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THE PRINCIPLE OF VIRTUAL WORK AND A SYMPLECTIC REDUCTION OF NON-LOCAL CONTINUUM MECHANICS

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1. Introduction

In this paper we derive a symplectic description for systems in continuum mechanics and a representation of the coresponding dynamics by exact one forms on a body manifold. In order to formulate the weak balance laws for the motion we start with the principle of virtual work and obtain thereby a generalization of the Hamiltonian approach. This makes it possible to include also non-hyperelastic media into the Hamiltonian description. For our considerations the virtual work functional is given in a very general form, i.e. we need not impose any restrictions on the constitutive laws. It allows in particular to include an arbitrary non-linear and non-local constitutive behavior of the material in view.

In the symplectic framework we study the effect of the group of rigid translations by means of the Marsden-Weinstein reduction with respect to that group. Thereby the deformation gradient appears as a natural geometric object on the reduced phase space. On the other hand the existence of a stress tensor can be shown to be a consequence of a (rigid) translational invariance for a general virtual work. In contrast to the classical approaches to continuum dynamics, we need not to impose any locality assumptions to find the stress tensor of such a system.

On the reduced phase space the balance law for the system turns into a weak equation, which relates the dynamics of the deformation gradient to the stress tensor. Remarkably we can describe this balance law by a differential equation on the level of exact $\mathbb{R}^n$ -valued one forms on the body manifold. By a further differentiation this implies the classical equations of continuum dynamics as well as appropriate boundary conditions for the stress.

The mathematical framework for the kinematics of a continuous medium is presented in section 2. We describe its dynamical behavior in terms of smooth curves of embeddings and consider

$$E(M,N) := \{\ J : M \longrightarrow N \mid J \ \text{ is a } \ C^\infty\text{– embedding }\ \} \qquad (1.1),$$

as the configuration space. Here the body M is a smooth orientable compact manifold with boundary and the smooth Riemannian manifold N describes the ambient space, the body moves in. The set $E(M,N)$ is endowed with the C^∞-topology and hence an infinite dimensional manifold.

A Riemannian structure on M is induced from a fixed metric $\langle,\rangle$ on N via pull back of the configuration J and yields a coresponding Riemannian volume form $\mu(J)$. With a density map $\rho : E(M,N) \to C^\infty(M,\mathbb{R})$, which satisfies the continuity equation with respect that induced volume form we obtain a (configuration independent) weak Riemannian structure on $E(M,N)$ written by

$$\mathcal{B}(J)(L,K) := \int_M \rho(J)\langle L,K\rangle\,\mu(J) \qquad \forall\, L,K \in T_J\,E(M,N) \qquad (1.2).$$

The symplectic form ω_N on TN, determined by the metric $\langle,\rangle$, yields a symplectic form on $TE(M,N)$, which coresponds to the metric $\mathcal{B}$ and reads as

$$\omega_{\mathcal{B}}(L_J)(\mathbf{X},\mathbf{Y}) \ = \ \int_M \rho(J)\,\omega_N(\mathbf{X}(J),\mathbf{Y}(J))\,\mu(J) \qquad \forall\,\mathbf{X},\mathbf{Y} \in \Gamma(TE(M,N)) \qquad (1.3).$$

A remarkable properties of these geometric quantities on the configuration space $E(M,N)$ is to be invariant under all isometries of the ambient manifold N.

In section 3 we formulate the principle of virtual work, in the context of (the infinite-dimensional manifold) $E(M,N)$. Therefore we define the work functional as a smooth map $\mathcal{F} : TE(M,N) \to \mathbb{R}$ such that the restriction $\mathcal{F}[J] = \mathcal{F}|_{T_J E(M,N)}$ is a linear map for each configuration J. Its value on some $L_J \in T_J E(M,N)$ has the physical interpretation as the work done by the system under that virtual displacement.
For the special case $N = \mathbb{R}^n$ one has $T_J E(M,\mathbb{R}^n) \cong C^\infty(M,\mathbb{R}^n)$ and the virtual work principle [Mau,AnOs] determines the motion via

$$\frac{d}{d\tau}\bigg|_{\tau=t} \mathcal{B}(J_\tau)(V_\tau, L) = \mathcal{F}[J_t](L) \qquad \forall L \in C^\infty(M,\mathbb{R}^n) \qquad (1.4),$$

where $V_t \in T_{J_t} E(M,\mathbb{R}^n)$ is the velocity field, coresponding to the curve J_t. We present a generalization of this dynamics to an arbitrary manifold N can be given and show how it can be understood as a generalized Hamiltonian motion on the symplectic manifold $\big(TE(M,N),\omega_{\mathcal{B}}\big)$. Therefore we need not restrict ourselves to a hyperelastic medium; for that special case the motion becomes Hamiltonian in the usual sense, with $\mathbf{D}U(J)(L) = -\mathcal{F}[J](L)$ for a smooth potential $U : E(M,N) \to \mathbb{R}$.
In all further investigations we consider $N = \mathbb{R}^n$, equipped with a fixed scalar product, and restrict the form of the virtual work by demanding a special L^2-representation for the linear functional given by

$$\mathcal{F}[J](L) = \int_M \langle \Phi[J], L\rangle\, \mu(J) + \int_{\partial M} \langle \varphi[J], L\rangle\, \mu_\partial(J) \qquad (1.5).$$

The physical interpretation of $\Phi : E(M,\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^n)$ is to represent the sum of the external body forces and the unbalanced constitutive force density and $\varphi : E(M,\mathbb{R}^n) \to C^\infty(\partial M,\mathbb{R}^n)$ characterizes the traction force density, caused by internal effects on boundary of the embedded body, and the external contact force density. We remark that such a description of $\mathcal{F}$ allows for an arbitrary non-linear and non-local constitutive behavior. On a real motion J_t the boundary force densities have to cancel each other, i.e. (1.4) is consistent only for $\varphi[J_t] = 0$.

In section 4 we study the effect of the group $\mathbb{R}^n$ of rigid translations acting by $J \mapsto J + Z$. To characterize the system under this group action we call a virtual work functional $\mathbb{R}^n$-invariant, if

$$\mathcal{F}[J](Z) = 0 \qquad \forall J \in E(M,\mathbb{R}^n) \qquad (1.6a)$$

$$\mathcal{F}[J+Z](L) = \mathcal{F}[J](L) \qquad \forall J \in E(M,\mathbb{R}^n) \quad \forall L \in C^\infty(M,\mathbb{R}^n) \qquad (1.6b)$$

holds for all $Z \in \mathbb{R}^n$. (1.6a) points out that the (integrated) total force on the embedding body vanishes for any configuration and (1.6b) expresses the homogeneity of the ambient space $\mathbb{R}^n$. Since the constitutive quality of a material should only depend on the internal distances (and orientations) between the points of the embedded body, it is a physically reasonable assumption that (at least) the internal part of any virtual work is $\mathbb{R}^n$-invariant.

Applying the Marsden-Weinstein reduction with respect to the action of the translation group on the phase space $TE(M, \mathbb{R}^n)$ we get for the momentum map

$$\mathcal{J}(V_J) = \int_M \rho(J) V_J \, \mu(J) \qquad (1.7).$$

Considered physically $\mathcal{J}(V_J)$ represents the total momentum Π of the moving body. To divide out the isotropy group, which is $\mathbb{R}^n$ for each orbit, we introduce the center of mass

$$\mathcal{S}_J := \int_M \rho(J) \cdot J \, \mu(J) \qquad (1.8),$$

such that $[J] \in E(M, \mathbb{R}^n)/\mathbb{R}^n$ is visualized by the relative configuration $J_0 = J - \mathcal{S}_J$. Hence an $\mathbb{R}^n$-invariance of the virtual work determines the dynamics on the reduced phase space $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ by fixing the center of mass and the total momentum of the moving continuum.

The geometric foundations, needed to describe the dynamics on $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$, are studied in section 5. For each tangent vector $L \in T_J E(M, \mathbb{R}^n)$ one has a decomposition

$$L = L_0 + C_L \quad \text{with} \quad C_L \in \mathbb{R}^n \quad \text{and} \quad \int_M \rho(J) \langle L_0, Z \rangle \, \mu(J) = 0 \quad \forall Z \in \mathbb{R}^n \quad (1.9),$$

such that an element of $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ is (uniquely) represented by a the pair (J_0, V_0). Alternatively such points are determined as pairs of differentials by (dJ, dV), where dJ can be identified with deformation gradient in classical continuum mechanics and dV is its time derivative. The metric $\mathcal{B}$ on $TE(M, \mathbb{R}^n)$ induces on $J^{-1}(\Pi)/\mathbb{R}^n$ a metric

$$\mathbb{B}\,(dK, dL) = \int_M \rho(dJ) \langle K_0, L_0 \rangle \, \mu(dJ) \qquad \forall\, L, K \in \mathcal{J}^{-1}(\Pi) \qquad (1.10)$$

and a coresponding symplectic form, denoted by $\omega_{\mathbb{B}}$. For an alternative description of $\mathbb{B}$ we observe that the boundary value problem

$$\rho(dJ) K_0 = \delta\kappa \quad , \quad d\kappa = 0 \quad \text{and} \quad \kappa(\mathcal{N}) = 0 \qquad (1.11),$$

has a unique solution, with $\delta : \mathcal{A}^1(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^n)$ for the co-differential on M and $\mathcal{N}$ for a the normal vector on ∂M. Using Stoke's theorem we then obtain

$$\mathbb{B}\,(dK, dL) = \int_M \mathcal{M}\big(\kappa, dL\big) \, \mu(dJ) \qquad (1.12),$$

were $\mathcal{M}$ is a $C^\infty(M, \mathbb{R})$-valued product on the space of all $\mathbb{R}^n$-valued one forms on M. In view of the virtual work functional on the reduced phase we prove that any $\mathbb{R}^n$-invariant one form $\mathbf{\Gamma} : TE(M, \mathbb{R}^n) \to \mathbb{R}$, obeying (1.5) accordingly, can be represented by

$$\Gamma[dJ](dL) = \int_M \mathcal{M}\big(d\mathcal{H}[dJ], dL\big) \, \mu(dJ) \qquad (1.13).$$

$\mathcal{H}[dJ] \in C^\infty(M; I\!R^n)$ is an $I\!R^n$-valued function, solving the Neumann problem $\Delta\mathcal{H} = \Phi$ with $d\mathcal{H}(\mathcal{N}) = \varphi$, which has an integrability condition equivalent to (1.6a).

Section 6 is concerned with the dynamics on the reduced phase space. We apply (1.13) to the internal virtual work $\mathcal{F}^0$ and by this establish the notion of the stress form $\alpha^{\mathcal{H}}[dJ] := d\mathcal{H}[dJ]$. Neglecting the effect of external body forces the induced Hamiltonian dynamics on $\mathcal{J}^{-1}(\Pi)/I\!R^n$ yields

$$\frac{d}{d\tau}\bigg|_{\tau=t} I\!B\,(dV_\tau, ddL) = \int_M \mathcal{M}(\alpha^{\mathcal{H}}[dJ_t], dL)\,\mu(dJ_t) - \int_{\partial M} \langle\varphi^{\text{cont}}[dJ_t], L\rangle\,\mu_\partial(dJ_t) \quad (1.14)$$

for the equation of motion, with $\varphi^{\text{cont}}[dJ]$ denoting the density of external contact forces. Comparing this dynamics to the classical formulation, the stress form $\alpha^{\mathcal{H}}[dJ]$ is to be identified by setting $\mathbf{A}^{\mathcal{H}}[dJ] = \alpha^{\mathcal{H}}[dJ] \circ (TJ)^{-1}$ with the Cauchy stress tensor in a material representation.
In contrast to standard approaches to continuum dynamics, which need to use some locality arguments to establish the notion of a stress tensor, our construction allows for an arbitrary non-locally constitutive behavior of the system. Another remarkable feature of (1.14) is to allow for a gauge freedom in representing the stress in terms of different one form $\tilde{\alpha}[dJ] = \alpha^{\mathcal{H}}[dJ] + \beta[dJ]$ on TM or by the coresponding Cauchy tensor $\tilde{\mathbf{A}}[dJ]$ without changing the physical content of that equation.

In section 7 we give up the covariant formulation and derive a local balance law on $\mathcal{J}^{-1}(\Pi)/I\!R^n$. To do so we solve the elliptic problem (1.11) for the inertia force field $\rho(dJ)\dot{V}_0$, accompanying a motion and denote the solution by $\omega(dJ)$. Representing $I\!B$ appropriately, i.e. by (1.12), we obtain with a standard localization procedure

$$\omega(dJ_t) = \alpha^{\mathcal{H}}[dJ_t] \quad (1.15)$$

for the dynamics on the reduced phase space. The divergence of this equation – rewritten it in the spatial representation – then yields

$$\rho(dJ_t)\,\frac{d}{d\tau}\bigg|_{\tau=t} \mathbf{V}_\tau = \mathrm{Div}\mathbf{A}^{\mathcal{H}}[dJ_t] + \mathbf{A}^{\mathcal{H}\mathcal{H}}[dJ_t](\Delta J_t) \quad (1.16).$$

For simple bodies the term ΔJ vanishes and (1.16) coincides with the standard equation of continuum dynamics; for shells or rods, however, this term reflects – as the mean curvature – the special geometry obtained by embedding a body with a nonvanishing co-dimension. Finally we observe that the global equation (1.14) also characterizes the behavior on ∂M, by enforcing the system to obey traction boundary conditions

$$d\mathcal{H}[dJ](\mathcal{N}) = \varphi^{\text{cont}}[dJ] \quad (1.17).$$

In section 8 we illustrate our results by an application to two simple cases :
First we study linear elasticity by writing for the virtual work functional

$$I\!F_{\text{lin}}[dJ](dL) = \int_M \mathcal{M}(C \circ dJ \circ c_r dI\!L)\,\mu(dJ) \quad (1.18)$$

where the map $C : M \to GL(n)$ and the bundle endomorphism $c \in \mathrm{End}(TM)$ both are smooth and configuration independent. The coresponding dynamics is given by

$$\omega(dJ_t) = \left(C \circ dJ_t \circ c\right)^{\mathrm{ex}} \qquad (1.19),$$

where $(C \circ dJ_t \circ c)^{\mathrm{ex}}$ denotes the exact part of the one form in view. For a homogeneous material this yields a modified wave equation of the form

$$\rho[dJ_t] \left.\frac{d}{d\tau}\right|_{\tau=t} V_\tau = \Delta^{\mathbf{C}} J_t \qquad (1.20),$$

where $\Delta^{\mathbf{C}}$ denotes the Laplacian, with respect to the induced metric $(C \circ dJ \circ c)^* \langle , \rangle$ on M. On the other hand we consider the virtual work form

$$I\!F_{\mathbf{p}}[dJ](dL) = \int_M \mathbf{p}[dJ]\, \mathcal{M}(dJ, dL)\, \mu(dJ) \qquad (1.21),$$

where $\mathbf{p}[dJ] : M \to I\!R$ is a smooth map. $I\!F_{\mathbf{p}}$ can be understood as the continuum analogon of the Gibbs form $\pi\, dV$ in classical thermodynamics. We note that it need not be integrable, but the analysis on $E(M, I\!R^n)$ presents a simple criterion to check this. In any case the generalized Hamiltonian formalism, yields for the dynamics on the reduced phase space

$$\rho[dJ_t] \left.\frac{d}{d\tau}\right|_{\tau=t} V_\tau = \mathbf{p}[dJ_t]\Delta\, J_t + dJ_t(\mathrm{grad}\, \mathbf{p}[dJ_t]) \qquad (1.21)$$

We terminate the paper by two appendices :
One, presenting details on the topology and geometry of the bundle $\mathcal{A}^1_E(M, TN)$ of TN valued differential forms on M, which gives the analytic base for our studies.
The second refers to the Laplace operators and the associated Neumann problem for functions on M, which is needed for the construction of $\omega_{I\!B}$ as well as for the representation of the virtual work in terms of $d\mathcal{H}[dJ]$.

2. Kinematic of bodies on the manifold E(M,N)

In this paper we describe the mechanical properties of a continuous medium in terms of embeddings of a Riemannian manifold, as presented e.g. in [HuMa]. For the the ambient space of these embeddings we take a connected oriented C^∞-manifold N which carries a Riemannian structure

$$\langle , \rangle : TN \times_N TN \longrightarrow I\!R \qquad (2.1).$$

With $T\pi_N : T^2N \to TN$ for the tangent map of the canonical projection we define by $\mathbf{V}(TN) := \mathrm{kern}(T\pi_N) \subset T^2N$ the vertical bundle of N. Then the covariant derivative with respect to the Levi-Civita connection of the metric $\langle , \rangle$ yields a smooth projection from T^2N to $\mathbf{V}(TN)$, which allows to determine the vertical component Y^{vert} of any vector $Y \in T^2N$.

The manifold N plays the role of the physical space, in which the deformable medium moves; it will be either an Euklidean space $\mathbb{R}^n$ or some constraint set in $\mathbb{R}^n$.
On the other hand we describe the material properties of the medium on a compact C^∞-manifold with boundary, called the body manifold M. Points of M are referred to as material points; they manifest themselves by their configurations in the ambient physical space N. By a configuration (or placement) of the body we then mean a smooth embedding $J : M \to N$ (where $\dim N \geq \dim M = r$) and call

$$E(M,N) := \{ J : M \to N \mid J \text{ is a } C^\infty - \text{ embedding } \} \tag{2.2}$$

the configuration space of the system. Since the set $C^\infty(M,N)$ of smooth maps from M into N, endowed with Whitney's C^∞-topology, is a Fréchet manifold (cf.[BSF]) and since $E(M,N)$ is open in $C^\infty(M,N)$, the configuration space is a Fréchet manifold, too.
Via pull-back by an embedding J we get an induced Riemannian structure on M, given by

$$m(J)(X,Y) := \langle TJ(X), TJ(Y) \rangle \qquad \forall X,Y \in \Gamma(TM) \qquad (2.3),$$

which also yields a Riemannian volume element $\mu(J)$ on M. Furthermore it makes sense to introduce the positively oriented (outward pointing) unite normal field – denoted by $\mathcal{N}$ – on the boundary $\partial M \subset M$ and observe that $\mu_\partial(J) := \mathbf{i}_\mathcal{N}\mu(J)$ becomes a naturally induced volume element on ∂M. Clearly $\mathcal{N}$ depends on the configuration J.

We call a smooth map $L : M \to TN$ with $\pi_N \circ L = f \in C^\infty(M,N)$ a "vector field along the map f". For a fixed f, the set of all such "vector fields along f" is a Fréchet space, also equipped with the C^∞-topology, which is denoted by $C^\infty_f(M,TN)$. This is precisely the tangent space $T_fC^\infty(M,N)$, hence the tangent bundle $TC^\infty(M,N)$ is identified with $C^\infty(M,TN)$.
Then the tangent bundle $TE(M,N)$ becomes an open submanifold of $C^\infty(M,TN)$, fibred over $E(M,N)$ by "composition with π_N", i.e.

$$TE(M,N) \cong C^\infty_E(M,TN) = \{ L : M \longrightarrow TN \mid \pi_N \circ L \in E(M,N) \} \qquad (2.4).$$

For the special case $N = \mathbb{R}^n$ the bundle TN is trivial and we have

$$TE(M,\mathbb{R}^n) = \{ (J,L) \mid J \in E(M,\mathbb{R}^n) \, , \; L \in C^\infty(M,\mathbb{R}^n) \} \qquad (2.5).$$

For the notion of the tangent space of any manifold – finite or infinite dimensional – we remark that there are two constructions possible; either by linearization of maps or by differentiation of curves on this manifold. Physically this reflects the observation that points in $TE(M,N)$ will appear under two different circumstances :
Either in the context of variational principles, where $L \in T_JE(M,N)$ receives the physical interpretation of a virtual displacement of the configuration $J \in E(M,N)$, or as velocity fields describing the evolution of embeddings. Therefore we consider $J_\cdot : (-\lambda,\lambda) \to E(M,N)$ as a smooth curve of embeddings and obtain

$$\left.\frac{d}{d\tau}\right|_{\tau=t} J_\tau \;=:\; V_t \in T_{J_t}E(M,N) \qquad (2.6)$$

as the time dependent velocity field on the deformable body. Usually in continuum dynamics such kind of velocity field, given in terms of a vector field over a map (the configuration J_t) is referred to as the velocity in the Lagrangian description. In the spatial (Eulerian) pictures, the corresponding field is a (proper) vector field on this image space $J_t(M) \subset N$, given by $\mathbf{V}_t := V_t \circ J_t^{-1}$ and the convective velocity $v_t \in \Gamma(TM)$ becomes $v_t := (TJ_t)^{-1} \circ V_t$ with $TJ_t : TM \longrightarrow TN$ for the tangent map of J_t.

As a further entity to describe the kinematics of a body we need to introduce the notion of a (Frechet-) smooth density map $\rho : E(M,N) \to C^\infty(M, I\!R)$. By assumption $\rho(J)$ is a positive valued function and its integral over M yields the total mass $\mathbf{m}$ of the body, i.e.

$$\int_M \rho(J)\,\mu(J) \;\equiv\; \mathbf{m} \qquad (2.7)$$

for any configuration $J \in E(M,N)$. Fixing a reference configuration $J_{\rm R} \in E(M,N)$ we may identify some positive valued function $\varrho \in C^\infty(M, I\!R)$, obeying $\int_M \varrho\,\mu(J_{\rm R}) = \mathbf{m}$ with the reference density, i.e. $\rho(J_{\rm R}) := \varrho$. To extend this density onto all of $E(M,N)$ we observe that there is for any $J \in E(M,N)$ a bundle map $f : TM \to TM$ such that $m(J)(f^2X,Y) = m(J_{\rm R})(X,Y)$ for all $X,Y \in \Gamma(TM)$. Then we chose $\rho(J) := \det(f)\cdot\rho(J_{\rm R})$. We remark that for classical approaches to continuum mechanics a description in terms of the reference density $\rho(J_{\rm R})$ is preferred. Here we use the configuration dependent density $\rho(J)$ since it is of an intrinsic geometrical nature and therefore is accessible to discuss deformations of densities analytically. Denoting by $\mathbf{D}$ the derivate on the Frechet-manifold $E(M,N)$ such $\rho(J)$ balances the configuration dependence of the volume $\mu(J)$ by requiring

$$\mu(J)\,\mathbf{D}\rho(J)(L) = -\,\rho(J)\,\mathbf{D}\mu(J)(L) = -\,\frac{1}{2}\,\rho(J)\mathrm{tr}_{\,m(J)}\mathbf{D}m(J)(L) \qquad (2.8),$$

where $L \in T_JE(M,N)$ is arbitrary and $\mathrm{tr}_{\,m(J)}$ denotes the trace formed with respect to the metric $m(J)$. If $\dim N = \dim M$ this yields the continuity equation

$$\mathbf{D}\rho(J)(L) = -\rho(J)\,\mathrm{div}_{\,\mu(J)}L \qquad (2.9).$$

The Riemannian volume $\mu(J)$, the coresponding density $\rho(J)$ and the metric $\langle\,,\,\rangle$ on N induces a (weak) Riemannian structure $\mathcal{B}$ on $E(M,N)$ by setting for any $J \in E(M,N)$

$$\mathcal{B}(J)(L,K) := \int_M \rho(J)\,\langle L,K\rangle\,\mu(J) \qquad \forall\, L,K \in T_J\,E(M,N) \qquad (2.10).$$

Clearly $\mathcal{B}(J)$ is a continuous, symmetric, positive-definite bilinear form. An important feature of that metric $\mathcal{B}$ is its (rather obvious) invariance property :

Proposition 2.1

Let $\mathcal{G}$ be a group of orientation-preserving isometries of N. Then $\mathcal{B}(J)$ is invariant under the left action of $\mathcal{G}$ on the $E(M,N)$ given by

$$\begin{aligned} \mathcal{G} \times E(M,N) &\longrightarrow E(M,N) \\ (g,J) &\longmapsto g \circ J \end{aligned} \qquad (2.11).$$

Proof :

From the continuity equation it is clear that $\rho(g \circ J)\,\mu(g \circ J) = \rho(J)\,\mu(J)$ and hence we obtain for any isometry $g \in \mathcal{G}$:

$$g^*\mathcal{B}(J)(L,K) \;=\; \int_M \rho(J)\,\langle Tg \circ L, Tg \circ K\rangle\,\mu(J) \;=\; \mathcal{B}(J)(L,K) \qquad \square$$

To equip $TE(M,N)$ with a symplectic structure we construct a one-form $\Theta_{\mathcal{B}}$ on $TE(M,N)$, naturally induced from this metric: Denoting by $\pi_E : TE(M,N) \to E(M,N)$ the projection, mapping any $L_J \in TE(M,N)$ to J, we define

$$\Theta_{\mathcal{B}}(L_J)(Y) := \mathcal{B}(J)(L, T\pi_E Y) \qquad \forall Y \in T_{L_J}TE(M,N) \qquad (2.12).$$

The symplectic form $\omega_{\mathcal{B}}$ on $TE(M,N)$ is minus the exterior derivative of $\Theta_{\mathcal{B}}$, i.e.

$$\omega_{\mathcal{B}}(L_J)(X,Y) = \mathbf{D}\big(\Theta_{\mathcal{B}}(L_J)(Y)\big)\,(X) - \mathbf{D}\big(\Theta_{\mathcal{B}}(L_J)(X)\big)\,(Y) + \Theta_{\mathcal{B}}(L_J)([X,Y]) \quad (2.13),$$

where $X,Y \in T_{L_J}TE(M,N)$. Since $T^2E(M,N) \cong C^\infty_E(M,T^2N)$, we obtain, cf. [BSF], for any two vector fields $\mathbf{X},\mathbf{Y} \in \Gamma(T^2E(M,N))$ and any $L_J \in TE(M,N)$

$$\mathbf{D}\big(\Theta_{\mathcal{B}}(L_J)(\mathbf{Y})\big)\,(\mathbf{X}) = \mathcal{B}(J)\big(\mathbf{Y}^{\text{vert}}, T\pi_E\,\mathbf{X}\big) + \mathcal{B}(J)\Big(L_J, \big(T(T\pi_E\,\mathbf{Y})(T\pi_E\mathbf{X})\big)^{\text{vert}}\Big) \quad (2.14)$$

and hence

$$\begin{aligned} \omega_{\mathcal{B}}(L_J)(\mathbf{X},\mathbf{Y}) &= \mathcal{B}(J)\big(\mathbf{Y}^{\text{vert}}, T\pi_E\,\mathbf{X}\big) - \mathcal{B}(J)\big(\mathbf{X}^{\text{vert}}, T\pi_E\,\mathbf{Y}\big) \\ &= \int_M \rho(J)\,\omega_N(\mathbf{X},\mathbf{Y})\,\mu(J) \end{aligned} \qquad (2.15),$$

where ω_N is the metric-induced symplectic structure on the finite dimensional manifold TN. This construction equips the phase space $TE(M,N)$ of a system in continuum mechanics with a weak symplectic structure. From proposition 2.1 it is obvious that $\Theta_{\mathcal{B}}$ and $\omega_{\mathcal{B}}$ inherit from $\mathcal{B}(J)$ its $\mathcal{G}$ invariance, if $\mathcal{G}$ is a group of isometries on the manifold N.

For an appropriate description of continuum mechanics in the context of the configuration space $E(M,N)$, we also need the notion of tensors in that framework. Therefore we consider two-point tensors [Eri,HuMa] over the body manifold M, as the natural generalizations of the vector fields over maps. Restricting the general definition to the case of our interest we define a two-point tensor $\mathbf{T}$ of type $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, shortly denoted as a (r,s)-type two-point tensor, at $p \in M$ over an embedding $J \in E(M,N)$ as a multilinear mapping

$$\mathbf{T} : \underbrace{(T_pM \times \ldots \times T_pM)}_{r-times} \times \underbrace{(T^*_{J(p)}N \times \ldots \times T^*_{J(p)}N)}_{s-times} \longrightarrow \mathbb{R} \qquad (2.16).$$

One can think of such a tensor having $r+s$ legs, r on M and s on N.
Of special interest in continuum mechanics are $(1,1)$-type two point tensors; e.g. the deformation tensor and the 1^{st} Piola-Kirchhoff stress tensor are of that kind. We define

$$\mathcal{A}^1_J(M,TN) := \{\, \alpha : TM \longrightarrow TN \mid \pi_N \circ \alpha \;=J\,,\; \alpha|_p : T_pM \to T_{J(p)}N \;\text{ linear } \forall p \in M \,\}$$

as the space of all linear bundle maps, which sit over a fixed embedding $J \in E(M,N)$. A remarkable feature of this space is to fit as well into the notion of (1,1)-type two point tensors as into the framework of bundle valued (one)-differential forms on M. For detail on the topology and geometry of $\mathcal{A}^1_J(M,TN)$ we refer to appendix A.

3. The principle of virtual work and the symplectic dynamics on E(M,N)

In order to formulate the (weak) balance laws of a system in continuum mechanics we start with the principle of virtual work [AnOs,Mau]. Its appropriate version in the framework of (the infinite-dimensional manifold) $E(M,N)$ as the configuration space is due to Epstein and Segev [EpSe], who introduced a one form $\mathcal{F}$ on $E(M,N)$ as the constitutive entity which determines the mechanical behavior of a system. We define such a one form as a smooth map

$$\begin{aligned} \mathcal{F} : TE(M,N) &\longrightarrow \mathbb{R} \\ L_J &\mapsto \mathcal{F}[J](L) \end{aligned} \qquad (3.1),$$

where the restriction $\mathcal{F}[J] = \mathcal{F}|_{T_JE(M,N)}$ is a linear map for each $J \in E(M,N)$. Recalling from (2.4) that at each configuration J the tangent space $T_JE(M,N)$ can be identified with the space $C^\infty_J(M,TN)$, $\mathcal{F}[J]$ is a continuous linear form on this fibre, i.e. an element of the topological dual $C^\infty_J(M,TN)' \cong \Gamma(J^*TN)'$. Loosely speaking, then, $\mathcal{F}$ is a smooth section of the "co-tangent bundle" of $E(M,N)$, but this will not be pursued here.

We will refer to this one form as the ***virtual work functional*** of the system. Its value on some tangent vector $L \in T_JE(M,N)$ has the physical interpretation as the work done by the system under the virtual displacement L. The ***principle of virtual work*** claims that the dynamics is determined by $\mathcal{F}$ as follows :
A smooth curve of embeddings $J_. : (-\lambda,\lambda) \to E(M,N)$ with $V_t \in T_{J_t}E(M,N)$ as its velocity field describes a motion of the system, iff it solves the equation

$$\frac{d}{d\tau}\bigg|_{\tau=t} \mathcal{B}(J_\tau)(V_\tau,\mathbf{L}) \;-\; \mathcal{B}(J_t)(V_t,(T\mathbf{L}(V_t))^{\text{vert}}) \;=\; \mathcal{F}[J_t](\mathbf{L}(J_t)) \qquad (3.2)$$

for all vector fields $\mathbf{L} : E(M,N) \to TE(M,N)$. This equation is the natural generalization of the classical principle of virtual work, cf. [AnOs,MuHa], which is restricted to the dynamics on a trivial ambient manifold, i.e. to $N = \mathbb{R}^n$. Since in that case the tangent bundle becomes $TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)$, one gets

$$\frac{d}{d\tau}\Big|_{\tau=t} \mathcal{B}(J_\tau)(V_\tau, L) = \mathcal{F}[J_t](L) \qquad \forall L \in C^\infty(M,\mathbb{R}^n) \tag{3.3}$$

for the equation of motion, instead of (3.2). If $TE(M,N)$ is nontrivial, however, the field L has to sit over the embedding J_t and we need to chose $L = \mathbf{L}(J_t)$ at any instant of time. Hence we have to subtract the extra term, coresponding to the time derivative of L.

The central result of this section will be to show, how the dynamics, obtained from the principle of virtual work – for a general ambient manifold N – can be understood as a symplectic motion. Therefore we first observe that the Hamiltonian description of continuum mechanics, cf. [Mar], appears as a specialization of (3.2) :

On the symplectic manifold $TE(M,N)$ we have a ***Hamiltonian system***, if there is a (conserved) energy functional $\mathbb{H} : TE(M,N) \to \mathbb{R}$, such that the dynamics is determined from its Hamiltonian vector field $\mathbf{X}_{\mathbb{H}}$, defined by

$$\mathbf{D}\mathbb{H}(J,V_J)(\mathbf{Y}) = \omega_{\mathcal{B}}(\mathbf{X}_{\mathbb{H}},\mathbf{Y}) \qquad \forall \mathbf{Y} \in \Gamma(T^2E(M,N)) \tag{3.4}$$

and the evolution of an observable $\mathbb{K}$, i.e. of any smooth function $\mathbb{K} : TE(M,N) \to \mathbb{R}$, is given by

$$\frac{d}{d\tau}\Big|_{\tau=t} \mathbb{K}(V_{J_\tau}) = \mathbf{D}\mathbb{K}(\mathbf{X}_{\mathbb{H}}) = \omega_{\mathcal{B}}(\mathbf{X}_{\mathbb{K}},\mathbf{X}_{\mathbb{H}}) \tag{3.5}$$

Clearly the existence of the Hamiltonian vector field $\mathbf{X}_{\mathbb{H}}$ is not guaranteed a priori. In particular, however, $\mathbb{H}(V_J) = \mathcal{E}(V_J)$ – with $\mathcal{E}(V_J) = \frac{1}{2}\mathcal{B}(J)(V_J,V_J)$ denoting the kinetic energy – yields a well defined vector field $\mathbf{X}_{\mathbb{H}} = \mathcal{S} \circ V_J$ where $\mathcal{S} : TN \to T^2N$ is the spray of the metric $\langle\, , \rangle$ on N.

Proposition 3.1

Let a system in continuum mechanics be determined by a virtual work principle, such that $\mathcal{F}$ is an exact one form on $E(M,N)$, i.e. there is a smooth potential $U : E(M,N) \to \mathbb{R}$, obeying

$$\mathbf{D}U(J)(L) = -\mathcal{F}[J](L) \qquad \forall J \in E(M,N) \quad \forall L \in TE_J(M,N) \tag{3.6}$$

If the energy functional $\mathbb{H}(V_J) = \mathcal{E}(V_J) + \pi_E^* U(V_J)$ – with $\pi_E^* U(V_J) = U(\pi_E V_J)$ – is such that the Hamiltonian vector field $\mathbf{X}_{\mathbb{H}}$ exists, the equation (3.2) is equivalent to the Hamiltonian dynamics.

Proof :
With (2.15) for the symplectic form the Hamiltonian vector field of – if it exists – obeys

$$\mathcal{B}(J)(\mathbf{Y}^{\text{vert}}, T\pi_E \mathbf{X}_{I\!H}) - \mathcal{B}(J)(\mathbf{X}_{I\!H}^{\text{vert}}, T\pi_E \mathbf{Y}) = \mathcal{B}(J)(V_J, \mathbf{Y}^{\text{vert}}) + \mathbf{D}U(\pi_E V_J)(T\pi_E \mathbf{Y})$$

where the first term on the right hand side comes from the kinetic energy $\mathcal{E}(V_J)$ by using the continuity equation. Then we get

$$\begin{aligned} \mathcal{B}(J)\,(T\pi_E \mathbf{X}_{I\!H}, \mathbf{Y}^{\text{vert}}) &= \mathcal{B}(J)\,(V_J, \mathbf{Y}^{\text{vert}},) \\ -\mathcal{B}(J)(\mathbf{X}_{I\!H}^{\text{vert}}, T\pi_E \mathbf{Y}) &= \mathbf{D}U(J)(T\pi_E \mathbf{Y}) \end{aligned} \qquad (3.7).$$

Considering the observable $I\!K^{\mathbf{L}}(V_J) := \mathcal{B}(J)(V_J, \mathbf{L}(J))$, where the vector field $\mathbf{L} \in \Gamma(TE(M,N))$ is arbitrary but fixed, we obtain

$$\mathbf{D}I\!K^{\mathbf{L}}(V_J)(\mathbf{X}_{I\!H}) = \mathcal{B}(J)(\mathbf{X}_{I\!H}^{\text{vert}}, \mathbf{L}) + \mathcal{B}(J)(V_J, (T\mathbf{L}(T\pi_E \mathbf{X}_{I\!H}))^{\text{vert}}) \qquad (3.8).$$

The coresponding Hamiltonian equation of motion

$$\left.\frac{d}{d\tau}\right|_{\tau=t} I\!K^{\mathbf{L}}(V_{J_\tau}) = -\mathbf{D}U(J_t)(\mathbf{L}(J_t)) + \mathcal{B}(J_t)(V_t, (T\mathbf{L}(V_t))^{\text{vert}}) \qquad (3.9)$$

coincides with the equation for the virtual work principle (3.2), since by assumption we have $\mathbf{D}U(J) \equiv -\mathcal{F}[J]$. □

By demanding (3.6) we restrict ourselves to a ***hyperelastic*** medium, for which the equivalence of the virtual work principle and the Hamiltonian description is shown. It is crucial to observe, however, that the hyperelasticity, i.e. the integrability of the one form $\mathcal{F}$ was redundant to derive the Hamiltonian form of the equations of motion in the proof above. Hence the dynamics of a system, subjected to a general virtual work functional $\mathcal{F}$ instead of a Hamiltonian $I\!H$, can be determined from a vector field $\mathbf{X}_{\mathcal{F}}$ given by

$$(\mathbf{D}\mathcal{E}(V_J) - \pi_E^* \mathcal{F}[J])(\mathbf{Y}) = \omega_{\mathcal{B}}(\mathbf{X}_{\mathcal{F}}, \mathbf{Y}) \qquad \forall \mathbf{Y} \in \Gamma(T^2E(M,N)) \qquad (3.10).$$

As in the Hamiltonian case the existence of $\mathbf{X}_{\mathcal{F}}$ is not guaranteed a priori, but from (3.5) and (3.7) the following is obvious :

Proposition 3.2

Let be $\mathcal{E}(V_J)$ the kinetic energy and $\mathcal{F}[J]$ be a virtual work form on $TE(M,N)$. A vector field solving (3.10) exists, iff the one form $\mathcal{F}$ lies in the range of the metric $\mathcal{B}$, i.e. there is a field $\mathbf{Z}_{\mathcal{F}} \in \Gamma(TE(M,N))$, such that $\mathcal{B}(J)(\mathbf{Z}_{\mathcal{F}}, \mathbf{Y}) = \mathcal{F}(J)(\mathbf{Y})$ for all $\mathbf{Y} \in \Gamma(TE(M,N))$. In that case the dynamics of a system, determined by the principle of virtual work (3.2), becomes a ***generalized Hamiltonian system*** in the sense that

$$\left.\frac{d}{d\tau}\right|_{\tau=t} I\!K(V_{J_\tau}) = -(\mathbf{D}\mathcal{E}(V_{J_t}) - \pi_E^* \mathcal{F}[J_t])(\mathbf{X}_{I\!K}) \qquad (3.11)$$

for any $I\!K : TE(M,N) \to I\!R$ with $\mathbf{X}_{I\!K}$ as the coresponding Hamiltonian vector field and vector field $\mathbf{X}_{\mathcal{F}}$ is given by $\mathbf{X}_{\mathcal{F}}(L) = \mathcal{S} \circ L - \mathbf{Z}_{\mathcal{F}}^{\text{vert}}(L)$ for $L \in TE(M,N)$, where $\mathbf{Z}_{\mathcal{F}}^{\text{vert}}$ means the vertical lift of $\mathbf{Z}_{\mathcal{F}}$.

Having shown the symplectic character of the motion (3.2) – with a general virtual work functional $\mathcal{F} : TE(M,N) \to I\!R^n$, which may characterizes a hyperelastic system or not – we limit ourselves in a twofold way :
First we choose $N = I\!R^n$ for the ambient manifold, such that we are in the framework of the classical virtual work principle (3.3). We do so to make the notion of translation invariance – which will be of central importance for the sequel – meaningful. A generalization of that concept to an arbitrary manifold is possible, by considering this a constraint set $N \subset I\!R^k$, but requires much more effort. For the boundaryless case see [Bi2], boundaries will be considered elsewhere.
On the other hand we assume for the one form $\mathcal{F}$ on $E(M, I\!R^n)$ a special L^2-representation – over the bounded manifold M – given by

$$\mathcal{F}[J](L) = \int_M \langle \Phi[J], L \rangle \, \mu(J) + \int_{\partial M} \langle \varphi[J], L \rangle \, \mu_\partial(J) \qquad (3.12),$$

where $\Phi : E(M, I\!R^n) \to C^\infty(M, I\!R^n)$ and $\varphi : E(M, I\!R^n) \to C^\infty(\partial M, I\!R^n)$ are (Fréchet) smooth maps. The functions $\Phi[J]$ and $\varphi[J]$ represent the density of forces, affecting the body and its surface. This representation is less restrictive then it seems : First the assumption on $\Phi(J)$ and $\varphi(J)$ to be of class C^∞ on M was made for sake of simplicity only. All result, derived below, also holds in the C^k-class or even for functions of Sobolev class. Secondly, derivatives of L which may appear for a general $\mathcal{F}[J]$ can be integrated out :

$$\sum_{j_1,\dots j_r} \int_M \langle \Theta_{j_1,\dots j_r}[J], \nabla_{j_1} \dots \nabla_{j_r} L \rangle \, \mu(J) = \sum_{j_1,\dots j_r} (-1)^r \int_M \langle \nabla_{j_k} \dots \nabla_{j_1} \Theta_{j_1,\dots j_r}[J], L \rangle \, \mu(J) + \text{boundary terms} \qquad (3.13).$$

Only for the effective boundary integral we really make a physical assumption in (3.12), by demanding that it does not depend on derivatives of the virtual displacement. To compare our representation with standard formulations for the virtual work, we split the force densities into their internal and external parts :

$$\Phi[J] = \Phi^{\mathrm{int}}[J] - \Phi^{\mathrm{ext}}[J] \qquad \text{and} \qquad \varphi[J] = \varphi^{\mathrm{int}}[J] - \varphi^{\mathrm{cont}}[J] \qquad (3.14).$$

Here the term $\Phi^{\mathrm{ext}}[J]$ is the density of external (long distance) forces affecting the body – often also denoted by $\rho(J)\Phi^{\mathrm{ext}}[J]$ – and $\varphi^{\mathrm{cont}}[J]$ describes the (external) contact forces on ∂M. Then the maps $\Phi^{\mathrm{int}} : E(M, I\!R^n) \to C^\infty(M, I\!R^n)$ and $\varphi^{\mathrm{int}} : E(M, I\!R^n) \to C^\infty(\partial M, I\!R^n)$ reflect the constitutive behavior of the material, described by (unbalanced) body and boundary forces, respectively.

It is essential to remark that this representation of $\mathcal{F}$ allows for an arbitrary non-local and non-linear constitutive behavior of the system. The form (3.12) for the virtual work $\mathcal{F}[J]$ furthermore differs from the one, which is studied in the literature, by the fact that we need not to specify the explicit form of the force densities a priori. Instead of choosing $\Phi^{\mathrm{int}}[J]$ as the divergence and $\varphi^{\mathrm{int}}[J]$ as the normal component of a stress tensor [AnOs], or starting with a internal work including based on n^{th}-order gradients [Mau], we are less restrictive. The tensorial character appears in our description as a consequence of a global

group invariance of the functional $\mathcal{F}$, which also need not to refer to locality assumptions, like Cauchy's tetrahedron construction or the Hamel-Noll theorem [Tru].
Finally we observe that the force fields $\Phi(J)$ and $\phi(J)$, respectively, are densities with respect to the configuration dependent measure $\mu(J)$ and not with respect to a fixed reference volume $\mu(J_{\mathrm{R}})$.

4. Translational invariance and the reduced phase space

The purpose of this section is to study the effect of symmetries on the configuration space $E(M, \mathbb{R}^n)$ on the form of the balance laws in continuum mechanics. We are motivated for doing so by considering classical field theories where symmetries cause the system to subject conservation law, e.g. of momentum and angular momentum, via Noether's theorem.
The symmetry group in question here is the group $\mathbb{R}^n$ of rigid translations on the ambient space $N = \mathbb{R}^n$, which naturally acts on the configuration space $E(M, \mathbb{R}^n)$ from the left by pointwise addition :

$$\begin{aligned} \mathbb{R}^n \times E(M, \mathbb{R}^n) &\longrightarrow E(M, \mathbb{R}^n) \\ (Z, J) &\mapsto J + Z \end{aligned} \qquad (4.1).$$

To characterize the behavior of a system in continuum mechanics under the action of this group we introduce the notion of $\mathbb{R}^n$-invariance of the virtual work, which will be fundamental to obtain a simplified description of the balance laws.

Definition

If the virtual work functional, i.e. the one form $\mathcal{F} : E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \to \mathbb{R}$ obeys for all $Z \in \mathbb{R}^n$ the condition

$$\mathcal{F}[J](Z) = 0 \qquad \forall J \in E(M, \mathbb{R}^n) \qquad (4.2a)$$

$$\mathcal{F}[J + Z](L) = \mathcal{F}[J](L) \qquad \forall J \in E(M, \mathbb{R}^n) \quad \forall L \in C^\infty(M, \mathbb{R}^n) \quad (4.2b)$$

then $\mathcal{F}$ is called an ***$\mathbb{R}^n$-invariant virtual work form.***

We have to remark that the conditions (4.2a) and (4.2b) are independent physical demands: The vanishing of the virtual work functional on all constant virtual displacements, (4.2a), means that there is no total force acting on the embedding body in any configuration, while (4.2b) expresses the homogeneity of the space $\mathbb{R}^n$, as far as this can be checked out with that body. It is not possible to understand (4.2a) as a consequence of (4.2b) – one easily constructs conter-examples – but for a hyperelastic system we can obtain (4.2a) as a consequence of (4.2b) by a Taylor expansion of the potential function $U(J)$.

Even if a given virtual work functional is not $\mathbb{R}^n$-invariant, but is representable by (3.12), we can proceed by subtracting the (integrated) total force

$$\Psi[J] = \int_M \Phi[J]\,\mu(J) \;+\; \int_{\partial M} \varphi[J]\,\mu_\partial(J) \qquad (4.3),$$

which is constant on M. This way we obtain :

Proposition 4.1

Let the virtual work functional $\mathcal{F}$ be of the form (3.12) and obey the homogeneity condition (4.2b). Then we (4.3) determines an $\mathbb{R}^n$-invariant functional given by

$$\widetilde{\mathcal{F}}[J](L) := \mathcal{F}[J](L) \;-\; \int_M \langle \Psi[J], L\rangle\,\mu(J) \qquad \forall\, L \in T_J E(M, \mathbb{R}^n) \qquad (4.4).$$

On the other hand it is physically reasonable to postulate that

$$\mathcal{F}^0[J](L) := \int_M \langle \Phi^{\mathrm{int}}[J], L\rangle\,\mu(J) + \int_{\partial M} \langle \varphi^{\mathrm{int}}[J], L\rangle\,\mu_\partial(J) \qquad (4.5)$$

describes an $\mathbb{R}^n$-invariant virtual work. This is a natural demand, since the constitutive qualities of a material are assumed to depend only on the internal distances (and orientations) between the points of the embedded body, and hence should not respond on rigid translations. The functional $\mathcal{F}^0[J]$ will be of central importance in section 6 to obtain a physical interpretation for the reduced dynamics, coresponding to (3.3) by ruling out the $\mathbb{R}^n$-invariance.

To perform the Marsden-Weinstein reduction on the phase space $TE(M, \mathbb{R}^n)$ we construct the momentum map $\mathcal{J}$, cf. [AbMa], with respect to the action (4.1) of the group $\mathbb{R}^n$. Therefore we consider $Z \in \mathbb{R}^n$ as an element in the coresponding Lie algebra and observe that its action on $TE(M, \mathbb{R}^n)$ is given by a vector field $\mathbf{Z}$, which is of the local form $\mathbf{Z} = (J, V_J, Z, 0)$. Since the one form $\Theta_{\mathcal{B}}$ and also the symplectic structure $\omega_{\mathcal{B}}$ are invariant under rigid translation – what is an immediate consequence of Proposition 2.1 – we obtain with (2.12)

$$< \mathcal{J}(V_J), Z >_* = \Theta_{\mathcal{B}}(V_J)(\mathbf{Z}) \;=\; \int_M \rho(J)\,\langle V_J, Z\rangle\;\mu(J) \qquad (4.6),$$

where $<,>_*$ denoted the dual pairing on $\mathbb{R}^n$. Hence we can investigate the system on the reduced phase space with the constraint set

$$\mathcal{J}^{-1}(\Pi) := \{\ (J, V_J) \in TE(M, \mathbb{R}^n) \mid \int_M \rho(J)\, V_J\, \mu(J) = \Pi\ \} \qquad (4.7).$$

This is a smooth submanifold of $TE(M, \mathbb{R}^n)$. Since $\mathbb{R}^n$ is Abelian it equals its isotropy group G_Π for each $\Pi \in (\mathbb{R}^n)^*$ and we can consider the equivalence classes of configurations $[J] \in E(M, \mathbb{R}^n)/\mathbb{R}^n$ given by

$$J' \in [J] \;\Leftrightarrow\; \exists\, Z \in \mathbb{R}^n \;\text{ such that }\; J' = J + Z \qquad (4.8).$$

These are visualized by introducing the center of mass

$$\mathcal{S}_J := \frac{1}{\mathbf{m}} \int_M \rho(J) \cdot J \, \mu(J) \qquad (4.9)$$

such that an $\mathbb{R}^n$-invariant system only depends on the relative configuration

$$J_0 = J - \mathcal{S}_J \qquad (4.10).$$

It is a remarkable feature of the harmonic decomposition of $E(M, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$, cf. appendix B, that the center of mass component $\mathcal{S}_J$ describes the unique $E(M, \mathbb{R}^n)/\mathbb{R}^n$ orthogonal component of the configuration J, where orthogonality is understood in the metric $\mathcal{B}(J)$.

On this base we apply the Marsden-Weinstein reduction of a mechanical system with symmetries to the case under consideration :

Theorem 4.2

Let a system in continuum mechanics be determined by the principle of virtual work (3.3), with an $\mathbb{R}^n$-invariant virtual work functional $\mathcal{F} : TTE(M, \mathbb{R}^n) \to \mathbb{R}$. Then the dynamics can be described on the reduced phase space $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ by

$$\begin{aligned} \int_M \rho([J_t])\, V_t\, \mu([J_t]) &= \Pi \\ \frac{d}{d\tau}\Big|_{\tau=t} \mathcal{B}([J_\tau])(V_\tau, L_\tau) &= \mathbb{F}([J_t])([L_t]) \end{aligned} \qquad \forall t \in (-\lambda, \lambda) \qquad (4.11)$$

with the *reduced virtual work functional*

$$\begin{aligned} \mathbb{F} &: E(M, \mathbb{R}^n)/\mathbb{R}^n \times C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow \mathbb{R} \\ \mathbb{F}([J])([L]) &= \mathcal{F}[J_0](L') \end{aligned} \qquad (4.12).$$

Here $J_0 \in [J]$ is the relative configuration (4.10) and L' is some representant of the class $[L] \in C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n$ formed as in (4.8), accordingly.

Proof :

We consider $Z \in \mathbb{R}^n$ as an element of the Lie algebra of the translation group and investigate the dynamics of the observable $\mathbb{J}_Z(V_J) = < \mathcal{J}(V_J), Z >_*$. Having a generalized Hamiltonian system (3.11) we have

$$\frac{d}{d\tau}\Big|_{\tau=t} \mathbb{J}_Z(V_\tau) = -(\mathbf{D}\mathcal{E}(V_t) - \pi_E^* \mathcal{F}[J_t])(\mathbf{Z}) = \mathcal{F}[J_t](Z) \qquad (4.13)$$

from the definition of the momentum map. We conclude from the $\mathbb{R}^n$-invariance of $\mathcal{F}$ that the dynamics leaves the level sets of the momentum map invariant, i.e. the motion is constraint to the subset $\mathcal{J}^{-1}(\Pi) \subset TE(M, \mathbb{R}^n)$.

Finally we divide out the translation group by considering $\mathcal{F}$ as functionally dependent on the relative configuration J_0. □

The physical content of that theorem is easily understood by observing that

$$\mathcal{J}(V_J) = \int_M \rho(J)\, V_J\, \mu(J) \tag{4.14}$$

is the total momentum of the body moving in $\mathbb{R}^n$, which is conserved since the total force vanishes by assumption (4.2a). On the other hand we can consider – by means of the assumed homogeneity of $\mathbb{R}^n$ – all constitutive properties as depending only on the relative configurations.

5. Geometry on the reduced phase space

According to the Marsden-Weinstein reduction the set $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ forms a symplectic manifold. Therefore we remark that the $\mathbb{R}^n$-action (4.1) induces a free and proper action on $TE(M, \mathbb{R}^n)$ and hence also on the submanifold $\mathcal{J}^{-1}(\Pi)$; thus the symplectic reduction theorem [MaWe] applies to the case under consideration.
To obtain an intrinsic description of the dynamics on the reduced phase space $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$, we observe that we have for each $L \in T_J E(M, \mathbb{R}^n)$ a unique decomposition

$$L = L_0 + C_L \quad \text{with} \quad C_L \in \mathbb{R}^n \quad \text{and} \quad \int_M \rho(J)\, \langle L_0, Z\rangle\, \mu(J) = 0 \quad \forall Z \in \mathbb{R}^n \tag{5.1}$$

Hence the component L_0 is $\mathcal{B}(J)$-orthogonal to all constants. We remark that this decomposition does not depend on the configuration J, as easily can be seen from the continuity equation (2.8).

Proposition 5.1

Let $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ be the reduced phase space of an $\mathbb{R}^n$-invariant system. Fixing Π, each of its elements is uniquely described as follows :
(i) By a pair (J_0, V_0) of maps $J_0 \in E(M, \mathbb{R}^n)$ and $V_0 \in C^\infty(M, \mathbb{R}^n)$ obeying

$$\int_M \rho(J)\, J_0\, \mu(J) = 0 \qquad \text{and} \qquad \int_M \rho(J)\, V_0\, \mu(J) = 0$$

(ii) By the pair of differentials

$$(dJ, dV) \quad \text{where} \quad J \in E(M, \mathbb{R}^n) \quad \text{and} \quad V \in C^\infty(M, \mathbb{R}^n)$$

Proof :
Since $TE(M, \mathbb{R}^n) = E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n)$ is a product bundle and the translation group $\mathbb{R}^n$ only acts on the first factor, we have from section 4

$$\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n = \{\, ([J], V) \mid [J] \in E(M, \mathbb{R}^n)/\mathbb{R}^n \ \text{and} \ V \in C^\infty(M, \mathbb{R}^n) \ \text{with} \int_M \rho(J)\, V\, \mu(J) = \Pi \,\} \tag{5.2}$$

For each representant $J' \in [J]$ and the component V of each point in $\mathcal{J}^{-1}(\Pi)$ we apply the decomposition (5.1), yielding

$$J' = J_0 + \mathcal{S}_{J'} \qquad \text{and} \qquad V = V_0 + \frac{1}{\mathbf{m}}\Pi \tag{5.3}$$

with $\mathcal{S}_{J'} \in \mathbb{R}^n$ for the center of mass (4.9), $\Pi \in \mathbb{R}^n$ for the total momentum (4.11) and $\mathbf{m}$ for the total mass (2.7) of the moving body.
Observing that the values of Π and $\mathcal{S}_J$ are redundant to describe points within $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ proves (i).
Since the exterior derivative $d : C^\infty(M, \mathbb{R}^n) \to \mathcal{A}^1(M, \mathbb{R}^n)$ vanishes exactly on those $\mathbb{R}^n$-valued maps on M, which are constants, this also shows (ii). □

Both representations for $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ allow for a clear the physical interpretation : Using (i) we describe each point by the relative configuration and velocity, taken with respect to the center of mass and the average velocity $\Pi/\mathbf{m}$.
For (ii) we observe that the differential dJ coresponds to the ***deformation gradient*** appearing in the usual treatment of continuum mechanics, cf. [HuMa], if we consider the one form $dJ \in \mathcal{A}^1(M, \mathbb{R}^n)$ as a two point tensor in the sense of section 2. Hence we are in the standard framework of elastodynamics with dV denoting the time derivative of the deformation gradient, i.e. $dV_t = \frac{d}{d\tau}\big|_{\tau=t} dJ_\tau$.
With respect to the Riemannian structure we observe that the decomposition (5.1) is $\mathcal{B}(J)$-orthogonal and obtain

$$\mathcal{B}(J)(L, K) = \int_M \rho(J) \langle L_0, K_0 \rangle \, \mu(J) + \mathbf{m} \cdot \langle C_L, C_K \rangle \tag{5.4}$$

for all fields $K, L \in C^\infty(M, \mathbb{R}^n)$ over $J \in E(M, \mathbb{R}^n)$. By construction the volume form $\mu(J)$ and the density map $\rho(J)$ only depend on the differential dJ. Identifying via proposition 5.1 the components (K_0, L_0) with their respective differentials (dK, dL) we get

$$\mathbb{B}\,(dK, dL) := \int_M \rho(dJ) \langle K_0, L_0 \rangle \, \mu(dJ) \tag{5.5}$$

for the Riemannian structure on the reduced phase space $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$. With (5.1) and the continuity equation this metric clearly becomes configuration independent. For an alternative representation of $\mathbb{B}$ we observe that for each (relative field) K_0 admits a unique one form $\kappa \in \mathcal{A}^1(M, \mathbb{R}^n)$ which is exact, i.e. $d\kappa = 0$, and solves the boundary value problem

$$\rho(dJ)\, K_0 = \delta\, \kappa \qquad \text{with} \qquad \kappa(\mathcal{N}) = 0 \tag{5.6}$$

Here $\delta : \mathcal{A}^1(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^n)$ denotes the co-differential taken with respect to the metric $m(J)$ on M and $\mathcal{N}$ is the outward pointing unite normal field on ∂M. This can be shown from the theory of elliptic operators on manifolds with boundaries [Hör,Mor] by observing that for $\kappa = dk$ equation (5.6) becomes a usual Neumann problem. For more details we refer to appendix B.

Using Stoke's theorem (B.6) we then obtain

$$\mathbb{B}(dK, dL) = \int_M \mathcal{M}(\kappa \ldots dL)\, \mu(dJ) \tag{5.7}$$

where the product $\mathcal{M}$ on $\mathcal{A}^1(M, \mathbb{R}^n)$ computes in terms of a local $m(J)$-orthonormal frame $\{E_1, \ldots, E_m\}$ on TM as

$$\mathcal{M}(\gamma_1, \gamma_2) := \sum_{j=1}^{r} \langle \gamma_1(E_j), \gamma_2(E_j) \rangle \tag{5.8}$$

Since $TE(M, \mathbb{R}^n)$ is a trivial bundle, cf. (2.5), any vector field $\mathbf{Y} \in \Gamma(T^2E(M, \mathbb{R}^n))$ has a global representation $\mathbf{Y} = (J, V, Y^1, Y^2)$ - with $Y^1, Y^2 \in C^\infty(M, \mathbb{R}^n)$. Denoting by $([\mathbf{X}], [\mathbf{Y}])$ a pair of arbitrary vector fields on $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$, the induced symplectic form on the reduced phase space, reads in terms of the metric $\mathbb{B}$ as

$$\omega_{\mathbb{B}}(dJ, dV)([\mathbf{X}], [\mathbf{Y}]) = \mathbb{B}(dX^1, dY^2) - \mathbb{B}(dY^1, dX^2) \tag{5.9}$$

In view of the representation of the reduced virtual work form $\mathbb{F}$ we now prove that any $\mathbb{R}^n$-invariant linear functional on $TE(M, \mathbb{R}^n)$ admits an integral representation, similar to (5.7), on $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$.

Proposition 5.2

Let Γ be a one form on $E(M, \mathbb{R}^n)$, which can be represented by a pair of smooth densities $(\Phi[J], \varphi[J])$ on M and ∂M, respectively, as

$$\Gamma[J](L) = \int_M \langle \Phi[J], L \rangle\, \mu(J) - \int_{\partial M} \langle \varphi[J], L \rangle\, \mu_\partial(J) \tag{5.10}$$

If it is $\mathbb{R}^n$-invariant in the sense, that (4.2) holds accordingly, Γ turns into a linear functional on the reduced phase which can be expressed in terms of a differential $d\mathcal{H}[dJ]$ as

$$\Gamma[dJ](dL) = \int_M \mathcal{M}(d\mathcal{H}[dJ], dL)\, \mu(dJ) \tag{5.11}$$

The function $\mathcal{H}[dJ] \in C^\infty(M, \mathbb{R}^n)$ is unique up to an additional constant.

Proof :

The invariance condition (4.2a) applied to $\Gamma[J]$ reads as

$$\int_M \Phi[J]\, \mu(J) + \int_{\partial M} \varphi[J]\, \mu_\partial(J) = 0 \tag{5.12}$$

since $Z \in \mathbb{R}^n$ is arbitrary. This is the integrability condition for the Neumann problem

$$\Delta \mathcal{H}[J] = \Phi[J] \qquad \text{with} \qquad d\mathcal{H}[J](\mathcal{N}) = \varphi[J] \tag{5.13}$$

where $\Delta = \delta\, d$ is the Laplace operator with respect to the Riemannian metric $m(J)$ on M. The solution $\mathcal{H}[J] \in C^\infty(M, I\!R^n)$ is unique up to an additive constant [Hör]. Replacing $\Phi[J] = \Delta\mathcal{H}[J]$ and using Stoke's theorem (B.6) we then get

$$\Gamma[J](L) = \int_M \mathcal{M}(d\mathcal{H}[J], dL)\, \mu(J) \tag{5.14}$$

and with proposition 5.1 the invariance (4.2b) yields the desired result. □

6. The equations of motion and the stress tensor

Applying proposition 5.2 to the internal part $\mathcal{F}^0[J]$ of some virtual work functional, cf. (4.5), which is $I\!R^n$-invariant from physical arguments we obtain

$$\begin{aligned} I\!F\,[dJ](L) \;=\; & \int_M \mathcal{M}(d\mathcal{H}[dJ], dL)\, \mu(dJ) \\ & + \int_M \langle \Phi^{\text{ext}}[dJ], L\rangle\, \mu - \int_{\partial M} \langle \varphi^{\text{cont}}[dJ], L\rangle\, \mu_\partial(dJ) \end{aligned} \tag{6.1}$$

The one form $\alpha^{\mathcal{H}}[dJ] := d\mathcal{H}[dJ] \in \mathcal{A}^1(M, I\!R^n)$, determined from the internal body and boundary forces is called the (integrable) ***stress form*** of the system [Bin,Sch]. Its physical significance will become clear after establishing the symplectic dynamics on the reduced phase space. For sake of simplicity we neglect the effect of external body forces from now on, i.e. we set $\Phi^{\text{ext}}[dJ] \equiv 0$. With

$$I\!E\,(dV) \;:=\; \frac{1}{2}\, I\!B\,(dV, dV) \;=\; \mathcal{E}(V) - \frac{1}{2\,\mathbf{m}}\, \langle \Pi, \Pi\rangle \tag{6.2}$$

for the kinetic energy on the reduced phase space we then obtain :

Theorem 6.1

Let the motion of a system in continuum mechanics be determined by an $I\!R^n$-invariant virtual work $\mathcal{F}$ which admits a generalized Hamiltonian description in the sense of proposition 3.2. Given an integral representation (3.12) for the $\mathcal{F}$ (with $\Phi^{\text{ext}} \equiv 0$), the motion becomes a generalized Hamiltonian system on $\mathcal{J}^{-1}(\Pi)/I\!R^n$. The Hamiltonian equation of motion demands for any $L \in C^\infty(M, I\!R^n)$

$$\left.\frac{d}{d\tau}\right|_{\tau=t} I\!B\,(dV_\tau, dL) = \int_M \mathcal{M}(\alpha^{\mathcal{H}}[dJ_t], dL)\, \mu(dJ_t) - \int_{\partial M} \langle \varphi^{\text{cont}}[dJ_t], L\rangle\, \mu_\partial(dJ_t) \tag{6.3}$$

Proof :
With the symplectic form $\omega_{\mathbb{B}}$ and the kinetic energy $\mathbb{E}(dV)$ the (generalized) Hamiltonian vector field $[\mathbf{X}_{\mathbb{H}}]$ on $\mathcal{J}^{-1}(\Pi)/\mathbb{R}^n$ is determined from the reduced virtual work $\mathbb{F}$ by

$$\mathbb{B}\left(dX^1_{\mathbb{F}}, dY^2\right) - \mathbb{B}\left(dY^1, dX^2_{\mathbb{F}}\right) = \mathbb{B}\left(dV, dY^2\right) - \mathbb{F}(dJ)(dY^1) \qquad \forall Y^1, Y^2 \in C^\infty(M, \mathbb{R}^n).$$

This yields $dX^1_{\mathbb{F}} = dV$, and considering the observable $\mathbb{K}^L(dJ, dV) := \mathbb{B}(dL, dV)$ – with $L \in C^\infty(M, \mathbb{R}^n)$ arbitrary but fixed (and time independent) – we get from (3.5) :

$$\left.\frac{d}{d\tau}\right|_{\tau=t} \mathbb{K}^L(dJ_\tau, dV_\tau) = \mathbb{F}(dJ_t)(dL) \tag{6.4}$$

Representing $\mathbb{F}$ by (6.1) then proves the assumption. □

Comparing the (weak) equation (6.3) with classical approaches to the principle of virtual work, cf. [AnOs,Mau], we observe that the stress form $\alpha^{\mathcal{H}}[dJ]$ is measured with respect to the induced volume form $\mu(dJ)$, while the 1^{th} Piola-Kirchhoff stress tensor appears in the virtual work functional measured with respect to a fixed reference volume $\mu(dJ_{\mathrm{R}})$. Observing, however, that $dJ : TM \to \mathbb{R}^n$ is a (pointwise) isometry with respect to the metrics $m(J)$ on M and $\langle\, , \rangle$ on $\mathbb{R}^n$ the stress form inherits the physical interpretation of the Cauchy stress tensor, represented on M.
To see this we pull $\alpha^{\mathcal{H}}[dJ]$ back by J^{-1} and define a coresponding tensor on $J(M)$ by

$$\begin{aligned} \mathbf{A}^{\mathcal{H}}[dJ] &: TJ(M) \times \mathbb{R}^n \to \mathbb{R} \\ \mathbf{A}^{\mathcal{H}}[dJ](X, Y) &:= \langle \alpha^{\mathcal{H}} \circ (TJ)^{-1}(X), Y \rangle \end{aligned} \tag{6.5}$$

holding for all $X \in \Gamma(TJ(M))$ and all $Y \in \Gamma(T\mathbb{R}^n)$. From this we immediately get

$$\int_M \mathcal{M}(\alpha^{\mathcal{H}}[dJ], dL)\mu(dJ) = \int_{J(M)} (\mathbf{A}^{\mathcal{H}}[dJ] : \mathbf{\Lambda})\mu_{\mathbb{R}^n} \tag{6.6}$$

where $\mathbf{\Lambda}$ denotes the tensor coresponding to the one form $dL \circ dJ^{-1}$ on $J(M)$ and $(\mathbf{A} : \mathbf{\Lambda})$ stands for the trace over the product of the two tensors. Using this representation for the first term of $\mathbb{F}$ (6.1) and transforming the (external) contact force $\varphi^{\text{cont}}[dJ]$ appropriately, equation (6.3) coincides with the principle of virtual work in terms of the Cauchy stress tensor $\mathbf{A}^{\mathcal{H}}[dJ]$, which can be found in the literature.
The Cauchy stress is given by a proper tensor, i.e. $\mathbf{A}^{\mathcal{H}}[dJ](X, Y)$ is local with respect to the fields X and Y. We remark, however, that our construction of the stress tensor allows for an arbitrary non-locally dependence on the configuration. This means that for $p \in M$ fixed, the value of $\mathbf{A}^{\mathcal{H}}[dJ](p)$ can depend on the deformation gradient globally and not only on its point value $dJ(p)$ and its derivatives at p. This is a remarkable fact, since all standard proves of the tensoriell nature of stress need to use some locality argument – like Cauchy's tetrahedron construction – whereas we obtain the same in a non-local framework by demanding just a rigid $\mathbb{R}^n$-invariance.
Another interesting feature of (6.3) as the equation of motion is that the stress form is integrable, i.e. it is given as the differential of a function by $\alpha^{\mathcal{H}}[dJ] = d\mathcal{H}[dJ]$. This

integrability, however, is not necessarily needed for an appropriate description : We may apply the transformation

$$\alpha^{\mathcal{H}}[dJ] \mapsto \tilde{\alpha}[dJ] = \alpha^{\mathcal{H}}[dJ] + \beta[dJ] \quad (6.7),$$

where $\beta \in \mathcal{A}^1(M, \mathbb{R}^n)$ obeys $\delta\beta[dJ] = 0$ and $\beta[dJ](\mathcal{N}) = 0$, without changing the physical content of equation (6.3). This means that $\tilde{\alpha}[dJ]$ determines the same dynamics as $\alpha^{\mathcal{H}}[dJ]$ does. That property, which is easily seen, since we have from Stoke's theorem (B.6)

$$\int_M \mathcal{M}(\beta[dJ], dL)\, \mu(dJ) = 0 \quad (6.8),$$

expresses a remarkable gauge freedom in choosing the stress for each system in continuum mechanics. On the other hand, our result shows, that for any given stress form $\tilde{\alpha}[dJ]$ only its exact part $\alpha^{\mathcal{H}}[dJ]$ contributes to the dynamics.

Obviously the integrability and the gauge freedom of the stress form induce coresponding properties on the level of the Cauchy stress tensor. It has to be mentioned, however, that the (integrable) tensor $\mathbf{A}^{\mathcal{H}}[dJ]$ will, in general, not be symmetric. To symmetrize it without changing its physical content, one can use the gauge freedom and construct an appropriate $\beta[dJ]$, such that $\tilde{\alpha}^{\mathcal{H}}[dJ] \circ dJ^{-1}$ becomes symmetric, cf. [Sch].

7. Localization of the Dynamics

We now give up the covariant description of the system by deriving from the variational equation of motion a local balance law. To compare in (6.3) the stress term with the dynamical one, we first get rid of its explicit time derivative by observing from the configuration independence of the metric $\mathbb{B}$ that

$$\frac{d}{d\tau}\bigg|_{\tau=t} \mathbb{B}(dV_\tau, dL) = \int_M \rho(J_t)\langle \dot{V}_0, L_0\rangle \mu(J) \quad (7.1).$$

Furthermore we can represent the metric via (5.7) by applying solving the elliptic problem (5.6) for the field $\rho(dJ)\dot{V}_0$, which coresponds to the (relative) inertia force of an actual motion J_t. This determines (uniquely) a one form $\omega(dJ) \in \mathcal{A}^1(M, \mathbb{R}^n)$ which obeys

$$\delta\omega(dJ) = \rho(dJ)\dot{V}_0 \quad , \quad d\omega(dJ) = 0 \quad \text{and} \quad \omega(dJ)(\mathcal{N}) = 0 \quad (7.2)$$

and the (weak) equation of motion on the reduced phase space transforms into

$$\int_M \mathcal{M}\Big(\omega(dJ) - \alpha^{\mathcal{H}}[dJ], dL\Big)\, \mu(dJ) = \int_{\partial M} \langle \varphi^{\text{cont}}[dJ], L\rangle\, \mu_\partial(dJ) \quad (7.3).$$

Since by assumption all quantities are smooth we can localize (7.3) as follows : We restrict the support of L to an arbitrary (connected) subset $U \subset M$ with $U \cap \partial M = \emptyset$ and observe that the integral of $\mathcal{M}$ over U yields a Riemannian structure on the subset of exact forms in $\mathcal{A}^1(U, \mathbb{R}^n)$, cf. (B.11). This yields as the local equation of motion

$$\omega(dJ_t) \;=\; d\mathcal{H}[dJ_t] \tag{7.4}$$

for any point in the interior of M. This local representation of the dynamics may give some new insight into the nature of the theory : It is a balance law on the level of exact one forms and hence stronger then the usual balance of local forces. To make such an observation applicable, however, a clear physical interpretation of $\omega(dJ)$ has to be worked out.
The classical description of dynamics is encoded in (7.4) as the trace over its spatial derivatives, i.e. by acting with the co-differential δ on it. For $\delta\omega(dJ)$ we re-obtain the inertia force and from the definition of the Cauchy stress tensor we have

$$\langle \delta d\mathcal{H}[dJ], X\rangle \;=\; \delta(\mathbf{A}^{\mathcal{H}}[dJ]\circ dJ)(X) \qquad \forall X \in \Gamma(T\mathbb{R}^n) \tag{7.5}$$

Using the product rule (B.5) for the operator δ and writing $\mathbf{V}_{J_t} = V_t \circ J_t^{-1}$ for the velocity field on $J_t(M)$ the equation of motion reads in its spatial representation as

$$\rho(dJ_t)\, \frac{d}{d\tau}\bigg|_{\tau=t} \mathbf{V}_{J_\tau} \;=\; \overline{\mathrm{Div}}\,\mathbf{A}^{\mathcal{H}} + \mathbf{A}^{\mathcal{H}}[dJ_t](\Delta J_t) \tag{7.6}$$

Here $\overline{\mathrm{Div}}$ denotes the divergence, taken with respect to the restricted metric $\langle,\rangle|_{J(M)}$ on $J(M) \subset \mathbb{R}^n$. Compared to the usual description of continuum mechanics, we see that (7.6) coincides for simple bodies, i.e. for embeddings of co-dimension zero, with the classical equations, since $\Delta J = 0$ in that case. In the general situation we have $\Delta J_t = \mathrm{trace}\,\mathbf{S}(J)$, where $\mathbf{S}(J)$ means the second fundamental form [BSF] of the embedded hypersurface $J(M)$, cf. (B.4). Then the term $\mathbf{A}^{\mathcal{H}}[dJ_t](\Delta J_t)$ reflects a force acting on the embedded surface in the normal direction, which is proportional to a characteristic geometric quantity on $J(M)$, e.g. for $\dim M = n-1$ to its mean curvature $\mathbf{H}(J) = \langle \mathbf{S}(J), \mathbf{n}\rangle$ with $\mathbf{n}$ denoting the normal field on $J(M) \subset \mathbb{R}^n$.
Finally we observe that the global equation of motion (7.3) also characterizes the behavior of the system on its boundary ∂M. To see this we apply Stoke's theorem what yields

$$\int_M \langle \delta\omega(dJ) - \Delta\mathcal{H}[dJ], L\rangle\, \mu(dJ) = \int_{\partial M} \langle (d\mathcal{H}[dJ] - \omega(dJ))(\mathcal{N}) - \varphi^{\mathrm{cont}}[dJ], L\rangle\, \mu_\partial(dJ) \quad (7.7).$$

Since we are in a smooth situation by assumption, the body integrals cancel up via (7.4). Furthermore we have $\omega(dJ)(\mathcal{N}) = 0$ and hence the system is enforced to obey on ∂M traction boundary conditions

$$\alpha^{\mathcal{H}}[dJ](\mathcal{N}) = \varphi^{\mathrm{cont}}[dJ] \quad (7.8).$$

In turn we remark that this is exactly the condition on the virtual work functional, represented by (3.12), to admit a generalized Hamiltonian description in the sense of proposition 3.2. In principle it is also possible to include boundary conditions of placement into this framework what, however, requires a modification of the configuration space $E(M, \mathbb{R}^n)$.

8. Two special Applications

We finish this paper by illustrating our formalism on the reduced phase space $J^{-1}(\Pi)/\mathbb{R}^n$ for two simple examples :

First we study linear elasticity by writing for the reduced virtual work functional

$$\mathbb{F}_{\text{lin}}[dJ](dL) = \int_M \mathcal{M}\big(C \circ dJ \circ c, dL\big)\, \mu(dJ) \qquad (8.1),$$

where the map $C : M \to GL(n)$ and the bundle endomorphism $c \in \text{End}(TM)$ both are smooth but configuration independent. The pair (C, c) determines the material properties of a linear elastic medium in the same way, as this will be done by the 4^{th} order elasticity tensor, say $\mathbf{C}$, in the classical tensorial description. Even though the material is not hyperelastic, a property which can be read off from the pair of maps (C, c), the generalized Hamiltonian formalism, presented above, can be applied. Doing so we obtain for the equation of motion on the reduced phase space

$$\omega(dJ_t) = \big(C \circ dJ_t \circ c\big)^{\mathbf{ex}} \qquad (8.2),$$

where the superscript "ex" means to consider only the exact part of the one form in view. Rewriting this in the form given by (6.11) the coresponding equation coincides with the well established representation of linear elastodynamics.

If the material is homogeneous, i.e. C and c are constant maps – the later in the sense that $\nabla_X c = 0$ for all $X \in \Gamma(TM)$ – there exists a diffeomorphism $\mathcal{C} : M \to M$, such that $C \circ dJ \circ c = d(C \circ J \circ \mathcal{C})$. In that case (8.2) turns into a modified wave equation of the form

$$\rho[dJ_t]\, \frac{d}{d\tau}\bigg|_{\tau=t} V_\tau = \Delta\big(C \circ J_t \circ \mathcal{C}\big) = \Delta^{\mathbf{C}} J_t \qquad (8.3),$$

where $\Delta^{\mathbf{C}}$ denotes the Laplace operator coresponding to the Levi-Civita connection of a metric g on M, which is the pull back of the scalar product $\langle , \rangle$ under the one form $C \circ dJ \circ c$. Explicitly this metric is given by

$$g(X,Y) = \langle C \circ dJ \circ c(X), C \circ dJ \circ c(Y) \rangle \qquad for al X, Y \in \Gamma(TM) \qquad (8.4).$$

In the non-homogeneous case a similar construction is possible. Representing the dynamics of a linear elastic medium by such a wave equation might be of some interest for applications and will be studied elsewhere.

On the other hand we consider the (reduced) virtual work form

$$\mathbb{F}_{\mathbf{p}}[dJ](dL) = \int_M \mathbf{p}[dJ]\, \mathcal{M}\big(dJ, dL\big)\, \mu(dJ) \qquad (8.5),$$

where $\mathbf{p}[dJ] : M \to \mathbb{R}$ is a smooth map. Doing so is motivated by the fact that the functional $\mathbf{V}[dJ] = \int_M \mu(dJ)$ measures the volume of the embedded body $J(M)$ and has

$$\mathbf{DV}[dJ] = \int_M \mathcal{M}\big(dJ, dL\big)\, \mu(dJ) \qquad (8.6)$$

as its differential, cf. [Bin]. Considering – as a motivation – classical thermodynamics, a system affected by pure volume work is determined by the Gibbs form $\pi\, dV$, where the pressure π is a $\mathbb{R}$-valued function of all variables of state. Based on the description of a state (in the sense of thermodynamics) by an element of the (reduced) configuration space $E(M,\mathbb{R}^n)/\mathbb{R}^n$, the functional

$$\mathbb{F}_{\text{vol}}[dJ](dL) = \pi[dJ] \int_M \mathcal{M}(dJ, dL)\, \mu(dJ) \tag{8.7}$$

represents the component of a total Gibbs form of any system, which coresponds to pure volume work.

The natural continuum analogon for $\pi\, dV$ is the virtual work form $\mathbb{F}_{\mathbf{p}}$ where – from the thermodynamical point of view – the total volume $\mathbf{V}[dJ]$ (as only one degree of freedom) is replaced by the volume form $\mu(dJ)$ as the relevant field of extensive variables. Then (8.6) presents the well defined kernel coresponding to $\mathbf{D}\mu[dJ]$. The function $\mathbf{p}[dJ] \in C^\infty(M,\mathbb{R})$ is supplied with the physical interpretation of the pressure field on the body in configuration dJ, such that (8.5) may be identified with the Gibbs form of that system.

A system described by that virtual work $\mathbb{F}_{\mathbf{p}}$ need not be hyperelastic, what means in turn that Gibbs form will not be integrable. The analysis on the Frechet manifold $E(M,\mathbb{R}^n)$, however, presents a simple criterion to check the integrability of the linear functional $\mathbb{F}_{\mathbf{p}}$: Iff one has

$$\mathbf{D}(\mathbb{F}_{\mathbf{p}}(dL))(dK) - \mathbf{D}(\mathbb{F}_{\mathbf{p}}(dK))(dL) = 0 \qquad \forall K, L \in \Gamma TE(M,\mathbb{R}^n) \qquad (8.8),$$

what can be expressed in terms of a relation on the constitutive dependence of the pressure $\mathbf{p} : E(M,\mathbb{R}^n)/\mathbb{R}^n \to C^\infty(M,\mathbb{R})$ the material is hyperelastic and the coresponding Gibbs form is integrable.

In either case the generalized Hamiltonian formalism yields for the equation of motion

$$\omega(dJ_t) \;=\; \left(\mathbf{p}[dJ_t]\, dJ_t\right)^{\text{ex}} \qquad (8.9).$$

Acting as in (6.11) with the co-differential δ on this equation and using (B.5) yields

$$\rho[dJ_t]\, \frac{d}{d\tau}\bigg|_{\tau=t} V_\tau \;=\; dJ_t\left(\operatorname{grad}\mathbf{p}[dJ_t]\right) \;+\; \mathbf{p}[dJ_t]\,\Delta\, J_t \qquad (8.10).$$

From the differential geometry of $J(M)$ one has $\Delta\, J = \operatorname{trace}\mathbf{S}(J)$, which equals the mean curvature of the embedded manifold. Since in the co-dimension zero case the second fundamental tensor vanishes, i.e. $\Delta\, J = 0$, this model covers the classical theory of (non-viscous) fluids. In a general situation (8.10) shows, how to include geometrical effects in a description of fluid dynamics, e.g. on deformable shells.

A remarkable specializations of this dynamics is the case, where the system is (approximately) determined by pure volume work (8.7), i.e. one has $\mathbf{p}[dJ] = \pi[dJ] \in \mathbb{R}$. Then the gradient term vanishes and the motion is given by a wave equation. Even if the pure volume work is no good approximation, such term with a constant pressure $\pi[dJ]$ is always present via the evaluation $\pi[dJ] = \mathbb{F}[dJ](\frac{dJ}{\mathbb{B}(dJ,dJ)})$. Similarly $\mathbf{p}[dJ]\, dJ$ can be obtained as a pointwise projection from any $\mathbb{F}$ such that $\pi[dJ]$ may be understood as an average.

Appendix A : Topology and Geometry of $\mathcal{A}^1_E(M,TN)$

Let $I\!E$ be some vector bundle over the compact Riemannian manifold N and define the set of all $I\!E$-valued one forms on M as

$$\mathcal{A}^1(M;I\!E) := \{\, \beta : TM \longrightarrow I\!E \mid \beta \text{ smooth and } \beta|_{T_pM} \text{ linear } \forall p \in M \,\} \qquad (A.1).$$

The requirement that $\beta \in \mathcal{A}^1(M;I\!E)$ should be linear along the fibres of TM means that there is a (smooth) map $f : M \to N$ such that $\beta|_{T_pM}$ is a linear map into the fibre $F_{f(p)}$ sitting over $f(p)$, i.e. that β is a bundle map $TM \to I\!E$ over f. In reverse this shows that

$$\mathcal{A}^1(M,I\!E) = \bigcup_{f\in C^\infty(M,N)} \Omega^1(M,f^*I\!E) \qquad (A.2),$$

where $\Omega^1(M,f^*I\!E)$ is – for fixed f – the Fréchet space of one forms with values in the pull-back bundle $f^*I\!E$ over M. It is clear from the construction that there is natural surjection

$$\pi_{\mathcal{A}^1} : \mathcal{A}^1(M,I\!E) \longrightarrow C^\infty(M,N) \qquad (A.3)$$

which is (set-theoretically !) locally trivial: Each $f \in C^\infty(M,N)$ has an open neighbourhood U_f such that there exists a fibre-preserving, fibrewise linear bijection

$$\varphi_f : \pi^{-1}_{\mathcal{A}^1}(U_f) \longrightarrow U_f \times \Omega^1(M,f^*I\!E) \qquad (A.4)$$

which also is topological on each fibre; thus, for each $g \in U_f$, the restriction of φ_f to $\pi^{-1}_{\mathcal{A}^1}(g)$ is a linear and topological isomorphism onto $\Omega^1(M,f^*I\!E)$. For all this we refer to [BiFi]. Restricting ourselves to $I\!E = TN$ we define

$$\mathcal{A}^1_E(M,TN) = \{\, \beta \in \mathcal{A}^1(M,TN) \mid \pi_N \circ \beta \in E(M,N) \,\} \qquad (A.5)$$

as the subset of all TN-valued one forms covering embeddings $J : M \to N$. The identification of a $(1,1)$-type two point tensor $\beta \in \mathcal{A}^1_J(M,TN)$ with a one form in $\mathcal{A}^1_E(M,TN)$ is obvious. Since this set is in the inverse image of $E(M,N)$ under the projection $\pi_{\mathcal{A}^1}$, i.e. $\mathcal{A}^1_E(M,TN) \subset \pi^{-1}_{\mathcal{A}^1}(E(M,N))$, it is an open submanifold and itself a (Fréchet) vector bundle with fibre $\mathcal{A}^1_J(M,TN)$ at J.
The bundle $\mathcal{A}^1_E(M,TN)$ can be equipped with a fibre metric as follows:
Let m be a Riemannian metric on M and $\{E_1,\dots,E_r\}$ a coresponding local orthonormal frame on TM. With $\langle\,,\,\rangle$ for the metric on N we define the product

$$\begin{aligned} \mathcal{M}(.,.) &: \mathcal{A}^1_J(M,TN) \times \mathcal{A}^1_J(M,TN) \longrightarrow C^\infty(M,I\!R) \\ \mathcal{M}(\beta_1,\beta_2) &:= \sum_{k=1}^{r} \langle \beta_1(E_j), \beta_2(E_j)\rangle \end{aligned} \qquad (A.6).$$

This definition is frame independent [Mat]. Since M is compact $\mathcal{M}(\beta_1,\beta_2)$ can be integrated, what yields a Riemannian structure on $\mathcal{A}^1_E(M,TN)$, given by

$$\int_M \mathcal{M}(\beta_1,\beta_2)\,\mu \qquad (A.7),$$

where μ is the Riemannian volume form accompanying the metric m. Obviously (**A.7**) is only meaningful for pairs (β_1, β_2) which cover the same embedding, i.e. $\pi_N \beta_1 = \pi_N \beta_2 = J$. Finally we observe that for the special case $N = \mathbb{R}^n$ the bundle $\mathcal{A}^1_E(M, TN)$ becomes trivial, i.e.

$$\mathcal{A}^1_E(M, T\mathbb{R}^n) = E(M, \mathbb{R}^n) \times \mathcal{A}^1(M, \mathbb{R}^n) \qquad (A.8).$$

Appendix B : The Laplace operator on M and the related Neumann problem

Via pull back by an embedding $J \in E(M, \mathbb{R}^n)$ the scalar product $\langle\, , \rangle$ on N determines a Riemannian structure on M given by

$$m(J)(X, Y) := \langle dJ(X), dJ(Y) \rangle \qquad \forall X, Y \in \Gamma(TM) \qquad (B.1).$$

Denoting by ∇ the coresponding Levi-Civita connection, the exterior derivative d : $C^\infty(M, \mathbb{R}^n) \longrightarrow \mathcal{A}^1(M, \mathbb{R}^n)$ is defined by

$$df(X) := \nabla_X f \qquad \forall X \in \Gamma(TM) \qquad (B.2).$$

Let $\{ E_1, \ldots, E_r \}$ be a local frame on TM which is orthonormal with respect to $m(J)$, the co-differential operator $\delta : \mathcal{A}^1(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^n)$ becomes

$$\delta\gamma := -\sum_{k=1}^{r} (\nabla_{E_k}\gamma)(E_k) = -\sum_{k=1}^{r} d(\gamma(E_k))(E_k) - \gamma(\nabla_{E_k} E_k)) \qquad (B.3).$$

The operator δ explicitly depends on the metric $m(J)$ but not on the choice of the frame [Mat]. In particular if $\gamma = dJ$ one has with $\mathbf{S}(J)$ for the second fundamental form of the embedded manifold $J(M) \subset \mathbb{R}^n$

$$\delta(dJ) = -\sum_{k=1}^{r} \mathbf{S}(J)(E_k, E_k) \qquad (B.4).$$

since $d(dJ(X))(Y) = dJ(\nabla_Y X) + \mathbf{S}(J)(X, Y)$ for all $X, Y \in \Gamma(TM)$. The field $\mathbf{S}(J)(X, Y)$ is normal to $J(M)$ $subset \mathbb{R}^n$ and vanishs for teh special case of dim$M = n$. Considering furthermore $\gamma = \mathbf{A}_J \circ dJ$, with $\mathbf{A}_J : J(M) \to \mathrm{Hom}(\mathbb{R}^n)$ we get as

$$\delta(\mathbf{A}_J \circ dJ) = -\sum_{k=1}^{r} d\left(\mathbf{A}_J(dJ(E_k))\right)(dJ(E_k)) + \mathbf{A}_J(\delta(dJ)) \qquad (B.5).$$

The first term on the right hand side coincides with the divergence $\overline{\mathrm{Div}}\, \mathbf{A}_J$, taken with respect to restriction of the metric $\langle , \rangle|$ on $TJ(M) \subset T\mathbb{R}^n$. This is clear since $(J^{-1})^* m(J) = \langle , \rangle|_{J(M)}$ by construction and hence $dJ(E_k)$ determines a (local) orthonormal frame on $J(M)$.

In an appropriate Sobolev extension of $C^\infty(M, \mathbb{R}^n)$ the operators d and δ are adjoint to each other up to boundary terms via Stoke's theorem

$$\int_M \langle \delta\gamma, L\rangle \, \mu(J) = \int_M \mathcal{M}(\gamma, dL) \, \mu(J) \; + \; \int_{\partial M} \langle \gamma(\mathcal{N}), L\rangle \, \mu(J) \qquad (B.6),$$

with $\mathcal{M}$ given as in (A.6) accordingly, $\gamma \in \mathcal{A}^1(M, \mathbb{R}^n)$ and $L \in C^\infty(M, \mathbb{R}^n)$.
For the Laplace operator, acting on zero forms on M, we set

$$\Delta := \delta d \; : \; C^\infty(M, \mathbb{R}^n) \longrightarrow C^\infty(M, \mathbb{R}^n) \qquad (B.7)$$

and consider the following two subspaces of $C^\infty(M, \mathbb{R}^n)$:

$$\begin{aligned} C_0^\infty(M, \mathbb{R}^n) &:= \{ \, K \in C^\infty(M, \mathbb{R}^n) \mid \int_M K \, \mu(J) = 0 \, \} \\ C_{\mathcal{N}}^\infty(M, \mathbb{R}^n) &:= \{ \, k \in C_0^\infty(M, \mathbb{R}^n) \mid dk(\mathcal{N}) = 0 \, \} \end{aligned} \qquad (B.8).$$

Then the existence and uniqueness theorem [Hör] concerning the Neumann problem

$$\Delta k \; = \; K \quad \text{and} \quad dk(\mathcal{N}) \; = \; 0 \qquad (B.9)$$

shows that the operator Δ is invertible for any $k \in C_{\mathcal{N}}^\infty(M, \mathbb{R}^n)$. Since the solution of that problem becomes unique by fixing an additive constant, the Laplacian acts as an isomorphism $\Delta : C_{\mathcal{N}}^\infty(M, \mathbb{R}^n) \to C_0^\infty(M, \mathbb{R}^n)$. Then for $\kappa := d(\Delta^{-1}K)$ the following is obvious :

Proposition B.1

The problem to find a one form $\kappa \in \mathcal{A}^1(M, \mathbb{R}^n)$ obeying

$$\delta \kappa \; = \; K \quad , \quad d\kappa = 0 \quad \text{and} \quad \kappa(\mathcal{N}) = 0 \qquad (B.10)$$

has a unique solution, provided that $K \in C_0^\infty(M, \mathbb{R}^n)$.

Let us further remark that the range of Δ, i.e. the space $C_{\mathcal{N}}^\infty(M, \mathbb{R}^n)$, admits some base of eigen-maps of Δ implying a Fourier expansion of any $k \in C_{\mathcal{N}}^\infty(M, \mathbb{R}^n)$, which is orthonormal with respect to the L^2-structure $\mathcal{B}(J)$.

The metric $\mathbb{B}$, introduced in section 5, defines a scalar product on $C_0^\infty(M, \mathbb{R}^n)$, cf. (5.5), as well as on $\mathcal{A}^1(M, \mathbb{R}^n)$, cf. (5.6). With respect to the later representation of $\mathbb{B}$ we observe, however, that the map

$$\begin{aligned} \mathbf{b} : \mathcal{A}^1(M, \mathbb{R}^n) &\longrightarrow \mathbb{R} \\ \gamma &\mapsto \int_M \mathcal{M}(\gamma, dL) \, \mu(J) \end{aligned} \qquad (B.11)$$

has as its kernel the space of all one forms, which are co-exact, i.e. obey $\delta\gamma = 0$, and vanish in normal direction ($\gamma(\mathcal{N}) = 0$). The solution theorem of the Neumann problem (B.9) then shows, that the kernel of $\mathbf{b}$ becomes $\{0\}$, if one restricts $\mathcal{A}^1(M, \mathbb{R}^n)$ to the subsets of those one forms, which obey (B.10).

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