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ABSTRACT

Among the foundations of continuum mechanics is the description of constitutive forces in terms of a symmetric tensor. Noll showed that this is a consequence of the axiom of material frame indifference [Nol,Tru], what in turn means a local invariance of the system under the Euklidean group. Here we will prove that the assumption of locality in this axiom is redundant to obtain the same result. We model a non-local system by means of a virtual work [AnOs,EpSe]. Under the global demand that this functional does not respond on rigid infintesimal motions, we show the existance of a symmetric stress tensor as a local result. As the mathematical tool for the localization we use the Hodge theory on manifolds with boundary.

1. Introduction

By means of differential geometric methods several progress has been made in the field of continuum mechanics within in the last two decades, cf. [AMR,Mar] and references therein. The purpose of this paper is to use this framework to study the symmetry under the Euklidean group of rigid motions and its consequences for the tensorial description of a system.

To formulate kinematics in continuum mechanics, we use the embeddings of a Riemannian manifold $\mathcal{B}$ into the $\mathbb{R}^n$ as ambient space, where $\mathcal{B}$ describes a material body, cf. [HuMa]. We denote by $E(\mathcal{B}, \mathbb{R}^n)$ the set of all embeddings $J : \mathcal{B} \to \mathbb{R}^n$, which itself carries the structure of an infinite-dimensional manifold. In the classical notation such J is also called a placement of the body and an element $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$ of the tangent space is referred to as a virtual displacements. In this setting dynamics means to formulate continuum mechanics in terms of curves $J(t)$ of embeddings. To explore the geometrical structure of the theory, however, we restrict ourselves to statics.

In order to formulate the balance laws for the system, we investigate the principle of virtual work [Hel,AnOs] in the geometric formulation of [EpSe] : They introduce the virtual work at the configuration J as a linear functional $\mathcal{F}_J$ on $T_J E(\mathcal{B}, \mathbb{R}^n)$. The work done to the system by some virtual displacement Λ then is given by its evaluation on that functional, i.e. by $\mathcal{F}_J(\Lambda)$, and the content of the (static) principle of virtual work is to demand :

$$J \in E(\mathcal{B}, \mathbb{R}^n) \text{ is an equilibrium configuration} \Leftrightarrow \mathcal{F}_J(\Lambda) = 0 \text{ for all } \Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n).$$

This form of the virtual work principle is general enough to include a description of all possible force fields affecting the body : The constitutive and external forces as well as tractions on the boundary. Denoting by $<,>_{\mathbb{R}^n}$ the scalar product on $\mathbb{R}^n$ we may write under considerable functional analytic restrictions

$$\mathcal{F}_J(\Lambda) = \int_{\mathcal{B}} < \Phi^{\text{int}}(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\mathcal{B}} < \psi_J, \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi_J, \Lambda >_{\mathbb{R}^n} \mu_{\partial} \tag{1.1}$$

where ψ_J and φ_J stand for the external body and boundary force, respectively. The primer integral describes the constitutive part of the virtual work, where the internal force field $\Phi^{\text{int}}(J)$ may have a general non-linear and non-local constitutive dependence on J. As a special case (1.1) includes the description of hyperelasticity, where the equations of continuum mechanics can be derived from a local energy functional or a Lagrangian [HuMa,TrTo].

Proceeding from the virtual work functional $\mathcal{F}_J$ we will investigate the effect of infinitesimal rigid motions on the equations of continuum mechanics : As a matter of experience we observe that at least a part of $\mathcal{F}_J$ vanishes on all virtual displacements, which are infinitesimal motions of the whole body in $\mathbb{R}^n$. To formalize this idea we introduce Lie algebra $e(n)$ of the Euclidean group describing all rigid translations and rotations in the ambient space $\mathbb{R}^n$. The action of some $g_{(z,C)} \in e(n)$ on an configuration $J \in E(\mathcal{B}, \mathbb{R}^n)$ is given as

$$g_{(z,C)}[J] = C\cdot(J+z) \tag{1.2}$$

where $C \in so(n)$ is a constant anti-symmetric matrix, $z \in I\!R^n$ and the multiplication and addition act pointwise. We now call $\mathcal{F}_J$ to be $e(n)$-invariant, iff

$$\mathcal{F}_J(g_{(z,C)}[J]) = 0 \qquad \forall g_{(z,C)} \in e(n) \tag{1.3}.$$

Considering the classical approach to continuum mechanics [Tru], the concept of an $e(n)$-invariant virtual work seems to be closely related to Noll's axiom of material frame indifference of working. Originally posed in terms of the mechanical power of a motion, this axiom is equivalent to the demand that no work is done against any virtual displacement, which is a rigid $e(n)$-action restricted to an arbitrary subbody. This means

$$\int_U < (\Phi^{\text{int}}(J) + \psi_J), g_{(z,C)}[J] >_{I\!R^n} \mu_{\mathcal{B}} = 0 \qquad \forall g_{(z,C)} \in e(n) \tag{1.4}$$

for any $U \subset \mathcal{B}$. We observe that this is a local demand, in contrast to (1.3), where only the integral over the whole body $\mathcal{B}$ has the desired property.
Noll's celebrate result [Nol,Tru] is to prove the existence of a symmetric stress tensor from the local assumption (1.4). The central result of this paper is to show that the locality is redundant for the existence of a symmetric stress tensor, but it suffies to start with the weaker global demand (1.3) on the virtual work $\mathcal{F}_J$.

After formalizing these ideas in section 2 we introduce in section 3 vector-valued differential form, needed as a technical tool for the proof. These forms may alternatively be considered as two-point tensors, well know in continuum mechanics [Eri,HuMa], e.g. to describe the deformation gradient or the 1^{st} Piola-Kirchhoff tensor. The motivation for using differential forms instead of tensor calculus lies in the fact that there is a Hodge theory, which serves as powerful tool for solving boundary value problems. Generalizing some classical results [Mo56,62] we derive a lemma concerning special boundary value problem for vector-valued differential forms.
In section 4 we then prove the tensorial character of the stress from the global demand of invariance of virtual work under infinitesimal rigid translations, which is a consequence of (1.3). To do so we need the solution of a Neumann problem for an $I\!R^n$-valued one form, which becomes the 1^{st} Piola-Kirchhoff tensor. Performing a Piola transformation in section 5 we then derive the weak form of Cauchy's equation for the coresponding stress tensor from the principle of virtual work (1.1). We observe that this tensor is not uniquely determined from the $\mathcal{F}_J$, but it owns a gauge freedom.
In section 6 we will solve another boundary value problem to show the existence of a symmetric stress tensor from the physical demand of invariance of the virtual work under rigid rotations. By Noll's theorem the existence of a symmetric stress tensor is equivalent to the local demand (1.3) of frame indifference. Hence the use of Hodge theory, required for our proof, may be considered as a localization of the global invariance. Finally we can prove a theorem, splitting any given $\mathcal{F}_J$ into a constant total force and a constant torque, plus a term describing deformation forces via a symmetric stress tensor.

2. The Virtual Work and the Euklidean Group in Continuum Mechanics

In this paper we will describe mechanical properties of a continuous medium in terms of embeddings of a Riemannian manifold, as presented e.g. in [HuMa]. For the physical space, i.e. the ambient space of the embeddings, we take the Euclidean $\mathbb{R}^n$; a generalization to other ambient manifolds is possible, but requires more effort [BiFi]. To fix the notation we introduce the following definitions :

By a body $\mathcal{B}$ we mean a compact orientable Riemannian C^k-manifold with boundary, where the dimension $\dim \mathcal{B} \leq n$. We denote by $G_{\mathcal{B}}$ the Riemannian metric on $\mathcal{B}$, by $\mathcal{N}$ the (outward pointing) unite normal field on the boundary $\partial\mathcal{B} \subset \mathcal{B}$ and have the Riemannian volume elements $\mu_{\mathcal{B}}$ on $\mathcal{B}$ and $\mu_{\partial} = \mathbf{i}_{\mathcal{N}}\mu_{\mathcal{B}}$ on $\partial\mathcal{B}$. Points of $\mathcal{B}$ are refered to as material points; they manifest themselves by their configurations in the ambient physical space $\mathbb{R}^n$. By a configuration (or placement) of the body $\mathcal{B}$ we then mean a C^k-embedding $J : \mathcal{B} \longrightarrow \mathbb{R}^n$ and call

$$E(\mathcal{B}, \mathbb{R}^n) := \{ \ J : \mathcal{B} \to \mathbb{R}^n \mid J \ \text{ is a } \ C^k-\text{ embedding } \ \} \tag{2.1}$$

the configuration space of the system. Although non-smooth configurations are important we restrict our interest to C^k-embeddings with $k \geq 2$ or $k = \infty$. The set $E(\mathcal{B}, \mathbb{R}^n)$ carries the structure of an infinite dimensional manifold with

$$TE(\mathcal{B}, \mathbb{R}^n) = \{ \ \Lambda : \mathcal{B} \longrightarrow T\mathbb{R}^n \mid \Lambda \ \text{ is a } \ C^k-\text{ map}, \ \Pi_{\mathbb{R}^n} \circ \Lambda \in E(\mathcal{B}, \mathbb{R}^n) \ \} \tag{2.2}$$

as tangent bundle [BSF,Mar]. A point in that bundle, i.e. some $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$, is called a virtual displacement of the configuration J.

In order to formulate the balance laws we start with the principle of virtual work [AnOs], first introduced in continuum mechanics as d'Alamberts principle by Hellinger [Hel]. The appropriate version of that principle in the framework of (the infinite-dimensional manifold) $E(\mathcal{B}, \mathbb{R}^n)$ as configuration space of a system is due to Epstein and Segev [EpSe]. They generalized the notion of classical mechanics [Arn] in a straightforward way calling an element $\mathcal{F}_J \in T_J^* E(\mathcal{B}, \mathbb{R}^n)$ a generalized force affecting the configuration J. The work done to the system under the action of a virtual displacement $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$ is given as the evaluation of $\mathcal{F}_J$ on that displacement, i.e. by the value $\mathcal{F}_J(\Lambda)$.

On this general level, where the co-vector $\mathcal{F}_J \in T_J^* E(\mathcal{B}, \mathbb{R}^n)$ describes all physical force fields attaching the body, i.e. the constitutive and external forces as well as tractions on the boundary, we formulate the principle of virtual work (for a static problem) as

$$J \in E(\mathcal{B}, \mathbb{R}^n) \text{ is an equilibrium configuration } \Leftrightarrow \ \mathcal{F}_J(\Lambda) = 0 \text{ for all } \Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n).$$

In principle it is possible to solve the weak boundary value problem of elastostatics by searching for equilibrium solutions in the above sense. For practical applications, or even for structural investigations, however, this approach is too abstract; the notion of a generalized force requires a refinement. Therefore we restrict the infinite-dimensional co-tangent space $T_J^* E(\mathcal{B}, \mathbb{R}^n)$ to the space of co-vectors having a special L^2-representation on the bounded manifold $\mathcal{B}$, i.e. to those linear functionals $\mathcal{F}_J : T_J E(\mathcal{B}, \mathbb{R}^n) \longrightarrow \mathbb{R}$ which are realized in the form

$$\mathcal{F}_J(\Lambda) = \int_{\mathcal{B}} < \Phi^{\rm int}(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\mathcal{B}} < \psi_J, \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \phi_J, \Lambda >_{\mathbb{R}^n} \mu_{\partial} \qquad (2.3).$$

Here $<,>_{\mathbb{R}^n}$ is the Euclidean scalar product on $\mathbb{R}^n$. Physically the functions $\psi_J \in C^k(\mathcal{B};\mathbb{R}^n)$ and $\phi_J \in C^k(\partial\mathcal{B};\mathbb{R}^n)$ are understood to characterize the external force density affecting body and the traction force density on its boundary in the configuration J, respectively. The primer integral then describes the effect of the (unbalanced) internal forces, associated to the configuration J on the virtual displacement Λ; its density $\Phi^{\rm int}(J) \in C^k(\mathcal{B};\mathbb{R}^n)$ is given from the constitutive laws of the material. To determine the constitutive function of a system means in this context to describe the internal force density by a (Frechet-)smooth map

$$\Phi^{\rm int} : E(\mathcal{B},\mathbb{R}^n) \longrightarrow C^k(\mathcal{B};\mathbb{R}^n) \qquad (2.4).$$

The virtual work (2.3) is a very general one, since the linear functional $\mathcal{F}_J$ encodes all information about the constitutive nature of the body as well as the external and boundary forces. Furthermore it allows for an arbitrary non-local and non-linear constitutive behavior of the material under consideration. A different constitutive behavior of the boundary material can also be described by adding an appropriate extra term in (2.3). To impose boundary conditions of placement, however, one has to modify the configuration space $E(\mathcal{B},\mathbb{R}^n)$. We remark that any hyperelastic model [HuMa,TrTu], where the balance laws are derived from a local energy functional $\mathcal{E}$ (or a Lagrangian), appears as a specialization of the description of continuum mechanics in terms of a virtual work. In such a case the virtual work will be given as the (Frechet derivative) $\mathbf{D}\mathcal{E} = \mathcal{F}_J$ with respect to the configuration J.
Knowing the constitutive function $\Phi^{\rm int}$ and the external force densities ψ_J and ϕ_J explicitly, the principle of virtual work can be used to determine the equilibrium solutions of the system. As shown by [AnOs] the usual balance laws in continuum mechanics are equivalent to this principle under some technical conditions. Also a description of dynamics might be included into this framework [BSS,BiSc].

Proceeding from this we will study the effect of symmetries on the configuration space $E(\mathcal{B},\mathbb{R}^n)$ for the form of the balance laws in continuum mechanics. We are motivated for doing so by considering classical field theories where symmetries cause the system to subject conservation law, e.g. of momentum and angular momentum, via Noether's theorem.
Here the symmetry group in question is the group of rigid changes of frame on $\mathbb{R}^n$, also called the the Euklidean group $E(n)$, which is the semi-direct product $\mathbb{R}^n \otimes_S SO(n)$, cf. [Thi]. An element $g_{(T,R)} \in E(n)$ is uniquely represented by a translation $T \in \mathbb{R}^n$ and a rotation $R \in SO(n)$ and its action on $E(\mathcal{B},\mathbb{R}^n)$ is induced from its natural $\mathbb{R}^n$-action by

$$\begin{aligned} E(n) \times E(\mathcal{B},\mathbb{R}^n) &\longrightarrow E(\mathcal{B},\mathbb{R}^n) \\ (g_{(T,R)}[J])(p) &= R\cdot(J(p)+T) \qquad \forall p \in \mathcal{B} \end{aligned} \qquad (2.5).$$

The coresponding action of the Lie algebra $e(n)$ is given by

$$\begin{aligned} e(n) \times E(\mathcal{B}, \mathbb{R}^n) &\longrightarrow E(\mathcal{B}, \mathbb{R}^n) \\ (g_{(z,C)}[J])(p) &= C \cdot (J(p) + z) \qquad \forall p \in \mathcal{B} \end{aligned} \tag{2.6}$$

where $z \in \mathbb{R}^n$ and $C \in so(n)$, is the space of all anti-symmetric $n \times n$ matrices, i.e. the Lie algebra of $SO(n)$. To characterize the behavior of a system in continuum mechanics under the action of the Euklidean group we now set :

Definition

The generalized force $\mathcal{F}_J$ defines a one form $\mathcal{F} : TE(\mathcal{B}, \mathbb{R}^n) \longrightarrow \mathbb{R}$ on the configuration space, called the virtual work form, if the external and boundary force densities ψ_J and ϕ_J depend Frechet smooth on J. Splitting this form smoothly into $\mathcal{F}_J(\Lambda) = F[J](\Lambda) + \widehat{F}[J](\Lambda)$ such that

$$F[J](g_{(z,C)}[J]) = 0 \qquad \forall J \in E(\mathcal{B}, \mathbb{R}^n) \;\; \forall g_{(z,C)} \in e(n) \tag{2.7}$$

the one form $F : TE(\mathcal{B}, \mathbb{R}^n) \longrightarrow \mathbb{R}$ defines an $e(n)$-invariant part of the virtual work. In considerable abuse of notation we also call $F[J]$ an $e(n)$-invariant virtual work. If the property (2.7) holds for the subgroup $\mathbb{R}^n \subset e(n)$ only we speak about an $\mathbb{R}^n$-invariant virtual work.

We remark that this does not define an infinitesimal $E(n)$-symmetry of the system, it might, however, be understood as a constraint, coming from that symmetry [BiSc] by using the momentum map technique [AbMa].
The notion of an $e(n)$-invariant virtual work $F[J]$ will be fundamental to obtain a simplified description of the balance laws. From a physical point of view it is obvious that the internal interactions should not respond on infinitesimal rigid Euklidean motions of the body as whole. Hence it is natural to postulate the constitutive part

$$F^{\text{int}}[J](\Lambda) := \int_{\mathcal{B}} < \Phi^{\text{int}}(J), \Lambda >_{\mathbb{R}^n} \tag{2.8}$$

of any virtual work functional $\mathcal{F}_J$ to be $e(n)$-invariant. We remark, however, that beside this natural splitting with respect to the internal and external interactions of the body there are other ways to determine an $e(n)$-invariant part, cf. theorem 4 below. In the general case an $e(n)$-invariant virtual work will be expressed in terms of two densities on $\mathcal{B}$ and $\partial\mathcal{B}$, respectively, as

$$F[J](\Lambda) = \int_{\mathcal{B}} < \Phi(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi(J), \Lambda >_{\mathbb{R}^n} \mu_{\partial} \tag{2.9},$$

and obeys the crucial property to vanishes on all virtual displacement $\Lambda = g_{(z,C)}[J]$, which are rigid infinitesimal Euklidean motion. A nice feature of that concept is to permit an explicit appearance of appropriate boundary terms in $F[J]$, what may be of some interest for applications.

Being familiar with the classical approach to continuum mechanics [Nol,Tru], the investigation of $e(n)$-invariance seems to be closely related to Noll's axiom of frame indifference of working : Originally posed in terms of the mechanical power of a motion, this axiom is in our notion equivalent to the demand

$$\int_U < \Phi(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial U} < \widetilde{\varphi}(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_\partial = 0 \quad \forall g_{(z,C)} \in e(n) \quad (2.10)$$

for any configuration $J \in E(\mathcal{B}, \mathbb{R}^n)$ and any subbody $U \subset \mathcal{B}$. Here $\widetilde{\varphi}(J)$ describes a force density on ∂U which is different from the surface force field $\varphi(J)$ on $\partial\mathcal{B}$ in (2.9).

The crucial difference to the $e(n)$-invariance (2.7) of the virtual work is that (2.10) demands an invariance to hold for any subbody $U \subset \mathcal{B}$. Hence Noll's frame indifference is of local nature. For a general theory, describing also a non-local constitutive behavior of the system this local axiom appears to be artificial and may also fail. To motivate it from physical considerations requires some more arguments like postulating the cut principle of Euler and Cauchy [Tru] or demanding only short distance interactions to have an effect [LaLi]. However, these extra demands are far from being obvious for a real physical system, as pointed out e.g. by [Krö].

Such problems were our motivation to replace Noll's axiom by the (global) demand of an $e(n)$-invariant virtual work, which is at least for the internal constitutive part $F^{\text{int}}[J]$ a natural claim. Then we can prove :

Main Result

Let $F[J]$ be an $e(n)$-invariant part (2.7) of the virtual work $\mathcal{F}_J = F[J] + \widehat{F}[J]$ of some system. Then $F[J]$ can entirely be described in terms of a symmetric stress tensor $\mathbf{A}(J)$ on $\mathcal{B}$ and the equations of elastostatics balance the divergence of the tensor $\mathbf{A}(J)$ and a (force) densities caused by $\widehat{F}[J]$.

The crucial point here is that the global rigid condition (2.7) suffies to prove the existence of a symmetric stress tensor. Under the stronger (local) assumption (2.10) the coresponding result is known as Noll's theorem [Tru]. Similar theorems have been derived by Green, Rivelin and Naghdi [GrRi], who replaced the working axiom (2.10) by starting with an $E(n)$-invariant energy functional $\mathcal{E}[J] \in C^\infty(\mathcal{B}; \mathbb{R})$, what is again a local invariance demand.

The proof of our result is based on Hodge theory on manifolds with boundaries, which makes it possible, to prove from the global demand (2.7), the existence of a symmetric stress tensor as a local result. In this sense the cut principle of Euler and Cauchy, which fills the gap between Noll's local axiom and the global invariance demand in the physical argumentation, may be understood as a reflection of Hodge theory. Before doing the constructions in detail we have to present some fundamental results of that theory for manifolds with boundaries.

3. Vector-valued differential forms and Hodge Theory

Considering $E(\mathcal{B}, \mathbb{R}^n)$ as the configuration space of continuum mechanics, two-point tensors over the body manifold $\mathcal{B}$ appear as natural objects [Eri]. Such tensors are the canonical generalizations of vector fields over maps. Restricting the general definition [HuMa] to the case of our interest we define a two-point tensor $\mathbf{T}$ of type $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, shortly denoted as a $(r,1)$-type two-point tensor, at $p \in \mathcal{B}$ over an embedding $J \in E(\mathcal{B}, \mathbb{R}^n)$ as a multilinear map

$$\mathbf{T} : \underbrace{(T_p\mathcal{B} \times \ldots \times T_p\mathcal{B})}_{r-times} \times T^*_{J(p)}\mathbb{R}^n \longrightarrow \mathbb{R} \qquad (3.1).$$

One can think of such a tensor having $r+1$ legs, one on $\mathbb{R}^n$ and r on $\mathcal{B}$. A remarkable feature of the skew-symmetric $(r,1)$-type two-point tensors is to fit into the notion of vector-valued differential forms. Those are defined for any Riemannian manifold M and any finite dimensional vector space V by :

Definition and Remark [GHV]

A V-valued differential form $\omega \in \Omega^r(M;V)$ of degree r over a m-dimensional manifold M is a smooth assignment of skew-symmetric r-linear maps to the points of M, where

$$\omega_p : \underbrace{(T_pM \times \ldots \times T_pM)}_{r-times} \longrightarrow V \qquad \forall\, p \in M \qquad (3.2).$$

The algebra of all V-valued forms on M is denoted by $\Omega(M;V) = \bigoplus_{r=1}^{m} \Omega^r(M;V)$.
There is a natural identification $\Omega(M;V) \cong \Omega(M;\mathbb{R}) \otimes V$, such that the algebraic and analytic structures on the algebra of usual ($\mathbb{R}$-valued) differential forms, carry over to $\Omega(M;V)$. In terms of a fixed scalar product $<,>_V$ that isomorphism can be given by means of the pairing

$$\begin{aligned} <<,>>:\ V \otimes \Omega^r(M;V) &\longrightarrow \Omega^r(M;\mathbb{R}) \\ << v,\omega >>(X_1,\ldots X_r) &:= < v, \omega(X_1,\ldots X_r) >_V \qquad \forall\,(X_1,\ldots X_r) \in \Gamma TM \end{aligned} \qquad (3.3).$$

Fundamental quantities in continuum mechanics as the deformation gradient or the 1st Piola-Kirchhoff tensor, are described by $(1,1)$-type two-point tensors on the body manifold and hence can also be considered as $\mathbb{R}^n$-valued one forms on $\mathcal{B}$. The use of vector-valued forms instead of the well known tensor language is motivated by the fact that the Hodge theory on the algebra of differential forms is a useful tools to solve boundary value problems on Riemannian manifolds, cf. [EbMa]. Thus the idea is to formulate problems in continuum mechanics in terms of V-valued forms $\omega \in \Omega(M;V)$ with $M = \mathcal{B}$ and $V = \mathbb{R}^n$ and use results from Hodge theory instead of solving those directly by tensor calculus.
To do so we introduce, in view of (3.3), the exterior derivative

$$\begin{aligned} d : \Omega^r(M;V) &\longrightarrow \Omega^{r+1}(M;V) \\ << v, d\omega >> &:= \mathbf{d} << v,\omega >> \qquad \forall\, v \in V \end{aligned} \qquad (3.4)$$

where $\mathbf{d}$ is the exterior derivative on the algebra $\Omega(M, I\!R)$ of real-valued forms. Similarly the Hodge $*$-operator on $\Omega(M, I\!R)$ induces an operator

$$\begin{aligned} \star : \Omega^r(M;V) &\longrightarrow \Omega^{m-r}(M;V) \\ << v, \star\omega >> &:= * << v,\omega >> \qquad \forall v \in V \end{aligned} \tag{3.5}$$

and we can define by $\delta := (-1)^{mr+1} \star d\star$ the co-differential $\delta : \Omega^{r+1}(M;V) \to \Omega^r(M;V)$. Like the co-differential on $\Omega(M, I\!R)$ this yields a nilpotent operator obeying $\delta^2 = 0$ on $\Omega(M;V)$. With G_M denoting the Riemannian metric on M each a one form $\omega \in \Omega^1(M;V)$ induces tensor $\omega^\sharp \in TM \otimes V$ by

$$G_M\Big(Y\,, < v, \omega^\sharp >_V\Big) := << v,\omega >> (Y) \qquad \forall Y \in \Gamma TM \quad \forall v \in V \tag{3.6}.$$

Then the co-differential coresponds to the divergence [AMR] by $\delta\omega = -\mathrm{div}\,\omega^\sharp$. In generalization of that property, the action of δ on $\Omega^r(M;V)$ can be expressed [Mat] by means of a local G_M-orthonormal frame $\{E_1, \dots E_m\}$ on TM as

$$(\delta\omega)(X_1, \dots X_r) := -\sum_{k=1}^{m} (\nabla_{E_k}\omega)(E_k, X_1, \dots X_r) \quad \text{with} \quad X_1, \dots X_r \in \Gamma TM \tag{3.7}$$

where ∇ is the Levi-Civita connection. Furthermore each space $\Omega^r(M;V)$ is equipped with a Riemannian structure, induced from the scalar product $<,>_V$ and the metric G_M by

$$\begin{aligned} <,>_{\Omega^r} : \Omega^r(M;V) \times \Omega^r(M;V) &\longrightarrow \Omega^0(M; I\!R) \\ < \omega, \eta >_{\Omega^r} &:= \sum_{1 \le j_1 < \dots < j_r}^{m} < \omega(E_{j_1}, \dots, E_{j_r}), \eta(E_{j_1}, \dots, E_{j_r}) >_V \end{aligned} \tag{3.8},$$

where the fields E_{j_r} run through a local orthonormal frame on TM. This definition is frame independent, it yields for $r = 0$ the scalar product $<,>_V$ and generalizes to the usual inner product $\omega \wedge *\eta = < \omega, \eta >_{\Omega^r} \mu_M$ on $\Omega^r(M; I\!R)$. With that product the space $\Omega^r(M;V)$ can be furnished with the structure of the Sobolev space $H^1\Omega^r(M;V)$, given as the completion of the space of smooth forms $\omega \in \Omega^r(M;V)$ with respect to the norm

$$\|\omega\|^2 = \int_M \Big(< \omega, \omega >_{\Omega^r} + < d\omega, d\omega >_{\Omega^{r+1}} \Big) \mu_M \tag{3.9}.$$

In the Sobolev space of H^1-forms over M the operators δ and d are adjoint to each other up to a boundary term, i.e. for any pair of $\omega \in H^1\Omega^1(M;V)$ and $\eta \in H^1\Omega^0(M;V)$ we have

$$\int_M < \omega, d\eta >_{\Omega^1} \mu_M = \int_M < \delta\omega, \eta >_{\Omega^0} \mu_M + \int_{\partial M} < \omega(\mathcal{N}), \eta >_{\Omega^0} \mu_\partial \tag{3.10},$$

which is a consequence of the Stokes theorem. By taking η to be constant this also yields the Gauß theorem in terms of differential forms.

Now we have introduced all structures, necessary to face the question of solving boundary value problems by means of Hodge theory. The Sobolev space $H^1\Omega^r(M;V)$ carries the same topology as the one, used in the book of Morrey [Mo62], and we get :

Theorem 1

Let M be a compact Riemannian C^k-manifold with boundary, and let $\mathcal{N}$ denote the (outward pointing) unite normal field on $\partial M \subset M$.

a) For any function $\Psi \in H^1\Omega^0(M;V)$, there is a decomposition

$$\Psi = \delta\beta_\Psi + c_\Psi \tag{3.11}$$

where $\beta_\Psi \in H^1\Omega^1(M;V)$ is a one form obeying $\beta_\Psi(\mathcal{N}) = 0$ and $c_\Psi \in V$ is a constant. If $\Psi \in C^{k-2}(M;V)$ then β_Ψ can be chosen of class C^{k-1}.

b) Given a r-form $\beta \in H^1\Omega^r(M;V)$ with $\beta|_{\partial M} \in H^1\Omega^r(\partial M;V)$, there exists a $(r+1)$-form $\xi \in H^1\Omega^{r+1}(M;V)$ obeying the boundary conditions

$$\begin{aligned} \xi|_{\partial M} &\equiv 0 \\ (\delta\xi)|_{\partial M}(X_1,\dots,X_r) &= \beta|_{\partial M}(X_1,\dots,X_r) \qquad \forall\, X_1,\dots X_r \in \Gamma T\partial M \end{aligned} \tag{3.12}$$

If $\beta|_{\partial M}$ is of class C^{k-2} on ∂M then ξ and $\delta\xi$ can be chosen of class C^{k-2} on M.

This theorem is a reformulation of two – at least for $V = I\!R$ – well established results. Part a) is usually refered to as the Kodeira decomposition of the function Ψ and the solvability of the problem (3.12) is due to [Mo56]. That result is not quoted literally, but taken from the proof of the lemma 6.2 there, where the assertion is given and explicitly used. By means of the identification $\Omega(M;V) \cong \Omega(M;I\!R) \otimes V$ the generalization to $H^1\Omega(M;V)$ is obvious. As a consequence we obtain on any compact Riemannian C^k-manifold M with boundary :

Lemma 1

a) Given a pair of vector-valued functions $\Phi \in H^1\Omega^0(M;V)$ and $\varphi \in H^1\Omega^0(\partial M;V)$, which obey the integrability condition

$$\int_M \Phi\,\mu_M + \int_{\partial M} \varphi\,\mu_\partial = 0 \tag{3.13}$$

then there is a one form $\alpha \in H^1\Omega^1(M;V)$ solving the boundary value problem

$$\begin{aligned} \delta\alpha &= \Phi \qquad \text{on } M \\ \alpha(\mathcal{N}) &= \varphi \qquad \text{on } \partial M \end{aligned} \tag{3.14}$$

If $\Phi \in C^{k-2}(M)$ and $\varphi \in C^{k-2}(\partial M)$ then α can be chosen of class C^{k-1} on M.

b) Given a V-valued function $\Theta \in H^1\Omega^0(M;V)$, which obeys the integrability condition

$$\int_M \Theta \, \mu_M = 0 \qquad (3.15),$$

then there is a one form $\gamma \in H^1\Omega^1(M;V)$ solving the boundary value problem

$$\begin{aligned} \delta\gamma &= \Theta && \text{on } M \\ \gamma|_{\partial M} &\equiv 0 && \text{on } \partial M \end{aligned} \qquad (3.16).$$

If $\Theta \in C^{k-3}(M)$ then γ can be chosen of class C^{k-2} on M.

Proof :
By the Kodeira decomposition (3.11) some $\beta_\Phi \in H^1\Omega^1(M;V)$ is determined from Φ, obeying

$$\Phi = \delta\beta_\Phi + c_\Phi \qquad \text{and} \qquad \beta_\Phi(\mathcal{N}) = 0 \qquad (3.17).$$

On the other hand one can choose for any $\varphi \in H^1\Omega^0(\partial M;V)$ some one-form $\phi \in H^1\Omega^1(M;V)$ such that $\phi(\mathcal{N}) = \varphi$ on ∂M and $\delta\phi \in H^1\Omega^0(M;V)$. Applying Kodeira's decomposition to the function $\delta\phi$ yields

$$\delta\phi = \delta\beta_\varphi + c_\varphi \qquad \text{with} \qquad \beta_\varphi(\mathcal{N}) = 0 \qquad (3.18).$$

Then the boundary value problem (3.14) is solved by the one-form

$$\alpha = \beta_\Phi + \phi - \beta_\varphi \qquad (3.19).$$

This can be seen by showing that the constants c_Φ and c_φ cancel each other : Using the integrability condition (3.13) and the Gauß theorem, cf. (3.10), we really get

$$\int_M (c_\Phi - c_\varphi)\, \mu_M = \int_M (\Phi - \delta\alpha)\, \mu_M = \int_M \Phi\, \mu_M + \int_{\partial M} \varphi\, \mu_\partial = 0 \qquad (3.20).$$

To prove b) we start similar as above and decompose Θ by (3.11). From the integrability condition (3.15) the constant c_Θ has to vanish and hence

$$\Theta = \delta\beta_\Theta \qquad \text{with} \qquad \beta_\Theta(\mathcal{N}) = 0 \qquad (3.21).$$

Then there exists by part b) of theorem 1 some $\xi_\Theta \in H^1\Omega^2(M;V)$ such that

$$(\delta\xi_\Theta)|_{\partial M}(X) = \beta_\Theta|_{\partial M}(X) \qquad \text{and} \qquad \xi_\Theta|_{\partial M} \equiv 0 \quad \forall\, X \in \Gamma T\partial M \qquad (3.22).$$

We choose $\gamma = \beta_\Theta - \delta\xi_\Theta$ and obtain

$$\begin{aligned} \delta\gamma &= \Theta && \text{on } M \\ \gamma|_{\partial M}(X) &= 0 && \text{on } \partial M \end{aligned} \qquad (3.23),$$

holding for all vector fields X along ∂M. It remains to show that also $\gamma|_{\partial M}(\mathcal{N}) = 0$. To do so we use the collar theorem [Hir], which guaranties that we can find near any $p \in \partial M$ a local orthonormal frame on TM of the form $\{\widetilde{\mathcal{N}}, \widetilde{E}_2, \ldots, \widetilde{E}_m\}$ with $\widetilde{\mathcal{N}}|_{\partial M} = \mathcal{N}$ and $\widetilde{E}_i|_{\partial M}$ tangential to ∂M. Then (3.7) yields

$$\begin{aligned}(\delta\xi_\Theta)|_{\partial M}(\mathcal{N}) &= -(\nabla_{\widetilde{\mathcal{N}}}\xi_\Theta)|_{\partial M}(\widetilde{\mathcal{N}}, \mathcal{N}) - \sum_{k=2}^{m}(\nabla_{\widetilde{E}_k}\xi_\Theta)|_{\partial M}(\widetilde{E}_k, \mathcal{N}) \\ &= -\sum_{k=2}^{m}\nabla_{\widetilde{E}_k}(\xi_\Theta(\widetilde{E}_k, \mathcal{N}))|_{\partial M}\end{aligned} \qquad (3.24),$$

using $(\xi_\Theta)|_{\partial M} \equiv 0$ from (3.22). Also due to that fact $(\xi_\Theta)|_{\partial M}$ is covariantly constant under the action of the vector fields $\widetilde{E}_k$ along ∂M, what proves $(\delta\xi_\Theta)|_{\partial M}(\mathcal{N}) = 0$. Since also $\beta_\Theta(\mathcal{N}) = 0$ by (3.21) the V-valued one form $\gamma|_{\partial M} = \beta_\Theta - \delta\xi_\Theta$ vanishes identically on ∂M. Finally the differentiability results directly read off from theorem 1. □

Both assertions of that lemma are not original. Corresponding problems for vector fields are solved by transforming (3.14) into an (elliptic) Neumann problem, cf. [Hör], and considering a boundary value problem for the divergence [vWa], respectively. For a discussion of these results in terms of our approach and for improved regularity assertions see [Sch].

4. Translational Invariance and the Notion of Stress

Having now the required tools from Hodge theory, we start the proof of our central result and derive a stress tensor formulation of continuum mechanics on the base of a general virtual work approach. Due to the product structure of the Euclidean group we can consider the translational and the rotational invariance separately. So we first use the invariance of $F[J]$ under global rigid translations as an integrability condition to show that any $\mathbb{R}^n$-invariant virtual work, given in the form (2.9), allows a tensorial description. Starting from Noll's axiom (2.10) such tensorial character of the stress is evident [Tru]. Under our (weaker) global assumption, however, we need Hodge theory to prove this local assertion :

Theorem 2

Let the body $\mathcal{B}$ be a Riemannian C^k-manifold with boundary and let a virtual work form on $E(\mathcal{B}, \mathbb{R}^n)$ be determined from a by $(\Phi(J), \varphi(J))$ of densities of Sobolov class $H^1\Omega^0$ on $\mathcal{B}$ and $\partial\mathcal{B}$, respectively, as

$$F[J](\Lambda) = \int_{\mathcal{B}} < \Phi(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi(J), \Lambda >_{\mathbb{R}^n} \mu_\partial \qquad (4.1).$$

If global rigid translations cause no work, i.e.

$$F[J](z) = \int_{\mathcal{B}} < \Phi(J), z >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi(J), z >_{\mathbb{R}^n} \mu_{\partial} = 0 \qquad \forall z \in \mathbb{R}^n \qquad (4.2),$$

there exists a $\mathbb{R}^n$-valued one form $\alpha(J) \in H^1\Omega^1(\mathcal{B};\mathbb{R}^n)$, called the stress form of the system, such that the virtual work becomes

$$F[J](\Lambda) = \int_{\mathcal{B}} < \alpha(J), d\Lambda >_{\Omega^1} \mu_{\mathcal{B}} \qquad (4.3).$$

Here $d\Lambda \in \Omega^1(\mathcal{B};\mathbb{R}^n)$ is the differential of the virtual displacement $\Lambda \in T_J E(\mathcal{B},\mathbb{R}^n)$ and $<,>_{\Omega^1}$ is the scalar product (3.8).
Furthermore $\alpha(J)$ is C^{k-2}-differentiable if $\Phi(J)$ and $\varphi(J)$ were of class C^{k-2}.

Proof :
Given the pair of functions $(\Phi(J), \varphi(J))$ we observe that the invariance condition (4.2) is equivalent to the integrability condition (3.13) since $z \in \mathbb{R}^n$ is arbitrary. Hence part a) of lemma 1 guaranties some $\alpha(J) \in H^1\Omega^1(\mathcal{B};\mathbb{R}^n)$ to exist, such that the virtual work becomes

$$F[J](\Lambda) = \int_{\mathcal{B}} < \delta\alpha(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \alpha(J)(\mathcal{N}), \Lambda >_{\mathbb{R}^n} \mu_{\partial} \qquad (4.4).$$

Since $< .,. >_{\Omega^0} = < .,. >_{\mathbb{R}^n}$ and $\Lambda \in T_J E(\mathcal{B},\mathbb{R}^n) \cong \Omega^0(\mathcal{B};\mathbb{R}^n)$ we can apply Stokes' theorem (3.10) to shift the operator δ, acting on $\alpha(J)$, to its adjoint d acting on Λ. Then the boundary terms cancel, what proves (4.3). □

This result, which is originally due to [Bin], enables us to link the virtual work description of continuum mechanics to the usual stress tensor formulation. The principle of virtual work, set up in section 2, rewrites in terms of the stress form $\alpha(J)$ as :

Corollary
Let the virtual work of a given system split into $\mathcal{F}_J = F[J] + \widehat{F}[J]$, where $F[J]$ is $\mathbb{R}^n$-invariant in the sense of (4.2). Then $J \in E(\mathcal{B},\mathbb{R}^n)$ describes an equilibrium configuration, iff for all $\Lambda \in T_J E(\mathcal{B},\mathbb{R}^n)$

$$\int_{\mathcal{B}} < \alpha(J), d\Lambda >_{\Omega^1} \mu_{\mathcal{B}} \; + \; \widehat{F}[J](\Lambda) \; = \; 0 \qquad (4.5)$$

Since by construction $\Omega^1(\mathcal{B},\mathbb{R}^n)$ is equal to the space of all (1,1)-type two-point tensor on $\mathcal{B}$, the stress form $\alpha(J)$ is to be interpreted as the 1^{st} Piola-Kirchhoff stress tensor of the system. To see this we start with the constitutive part of the virtual work $F^{\text{int}}[J]$ and derive a well established version of the equilibrium equation :

Let $U \subset \mathcal{B}$ be some open (connected) subset and assume that $\partial U \cap \partial\mathcal{B} = \emptyset$, for sake of simplicity. By U_ϵ we denote a family of open subset of $\mathcal{B}$, containing the closure of U, i.e. $\overline{U} \subset U_\epsilon \subset \mathcal{B}$, and require the volume of the set $U_\epsilon \backslash U$ to be bounded by *epsilon*. Then we choose a family of smooth virtual displacements

$$\Lambda_\epsilon^\lambda(p) = \begin{cases} \lambda & \text{on } U \\ \widetilde{\lambda}_\epsilon(p) & \text{on } U_\epsilon \backslash U \\ 0 & \text{on } \mathcal{B}\backslash U_\epsilon \end{cases} \tag{4.6}$$

where $\lambda \in \mathbb{R}^n$ is constant. Since $d\Lambda_\epsilon^\lambda = 0$ on U and on $\mathcal{B}\backslash U_\epsilon$, we obtain for the virtual work (4.3) by using Stokes' theorem

$$F^{\text{int}}[J](\Lambda_\epsilon^\lambda) = \int\limits_{U_\epsilon \backslash U} < \delta\alpha^{\text{int}}(J), \widetilde{\lambda}_\epsilon >_{\Omega^0} \mu_{\mathcal{B}} + \int\limits_{\partial(U_\epsilon \backslash U)} < \alpha^{\text{int}}(J)(\widehat{\mathcal{N}}_\epsilon), \widetilde{\lambda}_\epsilon >_{\Omega^0} \mu_\partial \tag{4.7}$$

where $\widehat{\mathcal{N}}_\epsilon$ denotes the outward pointing normal on $\partial(U_\epsilon \backslash U)$. By construction the boundary splits into $\partial(U_\epsilon \backslash U) = \partial U_\epsilon \cup \partial U$ and we have $-\widehat{\mathcal{N}}_\epsilon|_{\partial U} = \widetilde{\mathcal{N}}$, what is the outward pointing unite normal of U. Furthermore $\widetilde{\lambda}_\epsilon(p)$ vanishes on ∂U_ϵ and takes the constant value λ on ∂U. With (2.3) for the (total) virtual work $\mathcal{F}_J$ we then obtain from (4.5) for an equilibrium solution

$$\int\limits_{\partial U} < \alpha^{\text{int}}(J)(\widetilde{\mathcal{N}}), \lambda >_{\Omega^0} \mu_\partial = \int\limits_{U} < \psi_J, \widetilde{\lambda} >_{\Omega^0} \mu_{\mathcal{B}} + \int\limits_{U_\epsilon \backslash U} < (\delta\alpha^{\text{int}}(J) - \psi_J), \widetilde{\lambda}_\epsilon >_{\Omega^0} \mu_{\mathcal{B}} \tag{4.8.}$$

In the limit $\epsilon \to 0$ the term $< (\delta\alpha^{\text{int}}(J) - \psi_J), \widetilde{\lambda}_\epsilon >_{\Omega^0}$ remains bounded, hence the coresponding integral vanishes. Since $\lambda \in \mathbb{R}^n$ is arbitrary, the principle of virtual work finally yields for any subbody $U \subset \mathcal{B}$

$$J \text{ is an equilibrium configuration} \quad \Leftrightarrow \quad -\int\limits_{\partial U} \alpha^{\text{int}}(J)(\widetilde{\mathcal{N}})\, \mu_\partial + \int\limits_{U} \psi_J\, \mu_{\mathcal{B}} = 0 \tag{4.9.}$$

Hence we have obtained a well established formulation for the static integral equation of continuum mechanics, where the $\alpha^{\text{int}}(J)$ is to be considered as the 1^{st} Piola-Kirchhoff stress tensor.

We remark that the derivation of this equilibrium equation does not depend on the choice of $F^{\text{int}}[J]$ for the $e(n)$-invariant part of $\mathcal{F}_J$. Any other splitting would yield a similar result with an appropriate re-interpretation of ψ_J.

5. The Piola Transformation

Classical approaches to continuum mechanics favor a description in terms of usual tensors on $\mathcal{B}$ or $J(\mathcal{B})$ instead of two point tensors. Hence we have to study the Piola transformation in our framework. Therefore we restrict our consideration to a n-dimensional body $\mathcal{B}$. Then any embedding $J \in E(\mathcal{B}; I\!R^n)$ is a regular map, saying that dJ, the principle part of the tangent map $TJ = (J, dJ)$ is an isomorphism. It makes sense to introduce the adjoint $dJ^\dagger$ of the tangent map, which depends on the Riemannian metric $G_\mathcal{B}$ as well as on the scalar product $<,>_{I\!R^n}$, by writing

$$G_\mathcal{B}(W\,,\, dJ^\dagger w) \;:=\; < dJW\,,\, w >_{I\!R^n} \qquad \forall\, W \in T_p\mathcal{B} \quad \forall\, w \in I\!R^n \qquad (5.1).$$

If $\Delta(J)$ denotes the Jakobian determinant of the map J, the equation

$$\Delta(J)\cdot \mathbf{A}_\alpha(J)|_{J(p)}(v) \;:=\; \alpha(J)|_p(dJ^\dagger v) \qquad \forall\, v \in I\!R^n \qquad (5.2)$$

determines a well defined tensor $\mathbf{A}_\alpha(J) : I\!R^n \to I\!R^n$ over each point $J(p)$ in the image manifold $J(\mathcal{B})$. It is the inverse of that transformation, sending the tensor $\mathbf{A}_\alpha(J)$ into the stress form $\alpha(J)$, which is denoted as the Piola transformation [HuMa,TrTo] in continuum mechanics. To establish $\mathbf{A}_\alpha(J)$ as the Cauchy stress tensor we rewrite the virtual work (4.3) by pulling back the virtual displacement $\Lambda : \mathcal{B} \to I\!R^n$ to the $I\!R^n$-valued function $L = \Lambda \circ J^{-1}$ on $J(\mathcal{B}) \subset I\!R^n$. Then the differential becomes

$$d\Lambda|_p(E_i) = \big(\text{grad } L|_{J(p)} \circ dJ|_p\big)(E_i) \qquad (5.3)$$

where grad L is the vector gradient in the usual sense. With (3.8) for the scalar product $<,>_{\Omega^1}$ we get

$$\begin{aligned} F[J](\Lambda) &= \sum_{i=1}^{n} \int_{\mathcal{B}} < \mathbf{A}_\alpha(J) \circ (dJ^{-1})^\dagger(E_i), d\Lambda(E_i) >_{I\!R^n} \Delta(J)\, \mu_\mathcal{B} \\ &= \sum_{i=1}^{n} \int_{J(\mathcal{B})} G_\mathcal{B}\Big(E_i\,,\, dJ^{-1} \circ \mathbf{A}^*_\alpha(J) \circ (\text{grad } L) \circ dJ(E_i)\Big)\, \mu_{I\!R^n} \end{aligned} \qquad (5.4)$$

where $\mathbf{A}^*_\alpha(J)$ denotes the adjoint with respect to $<,>_{I\!R^n}$ and we notice that the Riemannian volume element transforms with the Jakobian as $J^*\mu_\mathcal{B} = \Delta(J)^{-1}\mu_{I\!R^n}$. Observing finally that $\{E_1, \ldots, E_n\}$ is a orthonormal base on $T_p\mathcal{B}$ we use the cyclic property of the trace to obtain for any $I\!R^n$-invariant virtual work :

$$F[J](L) = \int_{J(\mathcal{B})} \text{trace } \Big(\mathbf{A}^*_\alpha(J)\cdot \text{grad } L\Big)\mu_{I\!R^n} \qquad (5.5).$$

To make the Hodge theoretic results also available for investigations of the tensor $\mathbf{A}_\alpha(J)$ we establish the celebrate Piola identity for differential forms. Therefore we consider some real valued one form $\kappa \in \Omega^1(\mathcal{B}, I\!R)$ and observe that the induced vector fields $\kappa^\sharp$ and $(\kappa \circ dJ^\dagger)^\sharp$ on $\mathcal{B}$ and $J(\mathcal{B})$, respectively, are related by

$$< v, (\kappa \circ dJ^\dagger)^\sharp >_{I\!R^n} = \kappa(dJ^\dagger v) = < v, dJ\kappa^\sharp >_{I\!R^n} \qquad \forall\, v \in I\!R^n \tag{5.6}$$

We remark, that the $\sharp$-operator (3.6) is defined with respect to the (different) metrics $<,>_{I\!R^n}$ and $G_\mathcal{B}$ on the left and right hand side, respectively, and write the Piola identity as :

Lemma 2

The co-differential operator $\delta_\mathcal{B} : \Omega^1(\mathcal{B}, I\!R^n) \to \Omega^0(\mathcal{B}, I\!R^n)$ acting on the body manifold $\mathcal{B}$ and the coresponding operator $\delta_{I\!R^n} : \Omega^1(J(\mathcal{B}), I\!R^n) \to \Omega^0(J(\mathcal{B}), I\!R^n)$ on the embedded manifold $J(\mathcal{B})$ are related to each other via a Piola transformation by

$$\Delta(J) \cdot \delta_{I\!R^n}\big(\mathbf{A}_\alpha(J)\big) = \delta_\mathcal{B}\big(\alpha(J)\big) \quad \text{where} \quad \Delta(J) \cdot \mathbf{A}_\alpha(J) = \alpha(J) \circ dJ^\dagger \tag{5.7}$$

Proof :
Let $\mu_{I\!R^n}$ be the Riemannian volume form on $J(\mathcal{B})$ and $v \in I\!R^n$ be a constant vector. By using some standard properties of the Hodge $*$-operator [AMR] we obtain with (3.5) for the co-differential $\delta_{I\!R^n}$

$$(< v, \delta_{I\!R^n} \mathbf{A}_\alpha(J) >_{I\!R^n}) \mu_{I\!R^n} = d\Big(* < v, \mathbf{A}_\alpha(J) >_{I\!R^n}\Big) = d\Big(\mathbf{i}_{K^\sharp_\mathbf{A}} \mu_{I\!R^n}\Big) \tag{5.8}$$

where $K_\mathbf{A} := < v\,,\, \mathbf{A}_\alpha(J) >_{I\!R^n}$ is a $I\!R$-valued one form on $J(\mathcal{B})$. Replacing $\mathbf{A}_\alpha(J)$ by its Piola transformed we set $\kappa_\alpha := < v\,,\, \alpha(J) >_{I\!R^n} \in \Omega^1(\mathcal{B}, I\!R)$ and get from (5.6)

$$K^\sharp_\mathbf{A} = \Delta(J)^{-1} \cdot (< v, \alpha(J) >_{I\!R^n} \circ dJ^\dagger)^\sharp = \Delta(J)^{-1} \cdot dJ(\kappa^\sharp_\alpha) \tag{5.9}$$

Using $\Delta(J)^{-1} \mu_{I\!R^n} = J^* \mu_\mathcal{B}$ for the pull back of the volume form $\mu_\mathcal{B}$ this yields

$$d(\mathbf{i}_{K^\sharp_\mathbf{A}} \mu_{I\!R^n}) = J^* d(\mathbf{i}_{\kappa^\sharp_\alpha} \mu_\mathcal{B}) \tag{5.10}$$

We finish the proof by respelling (5.8) for $\kappa^\sharp_\alpha$ and observing that

$$(< v, J^*(\delta_\mathcal{B}\, \alpha(J)) >_{I\!R^n}) \cdot (J^* \mu_\mathcal{B}) = J^* d\Big(\mathbf{i}_{\kappa^\sharp_\alpha} \mu_\mathcal{B}\Big) = (< v, \delta_{I\!R^n} \mathbf{A}_\alpha(J) >_{I\!R^n})\, \mu_{I\!R^n} \qquad \square$$

As mentioned above, cf. (3.7), the co-differential operator and the divergence corespond to each other. Using that identification the Piola identity (5.7) reads as

$$\delta_\mathcal{B}\big(\alpha(J)\big) = \Delta(J)^{-1} \mathrm{div}_{I\!R^n} \mathbf{A}^\sharp_\alpha(J) \tag{5.11}$$

Then the equilibrium equation (4.9) for the 1^{st} Piola-Kirchhoff tensor $\alpha^{\text{int}}(J)$ transforms by using the Gauß theorem into

$$\int_{J(U)} \mathrm{div}_{I\!R^n} (\mathbf{A}^\sharp_\alpha)^{\text{int}}(J)\, \mu_{I\!R^n} + \int_{J(U)} \Psi(J)\, \mu_{I\!R^n} = 0 \qquad \forall U \subset \mathcal{B} \tag{5.12}$$

where $\Psi(J) := \Delta(J)^{-1}(\psi_J \circ J^{-1})$. This static form of the balance law of linear momentum for the Cauchy stress, as usually considered in continuum mechanics. For a direct derivation of that equation from the virtual work (5.5) we refer to [AnOs], where also possible functional analytic suptilities are studied in detail.
Finally we remark that the stress form $\alpha(J)$, and hence also the stress tensor $\mathbf{A}_\alpha(J)$ are not uniquely determined by the constructions made above : The stress form may be redefined to any $\widetilde{\alpha}(J)$, which coreponds to the same physical data $(\Phi(J), \varphi(J))$ by $\delta\widetilde{\alpha}(J) = \Phi(J)$ and $\widetilde{\alpha}(\mathcal{N}) = \varphi(J)$ and similarly one argues for $\mathbf{A}_\alpha(J)$. This gauge freedom coresponds to the fact, that only $\delta\alpha(J)$ or $\mathrm{div}\mathbf{A}^\sharp_\alpha(J)$ enter the equilibrium equations (4.9) or (5.12), respectively.
One can imagine several such modifications : From the mathematical point of view it seems natural to have $\alpha(J) \in \Omega^1(\mathcal{B}; \mathbb{R}^n)$ to be an exact one form, i.e. to be the gradient of some "stress function" $\mathcal{H}(J) \in \Omega^0(\mathcal{B}; \mathbb{R}^n)$. This is possible without further assumptions, as show in [Bin]. Considering continuum mechanics in its the classical formulation, however, a description of the Cauchy stress in terms of a symmetric tensor would be prefered.

6. The Symmetry of the Stress Tensor and a Natural Splitting of $\mathcal{F}_J$

To investigate the symmetry of the Cauchy stress we start from an $e(n)$-invariant virtual work of the form (2.9), which is necessarily also $\mathbb{R}^n$-invariant. Then theorem 2 guaranties the existence of a stress form $\alpha(J)$ and by performing an inverse Piola transformation and using (5.7) there is a tensor $\mathbf{A}_\alpha(J)$, solving the boundary value problem

$$\begin{aligned} \Delta(J)\cdot\delta_{\mathbb{R}^n}\mathbf{A}_\alpha(J) &= \Phi(J) \qquad \text{on } J(\mathcal{B}) \\ \Delta(J)\cdot\mathbf{A}_\alpha(J)\,\mathbf{n} &= \varphi(J) \qquad \text{on } J(\partial\mathcal{B}) \end{aligned} \tag{6.1}$$

Here $\mathbf{n}$ is the unite normal field along $J(\partial\mathcal{B})$, defined by $dJ^\dagger\mathbf{n} := \mathcal{N}$. By means of Hodge theory we now show the existence of a symmetric stress tensor $\widetilde{\mathbf{A}}_\alpha(J)$, using the remaining $so(n)$-invariance of $F[J](\Lambda)$, see (6.3) below, as an integrability condition.

Theorem 3

Let $\mathcal{B}$ be a n-dimensional Riemannian C^k-manifold with boundary and let the work done by any virtual displacement $L \in \Omega^0(J(\mathcal{B}), \mathbb{R}^n)$ be given by (5.5) as

$$F[J](L) = \int_{J(\mathcal{B})} \text{trace}\,\Big(\mathbf{A}^*_\alpha(J)\cdot\text{grad}\,L\Big)\,\mu_{\mathbb{R}^n} \tag{6.2}$$

where the Cauchy stress tensor $\mathbf{A}_\alpha(J)$ is determined from the forces $(\Phi(J), \varphi(J))$ by (6.1). If $F[J]$ is $e(n)$-invariant, infinitesimal rigid rotations of the whole body cause no work

$$\int_{J(\mathcal{B})} \text{trace}\Big(\mathbf{A}^*_\alpha(J)\cdot C\Big)\,\mu_{\mathbb{R}^n} = 0 \qquad \forall\, C \in so(n) \tag{6.3}$$

and there exists of a symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J) : \mathbb{R}^n \to \mathbb{R}^n$ obeying also (6.1).
Furthermore $\widetilde{\mathbf{A}}_\alpha(J)$ is C^{k-2} differentiable if $\Phi(J)$ and $\varphi(J)$ were of class C^{k-2}.

Proof :
Writing $2\mathbf{S}_\alpha(J) := \mathbf{A}_\alpha(J) - \mathbf{A}^*_\alpha(J)$ for the anti-symmetric part of the tensor $\mathbf{A}_\alpha(J)$ we understand this as a $so(n)$-valued zero form $\mathbf{S}_\alpha(J) \in \Omega^0(J(\mathcal{B}); so(n))$. Since $so(n)$ is the space of all anti-symmetric $n \times n$-matrices, the invariance condition (6.3) yields

$$\int_{J(\mathcal{B})} \mathbf{S}_\alpha(J)\, \mu_{\mathbb{R}^n} = 0 \qquad (6.4),$$

what we use as integrability condition to apply lemma 1. Thus there exists a one form $\sigma_\alpha \in \Omega^1(J(\mathcal{B}); so(n))$, solving the boundary value problem

$$\mathbf{S}_\alpha(J) = \delta\sigma_\alpha \qquad \text{with} \qquad \sigma_\alpha|_{J(\partial\mathcal{B})} \equiv 0 \qquad (6.5).$$

Now let x, y, z be some vector fields over $J(\mathcal{B})$, then

$$\begin{aligned} &< x, \Sigma_\alpha(y, z) >_{\mathbb{R}^n} := \qquad (6.6) \\ &\qquad < x, \sigma_\alpha(y)\, z >_{\mathbb{R}^n} - < x, \sigma_\alpha(z)\, y >_{\mathbb{R}^n} - < z, \sigma_\alpha(x)\, y >_{\mathbb{R}^n} \end{aligned}$$

defines an $\mathbb{R}^n$-valued two form $\Sigma_\alpha \in \Omega^2(J(\mathcal{B}); \mathbb{R}^n)$, since $\sigma_\alpha(x)$ is an anti-symmetric tensor. Its co-differential computes according to (3.7) as

$$\begin{aligned} < x, \delta\Sigma_\alpha(z) >_{\mathbb{R}^n} = -\Big(\sum_{i=1}^n \nabla_{e_i} < x, \Sigma_\alpha(e_i, z) >_{\mathbb{R}^n} - < \nabla_{e_i} x, \Sigma_\alpha(e_i, z) >_{\mathbb{R}^n} \\ - < x, \Sigma_\alpha(e_i, \nabla_{e_i} z) >_{\mathbb{R}^n}\Big) \qquad (6.7) \end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is a (local) orthonormal frame on $J(\mathcal{B})$. Expanding this by (6.6) yields

$$\begin{aligned} < x, \delta\Sigma_\alpha(z) >_{\mathbb{R}^n} &= < x, \delta\sigma_\alpha\, z >_{\mathbb{R}^n} \\ &+ \sum_{i=1}^n \Big(< x, \nabla_{e_i}(\sigma_\alpha(z)) e_i >_{\mathbb{R}^n} - < x, \sigma_\alpha(\nabla_{e_i} z)\, e_i >_{\mathbb{R}^n} \qquad (6.8) \\ &\quad + < z, \nabla_{e_i}(\sigma_\alpha(x)) e_i >_{\mathbb{R}^n} - < z, \sigma_\alpha(\nabla_{e_i} x)\, e_i >_{\mathbb{R}^n}\Big) \end{aligned}$$

and we obtain a symmetric tensor on $J(\mathcal{B})$ by setting

$$\widetilde{\mathbf{A}}_\alpha(J) := \mathbf{A}_\alpha(J) - \delta\Sigma_\alpha \qquad (6.9).$$

This is the symmetric part of $\mathbf{A}_\alpha(J)$ – by definition of $\mathbf{S}_\alpha(J)$ – modified by a symmetric correction term. Due to the nilpotence of the co-differential on $J(\mathcal{B})$ we furthermore have $\delta\widetilde{\mathbf{A}}_\alpha(J) = \delta\mathbf{A}_\alpha(J)$. Hence it remains to study the behavior of $\widetilde{\mathbf{A}}_\alpha(J)$ on $\partial\mathcal{B}$. Therefore we argue similarly as in section 3, cf. (3.24), by choosing the (local) orthonormal frame as $\{\widetilde{\mathbf{n}}, \widetilde{e}_2, \ldots, \widetilde{e}_n\}$ near the surface of the body, with $\widetilde{e}_i|_{J(\partial\mathcal{B})}$ tangential and $\widetilde{\mathbf{n}}|_{J(\partial\mathcal{B})}$ normal to $J(\partial\mathcal{B})$. By construction σ_α vanishes on $J(\partial\mathcal{B})$, thus we obtain

$$< x, \delta\Sigma_\alpha(\widetilde{\mathbf{n}}) >_{\mathbb{R}^n} = < \widetilde{\mathbf{n}}, \nabla_{\widetilde{\mathbf{n}}}(\sigma_\alpha(x))\,\widetilde{\mathbf{n}} >_{\mathbb{R}^n} + \sum_{i=2}^{n}\Big(< x, \nabla_{\widetilde{e}_i}(\sigma_\alpha(\widetilde{\mathbf{n}}))\,\widetilde{e}_i >_{\mathbb{R}^n} + < \widetilde{\mathbf{n}}, \nabla_{\widetilde{e}_i}(\sigma_\alpha(x))\,\widetilde{e}_i >_{\mathbb{R}^n}\Big) \qquad (6.10).$$

This expression vanishes on $J(\partial\mathcal{B})$ since $\sigma_\alpha(x)$ is anti-symmetric and – as a consequence of $\sigma_\alpha|_{J(\partial\mathcal{B})} \equiv 0$ – it is also covariantly constant under the fields $\widetilde{e}_i|_{J(\partial\mathcal{B})}$, which are vector fields along the boundary. We remark that in general $\widetilde{\mathbf{n}} \neq \mathbf{n}$ but $\delta\Sigma_\alpha(\widetilde{\mathbf{n}})|_{\partial\mathcal{B}} = 0$ guaranties that $\widetilde{\mathbf{A}}_\alpha(J)(\widehat{\mathbf{n}}) = \mathbf{A}_\alpha(J)(\widehat{\mathbf{n}})$ for any field $\widehat{\mathbf{n}}$ normal to $J(\partial\mathcal{B})$. □

Having proven this theorem our central result on the existence of a symmetric stress tensor, formulated in section 2, is established. Similar to section 4 one may argue that Noll's (local) working axiom (2.10) would yield the symmetry of $\mathbf{A}_\alpha(J)$ directly, but our weaker global demand (2.7) only gave rise to the integrated symmetry (6.3) for an $e(n)$-invariant virtual work. This, however, suffies to solve an appropriate boundary value problem (6.5) and construct the auxiliary quantity $\Sigma_{\alpha(J)}$ – introduced earlier in the context of relativistic field theories [Bel – which then yields the symmetric of $\widetilde{\mathbf{A}}_\alpha(J)$.
Now it is a matter of routine, to obtain the balance laws for the momentum and angular momentum, i.e. the equilibrium equations for the symmetric Cauchy stress tensor, in their final form as

$$\begin{aligned} &\int_{J(U)} \Big(\mathrm{div}_{\mathbb{R}^n}(\widetilde{\mathbf{A}}^\sharp_\alpha)^{\mathrm{int}}(J) + \mathbf{\Psi}(J)\Big)\mu_{\mathbb{R}^n} = 0 \\ &\int_{J(U)} \Big(\text{ trace } (\widetilde{\mathbf{A}}^{\mathrm{int}}_\alpha(J)\cdot C) + < \mathbf{\Psi}(J), C\cdot q >_{\mathbb{R}^n}\Big)\mu_{\mathbb{R}^n} = 0 \quad \forall C \in so(n) \end{aligned} \qquad (6.11)$$

where $q \in J(U)$ and $U \subset \mathcal{B}$ is an arbitrary subbody. Since by construction $\mathrm{div}\,\mathbf{A}^\sharp_\alpha(J) = \mathrm{div}\,\widetilde{\mathbf{A}}^\sharp_\alpha(J)$, the first equation is obvious from (5.12) and to prove the angular momentum balance one may follow literally the construction, made in order to derive (4.9).

Finally we re-investigate the splitting of a given virtual work functional $\mathcal{F}_J$ into an $e(n)$-invariant part and a rest :

Theorem 4

Let the virtual work functional $\mathcal{F}_J$ be given by (2.3). Then there exists a constant vector field $\mathbf{\Phi}_J \in \mathbb{R}^n$, a constant anti-symmetric tensor $\mathbf{D}_J \in so(n)$ and a symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J)$ on $J(\mathcal{B})$, such that

$$\mathcal{F}_J(L\circ J) = \frac{1}{\mathrm{vol}(\mathcal{B})}\Big\langle \mathbf{\Phi}_J, \int_{\mathcal{B}} (L\circ J)\,\mu_{\mathcal{B}}\Big\rangle_{\mathbb{R}^n} + \frac{1}{\mathrm{vol}(J(\mathcal{B}))}\text{ trace }\Big(\mathbf{D}_J\cdot\int_{J(\mathcal{B})}\text{grad } L\,\mu_{\mathbb{R}^n}\Big) + \int_{J(\mathcal{B})}\text{ trace }\Big(\widetilde{\mathbf{A}}_\alpha(J)\cdot\text{grad } L\Big)\,\mu_{\mathbb{R}^n} \qquad (6.12)$$

Proof:
Given a virtual work functional $\mathcal{F}_J$, the equation $\mathcal{F}_J(z) = < \mathbf{\Phi}_J, z >_{\mathbb{R}^n}$, holding for all $z \in \mathbb{R}^n$, uniquely defines a vector $\mathbf{\Phi}_J \in \mathbb{R}^n$, which is a constant field on $\mathcal{B}$. Then

$$\widetilde{F}[J](\Lambda) := \mathcal{F}_J(\Lambda) - \frac{1}{\text{vol}(\mathcal{B})} < \mathbf{\Phi}_J, \int_{\mathcal{B}} \Lambda \, \mu_{\mathcal{B}} >_{\mathbb{R}^n} \qquad (6.13)$$

is an $\mathbb{R}^n$-invariant part of $\mathcal{F}_J$, cf. (4.2), and we can use theorem 2 to express this functional via (4.3) in terms of a stress form $\alpha(J)$. Performing a Piola transformation we obtain a representation of $\widetilde{F}[J](\Lambda)$ in terms of the coresponding stress tensor $\mathbf{A}_\alpha(J)$. Then

$$\text{trace}\left(C \cdot \int_{J(\mathcal{B})} \mathbf{A}_\alpha(J) \, \mu_{\mathbb{R}^n}\right) = \text{trace}\left(\mathbf{D}_J \cdot C\right) \qquad (6.14),$$

holding for all $C \in so(n)$, uniquely defines an anti-symmetric matrix $\mathbf{D}(J) \in so(n)$. Arguing now similarly as above, the functional

$$F[J](\Lambda) := \widetilde{F}[J](\Lambda) - \frac{1}{\text{vol}(\mathcal{B})} \text{trace}\left(\mathbf{D}_J \cdot \int_{J(\mathcal{B})} \text{grad}\, L \, \mu_{\mathbb{R}^n}\right) \qquad (6.15)$$

vanishes for all $\Lambda = C \cdot J$ with constant $C \in so(n)$. Hence (6.15) defines an $e(n)$-invariant part of the given virtual work $\mathcal{F}_J$ and we apply theorem 3 to construct a coresponding symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J)$. □

This theorem yields a construction to encode the "maximal information" about the virtual work in its $e(n)$-invariant part given by (6.15), which can be expressed by a symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J)$ in (6.12). The quantities $\mathbf{\Phi}_J \in \mathbb{R}^n$ and $\mathbf{D}_J \in so(n)$ then have a natural interpretation as the total force and total torque, affecting the center of mass and inertia tensor of the body, respectively. To describe the dynamics of a body all these terms may vary in time with the configuration J. More details can be found elsewhere [BiSc].
This splitting of a general the virtual work, representing all constitutive and external forces, appears as something unusual in the classical treatment of continuum mechanics. It is not clear whether this might be useful for practical purposes, but it presents the following structural insight : To describe the motion of an deformable medium it suffies to know the total center of mass force $\mathbf{\Phi}_J$ and the tensor $\mathbf{D}_J$ of total torque, affecting the material body, plus the symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J)$. The later term will gives rise to deformation forces, affecting $\mathcal{B}$ as a deformable body, while the primer ones only yield a rigid body motion in space.

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