One–Forms on Spaces of Embeddings : A Frame Work for Constitutive Laws in Elasticity

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0. Introduction

The contribution to this volume comes from non–linear functional analysis and is motivated by a traditional part of physics. More precisely, the motivation arises from elasticity theory : Given a body, thought of as a smooth compact manifold M', possibly with boundary, moving and deforming in $\mathbb{R}^n$, the latter equipped with a fixed scalar product. The deformations shall be such that the diffeomorphism type does not change. Hence M' is the image under a smooth embedding of a smooth compact manifold M, possibly with boundary ∂M. Therefore the collection $E(M,\mathbb{R}^n)$ of all smooth embeddings of M into $\mathbb{R}^n$ is the collection of all the configuration under considerations. It is a Fréchet manifold if endowed with the C^∞–topology.

The quality of the deformable material shall be characterized by a smooth one–form F on $E(M,\mathbb{R}^n)$. This is to say that $F(J)(L)$ is linear in each distortion $L \in C^\infty(M,\mathbb{R}^n)$ at each configuration $J \in E(M,\mathbb{R}^n)$. The real $F(J)(L)$ is interpreted as the work the distortion L causes at the configuration J. An approach of this kind of elasticity is described e.g. in [E,S], [Bi,Sc,So] and [Bi 1] to [Bi 4]. Why we deviate from the classical setting will be explained in section 6, where we also relate our treatment of the subject to the usual one as presented in [L,L].

If either the deformations mentioned above are subjected to (smooth) constraints then the ambient space in which the body deforms is no longer $\mathbb{R}^n$, it is a smooth submanifold thereof. Therefore we are forced to consider E(M,N) the collection of all smooth embeddings of M into a smooth manifold N with general Riemannian metric $<\ ,\ >$. Since the tangent bundle TE(M,N) is no longer trivial the techniques to treat one–forms are more complicated. To allow integral representation of the forms under consideration we suppose M and N to be oriented.

This sort of integral representation we have in mind relies on the metrics $\mathfrak{G}$ and $\mathfrak{G}^\partial$ on E(M,N) respectively $E(\partial M,N)$ and on a so–called dot–metric $\mathfrak{g}$ on $\mathfrak{A}^1{}_E(M,TN)$, the latter being the collection of all TN–valued one–forms of M covering embeddings. We begin to describe these ingredients in a little more in detail.

First we note that E(M,N) is an open subset of $C^\infty(M,N)$ endowed with the C^∞–topology. Since $C^\infty(M,N)$ is a Fréchet manifold (cf.[Bi,Sn,Fi]) E(M,N) inherits this structure, too. The tangent space $T_J E(M,N)$ at each configuration $J \in E(M,N)$ can be identified with $C^\infty_J(M,TN)$, the collection of all smooth vector fields along J. Equipping this spaces with the C^∞–topology it turns into a nuclear Fréchet space. The tangent bundle of $E(M,N)$ is $C^\infty_E(M,TN)$; it consists of all maps in $C^\infty(M,TN)$ covering embeddings. The functional analytic structure on each fibre of $C^\infty_E(M,TN)$ guaranties us enough nowhere vanishing vector fields on $E(M,N)$ (cf. appendix 3.2), a crucial observation for the general representation theorem 4.3 in section 4.

In the second and third section we introduce the basic geometric ingredients to define an integral representation of those types of one–forms on $E(M,N)$ which depend at each $J \in E(M,N)$ on the first jet of the fields in $C^\infty_J(M,TN)$. The metric $<,>$ on N yields via integration on M two natural metrics $\mathfrak{G}$ on $E(M,N)$ respectively $\mathfrak{G}^\partial$ on $E(\partial M,N)$ given at each $J \in E(M,N)$ and each $j \in E(\partial M,N)$ by

$$\mathfrak{G}(J)(L_1,L_2) := \int_M <L_1,L_2> \mu(J) , \quad \forall\, L_1,L_2 \in C^\infty_J(M,TN)$$

and

$$\mathfrak{G}^\partial(j)(l_1,l_2) := \int_{\partial M} <l_1,l_2> i_n\mu(j) , \quad \forall\, l_1,l_2 \in C^\infty_j(\partial M,TN),$$

respectively. Both are invariant under the group Diff^+M of all smooth orientation preserving diffeomorphisms of M and under any group $\mathfrak{J}$ of orientation preserving isometries of N.

To formulate an integral representation involving the first jet dependence we consider in section 3 the so–called dot–metric on $\mathfrak{A}^1{}_E(M,TN)$, the collection of all smooth TN–valued one–forms covering embeddings. Endowed with the C^∞–topology $\mathfrak{A}^1{}_E(M,TN)$ is a vector bundle on $E(M,N)$ with fibre $\mathfrak{A}^1_J(M,TN)$. The dot–metric $\mathfrak{g}$ is defined by

$$\mathfrak{g}(J)(\alpha,\mathfrak{b}) = \int_M \alpha\cdot\mathfrak{b}\, \mu(J) , \qquad \forall\, \alpha,\mathfrak{b} \in \mathfrak{A}^1{}_E(M,TN),$$
$$\text{and} \quad \forall\, J \in E(M,N) ,$$

where $\alpha\cdot\mathfrak{b}$ is a smooth real valued function on M, based on the trace inner product of bundle endomorphisms of TM. These endomorphisms are obtained by representing $\mathfrak{a}$ and $\mathfrak{b}$ with respect to TJ. The dot–metric shares the invariance under $\mathrm{Diff}^+ M$ and $\mathfrak{J}$, too.

Next let us denote by ∇ the Levi–Civita connection on N. The collection $\mathfrak{L}_E(M,TN)$ consisting of all ∇L with $L \in C^\infty_E(M,TN)$ is a subspace of $\mathfrak{A}^1{}_E(M,TN)$. Then $\mathfrak{g}$ restricted to $\mathfrak{L}_E(M,TN)$ is a generalization of the classical Dirichlet integral (cf.[Bi 2]). We call a smooth one–form F on E(M,N) to be $\mathfrak{g}$–representable if there is a smooth map

$$\mathfrak{a} : E(M,N) \longrightarrow \mathfrak{A}^1{}_E(M,TN)\,,$$

such that

$$F(J)(L) = \int_M \mathfrak{a}(J)\cdot\nabla L\; \mu(J) \tag{0.1}$$

holds for all variables of F.

The next step in section 4 will be to show that associated with any $\mathfrak{g}$–representable one–form F there is a smooth vector field $\mathfrak{H}$ on E(M,N), satisfying

$$F(J)(L) = \int_M \nabla\mathfrak{H}(J)\cdot\nabla L\; \mu(J) \tag{0.2}$$

for all variables of F. The integrand now reflects the first jet dependence of F mentioned earlier. The existence of such a vector field $\mathfrak{H}$ is based on the fact that for each $J \in E(M,N)$ the TN–valued one–form $\mathfrak{a}(J)$ on M defines an elliptic boundary value problem (a Neumann–type of problem) of which the solvability is guaranteed by [Hö 2].

Converting the right hand side of (0.2) into

$$F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(J),L>\mu(J) + \int_{\partial M} <\nabla_n\mathfrak{H}(J),L> i_n\mu(J)\,, \tag{0.3}$$

where $\mathfrak{n}$ denotes the positively oriented normal on ∂M of unit length given by $J^*<\,,\,>$. Here $\Delta(J)$ denotes the Laplacean associated with ∇ and the metric $J^*<\,,\,>$ on M.

If F characterizes the physical properties of a medium deformable in N, then

$\Delta(J)\mathfrak{H}(J)$ and $\nabla_n\mathfrak{H}(J)$ are the force densities acting upon M and ∂M respectively.

In the following section we restrict our attention to $N = \mathbb{R}$ and $< , >$ being a fixed scalar product on $\mathbb{R}^n$. We will observe, that the first jet dependence of F is, in the physical interpretation, equivalent to say that we confine ourselves to those embeddings for which the center of mass is fixed.

Finally we show that the reason why we characterize the deforming medium in $\mathbb{R}^n$ via the notion of a one–form. The classical elasticity as described e.g. in [L,L] works on all the Riemannian metrics on M which are pull–backs of the fixed scalar product of $\mathbb{R}^n$ by all elements of $E(M,\mathbb{R}^n)$. It is not clear as to whether this collection of metrics endowed with the C^∞–topology is a manifold or not. It is the collection $\mathfrak{M}(M)$ of all Riemannian metrics provided the codimension of M in $\mathbb{R}^n$ is high enough as the celebrated theorem of Nash states (cf.[St]). Since in general N does usually not admit any non–trivial orientation preserving isometry groups we may not necessarily be able to work with the notion of a symmetric stress tensor (cf. below). This motivates us to lift the description of elasticity up to E(M,N) and to characterize there this medium by first jet depending one–forms.

As it is shown in e.g. [Bi 4] the description of elasticity given in [L,L] is included in ours. Moreover, a theorem in [S] shows that the smooth one–form F on $E(M,\mathbb{R}^n)$ can be replaced by a smooth stress tensor assignment provided F is invariant under SO(n) and no infinitesimal rigid motion causes any work.

1. Geometric preliminaries and the Fréchet manifold E(M,N)

Let M be a compact, oriented, connected smooth manifold with (oriented) boundary ∂M and N be a connected, smooth and oriented manifold with a Riemannian metric $< , >$. The Levi–Civita connection of $< , >$ on N is denoted by ∇ and by d in the euclidean case, i.e. if $N = \mathbb{R}^n$ and $< , >$ is assumed to be a fixed scalar product. For $J \in E(M,\mathbb{R}^n)$ we define a Riemannian metric on M by setting

$$(1.1) \qquad m(J)(X,Y) := <TJX,TJY> , \quad \forall X,Y \in \Gamma(TM)$$

and one on ∂M via the formula

$$(1.2) \qquad m(j)(X,Y) := <TjX,TjY> , \quad \forall X,Y \in \Gamma(T(\partial M))$$

(here $j := J|\partial M$). More customary are the notations $J^* < , >$ and $j^* < , >$ for m(J) and m(j) respectively.
We use ΓTQ to denote the collection of all smooth vector fields of any smooth manifold Q (with or without boundary). Moreover by $\pi_Q : TQ \to Q$ we mean the canonical projection.

Let $L : M \to TM$ be a smooth map. Then $f = \pi_N \circ L \in C^\infty(M,N)$ and L is a "vector field along f". For a fixed f, the set of all such "vector fields along f" is precisely the tangent space at f to $C^\infty(M,N)$ (cf. [Bi,Sn,Fi] and also below at the end of this section).

Next, let ∇ be a (linear) connection on N, i.e. in TN. There is the associated splitting of $T^2N = T(TN)$ into the canonically defined vertical bundle V(TN) and the horizontal bundle H(TN) defined by ∇ (cf.[G,H,V]). Since $V(TN) = \ker(T\pi_N)$, the fibre $V_v(TN)$ at the point $v \in TN$ is $T_v(T_qN)$ with $q = \pi_N v$ and hence, there is a natural isomorphism $\zeta_v : V_v(TN) \to T_{\pi_N v}N$ for every $v \in TN$. These isomorphisms yield a bundle map $\zeta : V(TN) \to TN$ covering the projection π_N. Lastly, let $P : T^2N \to V(TN)$ be the projection with kernel H(TN).

The covariant derivative ∇L of L is now defined as follows : For $X \in \Gamma(TM)$, $TL \cdot X$ is a map $M \to T^2N$ and we set

$$\nabla_X L := \zeta\, P(TL\cdot X)\,.$$

In our applications, ∇ will be the Levi–Cività connection of the Riemannian manifold $(N,<\,,\,>)$ and in this situation, the Levi–Cività connections of $(M,m(J))$, $(\partial M,m(j))$ respectively are obtained as follows :

$TN\,|\,J(M)$ splits into $TJ(TM)$ and its orthogonal complement $(TJ(TM))^{\perp}$ (the Riemannian normal bundle of J) and hence any $Z \in \Gamma(J(M),TN)$ has an orthogonal decomposition $Z = Z^{\top} + Z^{\perp}$, where the tangential component $Z^{\top}$ is a section of $TJ(TM)$ and so is of the form $Z^{\top} = TJ\cdot U$ for a unique $U \in \Gamma(TM)$.

If now $Y \in \Gamma(TM)$, then TJY is a smooth map $M \longrightarrow TN$ and therefore, the above covariant derivative $\nabla(TJY)$ is well–defined. We use this to define the vector field $\nabla(J)_X Y$ on M by the equation

$$TJ(\nabla(J)_X Y) = \nabla_X(TJY) - (\nabla_X(TJX))^{\perp}\,, \tag{1.3}$$

for all $X,Y \in \Gamma(TM)$. Moreover, if now $X,Y \in \Gamma T\partial M$, then

$$Tj(\nabla(j)_X Y) = TJ(\nabla(J)_X Y) - m(j)(W(j)X,Y)\cdot N(j) \tag{1.4}$$

defines a vector field $\nabla(j)_X Y$ on ∂M. Here $W(j)$, the Weingarten map, is defined as follows : By assumption, M is oriented and hence the normal bundle $(TM\,|\,\partial M)/T(\partial M)$ has a nowhere vanishing section s which is used to define the induced orientation of ∂M. Under the Riemannian structure $m(J)$, the normal bundle of ∂M is isomorphic to $T(\partial M)^{\perp}$ and as a consequence, this bundle now has a section $\mathfrak{n}$ of unit length which corresponds to a multiple of s by a non–vanishing positive function. This $\mathfrak{n}$ is the positive unit normal vector field along ∂M. With this, let $N(j) = TJ\cdot\mathfrak{n}$ and now set

$$TJ\cdot W(j)Z = (\nabla_Z N(j))^{\top}\,, \quad \forall\, Z \in \Gamma(T\partial M)\,. \tag{1.5}$$

As mentioned earlier, this determines $W(j)$ uniquely. Note here that $N = \mathbb{R}^n$, we may replace TJ and Tj by their "principal parts" dJ and dj respectively. In this particular case, we moreover define the *second fundamental form* $\mathfrak{f}(J)$ of J under the additional assumptions that $\partial M = \emptyset$ and $\dim(M) = n-1$, where now $N(j)$ is replaced by the positive unit normal field along J and $W(j)$ is defined as in (1.5). The two–tensor $\mathfrak{f}$ then is given by

$$\mathfrak{f}(j)(X,Y) = m(J)(W(j)X,Y) ,$$

for $J \in E(M,\mathbb{R}^n)$ and $X,Y \in \Gamma(T\partial M)$. Note finally that now $H(j) := \operatorname{tr} W(j)$ and $\kappa(j) = \det(W(j))$ are respectively the (unnormalized) *mean curvature* and the *Gaussian curvature* of $j(\partial M) \subset \mathbb{R}^n$. References for this section are e.g. [A,M,R], [Be,Go] and [G,H,V].

It is well–known that the set $C^\infty(M,N)$ of smooth maps from M into N endowed with Whitney's C^∞–topology is a Fréchet manifold (cf.e.g.[Bi,Sn,Fi]). For a given $K \in C^\infty(M,N)$, the tangent space $T_K C^\infty(M,N)$ is the Fréchet space $C^\infty_K(M,TN) = \{L \in C^\infty(M,TN) \,|\, \tau_N \circ L = K\} \cong \Gamma(K^*TN)$ and the tangent bundle $TC^\infty(M,N)$ is identified with $C^\infty(M,TN)$, the topology again being the C^∞–topology. In all this, M is assumed to be compact.

The set $E(M,N)$ of C^∞–embeddings $M \longrightarrow N$ is open in $C^\infty(M,N)$ and thus is a Fréchet manifold whose tangent bundle we denote by $C^\infty_E(M,TN)$; it is an open submanifold of $C^\infty(M,TN)$, fibred over $E(M,N)$ by "composition with τ_N". Moreover, if $\partial M = \emptyset$, $E(M,N)$ is a principal $\mathrm{Diff}(M)$–bundle under the obvious right $\mathrm{Diff}(M)$–action and the quotient $U(M,N) = E(M,N)/\mathrm{Diff}(M)$ is the manifold of "submanifolds of type M" of N (cf. the above reference, ch.5, and further literature quoted there).
Lastly, the set $\mathfrak{M}(M)$ of all Riemannian structures on M is a Fréchet manifold since it is an open convex cone in the Fréchet space of smooth, symmetric bilinear forms on M. Moreover, the maps

$$m : E(M,N) \longrightarrow \mathfrak{M}(M)$$

and

$$m : E(\partial M,N) \longrightarrow \mathfrak{M}(\partial M)$$

are smooth (cf.[Bi,Sn,Fi]).

By an E–valued one–form α on M, where E is a vector bundle over N, we mean a smooth map

$$\alpha : TM \longrightarrow E$$

for which $\alpha|T_pM$ is linear for all $p \in M$. We denote the set of such one–forms by $\mathfrak{A}^1(M,E)$ and now obtain the following description of its structure :

The requirement that $\alpha \in \mathfrak{A}^1(M,E)$ should be linear along the fibres of TM means that there is a (smooth) map $f : M \longrightarrow N$ such that $\alpha|T_pM$ is a linear map into $E_{f(p)}$ for $p \in M$, in other words, that α is a bundle map $TM \longrightarrow E$ over f :
There is $f \in C^\infty(M,N)$ such that $\pi_E \circ \alpha = f \circ \tau_M$ (where π_E , τ_M are the respective bundle projections). The set of such one–forms is naturally identified with the Fréchet space $A^1(M,f^*E)$. This shows that

$$\mathfrak{A}^1(M,E) = \bigcup_f \{A^1(M,f^*E) \mid f \in C^\infty(M,N)\} .$$

It is clear from the construction that there is a natural surjection

$$\beta : \mathfrak{A}^1(M,E) \longrightarrow C^\infty(M,N)$$

whose fibres are the Fréchet spaces $A^1(M,f^*E)$.

The map β is (set–theoretically !) locally trivial : $f \in C^\infty(M,N)$ has an open neighbourhood U_f such that there exists a fibre–preserving, fibrewise linear bijection

$$\varphi_f : \beta^{-1}(U_f) \longrightarrow U_f \times A^1(M,f^*E) ,$$

which also is topological on each fibre; thus, for each $g \in U_f$, the restriction of φ_f to $\beta^{-1}(g)$ is a linear and topological isomorphism onto $A^1(M,f^*E)$.

The assertion of local triviality can be established along the following lines (cf.[A]) :
One chooses a neighbourhood U_f of f in $C^\infty(M,N)$ which is diffeomorphic to some

open, *convex* neighbourhood of $0 \in T_f C^\infty(M,N) = \Gamma(f^* TN)$. By the very construction of the usual Fréchet manifold structure of $C^\infty(M,N)$, this is always possible (cf. e.g. [Bi,Sn,Fi], ch.5 and its references). Accordingly, there now exists a smooth contraction of U_f onto $\{f\}$, i.e. a smooth map $c : \mathbb{R} \times U_f \longrightarrow C^\infty(M,N)$, such that $c(1,.)$ is the identity of U_f, $c(t,U_f) \subset U_f$ for $0 \leq t \leq 1$, and $c(0,g) = f$ for every $g \in U_f$. In particular, every $g \in U_f$ is smoothly homotopic to f by a homotopy induced by c. Accordingly, the choice of a linear connection ∇ in E induces an isomorphism $g^* E \cong f^* E$ as in [G,H,V]; the corresponding isomorphisms $A^1(M,g^* E) \cong A^1(M,f^* E)$ now yield the desired trivialization φ_f.

Suppose next that U_1, U_2 are neighbourhoods of $f_{1,2}$ chosen as above and that $U_{1,2} := U_1 \cap U_2 \neq \emptyset$; let φ_i be the corresponding trivializations. Firstly, then, $U_{1,2} \times A^1(M,f^* E)$, $i=1,2$, will be open submanifolds of $U_i \times A^1(M,f_i^* E)$ and secondly, the compositions $\varphi_2 \varphi_1^{-1}$, $\varphi_1 \varphi_2^{-1}$ are diffeomorphisms of these two submanifolds. As a consequence, there exist a unique topology and differentiable structure on $\mathfrak{A}^1(M,E)$ with the following properties :
The sets $\beta^{-1}(U_f)$ obtained as above are *open submanifolds*, diffeomorphic to $U_f \times A^1(M,f^* E)$ under the maps φ_f. Thus, the model space for $\beta^{-1}(U_f)$ is the Fréchet space $T_f C^\infty(M,N) \times A^1(M,f^* E)$. Lastly, the construction shows that with this differentiable structure, $\mathfrak{A}^1(M,E)$ becomes a smooth Fréchet *vector bundle* over $C^\infty(M,N)$ with bundle projection β.

2. The metric $\mathfrak{G}$ on E(M,N)

The Riemannian structure $<\,,\,>$ of N induces a "Riemannian structure" $\mathfrak{G}$ on E(M,N) as follows: For $J \in E(M,N)$, let $\mu(J)$ be the Riemannian volume defined on M by the given orientation and the structure m(J). For any two tangent vectors $L_1, L_2 \in C^\infty_J(M,TN)$, we set

$$(2.1) \qquad \mathfrak{G}(J)(L_1,L_2) := \int_M <L_1,L_2>\mu(J)\,.$$

It is clear, that $\mathfrak{G}(J)$ is a continuous, symmetric, positive–definite bilinear form on $C^\infty_J(M,TN)$. In the same manner, one obtains the metric $\mathfrak{G}^\partial$ on $E(\partial M,N)$.

The metrics $\mathfrak{G}$ and $\mathfrak{G}^\partial$ possess some invariance properties which will become important later: Let Diff^+M be the group of orientation–preserving diffeomorphisms of M. As a subgroup of Diff M, it operates (freely) on the right on E(M,N) as well as on $E(\partial M,N)$ by

$$(2.2) \qquad \begin{array}{rcl} E(M,N) \times \mathrm{Diff}^+M & \xrightarrow{\Phi} & E(M,N) \\ (J,\varphi) & \longrightarrow & J \circ \varphi \end{array}$$

for a fixed φ, we also write $R_\varphi J$ for $J \circ \varphi$.

Similarly, if $\mathfrak{J}$ is any group of orientation–preserving isometries of N, then it operates on the left on E(M,N) as well as on $E(\partial M,N)$ by

$$(2.3) \qquad \begin{array}{rcl} \mathfrak{J} \times E(M,N) & \longrightarrow & E(M,N) \\ (g,J) & \longrightarrow & g \circ J \end{array}$$

for fixed g, we also write $L_g J$ for $g \circ J$.

The geometry of these actions will be dealt with elsewhere, but we need the following – rather obvious! – result for some basic invariance properties of one–forms on E(M,N) :

Proposition 2.1 :

Both $\mathfrak{G}$ and $\mathfrak{G}^\partial$ are invariant under Diff^+M and $\mathfrak{J}$.

Proof:

The Diff^+M–invariance is usual invariance of integration over M :

$$(2.4) \qquad R_\varphi^* \mathfrak{G}(J)(L_1,L_2) = \mathfrak{G}(J\circ\varphi)(L_1\circ\varphi, L_2\circ\varphi)$$
$$= \int_{\varphi(M)} <L_1,L_2> \circ\, \varphi\, \mu(J\circ\varphi)$$
$$= \mathfrak{G}(J)(L_1,L_2)\,.$$

Next, if $g \in \mathfrak{J}$, then $\mu(g\circ J) = \mu(J)$ and hence

$$(2.5) \qquad L_g^* \mathfrak{G}(J)(L_1,L_2) = \mathfrak{G}(g\circ J)(Tg\circ L_1, Tg\circ L_2)$$
$$= \int_M <Tg\circ L_1, Tg\circ L_2> \mu(g\circ J)$$
$$= \mathfrak{G}(J)(L_1,L_2)\,.$$

Similar arguments establish the claim for $\mathfrak{G}^\partial$.

□

3. The fibred space $\mathfrak{L}_E(M,TN)$ and its dot metric

To begin with, denote by $\mathfrak{A}^1{}_E(M,TN)$ the subset of $\mathfrak{A}^1(M,TN)$ consisting of all TN–valued one–forms covering embeddings $M \longrightarrow N$. This is the inverse image of $E(M,N)$ under the projection $\beta : \mathfrak{A}^1(M,TN) \longrightarrow C^\infty(M,N)$, hence is an open submanifold and, in fact, is itself a (Fréchet) vector bundle whose fibre at J we denote by $\mathfrak{A}^1{}_J(M,TN)$.

By construction of $m(J)$, TJ is fibrewise isometric and accordingly, the linear algebra outlined in appendix 3.1 (cf. below) may be used to write $\mathfrak{a} \in \mathfrak{A}^1{}_J(M,TN)$ in the form

$$\mathfrak{a} = c(\mathfrak{a},TJ)\cdot TJ + TJ\cdot A(\mathfrak{a},TJ) \tag{3.1}$$

for suitable bundle endomorphisms $c(\mathfrak{a},TJ)$ of $TN\,|\,J(M)$ and $A(\mathfrak{a},TJ)$ of TM; these endomorphisms are smooth and continuous linear functions of $\mathfrak{a}$. The second summand on the right can also be written as $\hat{A}(\mathfrak{a},TJ)TJ$ (cf. appendix 3.2), and so $\mathfrak{a} = c(\mathfrak{a},TJ) + \hat{A}(\mathfrak{a},TJ)$. The usual "trace inner product" for endomorphisms of TN then yields the dot product

$$\mathfrak{a}\cdot\mathfrak{b} := -\tfrac{1}{2}\,\mathrm{tr}\; c(\mathfrak{a},TJ)\cdot c(\mathfrak{b},TJ) + \mathrm{tr}\; A(\mathfrak{a},TJ)\cdot A^*(\mathfrak{b},TJ)\,, \tag{3.2}$$

A^* the adjoint of A formed fibre–wise with respect to $m(J)$, and we define

$$\mathfrak{g}(TJ)(\mathfrak{a},\mathfrak{b}) := \int_M \mathfrak{a}\cdot\mathfrak{b}\;\mu(J)\,. \tag{3.3}$$

This, yields a smooth and continuous, symmetric and positive–definite bilinear form on the Fréchet space $\mathfrak{A}^1{}_J(M,TN)$, the "dot metric".
We shall also need a subfibration of $\mathfrak{A}^1{}_E(M,TN)$, defined by

$$\mathfrak{L}_E(M,TN) := \{\nabla L\,|\,L \in C^\infty_E(M,TN)\}\,, \tag{3.4}$$

whose fibres we denote by $\mathfrak{L}_J(M,TN)$ $(= \mathfrak{L}_E(M,TN) \cap \mathfrak{A}^1{}_J(M,TN))$; evidently these are subspaces of the Fréchet spaces $\mathfrak{A}^1{}_J(M,TN)$; for more information, cf. appendix 3.2.

Next, we introduce the Laplacean $\Delta(J)$ which will depend on J via $m(J)$; cf.[Ma] and a sew remarks in appendix 3.2 :
For $K \in C^\infty{}_J(M,TN)$, we define the covariant divergence by

$$\nabla^*(J)K := 0\,, \tag{3.5}$$

as usual, while following [Ma], $\nabla^*(J)a$ for $a \in \mathfrak{A}^1{}_J(M,TN)$ is given locally by

$$\nabla^*(J)a := -\sum_{r=1}^{n} \nabla_{E_r}(a)(E_r)\,, \tag{3.6}$$

(E_r) a local orthonormal frame with respect to $m(J)$; $\nabla_X a = \nabla(J)_X a$ is defined in the standard manner by

$$(\nabla(J)_X a)(Y) = \nabla_X(aY) - a(\nabla(J)_X Y)\,, \quad \forall\, X,Y \in \Gamma(TM).$$

To see that this definition does not depend on the moving frames chosen we write a as a finite sum

$$a = \sum_i \gamma^i \otimes s_i\,, \tag{3.7}$$

with $\gamma^i \in A^1(M,\mathbb{R})$ and $s_i \in T_J E(M,N)$. Moreover, let $a(\gamma^i,J)$ be the smooth strong bundle endomorphism of TM such that

$$\nabla(J)_X(\gamma^i)(Y) = m(J)(a(\gamma^i,J)X,Y)\,, \tag{3.8}$$

holds for all pairs $X,Y \in \Gamma(TM)$ and for each i. In addition let $Y^i \in \Gamma(TM)$ for each i be such that

$$\gamma^i(X) = m(J)(Y^i,X)\,, \quad \forall\, X \in \Gamma(TM). \tag{3.9}$$

With these data it is a matter of routine to show that

$$\nabla^*(J)\alpha = \sum_i (\operatorname{tr} a(\gamma^i,J)\cdot s_i + \nabla_{Y^i} s_i) , \tag{3.10}$$

an expression independent of any moving frame.

Clearly if $\gamma \in A^1(M,\mathbb{R})$ and $\nabla = d$ then

$$d^*\gamma = -\operatorname{div}_J Y ,$$

provided that $\quad \gamma(X) = m(J)(Y,X) , \quad \forall X,Y \in \Gamma(TM).$

$\Delta(J)$ is then defined by

$$\Delta(J) := \nabla\nabla^*(J) + \nabla^*(J)\nabla , \tag{3.11}$$

The Laplacean $\Delta(J)$ is elliptic for any $J \in E(M,N)$ (cf.[Pa]). As we will see below it is self–adjoint with respect to $\mathfrak{G}(J)$ if $\partial M = \emptyset$. For each $K \in T_J E(M,N)$ equation (3.6) yields

$$\Delta(J)K = \nabla^*(J)\nabla K = -\sum_{r=1}^{n} \nabla_{E_r}(\nabla K)(E_r) . \tag{3.12}$$

Remark 3.1 :
Suppose that $\gamma \in A^1(M,\mathbb{R})$ and $\nabla = d$. Define the vector field Y on M by $\gamma(X) = m(J)(Y,X)$ ($\forall X \in \Gamma(TM)$). Then it is clear that $d^*\gamma = -\operatorname{div}_J Y$, div_J the classical divergence operator with respect to $\mu(J)$.

The following theorem will be a basic tool in our studies of one forms on $E(M,N)$:

Theorem 3.2 :
For any $J \in E(M,N)$, any $\alpha \in \mathfrak{A}^1{}_E(M,TN)$ and two $L_1, L \in C^\infty_J(M,TN)$ the following two relation hold

$$g(J)(\alpha,\nabla L) = \mathfrak{G}(J)(\nabla^*(J)\alpha,L) + \mathfrak{G}^\partial(j)(\alpha(n),l)\,, \tag{3.13}$$

and

$$g(J)(\nabla L_1,\nabla L) = \mathfrak{G}(J)(\Delta(J)L_1,L) + \mathfrak{G}^\partial(j)(\nabla_n L_1,l)\,, \tag{3.14}$$

where $j := J|\partial M$ and $l := L|\partial M$. Here ∇ denotes the Levi–Civita connection of the metric $<,>$ on N. Let $\mathfrak{K}_J := \{L \in C^\infty_J(M,TN)|\ \nabla L = 0\}$ for any $J \in E(M,N)$, then

$$L \in \mathfrak{K}_J \Leftrightarrow (\Delta(J)L = 0 \text{ and } \nabla_n L = 0). \tag{3.15}$$

In fact $\dim \mathfrak{K}_J < \infty$. Equation (3.14) implies in turn a Green's equation

$$\begin{aligned} &{}_M\!\int <\Delta(J)K,L>\ \mu(J) - {}_M\!\int <K,\Delta(J)L>\ \mu(J) \\ &= {}_{\partial M}\!\int <\nabla_n L,K>\ i_n\mu(J) - {}_{\partial M}\!\int <\nabla_n K,L>\ i_n\mu(J)\,. \end{aligned} \tag{3.16}$$

Here $i_n\mu(J)$ is the volume element on ∂M defined by $\mu(J)$. Moreover, if $\partial M = \emptyset$ then α is g–orthogonal to all of $\mathfrak{A}^1{}_E(M,TN)$, iff $\nabla^*(J)\alpha = 0$.

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Proof :
Writing any $L \in C^\infty(M,TN)$ relative to a given $J \in E(M,N)$ in the form

$$L = TJ\, X(L,J) + L^\perp, \tag{3.17}$$

with a unique $X(L,J) \in \Gamma(TM)$ (and $L^\perp$ being such that $L^\perp(p)$ is the component normal to TJT_pM for all $p \in M$), we have the following formula at hand :

$$\nabla_X L = TJ\,\nabla_X X(L,J) + (\nabla_X L)^\perp\,,\quad \forall X \in \Gamma(TM)\,. \tag{3.18}$$

From this equation we read off the coefficients in the decomposition (3.1) :

$$(3.19) \qquad c(\nabla L, T)TJ = (\nabla L)^{\perp} ,$$

as well as

$$(3.20) \qquad A(\nabla L, TJ) = \nabla X(L,J) + W(J,L), \qquad \forall L \in C^{\infty}(M,TN) \text{ and } \forall J \in E(M,N) .$$

Here $W(J,L)$ is given by $TJW(J,L)X = (\nabla L^{\perp}X)^{\top}$, where, once again, $\perp$ denotes the component in $TN|J(M)$ orthogonal to $TJ(TM)$, while $\top$ is the component tangential to $J(M)$, i.e. $TJ(TM)$.

For each $a \in \mathfrak{A}^1(M,TN)$ and for each $J \in E(M,N)$, we write on the other hand

$$(3.21) \qquad a = \bar{A}(\nabla a, TJ)TJ ,$$

with $\bar{A}(a,TJ) : TN|J(M) \longrightarrow TN|J(M)$ the smooth bundle endomorphism introduced above. Then for any moving frame (E_r) on M, orthonormal with respect to $m(J)$, we deduce

$$\begin{aligned} a\cdot\nabla L &= \sum_{r=1}^{m} <\bar{A}^{*}(a,TJ)\cdot\bar{A}(\nabla L,TJ)TJE_r ,TJE_r> \\ &= \sum_{r=1}^{m} <\bar{A}^{*}(a,TJ)\cdot\nabla_{E_r} L ,TJE_r> , \end{aligned}$$

$\bar{A}^{*}(a,TJ)$ being the adjoint of $\bar{A}(a,TJ)$ formed with respect to $< , >$. Hence

$$\begin{aligned} a\cdot\nabla L = \sum_{r=1}^{m} &<\nabla_{E_r}(\bar{A}^{*}(a,TJ)L),TJE_r> \\ &- \sum_{r=1}^{m} <L ,\nabla_{E_r}(\bar{A}(a,TJ))TJE_r> \end{aligned}$$

yields

$$\begin{aligned} a\cdot\nabla L = \sum_{i=1}^{m} &<\nabla_{E_r}(\bar{A}^{*}(a,TJ)L),TJE_r> \\ &+ <\nabla^{*}(J)a,L> + \sum_{i=1}^{m} <\bar{A}(a,TJ)\,\nabla_{E_r}(TJ)E_r,L> . \end{aligned}$$

Since $(\bar{A}^*(a,TJ)L)^\top = TJZ(a,L,J)$ for some well–defined $Z(a,L,J)$ and since $\nabla_{E_r}(TJ)E_r$ is pointwise normal to $TJTM$ the following series of equations are immediate :

$$\begin{aligned}(3.22)\qquad a\cdot\nabla L &= -\sum_{i=1}^{m} <\nabla_{E_r}(c(a,TJ)L),TJE_r> + \mathrm{div}_J Z(a,L,J)\\ &\quad + <\nabla^*(J)a,L> + \sum_{i=1}^{m} <c(a,TJ)\nabla_{E_r}(TJ)E_r,L^\top>\\ &= -\sum_{i=1}^{m} <\nabla_{E_r}(c(a,TJ)L^\perp),TJE_r>\\ &\quad -\sum_{i=1}^{m} <\nabla_{E_r}(c(a,TJ)L^\top),TJE_r>\\ &\quad + \mathrm{div}_J Z(a,L,J) + <\nabla^*(J)a,L>\\ &\quad + \sum_{i=1}^{m} <c(a,TJ)\nabla_{E_r}(TJ)E_r,L^\top>\\ &= -\sum_{i=1}^{m} <\nabla_{E_r}(c(aTJ)L^\perp),TJE_r>\\ &\quad + \mathrm{div}_J Z(a,L,J) + <\nabla^*(J)a,L>,\end{aligned}$$

where div_J the divergence operator associated with $m(J)$. Writing $c(a,TJ)L^\perp = TJ\ U(a,L,J)$, for some well defined $U(a,L,J) \in \Gamma(TM)$, we obtain

$$(3.23)\qquad a\cdot\nabla L = -\mathrm{div}_J U(a,L,J) + \mathrm{div}_J Z(a,L,J) + <\nabla^*(J)a,L>\,.$$

Here $U(a,L,J)$ is given by $TJ\ U(a,L,J) = c(a,TJ)\ L^\perp$. In case $a = \nabla K$, then (3.23) turns into

$$(3.24)\qquad \nabla K\cdot\nabla L = -\mathrm{div}_J U(K,L,J) + \mathrm{div}_J Z(K,L,J) + <\Delta(J)K,L>\,.$$

Integrating (3.23) and (3.24) and applying the theorem of Gauss yields the desired equations (3.13) and (3.14). Since ∇ and $\Delta(J)$ are elliptic (cf. appendix 3.2) $\dim \mathfrak{K}_J < \infty$ as shown, e.g. in [Pa] and [Hö 2]. The rest of the routine arguments in this proof are left to the reader.

□

We close this section by showing that the metric $\mathfrak{g}$ on the fibres of $\mathfrak{L}_E(M,TN)$ also possesses the *invariance under* Diff^+M and any group orientation–preserving isometries on N :
For any choice $\varphi \in \mathrm{Diff}^+M$, $J \in E(M,N)$ and $L \in C^\infty(M,TN)$ we form

$$(3.25) \qquad \nabla(L\circ\varphi) = \nabla L \circ T\varphi$$

and represent $\nabla(L\circ\varphi)$ with respect to $T(J\circ\varphi)$ yielding

$$(3.26) \qquad \nabla(L\circ\varphi) = c(\nabla(L\circ\varphi),T(J\circ\varphi))\cdot T(J\circ\varphi)A(\nabla(L\circ\varphi),T(J\circ\varphi)) .$$

Multiplying $\nabla(L\circ\varphi)$ with $(T\varphi)^{-1}$ and comparing the resulting coefficients of (3.26) with those of (3.1) shows

$$c(\nabla L,TJ)\circ\varphi = c(\nabla(L\circ\varphi),T(J\circ\varphi))$$

and

$$A(\nabla L,TJ)\circ\varphi = T\varphi\, A(\nabla(L\circ\varphi),T(J\circ\varphi))\cdot(T\varphi)^{-1} .$$

Now we verify

$$\begin{aligned}(3.27) \qquad & \mathfrak{g}(J\circ\varphi)(\nabla(L_1\circ\varphi),\nabla(L_2\circ\varphi)) \\ &= -\frac{1}{2}\int_M \mathrm{tr}\; c(\nabla L_1,TJ)\cdot c(\nabla L_2,TJ)\circ\varphi\; \mu(J\circ\varphi) \\ &\quad + \int_M \mathrm{tr}\; A(\nabla L_1,TJ)\cdot A^*(\nabla L_2,TJ)\circ\varphi\; \mu(J\circ\varphi) \\ &= \mathfrak{g}(J)(\nabla L_1,\nabla L_2) ,\end{aligned}$$

proving the Diff^+M–invariance of $\mathfrak{g}$ at TJ. To show the $\mathfrak{J}$–invariance we let $g \in \mathfrak{J}$ and only need to remark that

$$(3.28) \qquad \nabla(Tg\circ L) = Tg \circ \nabla L .$$

holds. The rest is obvious. Therefore we have :

Proposition 3.3 :
The metric $\mathfrak{g}$ on $\mathfrak{L}_E(M,TN)$ is invariant under Diff^+M and any group $\mathfrak{J}$ of orientation–preserving isometries on N.

Appendix 3.1:
As indicated earlier, we present here some of the linear algebra used in the construction of the dot product used in this section. The arguments may be interpreted as fibrewise considerations for bundle maps or, with some obvious changes in the formulation, as considerations at the level of section modules.

The aim is to show that the dot product "essentially" is induced by the classical trace inner product in endomorphism rings of euclidean spaces and to this end, we now consider euclidean spaces E,F with inner products $< , >$ and a fixed <u>isometry</u> α of E onto the subspace $E_1 \subset F$. For the sake of convenience, we write the elements of F as columns $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ with respect to the direct sum decomposition $F = E_1 \oplus E_1^{\perp}$; here, $e_1 \in E_1$ and $e_2 \in E_1^{\perp}$; let also $p_1 : F \longrightarrow E_1$, $p_2 : F \longrightarrow E_1^{\perp}$ be the respective orthogonal projections.

Any endomorphism D of F now is represented by a $2{\times}2$ – matrix

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

where $D_{11} \in L(E_1)$, $D_{22} \in L(E_1^{\perp})$, $D_{12} \in L(E_1^{\perp},E_1)$ and $D_{21} \in L(E_1,E_1^{\perp})$; the matrix acts on a column $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ by the usual rules of matrix algebra.

Next, let $\varphi \in L(E,F)$. We are going to write φ in the form

$$\varphi = c\alpha + \alpha A = c\alpha + \hat{A}\alpha \tag{3.29}$$

for suitable choices of $c \in L(F)$ and $A \in L(E)$ (or $\hat{A} \in L(F)$), both of them are linear functions of φ :

For $e \in E$, write $\varphi e = \begin{bmatrix} \varphi_1 e \\ \varphi_2 e \end{bmatrix}$; thus, $\varphi_1 = p_1\varphi$ and $\varphi_2 = p_2\varphi$. Firstly, since $E_1 = \mathrm{im}(\alpha)$, the expression $<\varphi e, \alpha f>$ (with $e, f \in E$) reduces to $<\varphi_1 e, \alpha f>$ and this bilinear form on E now can be written in the form $<Ae, f>$ for a unique $A \in L(E)$; in fact, since α is an isometry,

$$A = \alpha^{-1}\varphi_1 = \alpha^{-1}p_1\varphi .$$

There is a corresponding endomorphism A_1 of E_1, namely $A_1 = p_1\varphi\alpha^{-1}$ and the endomorphism $\hat{A}$ of F now is the extension by 0 of this map; in other words :

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{3.30}$$

Secondly, we wish to write $\varphi_2 = p_2\varphi$ in the form $c\alpha$ for some $c \in L(F)$ and it is clear that c is not automatically uniquely determined by this condition (unless $E_1 = F$), so that in the course of the construction, certain choices will have to be made. In a first step, let $c_1 : E_1 \to E_1^{\perp}$ be defined by $c_1 = P_2\varphi\alpha^{-1}$. Any extension of c_1 to an endomorphism of F then is of the form,

$$\begin{bmatrix} \beta & \gamma \\ c_1 & \delta \end{bmatrix}$$

and its action on αe is the map $\alpha e \longmapsto \begin{bmatrix} \beta\alpha e \\ c_1\alpha e \end{bmatrix}$; this forces the choice $\beta = 0$, but leaves γ, δ undetermined. The obvious choice for δ is 0 and with this, there now are three options for γ: $\gamma = 0$, $\gamma = c_1{}^*$ or $\gamma = -c_1{}^*$ (where "*" is the euclidean adjoint) and in all three cases, c will remain a linear function of φ. At this point, we make the choice $\gamma = -c_1{}^*$, so that we obtain

$$c = \begin{bmatrix} 0 & -c_1{}^* \\ c_1 & 0 \end{bmatrix}, \tag{3.31}$$

a skew–symmetric endomorphism of $F : c^* \longrightarrow -c$. In part, this choice is motivated by the usual splitting $so(F) = so(E_1) \oplus so(E_1{}^{\perp}) \oplus L(E_1,E_1{}^{\perp})$, cf. section 5.

Let now $\psi : E \longrightarrow F$ be a second linear map, written in the form $\psi = (D + \hat{B})\alpha$ under the construction just outlined. A simple calculation shows that

$$(c + A)(D + B)^* = -cD + \hat{A}\hat{B}^* + (c\hat{B}^* - \hat{A}^*D) ,$$

where the term in parentheses is *tracefree*. Moreover, the trace of $\hat{A}\hat{B}^*$ (in F) is easily seen to coincide with $tr_E(AB^*)$ since α is an isometry. Accordingly, the "trace inner product" in L(F) now reduces to $-tr_F(cD) + tr_E(AB^*)$. Thus, we see that the dot product $\varphi \cdot \psi$ in L(E,F) essentially is the inner product induced by the classical trace inner product under the construction $\varphi \mapsto c + \hat{A}$ – up to the factor $\frac{1}{2}$ in the first summand. We shall add some remarks on this point below, but firstly now indicate the application of the linear algebra outlined here to the actual constructions used in this section :
Pointwise, the role of α is played by TJ, that of φ by $\mathfrak{a} \in \mathfrak{A}^1_J(M,TN)$; accordingly $c(\mathfrak{a},TJ) = c$ and $A(\mathfrak{a},TJ) = A$. Note that this also shows that the bundle endomorphisms used above depend linearly on $\mathfrak{a}$.

Let us turn to the factor $\frac{1}{2}$ in equation (3.2); it appears because of the following reason : The endomorphism

$$-c(\mathfrak{a},TJ)\cdot c(\mathfrak{b},TJ)(J(p)) : T_{J(p)}N \longrightarrow T_{J(p)}N \tag{3.32}$$

of $T_{J(p)}N$ splits for each $p \in M$ into a direct sum of the two linear maps

$$-c(\mathfrak{a},TJ)\cdot c(\mathfrak{b},TJ)\,|\,TJT_pM$$

and

$$-c(\mathfrak{a},TJ)\cdot c(\mathfrak{b},TJ)\,|\,(TJT_pM)^{\perp} ,$$

both endomorphisms namely of TJT_pM and $(TJT_pM)^\perp$ respectively. Their traces are identical. Thus the factor $\frac{1}{2}$ allows us to take only the pointwise formed trace of

$$-c(a,TJ)\cdot c(b,TJ)\,|\,TJT_pM \tag{3.33}$$

into account. The endomorphism (3.33) can be pulled back to TM in the obvious manner. Hence in the dot product (3.2) contribute traces of endomorphisms of TM only.

Appendix 3.2 :

It was pointed out earlier that the fibres $C^\infty_J(M,TN)$ of $C^\infty(M,TN) = TC^\infty(M,N)$ are naturally isomorphic to the section spaces $\Gamma(J^*TN)$; similarly, $A^1_J(M,TN)$ is isomorphic to $A^1(M,J^*TN)$. On the other hand, if ∇ denotes e.g. the Levi–Cività connection of N, then there is the induced "pull–back connection" $J^*\nabla$ in J^*TN, obtained in the usual manner. It now is routine to verify that the following diagram commutes:

$$\begin{array}{ccc} C^\infty_J(M,TN) & \simeq & \Gamma(J^*TN) \\ \nabla\downarrow & & \downarrow J^*\nabla \\ A^1_J(M,TM) & \simeq & A^1_J(M,TM) \end{array} :$$

∇ simply "is" the induced connection in J^*TN. As a first consequence, one concludes that $\nabla_{\mathcal{W}}$ is a first–order *elliptic differential operator,* an observation of great importance for later sections.

In addition, J^*TN carries a natural Riemannian structure given by $<\,,\,>$ in TN; the connection $J^*\nabla$ is compatible with this metric. The Riemannian structure of J^*TN together with $\mu(J)$ now is used to obtain a pre–Hilbert space structure in $\Gamma(J^*TN)$ as well as in $A^1(M,J^*TN)$, etc., and hence under the isomorphisms in the above diagram, one obtains a formal adjoint $\nabla(J)^*$ of ∇. This operator coincides with the operator $\nabla^*(J)$ of this section and this shows that $\nabla^*(J)$ again is a first–order elliptic operator. Accordingly, the Laplacean $\Delta(J)$ as defined in the text now is seen to be a second–order elliptic operator. This will be true "at all levels", i.e. on the spaces $A^p_J(M,TN)$, $p \geq 1$, defined in the obvious manner. We omit the details here, but point out that the ellipticity of $\Delta(J)$ will be crucial later on.

Lastly, since ∇ is elliptic, its H^s (= Sobolev $W^{2,s}$–) extensions all are Fredholm maps and so have closed range. At "level 0", the symbol of ∇ is injective and one concludes now that the range $\mathfrak{L}_J(M,TN)$ of this ∇ is closed in $\mathfrak{A}^1_J(M,TN)$, hence itself a Fréchet space. In fact, one can argue that it is a split subspace and that $\mathfrak{L}_E(M,TN)$ is a Fréchet subbundle of $\mathfrak{A}^1_E(M,TN)$. The technical details of these claims will be dealt with elsewhere.

4. One forms on E(M,N)

Recall that the tangent bundle of E(M,N) is identified with $C^\infty_E(M,TN)$; accordingly, we define 1–forms on E(M,N) as follows:

A (scalar) *1–form on E(M,N)* is a smooth function

$$F : C^\infty_E(M,TN) \longrightarrow \mathbb{R}$$

with the property that for each $J \in E(M,N)$, the restriction $F(J) = F \mid C^\infty_J(M,TN)$ is linear in $L \in C_J(M,TN)$. In particular, F(J) is a continuous linear form on this fibre, i.e. an element of the topological dual $C^\infty_J(M,TN)' \simeq \Gamma(J^*TN)'$. Loosely speaking, then, F is a smooth section of the "cotangent bundle" $\bigcup_J C_J(N,TN)'$ of E(M,N), but this point–of–view will not be pursued any further here; cf.however below.

For our purposes, it will be sufficient to limit attention to a smaller class of such one–forms; in particular, their values will depend only on the one–jets of the elements of $C^\infty_E(M,TN)$. More precisely:

Definition 4.1:
The one–form F on E(M,N) is said to be $\mathfrak{g}$*–representable* if there exists a smooth section $\mathfrak{a} : E(M,N) \longrightarrow \mathfrak{A}^1_E(M,TN)$ of the bundle $(\mathfrak{A}^1_E(M,TN),\beta,E(M,N))$ such that

$$F(J)(L) = \int_M \mathfrak{a}(J) \cdot \nabla L\mu(J) = \mathfrak{g}(J)(\mathfrak{a}(J),\nabla L) \tag{4.1}$$

for $J \in E(M,N)$ and $L \in C^\infty_J(M,TN)$. The section $\mathfrak{a}$ is called the *(*$\mathfrak{g}$*–)kernel* of F .

For instance, suppose that $\mathfrak{H}$ is a smooth section of $C^\infty_E(M,TN)$ over E(M,N), i.e a smooth vector field. Then $\mathfrak{a}(J) = \nabla\mathfrak{H}(J)$ will provide a $\mathfrak{g}$–kernel and the right–hand side of (4.1) then will define a representable one–form. In fact, this example can be shown to characterize the representable one–forms, cf.below. Let us denote by $A^1\mathfrak{g}(E(M,N),\mathbb{R})$ the collection of all smooth $\mathfrak{g}$–representable one–forms on E(M,N).

Remark 4.2 :

Clearly, the existence of non–trivial 1–forms, in particular that of $\mathfrak{g}$–representable ones depends on the existence of not identically vanishing smooth sections of the bundles in question. Both $\mathfrak{A}^1_E(M,TN)$ and $C^\infty_E(M,TN) = TE(M,N)$ admit local sections since they are locally trivial over $E(M,N)$. Moreover, the model spaces $\Gamma(J^*TN)$ of $E(M,N)$ are *nuclear* Fréchet spaces obtained as countable inverse limits of Hilbert spaces, namely e.g. the H^s–completions of $\Gamma(J^*TN)$ for $s \in \mathbb{N}$. This implies that $E(M,N)$ admits enough "bump functions": Given the open neighbourhoods U,V of J with $\overline{V} \subset U$, there exist an open neighbourhood W of J and a smooth function f on $E(M,N)$ such that $\overline{W} \subset V$, together with $0 \leq f \leq 1$, $f|\overline{W} = 1$ and $f = 0$ on the complement of V. With this existence of non–zero sections of the above bundles is clear. The paracompactness of $E(M,N)$ (as subspace of the paracompact and locally metrizable, hence metrizable space $C^\infty(M,N)$!) can be used to obtain smooth partitions of unity, but we omit the details here and return to all these matters elsewhere.

We now show that any kernel $\mathfrak{a}$ of a smooth one–form F can be presented by $\nabla\mathfrak{H}$, where

$$\mathfrak{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$$

is a smooth vector field. This means that for any $J \in E(M,N)$

$$\int_M \mathfrak{a}(J)\cdot\nabla L\,\mu(J) = \int_M \nabla\mathfrak{H}(J)\cdot\nabla L\,\mu(J) \tag{4.2}$$

or equivalently

$$\mathfrak{g}(J)(\mathfrak{a}(J),\nabla L) = \mathfrak{g}(J)(\nabla\mathfrak{H}(J),\nabla L) \tag{4.3}$$

has to hold for all $L \in C^\infty_J(M,TN)$. To do so we are required to solve

$$\Delta(J)\mathfrak{H}(J) = \nabla^*\mathfrak{a} \tag{4.4}$$

and

$$\nabla_n\mathfrak{H}(J) = \mathfrak{a}(n)\,. \tag{4.5}$$

This is for each $J \in E(M,N)$ an elliptic boundary value problem (cf.[Pa] or [Hö 2] as well as appendix 3.2) and admits according to [Hö 2] a smooth solution $\mathfrak{H}(J)$ for each $J \in E(M,N)$. Since the solutions are smooth with respect to small perturbations of the system (cf.[Hö 2]), we may state :

Theorem 4.3 :
Any $F \in A^1_{\mathfrak{g}}(E(M,N),\mathbb{R})$ admits a smooth vector field

$$\mathfrak{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$$

for which

$$F(J)(L) = \int_M \nabla\mathfrak{H}(J)\cdot\nabla L \; \mu(J) \tag{4.6}$$

holds for all variables of F .

The following corollary is an easy consequence of proposition 2.1 :

Corollary 4.4 :
Let G and K be groups acting on M and on N for a given $J \in E(M,N)$ via the homomorphism

$$\Phi : G \longrightarrow \mathrm{Diff}^+ M \quad \text{and} \quad \Psi : K \longrightarrow \mathfrak{J}$$

respectively, where $\mathfrak{J}$ is an isometry group of N preserving the orientation. If $F \in A^1(E(M,N),\mathbb{R})$ is $\mathfrak{g}$–representable and invariant at J under Φ and Ψ respectively, then there is a smooth vector field $\mathfrak{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ such that

$$F(J)(L) = \int_M \nabla\mathfrak{H}(J)\cdot\nabla L \; \mu(J)$$

and

$$\mathfrak{H}(J \circ \Phi(g)) = \mathfrak{H}(J) \circ \Phi(g)\,, \quad \forall g \in G \tag{4.7}$$

as well as

(4.8) $$\mathfrak{H}(\Psi(k) \circ J) = T\Psi(k) \circ \mathfrak{H}(J)\,, \quad \forall\, k \in K$$

hold for all variables of F .

5. The special situation $N = \mathbb{R}^n$

We will show in this section that in case of $N = \mathbb{R}^n$ and $<\,,\,>$ being a fixed scalar product, the spaces $\mathfrak{L}_E(TM,T\mathbb{R}^n)$ allow much simpler and more detailed description. In particular (4.1) in the previous section admits a natural visualization. This enrichment is due to the algebraic structure of $\mathbb{R}^n$, which one hand yields the simplification as far as the triviality of $T\mathbb{R}^n$ is concerned. On the other hand the operation of $\mathbb{R}^n$ as a translation group of the linear space $\mathbb{R}^n$ allows us to split $\mathfrak{L}_E(TM,T\mathbb{R}^n)$ with respect to this action:

Based on the triviality of $T\mathbb{R}^n$ we first of all observe, that

(5.1) $$\mathfrak{L}_E(TM,T\mathbb{R}^n) = E(M,\mathbb{R}^n) \times \{dL \,|\, L \in C^\infty(M,\mathbb{R}^n)\} ,$$

which is a Fréchet manifold as seen directly by considering, for each $J \in E(M,\mathbb{R}^n)$ the bijection

(5.2) $$\mathfrak{L}_J(TM,T\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

given by

$$dL \longmapsto [L] .$$

The operation of $\mathbb{R}^n$ as translation group of the vector space $\mathbb{R}^n$ provides us with the action

(5.3) $$E(M,\mathbb{R}^n)\times\mathbb{R}^n \to E(M,\mathbb{R}^n)$$

given by

$$(J,u) \longmapsto J + u .$$

Let us study the orbit space : We consider

(5.4) $$E(M,\mathbb{R}^n) = E_0(M,\mathbb{R}^n)+\mathbb{R}^n ,$$

where

$$E_0(M,\mathbb{R}^n) := \{J_0 \in E(M,\mathbb{R}^n) \,|\, \int_M J_0\, \mu(J_0) = 0\} .$$

This set can be identified with the collection of all orbits of action (5.3). Since moreover, (5.6) $$E_0(M,\mathbb{R}^n) \longrightarrow \{dJ \mid J \in E(M,\mathbb{R}^n)\} \subset C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

given by

$$J \longmapsto dJ$$

is a bijection onto an open set of $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$, the orbit space $E_0(M,\mathbb{R}^n)$ of the action of the translation group $\mathbb{R}^n$ onto $E(M,\mathbb{R}^n)$ is thus a Fréchet manifold. Hence we read off from (5.4) :

(5.7) $$TE(M,\mathbb{R}^n) = TE_0(M,\mathbb{R}^n) + T\mathbb{R}^n .$$

Next let us determine $T_{J_0} E(M,\mathbb{R}^n)$ for any $J_0 \in E_0(M,\mathbb{R}^n)$. To this end we let $J(t)$ be a smooth parameterized family in $E(M,\mathbb{R}^n)$ which we decompose according to (5.4) into

(5.8) $$J(t) = J_0(t) + u(t) ,$$

with $u(t) \in \mathbb{R}^n$ for any real $t \in \mathbb{R}$. Let $J_0 := J_0(0)$ and $u_0 := U_0(0)$, then for $L := \dot{J}(0)$ we have

$$\frac{d}{dt} \int_M J(t)\, \mu(J(t))\big|_{t=0} = \int_M L\, \mu(J_0) + \int_M J_0 \operatorname{tr} Dm(J_0)(L)\, \mu(J_0)$$
$$= \int_M \dot{J}(0)\, \mu(J_0) + \dot{u}_0(0) \int_M \mu(J_0) + \int_M J_0 \operatorname{tr} Dm(J_0)(L)\, \mu(J_0) .$$

Decomposing L according to (5.7) into

(5.9) $$L = L_0 + z ,$$

with $L_0 \in T_{J_0} E(M,\mathbb{R}^n)$ and $z \in T_{u_0} \mathbb{R}^n$ yields immediately

$$\dot{J}_0(0) = L \quad \text{and} \quad \dot{u}_0(0) = z .$$

Due to

$$\int_M J_0(t)\, \mu(J_0(t)) = 0 ; \quad \forall\, t \in \mathbb{R} ,$$

we find

$$\int_M L_0\, \mu(J_0) = 0\,.$$

Introducing

$$(5.10)\qquad C^\infty_{J_0}(M,\mathbb{R}^n) := \{L_0 \in C^\infty(M,\mathbb{R}^n) \mid \int_M L_0\, \mu(J_0) = 0\}\,,$$

which naturally is linearly diffeomorphic to $C^\infty(M,\mathbb{R}^n)/\mathbb{R}^n$, we deduce

$$(5.11)\qquad T_{J_0} E_0(M,\mathbb{R}^n) = C^\infty_{J_0}(M,\mathbb{R}^n) \cong C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

and in turn obtain the splitting

$$(5.12)\qquad TE(M,\mathbb{R}^n) = E_0(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \oplus (\mathbb{R}^n \times \mathbb{R}^n)\,.$$

Any $F \in A^1(E(M,\mathbb{R}^n),\mathbb{R}^n)$ consequently splits for each $J \in E(M,\mathbb{R}^n)$ and for each $L \in C^\infty(M,\mathbb{R}^n)$ into

$$(5.13)\qquad F(J)(L) = F(J_0+u)(L_0) + F(J_0+u)(z)\,,$$

where J_0 , L_0 , u and z are as above.

Thus (4.1) in the previous section amounts in this case to say that

$$(5.14)\qquad F(J)(L) = F(J_0)(L_0)\,,$$

J_0 and L_0 being the components of J in $E_0(M,\mathbb{R}^n)$ and of L in $C^\infty_{J_0}(M,\mathbb{R}^n)$ respectively as introduced in (5.8) and (5.9). (In the application to continuum mechanics any one–form F satisfying (5.13) means, that F depends only on those embeddings for which the center of mass is fixed at $0 \in \mathbb{R}^n$.)

The following theorem (cf.[Bi 4]) describes in full generality the structure of $\mathfrak{g}$–representable one–forms for $N \equiv \mathbb{R}^n$ and $<\,,\,>$ being a fixed scalar product.

Theorem 5.1 :
Every $F \in A^1_g(E(M,\mathbb{R}^n),\mathbb{R})$ admits a smooth constitutive map

(5.15) $$\mathfrak{H} : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)\ ,$$

such that F can be expressed as

(5.16) $$F(J)(L) = \int_M <\Delta(J)\mathfrak{H}(J),L>\mu(J) + \int_{\partial M} <d\mathfrak{H}(J)(\mathfrak{n}),L> i_{\mathfrak{n}}\mu(J)\ ,$$

for each $J \in E(M,\mathbb{R}^n)$ and each $L \in C^\infty(M,\mathbb{R}^n)$. For all $J \in E(M,\mathbb{R}^n)$ the map $\mathfrak{H}$ defines $\Phi \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$ and $\varphi \in C^\infty(E(M,\mathbb{R}^n),C^\infty(\partial M,\mathbb{R}^n))$ respectively by

(5.17) $$\Phi(J) := \Delta(J)\ \mathfrak{H}(J)$$

and

(5.18) $$\varphi(J) := d\mathfrak{H}(J)(\mathfrak{n})\ ,$$

which satisfy due to the first jet dependence of F , the equation

(5.19) $$0 = \int_M \Phi(J)\ \mu(J) + \int_{\partial M} \varphi(J)\ i_{\mathfrak{n}}\mu(J)\ .$$

Given vice versa two smooth maps $\Phi \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$ and $\varphi \in C^\infty(E(M,\mathbb{R}^n),C^\infty(\partial M,\mathbb{R}^n))$, for which (5.19) holds as an integrability condition, then there exists a smooth map $\mathfrak{H} \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$ satisfying (5.17) and (5.18), which is uniquely determined up to a constant for each $J \in E(M,\mathbb{R}^n)$.

Δ

Remark 5.2 :
a) If $\Phi' \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$ and $\varphi' \in C^\infty(E(M,\mathbb{R}^n),C^\infty(\partial M,\mathbb{R}^n))$ are given arbitrarily, we may split off a constant and components Φ and φ satisfying (5.19). Then Φ and φ can be expressed as in (5.17) and (5.18).

b) To comment the interplay between linearity and non–linearity we point out this : Even if $\mathfrak{H}$ is of the form

$$\mathfrak{H}(J+K) = \mathfrak{H}(J) + D\mathfrak{H}(J)(K) ,$$

for any $K \in C^\infty(M,\mathbb{R}^n)$ for which $J+K \in E(M,\mathbb{R}^n)$, the map in (5.17) and (5.18) does not vary accordingly since the Laplacean varies more subtile on J (cf.(3.11)).

c) Introducing the Λ–product and the Hodge–star operator as done in [A,M,R] we may write

$$(5.20) \qquad \int_M dL_1 \cdot dL_2 \, \mu(J) = \int_M dL_1 \wedge {}^*dL_2 ,$$

for any pair $L_1, L_2 \in C^\infty(M,\mathbb{R}^n)$. This is easily seen by converting the right hand side of (5.20) into the right hand side of (5.16). In fact the equality holds on the level of the integrands (cf. [A]).

d) A theorem analogous to theorem 5.1 holds in the general case as well. We omit to state it because of the sake of simplicity.

Appendix 5.1 :

Here let us motivate (3.1) in the context of this section : Given two $I,J \in E(M,\mathbb{R}^n)$ which are in the same connected component. Then we may write

$$dJ = Q(J) \cdot dI ,$$

for some $Q(J) \in C^\infty(M,L(\mathbb{R}^n,\mathbb{R}^n))$. According to the pointwise performed polar decomposition (cf.[Bi,Sn,Fi]) the map Q(J) can be expressed by

$$Q(J) = g(J) \cdot \bar{f}(J) ,$$

where $g(J) \in C^\infty(M,SO(n))$ and $\bar{f}(J) \in C^\infty(M,L_s(\mathbb{R}^n,\mathbb{R}^n))$, the index s meaning self–adjoint with respect to $<\, ,\, >$. Moreover, for all $X,Y \in \Gamma(TM)$

$$(5.21) \qquad m(J)(X,Y) = <\bar{f}(J)dIX, \bar{f}(J)dIY>$$

$$= m(I)(f(J)X, f(J)Y) ,$$

where $f(J)$ is the square root of the strong bundle isomorphism $A'(J) \in L(TM,TM)$, defined by

$$m(J)(X,Y) = m(I)(A'(J)X,Y)\ , \quad \forall\, X,Y \in \Gamma(TM).$$

Defining $f'(J) \in C^\infty(M,L_s(\mathbb{R}^n,\mathbb{R}^n))$ by

$$f'(J)\cdot dI = dI\cdot f(J)\ ,$$

with $f'(J)\,|\,(T(JTM))^\perp = 0$, we conclude by (5.21)

$$dJ = g\cdot dI\cdot f\ .$$

Letting J depend on a smooth real parameter t with $J(0) = I$, we find

$$d\dot{J}(0) = \dot{g}(0)\, dI + dI\, \dot{f}(0)\ . \tag{5.22}$$

Thus there is a unique $C \in C^\infty(M,L_a(\mathbb{R}^n,\mathbb{R}^n))$, the index a meaning skew–adjoint, such that

$$\dot{g}(0)\, dI = c\cdot dI + dI\cdot C\ ,$$

with c as in (3.1). Collecting C and $\dot{f}(0)$ into $A(dJ,dI)$, yields

$$d\dot{J}(0) = c\cdot dI + dI\cdot A(dJ,dI) \tag{5.23}$$

the decomposition (3.1) in case of $a = d\dot{J}(0)$. Equation (5.23) then motivates the general decomposition (3.1). The meaning of the coefficients c, C and $\dot{f}$ are discussed e.g. in [Bi,Sc,So].

6. $\mathfrak{g}$–representable one–forms on $E(M,\mathbb{R}^n)$ as constitutive laws

In this part of the paper we link the formalism developed earlier to classical elasticity as presented e.g. in [L,L]. In doing so, we work in a C^∞–setting. First of all we introduce the work caused by deforming a body. The body being identified with the manifold M with boundary enjoying the properties of the previous sections. To this end we consider the derivative of the map $m : E(M,\mathbb{R}^n) \longrightarrow \mathfrak{M}(M)$, at any $J \in E(M,\mathbb{R}^n)$ in the direction of any $L \in C^\infty(M,\mathbb{R}^n)$. It is determined by

$$(6.1) \qquad Dm(J)(L)(X,Y) = \langle dJX,dLY\rangle + \langle dLX,dJY\rangle , \quad \forall\, X,Y \in \Gamma(TM).$$

Writing $Dm(J)(L)$ with respect to $m(J)$ yields the strong smooth bundle endomorphism

$$(6.2) \qquad B(dL,dJ) : TM \longrightarrow TM .$$

Hence, $B(dL,dJ)$ is the symmetric part of $A(dL,dJ)$ a coefficient appearing in (3.1). This is easily seen by using (3.1) and (6.1), the tensor

$$m(J)(B(dL,dJ)..,..) = \frac{1}{2}\, Dm(J)(L)$$

is called the linearized *deformation tensor.*

Let us assume that some smooth map

$$\mathfrak{T} : m(E(M,\mathbb{R}^n)) \longrightarrow S^2(M)$$

is prescribed, where the range is the collection of all symmetric two tensors on M endowed with the C^∞–topology. $\mathfrak{T}(m(J))$ is called the *stress tensor* at $m(J)$.
$\mathfrak{T}(m(J))$ determines a uniquely defined smooth strong bundle map of TM, such that

$$(6.3) \qquad \mathfrak{T}(m(J))(X,Y) = m(J)(\hat{\mathfrak{T}}(dJ)X,Y) , \quad \forall\, X,Y \in \Gamma(TM).$$

We define

$$\text{(6.4)} \qquad F_m(m(J))(\tfrac{1}{2}\,Dm(J)(L)) := \int_M \operatorname{tr}\,(\hat{\mathfrak{T}}(m(J))\cdot B(dL)dJ)\;\mu(J)\,,$$

for any $m(J) \in m(E(M,\mathbb{R}^n))$ and any $Dm(J)(L) \in Dm(E(M,\mathbb{R}^n))(C^\infty(M,\mathbb{R}^n))$.

It is not clear as to whether $m(E(M,\mathbb{R}^n))$ is a manifold or not. It is one if the codimension of M in $\mathbb{R}^n$ is high enough (cf.[St]). Hence the usual techniques in analysis and differential geometry cannot by applied with caution to this topological space. However, $E(M,\mathbb{R}^n)$ is a Fréchet manifold and it makes sense to lift (6.4) to $E(M,\mathbb{R}^n)$ by introducing the one–form

$$F : E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

given by

$$F(J)(L) = F_m m(J)(\tfrac{1}{2}\,Dm(J)(L))\,,$$

for any of the variables of F. It makes also sense to require that F is smooth even though smoothness is not defined for F_m. As shown in [Bi 4] there is a map

$$\mathfrak{H} : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)\,,$$

for which

$$\text{(6.5)} \qquad F(J)(L) = \int_M d\mathfrak{H}(dJ)\cdot dL\;\mu(J)$$

holds for all variables of F. Hence, prescribing the stress tensor at each configuration in $m(E(M,\mathbb{R}^n))$ yields a $\mathfrak{g}$–representable one–form F. Since $\mathfrak{T}$ is a constitutive entity in elasticity, we call F a constitutive law (cf.[E,S]). Equation (6.5) is the motivation for calling any $F \in A^1_{\mathfrak{g}}(E(M,\mathbb{R}^n),\mathbb{R})$ a constitutive law.

As shown in [S], given any $\mathfrak{g}$–representable one–form F invariant under the natural action of the euclidean group of $\mathbb{R}^n$ on $E(M,\mathbb{R}^n)$, satisfying an additional condition, there is a map $\mathfrak{T}$ such that (6.4) holds. The additional condition amounts to say that no rigid motion in $\mathbb{R}^n$ causes any work.

The force densities associated with any constitutive law F with $\mathfrak{g}$–kernel $d\mathfrak{H}$ are given at each $J \in E(M,\mathbb{R}^n)$ by

(6.6) $$\Delta(J)\mathfrak{H}(J) \quad \text{on} \quad M$$

and

(6.7) $$d\mathfrak{H}(J)(\mathbf{n}) \quad \text{on} \quad \partial M$$

cf.[Bi 4]). Thus, the formalism presented in these notes refines the usual treatment of elasticity and carries over to any ambient manifold N (cf. Remark 5.2 d) in the previous section). If $N \subset \mathbb{R}^n$, then it may reflect constraints a deformation of a body in $\mathbb{R}^n$ has to satisfy.

If N has no non–trivial isometry group, then there is in general no natural symmetric stress–tensor available at each configuration. Hence the generality of the mechanism presented here, which describes all the deformable media admitting smooth force densities at each configuration acting upon M and ∂M respectively seems to be necessary.

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