^2_{L^2} - A(\Psi) + m^2\|\Psi\|^2_{L^2} \ , \tag{2.27}$$

where

$$A(\Psi) = \frac{1}{2} \sum_{j,k=1}^{3} \langle[\gamma^j, \gamma^k]\partial_k\Psi, \partial_j\Psi\rangle_{L^2} \ . \tag{2.28}$$

Integration by parts shows that $A(\Psi)$ vanishes for all Ψ in $C^\infty(\mathbb{R}^3, V_G) \cap H^1(\mathbb{R}^3, V_G)$. Thus, by a density argument, $A(\Psi) = 0$ for all $\Psi \in H^1(\mathbb{R}^3, V_G)$. Therefore

$$C_3\|\mathcal{D}\Psi\|_{L^2} \leq \|\Psi\|_{H^1} \leq C_4\|\mathcal{D}\Psi\|_{L^2} \tag{2.29}$$

and

$$\|\Psi\|_{H^2} \leq C_5\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2}\big) \ . \tag{2.30}$$

Since $\exp(t\mathcal{D})$ is a unitary operator, which commutes on the domain $\mathbf{D}_D$ with its generator $\mathcal{D}$, cf. [15], we can estimate for all $\Psi \in H^2(\mathbb{R}^3, V_G)$:

$$\begin{aligned} \|\exp(t\mathcal{D})\Psi\|_{H^2} &\leq C_5\big(\|\mathcal{D}^2\exp(t\mathcal{D})\Psi\|_{L^2} + \|\mathcal{D}\exp(t\mathcal{D})\Psi\|_{L^2}\big) \\ &= C_5\big(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2}\big) \leq C_6\|\Psi\|_{H^2} \ . \end{aligned} \tag{2.31}$$

Hence $\exp(t\mathcal{D})$ acts continuously in the Hilbert space $H^2(\mathbb{R}^3, V_G)$. Q.E.D.

Corollary 2.4

The linear operator

$$\mathcal{S} = 0 \oplus \mathcal{T} \oplus \mathcal{D} \tag{2.32}$$

with domain $\mathbf{D} = \mathbf{H}_L \times \mathbf{D}_T \times \mathbf{D}_D$, corresponding to the dynamical system (2.12), (2.13) and (2.14), generates a one parameter group $\exp(t\mathcal{S})$ of continuous transformations in $\mathbf{H} = \mathbf{H}_L \times \mathbf{H}_T \times \mathbf{H}_D$. The space

$$\mathbf{P} = \{(A, E, \Psi) | A \in H^2(\mathbb{R}^3, \mathfrak{g}),\ E \in H^1(\mathbb{R}^3, \mathfrak{g}),\ \Psi \in H^2(\mathbb{R}^3, V_G)\} \tag{2.33}$$

is preserved by the action of $\exp(t\mathcal{S})$ in $\mathbf{H}$. The restriction of $\exp(t\mathcal{S})$ to $\mathbf{P}$ is a continuous one parameter group $\mathcal{U}(t)$ of continuous transformations in $\mathbf{P}$,

$$\mathcal{U}(t) = \exp(t\mathcal{S})|_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{P} \text{ such that } (A, E, \Psi) \mapsto \mathcal{U}(t)(A_0, E_0, \Psi_0) \tag{2.34}$$

and $\mathcal{U}(t)(A_0, E_0, \Psi_0)$ is the unique solution of the linear evolution equations (2.12), (2.13) and (2.14) with initial condition (A_0, E_0, Ψ_0).

Having solved the linearized problem, we can rewrite the coupled nonlinear equations (1.2), (1.3) and (1.4) in an abstract form as

$$\frac{d}{dt}(A, E, \Psi)_t = \mathcal{S}(A, E, \Psi)_t - \mathcal{F}((A, E, \Psi)_t)\ . \tag{2.35}$$

Here $\mathcal{F}$ describes the nonlinearity of the theory and is given by

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \qquad \text{where} \qquad \begin{cases} \mathcal{F}_1(A, E, \Psi) &= \big(0\,;\ [A\times, B] + \operatorname{curl}[A\times, A]\,;\ 0\big) \\ \mathcal{F}_2(A, E, \Psi) &= \big(0\,;\ -J\,;\ \gamma^0\gamma^j A_j \Psi\big) \\ \mathcal{F}_3(A, E, \Psi) &= \big([\Phi, A]\,;\ [\Phi, E]\,;\ [\Phi, \Psi]\big)\ . \end{cases} \tag{2.36}$$

In order to solve the system (2.35) we apply the method of nonlinear semigroups. It requires the knowledge of some analytic properties of the nonlinearity.

Proposition 2.5

The nonlinear part of the Yang-Mills-Dirac system, given by Eq. (2.36), is a map $\mathcal{F} : \mathbf{P} \longrightarrow \mathbf{P}$. It is continuous and smooth with respect to the norm

$$\|(A, E, \Psi)\|^2_{\mathbf{P}} = \|A\|^2_{H^2} + \|E\|^2_{H^1} + \|\Psi\|^2_{H^2}\ . \tag{2.37}$$

Proof.

The continuity and smoothness was proved for the component $\mathcal{F}_1$ in [3], and for the minimal coupling component $\mathcal{F}_2$ in [16]. The proof given there under the bag boundary conditions literally generalizes to $\mathbb{R}^3$. For the component $\mathcal{F}_3$ we get with the estimates of Lemma A.3 and (2.6)

$$\begin{aligned} \|\mathcal{F}_3\|^2_{\mathbf{P}} = \|[\Phi, A]\|^2_{H^2} + \|[\Phi, E]\|^2_{H^1} + \|[\Phi, \Psi]\|^2_{H^2} &\leq C\|\Phi\|^2_{\mathcal{B}^2}\big(\|A\|_{H^2} + \|E\|_{H^1} + \|\Psi\|_{H^2}\big)^2 \\ &\leq C'\|E\|^2_{H^1}\|(A, E, \Psi)\|^2_{\mathbf{P}}\ . \end{aligned} \tag{2.38}$$

This proves the continuity of $\mathcal{F}_3 : \mathbf{P} \to \mathbf{P}$. To show differentiability we write (a, e, ψ) for an arbitrary infinitesimal variation and evaluate

$$D\mathcal{F}_3(A, E, \Psi)(a, e, \psi) = \big([\phi, A] + [\Phi, a]\,;\, [\phi, E] + [\Phi, e]\,;\, [\phi, \Psi] + [\Phi, \psi]\big) \tag{2.39}$$

where $\Delta\phi = -\operatorname{div} e$. Since (a, e, ψ) are of the same Sobolev class as (A, E, Ψ) we can estimate similarly as in (2.38)

$$\|D\mathcal{F}_3(A, E, \Psi)(a, e, \psi)\|_{\mathbf{P}}^2 \leq C\big(\|a\|_{H^2} + \|e\|_{H^1} + \|\psi\|_{H^2}\big)^2 \|(A, E, \Psi)\|_{\mathbf{P}}^2 \,. \tag{2.40}$$

This proves that $\mathcal{F}_3 : \mathbf{P} \to \mathbf{P}$ is differentiable. Higher order differentiability is shown accordingly. Q.E.D.

The result of Proposition 2.5 enables us to infer the existence and uniqueness of solutions of minimally coupled Yang-Mills and Dirac equations from the corresponding results for nonlinear semigroups, cf. [17] :

Theorem 2.6

For every initial condition $(A_0, E_0, \Psi_0) \in \mathbf{P}$ there exists a unique maximal $T \in (0, \infty]$ and a unique curve $(A(t), E(t), \Psi(t))$ in $C^1([0, T), \mathbf{P})$ satisfying the Yang-Mills and Dirac equations (1.2), (1.3) and (1.4). If $T < \infty$, then

$$\lim_{t \to T} \|(A, E, \Psi)_t\|_{\mathbf{P}} = \infty \,. \tag{2.41}$$

Observe that the time evolution of the Yang-Mills-Dirac system discussed here gives rise to local diffeomorphisms of the phase space $\mathbf{P}$. To see this, we consider the map

$$(A, E, \Psi)_0 \longmapsto (A, E, \Psi)_t = \mathcal{U}(t)(A, E, \Psi)_0 + \int_0^t \mathcal{U}(t - s)\mathcal{F}((A, E, \Psi)_s)ds \,. \tag{2.42}$$

By differentiation of this map in the direction of a vector (a, e, ψ) in $\mathbf{P}$ we obtain

$$\begin{aligned} &\big((A, E, \Psi)_0, (a, e, \psi)\big) \longmapsto \\ &\qquad \mathcal{U}(t)(a, e, \psi) + \int_0^t \mathcal{U}(t - s)D\mathcal{F}((A, E, \Psi)_s)(a, e, \psi)ds \,, \end{aligned} \tag{2.43}$$

which is continuous, since $\mathcal{F}$ is smooth. A corresponding argument for the higher derivatives implies that the time evolution (2.42) is smooth. Since the dynamics is reversible, this shows that it is a local diffeomorphism. It should be emphasized that this diffeomorphism is not a symplectomorphism. To obtain a Hamiltonian evolution one has to modify the gauge condition of Proposition 2.1, cf. [18].

If the initial conditions for the Yang-Mills-Dirac system are more regular, say in

$$\mathbf{P}^k = \{(A, E, \Psi) | A \in H^{k+1}(\mathbb{R}^3, \mathfrak{g}),\ E \in H^k(\mathbb{R}^3, \mathfrak{g}),\ \Psi \in H^{k+1}(\mathbb{R}^3, V_G)\} \tag{2.44}$$

with $k \geq 1$, then the time evolution maintains this regularity. To see this, note that

$$\mathbf{D}_T^k = \{(A^T, E^T) \in H^{k+1}(\mathbb{R}^3, \mathfrak{g}) \times H^k(\mathbb{R}^3, \mathfrak{g})\} \tag{2.45}$$

is the domain of the k-th power of the operator $\mathcal{T}$. Moreover, by repeating the arguments of Proposition 2.3(ii), it follows that the domain of $\mathcal{D}^k$ is

$$\mathbf{D}_D^k = \{\Psi \in H^k(\mathbb{R}^3, V_G)\} \ . \tag{2.46}$$

It is straightforward to show that $\mathcal{F} : \mathbf{P}^k \to \mathbf{P}^k$ is continuous and smooth. Therefore we can conclude with [19] :

Corollary 2.7

For every initial condition $(A_0, E_0, \Psi_0) \in \mathbf{P}^k$ the solution of Eqs. (1.2), (1.3) and (1.4) is a curve $(A(t), E(t), \Psi(t))$ in $C^1([0,T), \mathbf{P}^k)$.

3. Gauge symmetries.

The group $GS(\mathbf{P})$ of gauge symmetries of the Yang-Mills-Dirac system in the phase space $\mathbf{P}$ is the connected group of gauge transformations

$$A \mapsto \varphi A \varphi^{-1} + \varphi \operatorname{grad} \varphi^{-1} \ , \quad E \mapsto \varphi E \varphi^{-1} \ , \quad \Psi \mapsto \varphi \Psi \ , \tag{3.1}$$

where φ is a map from $\mathbb{R}^3$ to the structure group G, which preserve $\mathbf{P}$. The infinitesimal action of the elements ξ of the Lie algebra $gs(\mathbf{P})$ of $GS(\mathbf{P})$ is given by

$$A \mapsto A - D_A \xi \ , \quad E \mapsto E - [E, \xi] \ , \quad \Psi \mapsto \Psi + \xi \Psi \ , \tag{3.2}$$

where

$$D_A \xi = \operatorname{grad} \xi + [A, \xi] \tag{3.3}$$

is the covariant differential of ξ with respect to the connection defined by A. Since the Yang-Mills potentials A in $\mathbf{P}$ are of Sobolev class $H^2(\mathbb{R}^3, \mathfrak{g})$, it follows that $\xi \in gs(\mathbf{P})$ only if $\operatorname{grad} \xi \in H^2(\mathbb{R}^3, \mathfrak{g})$. This suggests the following :

Proposition 3.1

The set of infinitesimal gauge symmetries of $\mathbf{P}$ is the Hilbert-Lie algebra

$$gs(\mathbf{P}) = \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \ . \tag{3.4}$$

The scalar product in $gs(\mathbf{P})$ is given by (A.14). The action of $gs(\mathbf{P})$ in $\mathbf{P}$ is continuous.

Proof.

The estimates of Lemma A.3 imply that

$$\begin{aligned} \|[A, \xi]\|_{H^2} \leq C \, \|\xi\|_{\mathcal{B}^3} \, \|A\|_{H^2} \ , \quad \|[E, \xi]\|_{H^1} &\leq C \, \|\xi\|_{\mathcal{B}^3} \, \|E\|_{H^1} \\ \text{and} \qquad \|\xi \Psi\|_{H^2} &\leq C \, \|\xi\|_{\mathcal{B}^3} \, \|\Psi\|_{H^2} \end{aligned} \tag{3.5}$$

for $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$. Therefore the infinitesimal action (3.2) of each $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ preserves $\mathbf{P}$. This implies that $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \subseteq gs(\mathbf{P})$. By the argument above $(\operatorname{grad} \xi)$ has to be in

$H^2(\mathbb{R}^3, \mathfrak{g})$ in order to have $\xi \in gs(\mathbf{P})$. With the definition of $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$, cf. (2.3), this proves (3.4). Moreover,

$$\|[\xi, \eta]\|_{\mathcal{B}^3} \leq \|\xi\|_{\mathcal{B}^3} \, \|\eta\|_{\mathcal{B}^3} \tag{3.6}$$

which proves that $gs(\mathbf{P})$ is a Banach-Lie algebra. Since $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ is a Hilbert space by Theorem A.2, $gs(\mathbf{P})$ is a Hilbert-Lie algebra. Finally, the continuity of the action of $gs(\mathbf{P})$ in $\mathbf{P}$ follows from the estimates (3.5). Q.E.D.

Let $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ denote the space of all smooth maps $\xi : \mathbb{R}^3 \to \mathfrak{g}$ which are constant outside a compact set, and let $C_0^\infty(\mathbb{R}^3, \mathfrak{g})$ be the subspace of compactly supported maps. From the decomposition results of [8] we infer that

$$gs(\mathbf{P}) = gs(\mathbf{P})_0 \oplus \mathfrak{g} \ , \tag{3.7}$$

where $gs(\mathbf{P})_0$ is the closure of $C_0^\infty(\mathbb{R}^3, \mathfrak{g})$ in the topology given by the norm (2.4). By Theorem A.2, $gs(\mathbf{P})_0 \subset C^1(\mathbb{R}^3, \mathfrak{g})$, so that all infinitesimal gauge transformations in $gs(\mathbf{P})$ are C^1-maps from $\mathbb{R}^3$ the structure Lie algebra $\mathfrak{g}$. Moreover $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ is dense in $gs(\mathbf{P})$, cf. Lemma A.1.

The topology of the gauge group on non-compact manifolds with a Sobolev-Lie algebra has been studied in [1] and [20]. Here we adapt the approach of [1] to our case of a $\mathcal{B}^3$ Hilbert-Lie algebra. The set $C_c^\infty(\mathbb{R}^3, G)$ is a group under pointwise multiplication with the identity denoted by e. If we consider G as a subset of the space M_k^k of $k \times k$ matrices, $C_c^\infty(\mathbb{R}^3, G) \subset C_c^\infty(\mathbb{R}^3, M_k^k)$ and it can be topologized by the norm $\| \cdot \|_{\mathcal{B}^3}$, given by (2.4).

One parameter subgroups of $C_c^\infty(\mathbb{R}^3, G)$ are of the form $\exp(t\xi)$, where ξ is in the dense subalgebra $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ of $gs(\mathbf{P})$. The topology of $gs(\mathbf{P})$ induces a uniform structure in $C_c^\infty(\mathbb{R}^3, G)$, with a neighbourhood basis at e consisting of the sets

$$N_\epsilon = \{\exp(\xi) \mid \xi \in C_c^\infty(\mathbb{R}^3, \mathfrak{g}) \, , \, \|\xi\|_{\mathcal{B}^3} < \epsilon\} \qquad \text{with } \epsilon > 0 \ . \tag{3.8}$$

In order to show that the completion of $C_c^\infty(\mathbb{R}^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication, we need to show :

Proposition 3.2

The mapping $\exp(\xi) \mapsto \exp(\xi)^{-1}$ is uniformly continuous relative to N_1. That is, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $\exp(\xi) \in N_1$,

$$\exp(\xi)^{-1} N_\delta \exp(\xi) \subseteq N_\epsilon \ . \tag{3.9}$$

Proof.

Let $\varphi \in N_\epsilon \subset C^\infty(\mathbb{R}^3, M_k^k)$ then

$$\varphi = \exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \, \xi^n \ , \tag{3.10}$$

and

$$\operatorname{grad} \varphi = \operatorname{grad} \exp(\xi) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n!} \, \xi^k \, (\operatorname{grad} \xi) \, \xi^{n-k-1} \ . \tag{3.11}$$

Using the estimates of Lemma A.3 this implies that

$$\|\mathrm{grad}\,\varphi\|_{H^2} \leq \exp(C\|\xi\|_{\mathcal{B}^3})\,\|\mathrm{grad}\,\xi\|_{H^2} \tag{3.12}$$

and

$$\|\varphi\|_{\mathcal{B}^3} \leq \sum_{n=0}^{\infty} \frac{1}{n!}\left(C\|\xi\|_{\mathcal{B}^3}\right)^n < \mathrm{e}^{C\epsilon}\,. \tag{3.13}$$

For each $\zeta \in gs(\mathbf{P})$ we then obtain by using Lemma A.3 once more :

$$\|\exp(\xi)^{-1}\,\zeta\,\exp(\xi)\|_{\mathcal{B}^3} \leq C^2\|\exp(-\xi)\|_{\mathcal{B}^3}\|\,\zeta\,\|_{\mathcal{B}^3}\|\exp(\xi)\|_{\mathcal{B}^3} < C^2\mathrm{e}^{2C\epsilon}\|\,\zeta\,\|_{\mathcal{B}^3}\,. \tag{3.14}$$

This proves (3.9) with $\delta = \epsilon(C\mathrm{e}^{C\epsilon})^{-2}$. Q.E.D.

By a result of [21], Proposition 3.2 implies that the completion of $C_c^\infty(I\!R^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication. It is a Hilbert-Lie group, whose Lie algebra is canonically isomorphic to the Hilbert-Lie algebra $gs(\mathbf{P})$. In view of this we set :

Definition 3.3

The Hilbert-Lie group $GS(\mathbf{P})$ of gauge symmetries is the completion of the group $C_c^\infty(I\!R^3, G)$ in the uniform structure defined by the topology of its Lie algebra $gs(\mathbf{P})$.

The exponential map $\exp : gs(\mathbf{P}) \to GS(\mathbf{P})$ maps the unit ball in $gs(\mathbf{P})$ onto the neighbourhood of identity in $GS(\mathbf{P})$ given by the completion $\overline{N}_1$ of N_1. Since G is connected, it follows that $C_c^\infty(I\!R^3, G)$ is connected, and $GS(\mathbf{P})$ is connected. Therefore, $GS(\mathbf{P})$ is the union of the sets

$$N_1^m = \{\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_m | \varphi_i \in N_1\}\,. \tag{3.15}$$

The inequality (3.12) together with (3.15) implies that, for each $\varphi \in GS(\mathbf{P})$,

$$\mathrm{grad}\,\varphi \in H^2(I\!R^3, \mathfrak{g})\,. \tag{3.16}$$

Moreover, since G is compact, it is bounded in M_k^k, and the Sobolev embedding theorem implies that each $\varphi \in GS(\mathbf{P})$ is a bounded continuous map. Hence, $\|\varphi\|_{\mathcal{B}^3}$ is finite for every φ in $GS(\mathbf{P})$. We can give an alternative characterization of the topology of $GS(\mathbf{P})$.

Proposition 3.4

A sequence $\varphi_k \in GS(\mathbf{P})$ converges to φ in $GS(\mathbf{P})$ if and only if the sequence of maps $\varphi_k : I\!R^3 \to G$ converges to φ in the topology defined by the norm $\|\cdot\|_{\mathcal{B}^3}$.

Proof.

Suppose that φ_k converges to φ in the uniform topology of $GS(\mathbf{P})$. For sufficiently large k,

$$\varphi_k = \varphi\exp(\xi_k)\,, \tag{3.17}$$

where the sequence ξ_k converges to zero in the topology of $gs(\mathbf{P})$. The estimate (A.24) implies that

$$\|\varphi_k - \varphi\|_{\mathcal{B}^3} \leq C\|\varphi\|_{\mathcal{B}^3}\left\|\sum_{n=1}^{\infty}\frac{1}{n!}(\xi_k)^n\right\|_{\mathcal{B}^3} \leq C\|\varphi\|_{\mathcal{B}^3}\left|1 - \mathrm{e}^{C\|\xi_k\|_{\mathcal{B}^3}}\right|\,. \tag{3.18}$$

For $\xi_k \to 0$ in the norm topology of $gs(\mathbf{P})$ the right hand side converges to zero. Therefore $\varphi_k \to \varphi$ in the topology defined by $\|\cdot\|_{\mathcal{B}^3}$.

Conversely, suppose that $\|\varphi_k - \varphi\|_{\mathcal{B}^3} \to 0$. Then

$$\|e - \varphi^{-1}\varphi_k\|_{\mathcal{B}^3} \leq \epsilon_k \tag{3.19}$$

with $\epsilon_k \to 0$ as k goes to infinity. Eq. (3.17) yields

$$\xi_k = \log(\varphi^{-1}\varphi_k) = -\sum_{n=1}^{\infty} \frac{(e - \varphi^{-1}\varphi_k)^n}{n} \tag{3.20}$$

for k sufficiently large. Therefore, by (A.24),

$$\|\xi_k\|_{\mathcal{B}^3} \leq \sum_{n=1}^{\infty} \frac{\left(C\|e - \varphi^{-1}\varphi_k\|_{\mathcal{B}^3}\right)^n}{Cn} \leq -\frac{1}{C}\log(1 - C\epsilon_k) \ . \tag{3.21}$$

This implies that $\xi_k \to 0$ in the topology of $gs(\mathbf{P})$, and hence $\varphi_k \to \varphi$ in the uniform topology of $GS(\mathbf{P})$. Q.E.D.

Theorem 3.5

The action of $GS(\mathbf{P})$ in $\mathbf{P}$, given by (3.2), is continuous and proper.

Proof.

Let φ_n be a sequence in $GS(\mathbf{P})$ converging to φ, and $p_n = (A_n, E_n, \Psi_n)$ a sequence in $\mathbf{P}$ converging to $p = (A, E, \Psi)$. From (3.2) we obtain by using the estimate (A.23) and the fact that the inversion $\varphi \mapsto \varphi^{-1}$ in $GS(\mathbf{P})$ is continuous :

$$\begin{aligned}
&\|\left(\varphi_n A_n \varphi_n^{-1} + \varphi_n \mathrm{grad}\, \varphi_n^{-1}\right) - \left(\varphi A \varphi^{-1} - \varphi \mathrm{grad}\, \varphi^{-1}\right)\|_{H^2} \qquad\qquad (3.22)\\
&\leq \|\varphi_n A_n \varphi_n^{-1} - \varphi_n A \varphi_n^{-1}\|_{H^2} + \|\varphi_n A \varphi_n^{-1} - \varphi_n A \varphi^{-1}\|_{H^2} + \|\varphi_n A \varphi^{-1} - \varphi A \varphi^{-1}\|_{H^2} \\
&\qquad + \|\varphi_n \mathrm{grad}\, \varphi_n^{-1} - \varphi_n \mathrm{grad}\, \varphi^{-1}\|_{H^2} + \|\varphi_n \mathrm{grad}\, \varphi^{-1} - \varphi \mathrm{grad}\, \varphi^{-1}\|_{H^2} \\
&\leq C\Big(\|\varphi_n\|^2_{\mathcal{B}^3}\|A_n - A\|_{H^2} + \left(\|\varphi_n\|_{\mathcal{B}^3} + \|\varphi\|_{\mathcal{B}^3}\right)\|A\|_{H^2}\|\varphi_n - \varphi\|_{\mathcal{B}^3} \\
&\qquad + \|\varphi_n\|_{\mathcal{B}^3}\|\mathrm{grad}\, \varphi_n - \mathrm{grad}\, \varphi\|_{H^2} + \|\varphi_n - \varphi\|_{\mathcal{B}^3}\|\mathrm{grad}\, \varphi\|_{H^2}\Big) \ .
\end{aligned}$$

Writing symbolically φp for the action of $\varphi \in gs(\mathbf{P})$ on $p \in \mathbf{P}$, and $(\varphi p)_A$ for its A component, this implies that

$$\|(\varphi_n p_n)_A - (\varphi p)_A\|_{H^2} \leq C'\left(\|A_n - A\|_{H^2} + \|\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\right) , \tag{3.23}$$

since $\|\varphi_n\|_{\mathcal{B}^3}$ is bounded. Correspondingly we estimate with (A.22) and (A.23),

$$\begin{aligned}
\|(\varphi_n p_n)_E - (\varphi p)_E\|_{H^1} &\leq C'\left(\|E_n - E\|_{H^1} + |\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\right) \\
\|(\varphi_n p_n)_\Psi - (\varphi p)_\Psi\|_{H^2} &\leq C'\left(\|\Psi_n - \Psi\|_{H^2} + \|\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\right) .
\end{aligned} \tag{3.24}$$

Therefore $\|\varphi_n p_n - \varphi p\|_{\mathbf{P}} \to 0$ as $n \to \infty$, which proves the continuity of the action.

Let $p_n = (A_n, E_n, \Psi_n)$ converge in $\mathbf{P}$ to $p = (A, E, \Psi)$, and φ_n be a sequence in $GS(\mathbf{P})$ such that $\varphi_n p_n$ converges to $\widetilde{p} \in \mathbf{P}$. To prove properness of the action it is to show that φ_n converges to $\varphi \in GS(\mathbf{P})$ and $\widetilde{p} = \varphi p$. The argument used in [16] for compact domains implies that, for every compact domain $M \subset I\!R^3$, the restrictions $\varphi_n|M$ converge in $H^2(M)$ to a map $\varphi_M \in H^2(M)$. Since, $M \subseteq \widetilde{M}$ implies that $\varphi|\widetilde{M}$ restricted to M coincides with φ_M, it follows that there exists a continuous map $\varphi : I\!R^3 \rightarrow G$ such that φ_M is the restriction of φ to M. The proof that $\operatorname{grad}\varphi_n$ converges to $\operatorname{grad}\varphi$ in the H^2 topology is the same as in the compact case, [16]. Hence, Proposition 3.4 implies that φ_n converges to φ in the uniform topology. Q.E.D.

Let $C_0^\infty(I\!R^3, G)$ be the subgroup of $C_c^\infty(I\!R^3, G)$ consisting of maps $\varphi : I\!R^3 \rightarrow G$ which are the identity in G outside a compact set. Its closure in the uniform topology discussed above defines a closed subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$. The subalgebra $gs(\mathbf{P})_0$ of $gs(\mathbf{P})$, defined by (3.7), is an ideal and hence $GS(\mathbf{P})_0$ is a normal subgroup of $GS(\mathbf{P})$.

Proposition 3.6

$GS(\mathbf{P})_0$ is a Hilbert-Lie group with Lie algebra $gs(\mathbf{P})_0$. The action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is free and proper.

Proof.

To show that the infinitesimal action is also free suppose that $\xi_0 \in gs(\mathbf{P})_0$ has a fixed point (A, E, Ψ). By (3.2)

$$D_A \xi_0 = 0 \ , \tag{3.25}$$

that is, ξ_0 is covariantly constant with respect to the connection given by A. Since the scalar product in $\mathfrak{g}$ is ad-invariant, this implies that ξ_0 is constant. This contradicts the assumption $\xi_0 \in gs(\mathbf{P})_0$, which proves that the action of $gs(\mathbf{P})_0$ is free. Since $GS(\mathbf{P})_0$ is connected, every $\varphi \in GS(P)_0$ is of the form

$$\varphi = (\exp \xi_1) \cdot (\exp \xi_2) \ldots (\exp \xi_n) \tag{3.26}$$

for some $\xi_1, \ldots \xi_n$ in $gs(\mathbf{P})_0$. Therefore the action of $GS(\mathbf{P})_0$ is free.

The result of Proposition 3.4 implies that the Lie algebra of $GS(\mathbf{P})_0$ is the closure of $C_0^\infty(I\!R^3, \mathfrak{g})$ in the $\mathcal{B}^3$ topology. By the decomposition (3.7) this coincides with $gs(\mathbf{P})_0$. Since $GS(\mathbf{P})_0$ is a closed subgroup of $GS(\mathbf{P})$ which acts properly in $\mathbf{P}$, it follows that the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper. Q.E.D.

The extended phase space $\mathbf{P}$ is endowed with a 1-form θ given by

$$\langle \theta(A, E, \Psi) | (a, e, \psi) \rangle = \int_{I\!R^3} (E \cdot a + \Psi^\dagger \psi) d_3 x \ , \tag{3.27}$$

for $(a, e, \psi) \in T\mathbf{P}$, where $E \cdot a = -\mathrm{tr}(Ea)$. The exterior differential $\omega = d\theta$ of θ is a weakly symplectic form on $\mathbf{P}$, that is ω is non-degenerate and closed, but the induced mapping $\flat : T\mathbf{P} \rightarrow T^*\mathbf{P}$ defined by $u^\flat(v) = \omega(u, v)$ is not onto. Here $T^*\mathbf{P}$ denotes the cotangent bundle of $\mathbf{P}$, that is the topological dual of the tangent bundle $T\mathbf{P}$.

The action of $gs(\mathbf{P})$ in $\mathbf{P}$ is Hamiltonian with the momentum map $\mathcal{J}$ given by

$$\langle \mathcal{J}(A,E,\Psi)|\xi\rangle = \langle \theta|\xi_{\mathbf{P}}(A,E,\Psi)\rangle = \int_{I\!R^3} (-E\cdot \dot{D}_A\xi + \Psi^\dagger \xi\Psi) d_3x \ . \tag{3.28}$$

Each ξ in $gs(\mathbf{P})_0$ is the limit of a sequence ξ_n of smooth and compactly supported elements of $gs(\mathbf{P})_0$. The continuity of the momentum map $\mathcal{J}$ implies that

$$\langle \mathcal{J}(A,E,\Psi)|\xi\rangle = \lim_{n\to\infty} \langle \mathcal{J}(A,E,\Psi)|\xi_n\rangle = -\lim_{n\to\infty} \int_{I\!R^3} \left(\operatorname{div} E + [A;E] - J^0\right)\cdot \xi_n d_3x, \tag{3.29}$$

which follows by integration by parts. Therefore, for every ξ in $gs(\mathbf{P})_0$, the momentum $\langle \mathcal{J}(A,E,\Psi)|\xi\rangle$ vanishes for all (A,E,Ψ) satisfying the constraint equation (1.5). On the other hand, if $\xi : I\!R^3 \to \mathfrak{g}$ is a constant map, then there exists $(A,E,\Psi)\in \mathbf{C}$ such that $\langle \mathcal{J}(A,E,\Psi)|\xi\rangle$ does not vanish. Hence, we have obtained a geometric characterization of $gs(\mathbf{P})_0$ as

$$gs(\mathbf{P})_0 = \{\xi \in gs(\mathbf{P}) \mid \langle \mathcal{J}(\mathbf{A},\mathbf{E},\Psi)|\xi\rangle = 0 \quad \forall (\mathbf{A},\mathbf{E},\Psi)\in \mathbf{C}\} \ . \tag{3.30}$$

Let $\mathcal{J}_0$ be the restriction of the momentum mapping $\mathcal{J}$ to the subalgebra $gs(\mathbf{P})_0$. That is, $\mathcal{J}$ is the map from $\mathbf{P}$ to $gs(\mathbf{P})_0^*$ such that

$$\langle \mathcal{J}_0|\xi\rangle = \langle \mathcal{J}|\xi\rangle \tag{3.31}$$

for all $\xi \in gs(\mathbf{P})_0$. It follows from Eq. (3.30) that the constraint set $\mathbf{C}$ is contained in the zero level of $\mathcal{J}_0$. Conversely, the vanishing of $\langle \mathcal{J}_0|\xi\rangle$ for all smooth compactly supported maps ξ from $I\!R^3$ to the Lie algebra $\mathfrak{g}$ implies the constraint equations. This follows from the Fundamental Theorem of the Calculus of Variations and Eq. (3.29). Since the momentum mapping $\mathcal{J}_0$ is continuous and every $\xi \in gs(\mathbf{P})_0$ is the limit of a sequence of smooth and compactly supported elements ξ_n it follows that the zero level of $\mathcal{J}_0$ is contained in $\mathbf{C}$. Hence, we have proved that

$$\mathbf{C} = \mathcal{J}_0^{-1}(0). \tag{3.32}$$

We define the reduced phase space to be the space $\check{\mathbf{P}}$ of the $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$,

$$\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0 \ , \tag{3.33}$$

and denote by ρ the canonical projection from $\mathbf{C}$ to $\check{\mathbf{P}}$. Since $\mathbf{C}$ is a closed subset of $\mathbf{P}$ and the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper and preserves $\mathbf{C}$, it follows that the quotient topology in $\check{\mathbf{P}}$ is Hausdorff. The differentiable structure of $\check{\mathbf{P}}$ will be analysed in another paper, [22].

It follows from Eq. (3.7) that $gs(\mathbf{P})_0$ is an ideal in $gs(\mathbf{P})$ and that the quotient algebra

$$colour(\mathbf{P}) = gs(\mathbf{P})/gs(\mathbf{P})_0 \tag{3.34}$$

is isomorphic to $\mathfrak{g}$. For $\xi \in gs(\mathbf{P})$ and $(A,E,\Psi)\in\mathbf{C}$, the momentum $\langle \mathcal{J}(A,E,\Psi)|\xi\rangle$ depends only on the class $[\xi]$ in $colour(\mathbf{P})$ and on the $GS(\mathbf{P})_0$ orbit through (A,E,Ψ). It is interpreted as the colour charge in the physical state $\rho(A,E,\Psi)$ in the direction of $[\xi]\in colour(\mathbf{P})$. It should be noted that in the decomposition (3.7) of $gs(\mathbf{P})$ the second term

$\mathcal{G}$ is not an ideal. Hence, the notion of the "constant infinitesimal gauge transformations" makes invariant sense only as an element of the quotient algebra $colour(\mathbf{P})$, [9].

Appendix : Decompositions and estimates for Beppo Levi spaces

Let $\mathcal{S}$ denote the Schwartz space of smooth fast falling test functions on $\mathbb{R}^3$. The Fourier transformation $X \mapsto \mathcal{F}(X)$ is a homeomorphism from $\mathcal{S}$ to $\mathcal{S}$ which extends to a unitary map from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Given a vector field $X \in \mathcal{S}$, one has

$$\mathcal{F}(\operatorname{div} X)(p) = \sum_{i=1}^{3} p_i \mathcal{F}(X_i)(p) \qquad \text{and} \qquad \mathcal{F}(\operatorname{curl} X)_l = \sum_{i,j=1}^{3} \epsilon_{ijl} p_i \mathcal{F}(X_j)(p) \,. \tag{A.1}$$

This implies a splitting of $\mathcal{F}(X) = \mathcal{F}(X)^L + \mathcal{F}(X)^T$ with the components given as

$$\mathcal{F}(X)^L_j(p) = \frac{p_j}{|p|^2}\mathcal{F}(\operatorname{div} X)(p) \qquad \text{and} \qquad \mathcal{F}(X)^T_j(p) = \left(\frac{p}{|p|^2} \times \mathcal{F}(\operatorname{curl} X)(p)\right)_j \,. \tag{A.2}$$

The Helmholtz decomposition $X = X^L + X^T$ is defined via inverse Fourier transformation

$$X^L = \mathcal{F}^{-1}(\mathcal{F}(X)^L) \qquad \text{and} \qquad X^T = \mathcal{F}^{-1}(\mathcal{F}(X)^T) \tag{A.3}$$

on $\mathcal{S}$. It extends to a decomposition for vector fields in $L^2(\mathbb{R}^3)$. Moreover (A.2) implies that

$$\begin{aligned} \|X^L\|^2_{H^k} &= \int (1+|p|^2)^k |\mathcal{F}(X)^L(p)|^2 d_3p \\ &\le \int (1+|p|^2)^{k-1} |\mathcal{F}(X)^L(p)|^2 d_3p + \int (1+|p|^2)^{k-1} |\mathcal{F}(\operatorname{div} X)(p)|^2 d_3p \\ &\le \|X^L\|^2_{H^{k-1}} + \|X\|^2_{H^{k-1}} \end{aligned} \tag{A.4}$$

for $k \ge 1$. Similarly

$$\|X^T\|^2_{H^k} \le \|X^T\|^2_{H^{k-1}} + \|\operatorname{curl} X\|^2_{H^{k-1}} \,. \tag{A.5}$$

In order to solve the Laplace equation on $\mathbb{R}^3$ one needs to introduce the weighted Sobolev space $H^1_{-1}(\mathbb{R}^3, V)$ and $H^2_{-2}(\mathbb{R}^3, V)$, where V is a finite dimensional vector space. With the weight function $\rho = \sqrt{1+|x|^2}$ these spaces are defined as the respective completions of $C_0^\infty(\mathbb{R}^3, V)$ in the norms

$$\|g\|^2_{H^1_{-1}} := \int_{\mathbb{R}^3} |\rho^{-1} g|^2 d_3x + \sum_{j=1}^{3} \int_{\mathbb{R}^3} |\partial_j g|^2 d_3x \qquad \text{and} \tag{A.6}$$

$$\|g\|^2_{H^2_{-2}} := \int_{\mathbb{R}^3} |\rho^{-2} g|^2 d_3x + \sum_{j=1}^{3} \int_{\mathbb{R}^3} |\rho^{-1}\partial_j g|^2 d_3x + \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} |\partial_k \partial_j g|^2 d_3x \,.$$

By this definition, the derivatives are continuous as maps $\partial_j : H^2_{-2}(I\!R^3, V) \to H^1_{-1}(I\!R^3, V)$. The space $H^1_{-1}(I\!R^3, V)$ does not contain the constants, since for all $c \in V$

$$\|c\|^2_{H^1_{-1}} = |c|^2 \int_{I\!R^3} \rho^{-2} d_3x = \infty \quad \text{if } c \neq 0 \ . \tag{A.7}$$

Let $BL_1(L^2(I\!R^3, V))$ be the first Beppo Levi space of V-valued distributions which have a square integrable gradient, [7]. The following result can be found in a paper of Aikawa [8]:

Lemma A.1

The space $BL_1(L^2(I\!R^3, V))$ can be topologized by the norm

$$\|g\|_{\mathcal{B}^1} = \int_{D_1} |g| d_3x + \|\mathrm{grad}\, g\|_{L^2} \ . \tag{A.8}$$

It has a direct sum decomposition

$$BL_1(L^2(I\!R^3, V)) = \overline{\mathcal{D}_1} \oplus V \tag{A.9}$$

where V is considered as the space of constant functions from $I\!R^3$ to V and $\overline{\mathcal{D}_1}$ is the closure of the space $C_0^\infty(I\!R^3, V)$ of smooth compactly supported functions in the topology of $BL_1(L^2(I\!R^3, V))$ given by the norm (A.8).

The intersection of k Beppo Levi spaces we denote by

$$\mathcal{B}^k(I\!R^3, V) := \bigcap_{m=1}^{k} BL_k(L^2(I\!R^3, V)) \ . \tag{A.10}$$

This space is topologized by the norm

$$\|g\|_{\mathcal{B}^k} = \int_{D_1} |g| d_3x + \|\mathrm{grad}\, g\|_{H^{k-1}} \ . \tag{A.11}$$

Theorem A.2

(i) The space $\mathcal{B}^k(I\!R^3, V)$ splits into

$$\mathcal{B}^k(I\!R^3, V) = \overline{\mathcal{D}_k} \oplus V \tag{A.12}$$

where $\overline{\mathcal{D}_k}$ is the closure of the space $C_0^\infty(I\!R^3, V)$ of smooth compactly supported functions in the topology given by the norm (A.11). Each $g \in \mathcal{B}^k(I\!R^3, V)$ uniquely decomposes into

$$g = g_0 + c_g \qquad \text{where} \qquad g_0 \in \overline{\mathcal{D}^k} \ \text{ and } \ c_g \in V \ . \tag{A.13}$$

(ii) The space $\mathcal{B}^k(I\!R^3, V)$ is a Hilbert space with the scalar product

$$\ll f, g \gg_{\mathcal{B}^k} = \ll c_f, c_g \gg_V + \ll \rho^{-1} f_0, \rho^{-1} g_0 \gg_{L^2} + \ll \operatorname{grad} f, \operatorname{grad} g \gg_{H^{k-1}} \quad (A.14)$$

where $\ll, \gg_V$ and $\ll, \gg_{H^j}$ denote the scalar products in V and $H^j(I\!R^3, V)$.

(iii) For $k \geq 2$, each $f \in \mathcal{B}^k(I\!R^3, V)$ is continuous and C^{k-2}-differentiable. Let $g^{(\alpha)}$ denotes the partial derivative corresponding to a multi-index α, then

$$\sum_{|\alpha| \leq k-2} \|g^{(\alpha)}\|_{L^\infty} \leq C \|g\|_{\mathcal{B}^k} . \quad (A.15)$$

Proof.

(i) The decomposition (A.12) is obvious by intersecting (A.9) with $\mathcal{B}^k(I\!R^3, V)$.

(ii) On the space $C_0^\infty(I\!R^3, V)$ the $\mathcal{B}^1$-norm (A.8) is equivalent to the weighted Sobolev norm induced by the scalar product

$$\ll f, g \gg_{\mathcal{H}^1_{-1}} = \ll \rho^{-1} f_0, \rho^{-1} g_0 \gg_{L^2} + \ll \operatorname{grad} f, \operatorname{grad} g \gg_{L^2} . \quad (A.16)$$

This follows from the weighted Poincaré inequality for the weight function ρ, cf. [23], which states that there is a constant $C_p > 0$ such that

$$\|f_0\|^2_{L^2_{-1}} \leq C_p \|\operatorname{grad} f_0\|^2_{L^2_0} \leq C_p \|\operatorname{grad} f_0\|^2_{\mathcal{B}^1} \qquad \forall f_0 \in C^\infty(I\!R^3, V) . \quad (A.17)$$

Conversely

$$\frac{1}{4}\Big(\int_{D_1} |f_0|\, d_3x\Big)^2 \leq \Big(\int_{I\!R^3} \rho^{-2} |f_0|\, d_3x\Big)^2 \leq \|f_0\|^2_{H^1_{-1}} , \quad (A.18)$$

which implies that $\overline{\mathcal{D}_1} = H^1_{-1}(I\!R^3, V)$.

The finite dimensional subspace $V \subset \mathcal{B}^1(I\!R^3, V)$ is split. Therefore the scalar product on $\mathcal{B}^1(I\!R^3, V)$ given by (A.16) induces a norm which is equivalent to the norm given by (A.8). The result for $\mathcal{B}^k(I\!R^3, V)$ then is obvious.

(iii) To prove the embedding result of the Sobolev type consider the Fourier transform $\mathcal{F}(g)$ of $g \in C_0^\infty(I\!R^3, V)$. Then

$$g(x) = \int e^{ipx}\, \mathcal{F}(g)(p) \left(|p|^2(1+|p|^2)^{k-1}\right)^{1/2} \left(|p|^2(1+|p|^2)^{k-1}\right)^{-1/2} d_3p . \quad (A.19)$$

Using the Cauchy-Schwarz inequality we estimate

$$|g(x)|^2 \leq \Big(\int |p\,\mathcal{F}(g)(p)|^2 (1+|p|^2)^{k-1}\, d_3p\Big)\Big(\int \frac{(1+|p|^2)^{1-k}}{|p|^2}\, d_3p\Big) . \quad (A.20)$$

Since $\int_{I\!R} (1+|p|^2)^{1-k} dp < \infty$ for $k > \frac{3}{2}$ and $|p\,\mathcal{F}(g)(p)|^2 = |\mathcal{F}(\operatorname{grad} g)(p)|^2$ this implies that

$$|g(x)|^2 \leq C \|\operatorname{grad} g\|^2_{H^{k-1}} \leq C \|\operatorname{grad} g\|^2_{\mathcal{B}^k} . \quad (A.21)$$

This shows that each $f \in B^k(I\!R^3, V)$ is continuous and uniformly bounded. For the higher order derivatives the argument applies correspondingly. Q.E.D.

Lemma A.3

Let f and g be maps from $I\!R^3$ to normed vector spaces, and $f \cdot g$ any pointwise multiplication with values in a normed vector space. If $f \in \mathcal{B}^k(I\!R^3, V)$ and $k \geq 2$, the following estimates hold :

$$\|f \cdot g\|_{H^1} \leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1} \qquad \forall g \in H^1(I\!R^3, W) , \tag{A.22}$$
$$\|f \cdot g\|_{H^2} \leq C_2 \|f\|_{\mathcal{B}^k} \|g\|_{H^2} \qquad \forall g \in H^2(I\!R^3, W) , \tag{A.23}$$
$$\|f \cdot g\|_{\mathcal{B}^k} \leq C_3 \|f\|_{\mathcal{B}^k} \|g\|_{\mathcal{B}^k} \qquad \forall g \in \mathcal{B}^k(I\!R^3, W) . \tag{A.24}$$

Proof.

By Theorem A.2, $f \in \mathcal{B}^k(I\!R^3, V)$ implies that $\|f\|_{L^\infty}$ is finite, and hence

$$\|f \cdot g\|_{L^2} \leq \|f\|_{L^\infty} \|g\|_{L^2} \qquad \forall g \in H^1(I\!R^3, W) . \tag{A.25}$$

With an appropriately defined pointwise product $\cdot$ on the right hand side we have

$$\operatorname{grad}(f \cdot g) = \operatorname{grad}(f) \cdot g + f \cdot \operatorname{grad}(g) . \tag{A.26}$$

If $f \in \mathcal{B}^k(I\!R^3, V)$ then $\operatorname{grad}(f) \in H^1(I\!R^3, V)$ and

$$\|\operatorname{grad}(f) \cdot g\|_{L^2} \leq \|\operatorname{grad} f\|_{H^1} \|g\|_{H^1} . \tag{A.27}$$

Together with (A.15) and (A.25) this implies that

$$\|f \cdot g\|_{H^1} \leq \|f\|_{L^\infty} \left(\|g\|_{L^2} + \|\operatorname{grad} g\|_{L^2} \right) + \|\operatorname{grad} f\|_{H^1} \|g\|_{H^1} \leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1} , \tag{A.28}$$

which proves (A.22). Differentiating (A.26), we get

$$D \operatorname{grad}(f \cdot g) = D \operatorname{grad}(f) \cdot g + 2 \operatorname{grad}(f) \cdot \operatorname{grad}(g) + f \cdot D \operatorname{grad}(g) . \tag{A.29}$$

Therefore, for $g \in H^2(I\!R^3, \mathcal{g})$,

$$\|D \operatorname{grad}(f \cdot g)\|_{L^2} \leq \|D \operatorname{grad} f\|_{L^2} \|g\|_{H^2} + 2 \|\operatorname{grad} f\|_{H^1} \|g\|_{H^2} + \|f\|_{L^\infty} \|g\|_{H^2} . \tag{A.30}$$

With (A.22) and (A.15) this implies that

$$\|f \cdot g\|_{H^2} \leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1} + 3 \|\operatorname{grad} f\|_{H^1} \|g\|_{H^2} + \|f\|_{L^\infty} \|g\|_{H^2} , \tag{A.31}$$

which proves (A.23). Finally the estimates above yield

$$\begin{aligned} \|f \cdot g\|_{\mathcal{B}^2} \leq \|f\|_{L^\infty} \Big(\int_{D_1} |g| d_3 x + \|\operatorname{grad} g\|_{L^2} + \|D\operatorname{grad} g\|_{L^2} \Big) + 2 \|\operatorname{grad} f\|_{H^1} \|\operatorname{grad} g\|_{H^1} \\ + \|g\|_{L^\infty} \left(\|\operatorname{grad} g\|_{L^2} + \|D\operatorname{grad} g\|_{L^2} \right) . \end{aligned} \tag{A.32}$$

Since $\|f\|_{L^\infty} \leq C \|f\|_{\mathcal{B}^2}$ this proves

$$\|f \cdot g\|_{\mathcal{B}^2} \leq C_3 \|f\|_{\mathcal{B}^2} \|g\|_{\mathcal{B}^2} . \tag{A.33}$$

For $k > 2$ the estimate (A.24) is shown accordingly. Q.E.D.

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