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ON THE IRREDUNDANT PART OF THE FIRST PIOLA-KIRCHHOFF STRESS TENSOR

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E. BINZ

Lehrstuhl für Mathematik I
Universität Mannheim
Schloß, D - 6800 Mannheim
W. - Germany

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E.Binz

Introduction

Let us assume a given medium moves and deforms in an ambient smooth and oriented Riemannian manifold N with metric $\langle\ ,\ \rangle$. This medium at hand is supposed to maintain the shape of a compact smooth orientable and connected manifold M with boundary. Clearly $\dim M \leq \dim N$. By a configuration j of the medium we mean a smooth embedding of M into N. The configuration space is $E(M,N)$, the collection of all smooth embeddings of M into N endowed with the C^∞-topology.

There are several types of ingredients to characterize the physical quality of a medium (cf. [M,H]) of this type. At first we adopt the characterization via the first Piola-Kirchhoff stress tensor α, a $(1,1)$-tensor. To keep the full generality we assume that α depends on the actual configuration j, possibly in a non local way. Geometrically speaking $\alpha(j)$ is a bundle map of TM into TN over j. Since $\alpha(j)$ is not specified with respect to a fixed configuration, we omit to include the volume correction as usually done. If $\dim N = \dim M$ then $\alpha(j)$ together with the pullback metric $j^\star(<\ ,\ >)$ yield for each $j \in E(M,N)$, a $(2,0)$-tensor, the stress tensor describing the medium as well as $\alpha(j)$ does. If $\dim M < \dim N$ this is not true, however (cf sec.6).

Given now a virtual infinitesimal distortion at j, then the virtual work $F(j)$ (cf. [M,H], [He], [E,S]) caused by it can be computed with the help of $\alpha(j)$. In fact $\alpha(j)$ determines the force densities $\Phi(j)$ on M and $\varphi(j)$ on ∂M for each $j \in E(M,N)$ by $\Phi(j) = \nabla^\star\alpha(j)$ and $\varphi(j) = \nabla_{\mathbf{n}}\alpha(j)$, respectively. Here ∇ is the covariant derivative on N and $\nabla^\star$ the associate covariant divergence. The vector field $\mathbf{n}$ along ∂M is the positively oriented unit normal of ∂M in M formed with respect to the pull back metric $j^\star <\ ,\ >$.

The main purpose of this notes is to exhibit (in absence of exterior force densities) the irredundant part of $\alpha(j)$ that determines the force densities mentioned and the virtual work caused by any infinitesimal distortion at j. This irredundant part is itself a bundle map of TM into TN along j; it is of the form $\nabla\mathcal{H}(j)$, where $\mathcal{H}(j): M \longrightarrow TN$ is a smooth TN-valued vector field along j. Here ∇ is the Levi-Cività connection on N. Clearly the irredundant part $\nabla\mathcal{H}(j)$ is a first Piola-Kirchhoff stress tensor, too. However, $\nabla\mathcal{H}(j)$ and $\alpha(j)$ are not identical in general, even though they cause the same force densities (cf. [Bi 1] to [Bi 4]). In fact if the ambient manifold is Euclidean, $\nabla\mathcal{H}(j)$ is the exact part of $\alpha(j)$, exhibited via Hodge's theory. This is a special case of the following:

As just pointed out $\mathcal{H}(j)$ characterizes the medium as well as $\alpha(j)$ does, in the sense that $\alpha(j)$ and $\mathcal{H}(j)$ both cause the same force densities $\Phi(j)$ and $\varphi(j)$. This map $\mathcal{H}$ can be determined directly by these densities. In fact, given any pair of force densities $\Phi(j)$ and $\varphi(j)$ (coming from $\alpha(j)$ or not) then $\mathcal{H}(j)$ can be established as the solution of the Neumann problem, consisting of the equations $\Phi(j) = \Delta(j)\mathcal{H}(j)$ on M and $\varphi(j) = \nabla_{\mathbf{n}}\mathcal{H}(j)$ on ∂M. Here $\Delta(j)$ is the Laplacian determined by ∇ and the pullback metric $j^\star <,>$. The vector field $\mathbf{n}$ along ∂M is the positively oriented unit normal of ∂M in M. This solution exists provided that $\Phi(j)$ and $\varphi(j)$ satisfy the integrability conditions.
Thus $\Phi(j)$ and $\varphi(j)$ determine, via the solution $\mathcal{H}(j)$ of a Neumann problem, a minimal type of Piola-Kirchhoff stress tensor, namely $\nabla\mathcal{H}(j)$. This is established in sec.5. In sec.6 we relate this point of view with the usual setting of a $(2,0)$-stress tensor, the well known stress tensor. It will be apparent by a result in [Sch] that not only $\nabla\mathcal{H}$ but also α itself has a physical significance. In particular if $\dim M = \dim N$ the stress-tensor defined by α might be symmetric, while the stress-tensor caused by $\nabla\mathcal{H}$ might miss this property. The significance of the stress tensor of α in this case is thus the superior visualization of the medium over the more involved description in terms of $\nabla\mathcal{H}$. However, the irredundant part is present in this case also as a physical entity. It is hence worthwhile to study it in its full generality.

The rest of the paper is concerned with showing the specific features of working with $\nabla\mathcal{H}(j)$ for any configuration $j \in E(M,N)$. In particular we exhibit the volume and the area sensitive parts of $\nabla\mathcal{H}(j)$ (cf. sec.7) as well as the parts which are sensitive to the Ricci and scalar curvature of the pull back metric $j^\star <,>$ on M (cf. sec.8). In particular the influence of these curvatures to pressure and capillarity is studied.
Since $\Phi(j) = \Delta(j)\mathcal{H}(j)$ we may use in case of $\partial M = \emptyset$ the complete eigen system of the Laplacian given by $\Delta(j_0)$ at a fixed reference configuration j_0 to obtain a Fourier expansion of the force density $\Phi(j)$ pulled back to j_0. Given a first Piola Kirchhoff stress tensor α and assuming that all but ν many of these coefficients vanish in some (closed) neighbourhood $\mathcal{W}$ of j_0 in $E(M,N)$ we exhibit under a regularity assumption and a boundary condition an exact and non-exact part $d\mathcal{U}$, respectively Ψ, of the virtual work F. This is done by using Hodge theory on a ν-dimensional foliation of $\mathcal{W}$ which is caused by the requirement that only the first ν of these Fourier coefficients are taken in account. $d\mathcal{U}$ represents the hyperelastic part (cf. [M,H]) of the medium characterized by α. Based on the theory of integrating factors a temperature like map $T : \mathcal{W} \longrightarrow \mathbb{R}$ is constructed (from a formal point of view) in a special case.

Finally we show in sec.11 a dynamics for boundary less manifolds based on a symplectic structure on $TE(M,N)$, study the influence of symmetry groups in sec.12 and lastly consider motions along a fibre of the $Diff M$-principal bundle and obtain Euler's equation of a perfect fluid as a special case.

The first four sections are devoted to the geometric background in order to treat this sort of approach of describing deformable media in this generality, in a rigorous fashion. The term smoothness on Fréchet manifolds refers to the one introduced in [Bi,Sn,Fi] or [Fr,Kr].

1. Geometric preliminaries and the Fréchet manifold $E(M,N)$

Let M be a compact, oriented, connected smooth manifold with boundary ∂M and N be a connected, smooth and oriented manifold without boundary equipped with a fixed Riemannian metric $<\ ,\ >$. For any $j \in E(M,N)$ we define a Riemannian metric $m(j)$ on M by setting

$$m(j)(X,Y) := < TjX, TjY >, \qquad \forall X,Y \in \Gamma(TM). \tag{1.1}$$

(More customary is the notation $j^\star <\ ,\ >$ instead of $m(j)$.) We use $\Gamma(\mathbf{E})$ to denote the collection of all smooth sections of any smooth vector bundle $\mathbf{E}$ over a manifold Q with $\pi_Q : \mathbf{E} \longrightarrow Q$ the canonical projection.
The positively oriented unit normal of ∂M in M is called $\mathbf{n}$. Let moreover $j_\partial := j|\partial M$. The metric on ∂M induced by $m(j)$ is called $m(j_\partial)$. The Weingarten map of $\mathbf{n}$ formed with respect to $m(j_\partial)$ is called $W_\partial(j)$ and its trace $H_\partial(j)$ referred to as the Weingarten map and the mean curvature, respectively, of ∂M. Let $\mu(j)$ and $\mu(j_\partial)$ be the Riemannian volume on M and ∂M respectively defined by the given orientations and the metrics $m(j)$ and $m(j_\partial)$, respectively. Hence $m(j_\partial) = (j_\partial)^\star m(j)$ and $\mu(j) = (j_\partial)^\star \mu(j)$.

Let ∇ be the Levi-Cività connection of the Riemannian manifold $(N, <\ ,\ >)$. In this situation, the Levi-Cività connection $\nabla(j)$ of $(M, m(j))$ is obtained as follows:

$TN|j(M)$ splits into $Tj(TM)$ and its orthogonal complement $(Tj(TM))^\perp$ (the Riemannian normal bundle of j) and hence any $Z \in \Gamma(TN|j(M))$ has an orthogonal decomposition $Z = Z^\top + Z^\perp$, where the tangential component $Z^\top$ is of the form $Z^\top = Tj \cdot V$ for a unique $V \in \Gamma(TM)$.

If now $Y \in \Gamma(TM)$, then $TjY : M \longrightarrow TN$ is smooth and therefore, the above covariant derivative $\nabla(TjY)$ is well-defined. We use this to define the vector field $\nabla(j)_X Y$ on M by the equation

$$Tj(\nabla(j)_X Y) = \nabla_X(TjY) - (\nabla_X(TjX))^\perp \quad . \tag{1.2}$$

for all $X, Y \in \Gamma(TM)$. In fact $\nabla(j)$ is the Levi - Cività connection of $m(j)$ which is called d in case N is Euclidean, i.e if $N = \mathbb{R}^n$ and $<\ ,\ >$ is a fixed scalar product. Instead of $(\nabla_X(T_jX))^\perp$ we write $S(j)(X,Y)$ and call $S(j)$ the second fundamental tensor of j. The Levi-Cività connection of $m(j_\partial)$ on ∂M is denoted by $\nabla^\partial(j)$.

It is well-known that the set $C^\infty(M,N)$ of smooth maps from M into N endowed with Whitney's C^∞-topology is a Fréchet manifold (cf.e.g.[Bi,Sn,Fi] or [Fr,Kr]). For a given $f \in C^\infty(M,N)$, the tangent space $T_f C^\infty(M,N)$ is the Fréchet space $C^\infty_f(M,TN) := \{l \in C^\infty(M,TN) | \pi_N \circ l = f\} \cong \Gamma(f^\star TN)$ and the tangent bundle $TC^\infty(M,N)$ is identified with $C^\infty(M,TN)$, the topology again being the C^∞-topology.

The set $E(M,N)$ of all C^∞-embeddings from M to N is open in $C^\infty(M,N)$ and thus is a Fréchet manifold whose tangent bundle we denote by $C^\infty_E(M,TN)$. It is an open submanifold of $C^\infty(M,TN)$, fibred over $E(M,N)$ by "composition with π_N". Moreover, if $\partial M = \emptyset$, the Fréchet manifold $E(M,N)$ is a principal *Diff* M-bundle under the obvious right *Diff* M-action and the quotient $U(M,N) := E(M,N)/\mathit{Diff}\ M$ is the manifold of "submanifolds of type M" of N (cf. [Bi,Sn,Fi], ch.5, and [Bi,Fi 1]).

The set $\mathcal{M}(M)$ of all Riemannian structures on M is a Fréchet manifold since it is an open convex cone in the Fréchet space $S^2(M)$ of smooth, symmetric bilinear forms on M. Moreover, the map

$$m : E(M,N) \longrightarrow \mathcal{M}(M)$$

is smooth (cf.[Bi,Sn,Fi]).

Lastly, by an $\mathbf{E}$-valued one-form a on M, where $\mathbf{E}$ is a vector bundle over N, we mean a smooth map

$$a : TM \longrightarrow \mathbf{E}$$

for which $a|T_pM$ is linear for all $p \in M$. We denote the set of such one-forms by $\mathcal{A}^1(M,\mathbf{E})$ and now obtain the following description of its structure: The requirement that $a \in \mathcal{A}^1(M,\mathbf{E})$ should be linear along the fibres of TM means that there is a (smooth) map $f : M \longrightarrow N$ such that $a|T_pM$ is a linear map into $\mathbf{E}_{f(p)}$ for $p \in M$, in other words, that a is a bundle map from TM to $\mathbf{E}$ over f: This map $f \in C^\infty(M,N)$ satisfies $\pi_{\mathbf{E}} \circ a = f \circ \pi_M$ (where π_E, π_M are the respective bundle projections). The set of such one-forms is naturally identified with the Fréchet space $\mathcal{A}^1(M, f^\star\mathbf{E})$. This shows that

$$\mathcal{A}^1(M,\mathbf{E}) = \bigcup_f \{\mathcal{A}^1(M, f^\star\mathbf{E}) | f \in C^\infty(M,N)\}.$$

It is clear from the construction that there is a natural surjection

$$\Pi : \mathcal{A}^1(M,\mathbf{E}) \longrightarrow C^\infty(M,N)$$

whose fibres are the Fréchet spaces $\mathcal{A}^1(M, f^\star\mathbf{E})$, in fact $\mathcal{A}^1(M, f^\star\mathbf{E})$ is a vector bundle over $C^\infty(M,N)$ with projection Π (cf.[Bi,Fi 2]).

In the following three sections we will introduce three metrics on some special types of infinite dimensional manifolds and will prepare in this way the geometric background of the description of smoothly deformable media as well as a dynamics in case $\partial M = 0$.

2. The metrics $\mathcal{G}$ and $\mathcal{B}$ on $E(M,N)$

The Riemannian structure $< \, , \, >$ of N induces a "Riemannian structure" $\mathcal{G}$ on $E(M,N)$ as follows: For $j \in E(M,N)$, let $\mu(j)$ and $\mu(j_\partial)$ be the Riemannian volume on M and ∂M respectively defined by the given orientations and the metric $m(j)$. For any two tangent vectors $l_1, l_2 \in C_j^\infty(M,TN)$, we set

$$\mathcal{G}(j)(l_1,l_2) := \int_M \langle l_1, l_2 \rangle \, \mu(j) \qquad \text{and} \qquad \mathcal{G}^\partial(j)(l_1,l_2) = \int_{\partial M} \langle l_1, l_2 \rangle \, \mu(j_\partial) \, . \tag{2.1}$$

It is clear that $\mathcal{G}(j)$ is a continuous, symmetric, positive-definite bilinear form on $C_j^\infty(M,TN)$ for each $j \in E(M,N)$.

The metric $\mathcal{G}$ possesses some invariance properties (which will become important in sec. 11): Let $Diff^+ M$ be the group of orientation-preserving diffeomorphisms of M. As a subgroup of $Diff\ M$, it operates (freely) on the right on $E(M,N)$ by

$$\begin{aligned} \phi : E(M,N) \times Diff^+ M &\longrightarrow E(M,N) \\ (j,\varphi) &\longmapsto j \circ \varphi. \end{aligned} \tag{2.2}$$

For a fixed φ, we also write $R_\varphi j$ for $j \circ \varphi$.

Similarly, if $\mathcal{J}$ is any group of orientation-preserving isometries of N, then it operates on the left on $E(M,N)$ by

$$\begin{aligned} \mathcal{J} \times E(M,N) &\longrightarrow E(M,N). \\ (g,j) &\longmapsto g \circ j \end{aligned} \tag{2.3}$$

We need the following rather obvious result (cf.[Bi,Fi 2]) for some basic invariance properties of one-forms on $E(M,N)$:

Proposition 2.1

$\mathcal{G}$ is invariant under both $Diff^+ M$ and $\mathcal{J}$. •

The dynamics of boundary less deformable media to be introduced later relies on the metric $\mathcal{B}$ on $E(M,N)$. This metric will be based on a density map ρ. A smooth map

$$\rho : E(M,N) \longrightarrow C^\infty(M, \mathbb{R})$$

is called a ***density map*** if the following is satisfied:

$$\rho(j)(p) > 0 \qquad \forall j \in E(M,N) \text{ and } \forall p \in M. \tag{2.4}$$

$$d\!\!l\rho(j)(k) = -\frac{\rho(j)}{2} tr_{m(j)} d\!\!l m(j)(k) \qquad \forall j \in E(M,N) \text{ and } \forall k \in C_j^\infty(M,TN). \tag{2.5}$$

$tr_{m(j)}$ denotes the trace formed with respect to $m(j)$. The symbol $d\!\!l$ denotes the differential of maps of which the domain is a Fréchet manifold and which assume values in a Fréchet space. If both domain and range are finite dimensional the usual d replaces $d\!\!l$. We will construct a density map next. For any $j' \in E(M,N)$ we express $m(j')$ via a smooth strong bundle endomorphism $f^2(j')$ of TM, selfadjoint with respect to $m(j)$, as

$$m(j')(v_p, w_p) = m(j)(f^2(j')(p)v_p, w_p) \qquad \forall v_p, w_p \in T_pM \text{ and } \forall p \in M \tag{2.6}$$

and observe that the Riemannian volume forms $\mu(j)$ and $\mu(j')$ are linked by

$$\mu(j') = \det\ f(j')\mu(j). \tag{2.7}$$

Fixing a map $\rho(j) \in C^\infty(M, \mathbb{R})$ for some fixed $j \in E(M,N)$ satisfying (2.4) then

$$\rho : E(M,N) \longrightarrow C^\infty(M, \mathbb{R})$$

given for any $j' \in E(M,N)$ by

$$\rho(j') := \det f(j')^{-1} \rho(j) \tag{2.8}$$

satisfies both (2.4) and (2.5).

To construct the metric $\mathcal{B}$ we fix a density map ρ on $E(M,N)$ once and for all (unless specified otherwise).

The metric $\mathcal{B}$ is then defined by

$$\mathcal{B}(j)(l_1,l_2) := \int_M \rho(j) < l_1, l_2 > \mu(j) \tag{2.9}$$

for each $j \in E(M,N)$ and for each pair $l_1, l_2 \in C^\infty(M,TN)$. This metric depends smoothly on all of its variables. For its covariant derivative and geodesics see appendix A.

3. The fibred space $\mathcal{L}_E(M,TN)$ and its dot metric

To prepare the geometric realm for the introduction of the first Piola-Kirchhoff stress tensor characterizing a medium deforming in the Riemannian manifold N, we need to introduce a space of special TN-valued one-forms. To begin this preparation, we denote by $\mathcal{A}^1_E(M,TN)$ the subset of $\mathcal{A}^1(M,TN)$ consisting of all TN-valued one-forms covering smooth embeddings from M to N. This is the inverse image of $E(M,N)$ under the projection $\Pi : \mathcal{A}^1(M,TN) \longrightarrow C^\infty(M,N)$, hence is an open submanifold and, in fact, is itself a (Fréchet) vector bundle whose fibre at j we denote by $\mathcal{A}^1_j(M,TN)$.
By construction of $m(j)$, the map Tj is fibrewise isometric. This allows us to write $a \in \mathcal{A}^1_j(M,TN)$ in the form

$$\alpha = c(\alpha, Tj) \cdot Tj + Tj \cdot A(\alpha, Tj) \tag{3.1}$$

for a suitable bundle endomorphism $c(\alpha,Tj)$ of $TN|j(M)$, skew adjoint with respect to $< \, , \, >$ and mapping $TjTM$ into its normal bundle $(TjTM)^\perp$ and vice versa and where $A(\alpha,Tj)$ is a strong bundle endomorphism of TM. These endomorphisms are uniquely determined and are smooth continuous linear functions of α as shown in [Bi 4]. The usual "trace inner product" for endomorphisms of TN and of TM then yields for any $j \in E(M,N)$ the ***dot product***

$$\alpha \cdot \beta := -\tfrac{1}{2} tr\ c(\alpha,Tj) \cdot c(\beta,Tj) + tr\ A(\alpha,Tj) \cdot A^\star(\beta,Tj) \ ,$$

for any two $\alpha, \beta \in \mathcal{A}^1_j(M,TN)$. Here $A^\star$, the adjoint of A, is formed fibrewise with respect to $m(j)$. We define

$$\mathcal{G}(Tj)(\alpha,\beta) := \int_M \alpha \cdot \beta \ \mu(j) \ , \tag{3.2}$$

$\mathcal{g}$ a smooth continuous, symmetric and positive-definite bilinear form on the Fréchet space $\mathcal{A}^1_j(M,TN)$, the ***dot metric***. It is a generalization of the classical ***Dirichlet integral*** (which will be apparent by the theorems (3.1) and (4.2) below). For the sake of simplicity we will write $\mathcal{g}(j)$ instead of $\mathcal{g}(Tj)$.

We shall also need a subfibration of $\mathcal{A}^1_E(M,TN)$ defined by

$$\mathcal{L}_E(M,TN) := \{\nabla l | l \in C^\infty_E(M,TN)\} \tag{3.3}$$

whose fibres we denote by $\mathcal{L}_j(M,TN)(= \mathcal{L}_E(M,TN) \cap \mathcal{A}^1_j(M,TN))$; evidently these are subspaces of the Fréchet spaces $\mathcal{A}^1_j(M,TN)$; cf.[Bi,Fi 2].

Next, we introduce the Laplacian $\Delta(j)$ which will depend on j via $m(j)$; (cf.[Ma]):
For $k \in C^\infty_j(M,TN)$, we define the covariant divergence by

$$\nabla^\star(j)k := 0 \tag{3.4}$$

as usual, while following [Ma], $\nabla^\star(j)\alpha$ for $\alpha \in \mathcal{A}^1_j(M,TN)$ is given pointwise by

$$\nabla^\star(j)\alpha := -\sum_{r=1}^{n} \nabla_{E_r}(\alpha)(E_r)\ , \tag{3.5}$$

(E_r) a local orthonormal frame with respect to $m(j)$. In (3.5) we have used $\nabla_X\alpha$ to denote the more informative symbol $\nabla(j)_X\alpha$, defined in the standard manner by

$$\nabla(j)_X(\alpha)(Y) = \nabla_X(\alpha Y) - \alpha(\nabla(j)_X Y) \qquad \forall X,Y \in \Gamma(TM)\ .$$

The definition of $\Delta(j)$ does not depend on the moving frames chosen (cf. [Bi,Fi 2]).

The following theorem (cf. [Bi 4], [Bi,Fi 2] and [L,M] for the last assertion) will be a basic tool in our studies of one-forms on $E(M,N)$ (cf. [Bi 4] and [Bi,Fi 2]). It relates the metric $\mathcal{G}$ with $\mathcal{g}$ (the Dirichlet integral as (3.7) shows):

Theorem 3.1

For any $j \in E(M,N)$, any $\alpha \in \mathcal{A}^1_E(M,TN)$ and any two $h,l \in C^\infty_j(M,TN)$ the following relations hold

$$\mathcal{g}(j)(\alpha,\nabla l) \ = \ \mathcal{G}(j)(\nabla^\star(j)\alpha, l) \ + \ \mathcal{G}^\partial(j)(\alpha(\mathbf{n}), l) \tag{3.6}$$

and

$$\mathcal{g}(j)(\nabla h,\nabla l) \ = \ \mathcal{G}(j)(\Delta(j)h, l) \ + \ \mathcal{G}^\partial(j)(\nabla_{\mathbf{n}}h, l)\ . \tag{3.7}$$

Here ∇ denotes the Levi-Cività connection of the metric $<\ ,\ >$ on N. The dimension of $\mathbf{K}_j := \{l \in C^\infty_j(M,TN) | \nabla l = 0\}$ for any $j \in E(M,N)$ is finite. •

We close this section by stating invariance properties. For the rather straight forward proof we refer to [Bi,Fi 2].

Proposition 3.2

The metric $\mathcal{g}$ on $\mathcal{L}_E(M,TN)$ is invariant under $Diff^+M$ and any group $\mathcal{J}$ of orientation-preserving isometries on N. •

4. The first Piola-Kirchhoff stress tensor and the virtual work

Let us consider a special situation: We assume that the medium moves and deforms in $\mathbb{R}^n$, equipped with a fixed scalar product. The configuration of the medium may vary rapidly. At each configuration $j \in E(M,N)$ we characterize the quality of the medium by the smooth first Piola-Kirchhoff stress tensor (cf. [M,H], p.135)

$$\alpha(j) : TM \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n ,$$

a two point tensor. Clearly $\alpha(j)$ is a smooth $\mathbb{R}^n$-valued one-form if the values will be projected onto the second factor.

Remark:

If we fix a reference configuration $j_0 \in E(M,N)$ then

$$\int \alpha(j) \cdot dl\mu(j) = \int (\alpha(j) \det f(j)) \cdot dl\mu(j_o)$$

for all $j \in E(M,N)$ and all $f \in C^\infty_j(M,TN)$. In fact $\alpha(j) \det f(j)$ is usually called the first Piola-Kirchhoff stress tensor. However since we work without reference configurations in general we call α the second Piola-Kirchhoff stress tensor.

Let us assume from now on that

$$\alpha : E(M,\mathbb{R}^n) \longrightarrow \mathcal{A}^1(M,\mathbb{R}^n)$$

is smooth (the range carries the C^∞-topology). The medium characterized by α is thus called a ***smoothly deformable medium***. The virtual work determined by α and an (infinitesimal virtual) distortion l will be nothing else but a generalization of the classical Dirichlet integral, as shown as follows: By (3.1), we can represent $\alpha(j)$ as

$$\alpha(j) = c_{\alpha(j)} \cdot dj + dj \cdot A_{\alpha(j)} \tag{4.1}$$

in a unique way. Any infinitesimal distortion $l \in C^\infty(M,\mathbb{R}^n)$ yields the one-form

$$dl : TM \longrightarrow \mathbb{R}^n$$

and admits the representation

$$dl = c_{dl} \cdot dj + dj \cdot A_{dl} . \tag{4.2}$$

The virtual work $F(dj)(dl)$ is then given by the Dirichlet integral (cf. sec.3)

$$F(dj)(dl) := \int_M \alpha(j) \cdot dl\, \mu(j) \equiv \mathcal{G}(dj)(\alpha(j), dl) \tag{4.3}$$

where

$$\alpha(j) \cdot dl \;=\; -\tfrac{1}{2}\mathrm{tr}\, c_{\alpha(j)}\, c_{dl} \;+\; \mathrm{tr}\, A_{\alpha(j)} \cdot A^*_{dl} \tag{4.4}$$

with $*$ denoting the fibre wise adjoint of $A_{dl} : TM \longrightarrow TM$ formed with respect to $m(j)$. The virtual work

$$\overline{F} : E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \longrightarrow \mathbb{R}$$

given by

$$\overline{F}(j\;,\;l) \;=\; \int \alpha(j) \cdot dl\, \mu(j) \tag{4.5}$$

is a smooth one-form on $E(M, N)$. This work $\overline{F}$ has two special independent properties: These are for $j \in E(M, N)$ and any $l \in C^\infty_j(M, TN)$

$$1) \quad \overline{F}(j+z)(l) \;=\; \overline{F}(j)(l) \qquad \forall z \in \mathbb{R}^n$$

and

$$2) \quad \overline{F}(j)(l+z) \;=\; \overline{F}(j)(l) \qquad \forall z \in \mathbb{R}^n\,.$$

Let us investigate them somewhat closer. The first one is certainly the invariance under the obvious action on $E(M, \mathbb{R}^n)$ of the translation groups $\mathbb{R}^n$ of the vector space $\mathbb{R}^n$. Factoring out this action on $E(M, \mathbb{R}^n)$ yields again a Fréchet manifold, called $E(M, \mathbb{R}^n)/_{\mathbb{R}^n}$ (the ***reduced configuration space***). It admits a natural representation via the center of mass as seen as follows (cf. [Bi 4]).

Specifying a density map ρ (cf. section 2) on $E(M, \mathbb{R}^n)$ we introduce the ***center*** z_m ***of mass*** by

$$z_m(j) := \int \rho(j)\, j\, \mu(j)$$

for any $j \in E(M, \mathbb{R}^n)$. Fixing the center of mass at zero, then

$$\begin{array}{rcl} \{\, j \in E(M, \mathbb{R}^n) \,|\, z_m(j) = 0 \,\} & \longrightarrow & E(M, \mathbb{R}^n)/_{\mathbb{R}^n} \\ j & \longmapsto & [j] \end{array} \tag{4.6}$$

is a diffeomorphism. Here $[j]$ denotes the equivalence class of j formed with respect to the action of the translation group $\mathbb{R}^n$ on $E(M, \mathbb{R}^n)$.

Clearly the differential

$$d : E(M, \mathbb{R}^n) \longrightarrow \{\, dj \,|\, j \in E(M, \mathbb{R}^n) \,\}$$

(the second factor carries the C^∞-topology) induces a diffeomorphism

$$\begin{array}{rcl} \widetilde{d} : E(M, \mathbb{R}^n)/_{\mathbb{R}^n} & \longrightarrow & \{\, dj \,|\, j \in E(M, \mathbb{R}^n) \,\} \\ [j] & \longmapsto & dj \end{array} \tag{4.7}$$

since $[j] = \{ j+z | z \in \mathbb{R}^n \}$ for any $[j] \in E(M, \mathbb{R}^n)/_{\mathbb{R}^n}$. Thus we identify $E(M, \mathbb{R}^n)/_{\mathbb{R}^n}$ with $\{ dj \mid j \in E(M, \mathbb{R}^n) \}$. Obviously the differential

$$\{ j \in E(M, \mathbb{R}^n) \mid z_m(j) = 0 \} \xrightarrow{d} \{ dj \mid j \in E(M, \mathbb{R}^n) \}$$

is a (smooth) diffeomorphism too. Therefore, the reduced configuration space of the action of the translation group $\mathbb{R}^n$ on $E(M, \mathbb{R}^n)$ can be described either by

$$E(M, \mathbb{R}^n)/_{\mathbb{R}^n} \quad \text{or} \quad \{ j \in E(M, \mathbb{R}^n) \mid z_m(j) = 0 \} \quad \text{or} \quad \{ dj \mid j \in E(M, \mathbb{R}^n) \} .$$

Thus we may identify all the three and will use either symbol according to the convenience. The reduced phase space is hence

$$E(M, \mathbb{R}^n)/_{\mathbb{R}^n} \times C^\infty(M, \mathbb{R}^n)/_{\mathbb{R}^n} ,$$

the second factor being the usual factor space of $C^\infty(M, \mathbb{R}^n)$ modulo $\mathbb{R}^n$ (identified with $\{ dl \mid l \in C^\infty(M, \mathbb{R}^n) \}$) endowed with the C^∞-topology.
Thus our work $\overline{F}$, defined on $TE(M, \mathbb{R}^n) = E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n)$ factors to F as follows

$$\begin{array}{ccc} E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) & \xrightarrow{\overline{F}} & \mathbb{R} \\ {\scriptstyle d\times d} \downarrow & & \nearrow_F \\ \{dj | j \in E(M, \mathbb{R}^n)\} \times \{dl | l \in C^\infty(M, \mathbb{R}^n)\} & & \end{array}$$

These considerations reversed yield immediately

Lemma 4.1

The characterization of a smoothly deformable medium by a smooth assignment

$$\alpha : E(M, \mathbb{R}^n) \longrightarrow \mathcal{A}^1(M, \mathbb{R}^n)$$

of a first Piola-Kirchhoff stress tensor is equivalent with specifying a smooth one-form (the virtual work)

$$\overline{F} : E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \longrightarrow \mathbb{R}$$

with the following properties:

1) $\overline{F}$ is invariant under the action of the translation group $\mathbb{R}^n$ on $E(M, \mathbb{R}^n)$
2) $\overline{F}(j)(l + z) = F(j)(l) \quad \forall j \in E(M, \mathbb{R}^n), \ \forall l \in C^\infty(M, \mathbb{R}^n)$ and $\forall z \in \mathbb{R}^n$
3) $\overline{F}$ admits an integral representation

$$\overline{F}(j)(l) = \int \alpha(j) \cdot dl\, \mu(j) = g(dj)(\alpha(j), dl)$$

for all variables of $\overline{F}$ where $\alpha : E(M, \mathbb{R}^n) \longrightarrow \mathcal{A}^1(M, \mathbb{R}^n)$ a smooth density. •

In the sequel we will identify $\overline{F}$ with F given by (4.5) and (4.3), respectively.

Remark:
For the sake of shortness we will frequently use the term "stress form" for $\alpha(j)$ rather than "a first Piola-Kirchhoff stress tensor".

Now we generalize the whole setting as follows. $(\mathbb{R}^n, <,>)$ will be replaced by a Riemannian manifold with $(N, <,>)$ with a fixed Riemannian metric $<,>$.
The configuration space is now $E(M,N)$, its tangent space $C^\infty_E(M,TN)$ (cf. sec.1).

Again we will characterize a smoothly deformable medium moving and deforming in N at each configuration $j \in E(M,N)$ by a two point tensor $\alpha(j)$ (depending smoothly on j) for which the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\alpha(j)} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{j} & N \end{array}$$

commutes. An infinitesimal deformation $l \in T_jE(M,N)$ is a smooth map $l: M \longrightarrow TN$ covering the embedding $j: M \longrightarrow N$. Instead of dl we now have $\nabla l: TM \longrightarrow TN$ (with ∇ the Levi Civita connection of N) for which the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{\nabla l} & TN \\ {\scriptstyle \pi_M}\downarrow & {\scriptstyle l}\nearrow & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{j} & N \end{array}$$

The collection of all ∇l with $l \in C^\infty_E(M,TN)$ is called $\mathcal{L}_E(M,N)$ (cf. sec.3). As $E(M,\mathbb{R}^n)$ maps d into $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$, also $E(M,N)$ is mapped naturally into $\mathcal{L}_E(M,N) \subset \mathcal{A}^1_E(M,TN)$, as seen as follows:

Given any $j \in E(M,N)$ we solve the Neumann problem

$$\Delta(j)\bar{j} = \nabla^* Tj = -\sum_{s=1}^{\dim M} S(j)(E_s,E_s)$$

with the boundary condition

$$\nabla_{\mathbf{n}}\bar{j} = Tj\mathbf{n}$$

with $E_1,\ldots,E_{\dim M}$ being an orthonormal moving frame on M and with $S(j)$ being the second fundamental tensor of j in N (cf. sec.1). The vector field $\nabla^* Tj$ along j is called the ***mean curvature field*** of j (cf. [L]).
Since the integrability condition is satisfied, there is, according to [Hö], a solution, which is unique provided, we require that $\bar{j}$ is in the complement of $\mathbf{K}_j$ (cf. theorem 3.1).
Let us do so from now on. In fact the map from $E(M,N)$ into $C^\infty_E(M,TN)$ sending any j into $\bar{j}$ is smooth. As easily verified, the following equation

$$\mathcal{g}(j)(\nabla\bar{j},\nabla l) = \mathcal{G}(j)(\nabla^* Tj, l) + \mathcal{G}^\partial(j)(Tj\mathbf{n}, l) = \mathcal{g}(Tj,\nabla l)$$

holds for any $l \in C^\infty_E(M,TN)$. It implies

$$Tj = \nabla \bar{j} .$$

The map from $E(M,N)$ into $\mathcal{A}^1_E(M,TN)$ assigning to any j, the TN-valued one-form $\nabla\bar{j}$ on M, is smooth as well. Thus as we have worked with $T\left(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}\right)$ we work with $T\mathcal{L}_E(M,N)$ in the general situation.

The virtual work $F(j)(l)$ at any $j \in E(M,N)$ caused by any virtual infinitesimal deformation $l \in C^\infty_j(M,TN)$ is defined by

$$F(j)(l) := \int \alpha(j) \cdot \nabla l \, \mu(j) \equiv \mathcal{G}(j)(\alpha(j), \nabla l) . \tag{4.8}$$

Again

$$F : TE(M,N) \longrightarrow \mathbb{R}$$

is a smooth one form on $E(M,N)$ (which admits a stress form).

Remark:
Given a virtual work F with stress form α, we will assume throughout these notes, that $\alpha(j)$ in fact depends for any $j \in E(M,N)$ only on $\nabla\bar{j}$ rather than on j itself.

As theorem 5.2 in the next section shows the characterization of the medium deforming in an ambient Riemannian manifold by α is rather general.

The question arises (also in the case of an Euclidean ambient space) as to whether the stress form α is unique. In fact it is not, as we will see in the next section.

5. The part of the stress which is irredundant with respect to the virtual work, the notion of a constitutive law

To find the part of any specification of a stress form $\alpha(j)$ at any configuration $j \in E(M,N)$ which causes the same force densities and hence is irredundant with respect to the virtual work, we study the problem in a pure mathematical frame work. We begin with the following definition:

Definition 5.1
The one-form F on $E(M,N)$ is said to be ***$\mathcal{G}$-representable*** if there exists a smooth section $\alpha : E(M,N) \longrightarrow \mathcal{A}^1_E(M,TN)$ of the bundle $(\mathcal{A}^1_E(M,TN), \Pi, E(M,N))$, such that

$$F(j)(l) = \int_M \alpha(j) \cdot \nabla l \mu(j) = \mathcal{G}(j)(a(j), \nabla l) \tag{5.1}$$

for $j \in E(M,N)$ and $l \in C^\infty_j(M,TN)$. The section α is called a ***$\mathcal{G}$-density*** of F.

Suppose, for instance, that $\mathcal{H}$ is a smooth section of $C_E^\infty(M,TN)$ over $E(M,N)$, i.e a smooth vector field on $E(M,N)$; for the existence of such fields cf. [Bi,Fi 2]. Then $\alpha(j) = \nabla\mathcal{H}(j)$ will provide a $\mathcal{g}$-density and the right-hand side of (5.1) will define a $\mathcal{g}$ - representable one-form. In fact, this example can be shown to characterize the representability of one-forms: Let us denote by $\mathcal{A}^1_{\mathcal{g}}(E(M,N),\mathbb{R})$ the collection of all smooth $\mathcal{g}$-representable one-forms on $E(M,N)$. We now show that any $\mathcal{g}$-density α of a smooth one-form $F \in A^1_{\mathcal{g}}(E(M,N),\mathbb{R})$ can be replaced by $\nabla\mathcal{H}$ where $\mathcal{H} : E(M,N) \longrightarrow C_E^\infty(M,TN)$ is a smooth vector field. This means that for any $j \in E(M,N)$ the following formulas

$$\int_M \alpha(j)\cdot\nabla l\ \mu(j) = \int_M \nabla\mathcal{H}(j)\cdot\nabla l\ \mu(j) + \int \langle\nabla_{\mathbf{n}}\mathcal{H}(j), l\rangle\mu(j_\partial) \tag{5.2}$$

or equivalently,

$$\mathcal{g}(j)(\alpha(j),\nabla l) = \mathcal{g}(j)(\nabla\mathcal{H}(j),\nabla l) \tag{5.3}$$

have to hold for all $l \in C_j^\infty(M,TN)$. To prove this, we are required to solve the Neumann problem

$$\Delta(j)\mathcal{H}(j) = \nabla^\star\alpha \qquad \text{and} \qquad \nabla_{\mathbf{n}}\mathcal{H}(j) = \alpha(\mathbf{n})\ . \tag{5.4}$$

Here $\nabla^\star$ denotes the covariant divergence. It depends on ∇ and $\nabla(j)$.

The existence of a solution $\mathcal{H}(j)$ is guaranteed for each $j \in E(M,N)$ by the theory of elliptic problems comprehensively described in [Hö] (cf. also [Bi,Fi 2]). Choosing $\mathcal{H}(j)$ for any $j \in E(M,N)$ in the complement of $\mathbf{K}_j$, the vector field $\mathcal{H} : E(M,N) \longrightarrow C_E(M,TN)$ is uniquely determined by α and thus is smooth.

Let us formulate this result as follows:

Theorem 5.2
For any $\alpha : E(M,N) \longrightarrow \mathcal{A}^1(M,TN)$ the map

$$\alpha(j) : TM \longrightarrow TN$$

at any $j \in E(M,N)$ uniquely splits into

$$\alpha(j) = \nabla\mathcal{H}(j) + \beta(j) \qquad \forall j \in E(M,N) \tag{5.5}$$

with $\mathcal{H}(j) \in C_j^\infty(M,TN)$ being in the complement of $\mathbf{K}_j$ and where $\mathcal{g}(j)(\beta(j),\nabla l) = 0$ for all $l \in C_j^\infty(M,TN)$. In fact both $\mathcal{H}$ and β vary smoothly with j. •

Using the identity (5.3) we may therefore state

Theorem 5.3
Any $F \in \mathcal{A}^1_{\mathcal{g}}(E(M,N),\mathbb{R})$ admits a smooth vector field $\mathcal{H} : E(M,N) \longrightarrow C_E^\infty(M,TN)$ for which

$$F(j)(l) = \int_M \nabla\mathcal{H}(j)\cdot\nabla l\ \mu(j) \tag{5.6}$$

and hence

$$F(j)(l) = \int_M < \Delta(j)\mathcal{H}(j), l > \mu(j) + \int_{\partial M} \langle \nabla_{\mathbf{n}} \mathcal{H}(j), l \rangle \, \mu(j) \qquad (5.7)$$

hold for all variables of F. The map $\mathcal{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ is uniquely determined by α and is smooth provided $\mathcal{H}(j)$ is chosen to be in the complement of $\mathbf{K}_j$ for any $j \in E(M,N)$. •

The consequence of our studies so far in the realm of characterizing smoothly deformable media is the following

Theorem 5.4

Let be given any smoothly deformable medium characterized by a smooth stress form assignment α. This medium is characterized as well by a smooth vector field

$$\mathcal{H} : E(M,N) \longrightarrow C^\infty_E(M,TN)$$

uniquely determined by α and enjoys the property

$$\mathcal{g}(j)(\alpha(j), \nabla l) = \mathcal{g}(j)(\nabla \mathcal{H}(j), \nabla l) = \mathcal{G}(j)(\Delta(j)\mathcal{H}(j), l) + \mathcal{G}^\partial(\nabla_{\mathbf{n}} \mathcal{H}(j), l) = F(j)(l) \quad (5.8)$$

$\forall j \in E(M,N)$ and $\forall l \in C^\infty_E(M,TN)$ where

$$\Delta(j)\mathcal{H}(j) = \nabla^\star \alpha(j) \qquad \text{and} \qquad \alpha(j)(\mathbf{n}) = \nabla_{\mathbf{n}} \mathcal{H}(j) \qquad \forall j \in E(M,N)\,. \qquad (5.9)$$

The maps $\nabla^\star \alpha(j) : M \longrightarrow TN$ and $\alpha(\mathbf{n}) : \partial M \longrightarrow TN$ are called the ***force densities*** associated with $\alpha(j)$ on M and ∂M, respectively. Given vice versa smooth fields $\Phi : E(M,N) \longrightarrow C_E(M,TN)$ and $\varphi : E(M,N) \longrightarrow C^\infty_E(\partial M, TN)$ of force densities, then there is a unique smooth constitutive field $\mathcal{H} : E(M,N) \longrightarrow C_E(M,TN)$ such that $\mathcal{H}(j)$ satisfies

$$\Phi(j) = \Delta(j)\mathcal{H}(j) \qquad \text{and} \qquad \varphi(j) = \nabla_{\mathbf{n}} \mathcal{H}(j) \qquad \forall j \in E(M,N) \qquad (5.10)$$

and is in the complement of $\mathbf{K}_j$, provided that $\Phi(j)$ and $\varphi(j)$ satisfy the integrability conditions. $\nabla \mathcal{H}$ is a stress form characterizing the medium. •

Remark:

The (finite dimensional) cokernel of the Neumann problem at each $j \in E(M,N)$ consists of all $\Phi \in C^\infty_j(M,TN)$ and $\varphi \in C^\infty_j(\partial M, TN)$ such that $\Delta(j)\mathcal{H} = \Phi$ and $\nabla_{\mathbf{n}} \mathcal{H} = \varphi$ has no solution. The integrability conditions make sure that Φ and φ are in the complement of the cokernel. We will assume from now on that $\mathcal{H}(j)$ for each $j \in E(M,N)$ is in the complement of $\mathbf{K}_j$ and hence is uniquely determined. We make thus no use of of the gauche freedom of adding to $\mathcal{H}$ maps $\mathcal{H}'$ for which $\nabla \mathcal{H}' = 0$.

Definition 5.5
By a ***constitutive law*** we mean either some $F \in \mathcal{A}'_{\mathcal{G}}(E(M,N), \mathbb{R})$ or a $\mathcal{G}$-density α of F. The vector field $\mathcal{H}$ given by (5.9) is called a ***constitutive field.***

To construct examples of a constitutive field associated with well known sorts of virtual work let us consider the maps

$$\mathcal{V} : E(M,N) \longrightarrow \mathbb{R} \qquad \text{and} \qquad \mathcal{A} : E(M,N) \longrightarrow \mathbb{R} \ ,$$

the volume of $j(M)$ and the area of $j(\partial M)$, respectively. They are given by

$$\mathcal{V}(j) = \int \mu(j) \qquad \text{and} \qquad \mathcal{A}(j) = \int \mu(j) \qquad \forall j \in E(M,N)$$

respectively. According to [Bi,Sn,Fi]

$$d\mu(j)(l) = \tfrac{1}{2} \mathrm{tr}_{\,m(j)} dm(j)(j)\, \mu(j)$$

and hence

$$d\mathcal{V}(j)(l) = \tfrac{1}{2} \int_M \mathrm{tr}_{\,m(j)} dm(j)(l)\, \mu(j) = \int_M Tj \cdot \nabla l\, \mu(j) \tag{5.11}$$

holds for any $j \in E(M,N)$ and any $l \in C^\infty_j(M, TN)$. Similarly

$$d\mathcal{A}(j)(l) = \int_{\partial M} T(j_\partial) \cdot \nabla(l_\partial)\, \mu(j_\partial) = \int_{\partial M} \langle \Delta(j)(j_\partial), \nabla(l_\partial) \rangle\, \mu(j_\partial)$$

with $\Delta(j_\partial)$ the Laplacian of the Riemannian manifold $(\partial M\,,\, m(j_\partial))$ with $m(j_\partial)$ the Riemannian metric on ∂M, induced by $j_\partial := j|\partial M$. Moreover $l_\partial := l|\partial M$ for each $l \in C^\infty_E(M,N)$.
Solving the elliptical boundary value problem (Višik problem)

$$\Delta(j)\mathcal{H}_{\mathcal{A}}(j) = 0 \qquad\qquad \nabla \mathbf{n} \mathcal{H}_{\mathcal{A}}(j) = \Delta(j_\partial) j_\partial \tag{5.12}$$

admitting a solution $\mathcal{H}_{\mathcal{A}}$ (cf. [Hö]) smooth in j, then we have shown the following:

Lemma 5.6

$$\begin{aligned} d\mathcal{V}(j)(l) &= \int_M Tj \cdot \nabla l\, \mu(j) = \int_M \nabla \bar{j} \cdot \nabla l\, \mu(j) \\ &= \int_M \langle \Delta(j)\bar{j}, l \rangle\, \mu(j) + \int_{\partial M} \langle \nabla_{\mathbf{n}} \bar{j}, l \rangle\, \mu_\partial(j) \end{aligned}$$

and similarly

$$\begin{aligned} d\mathcal{A}(j)(l) &= \int_M \nabla \mathcal{H}_{\mathcal{A}}(j) \cdot \nabla l\, \mu(j) = \int_{\partial M} \langle \nabla_{\mathbf{n}} \mathcal{H}(j), l \rangle\, \mu(j_\partial) \\ &= \int_{\partial M} \langle \Delta(j)(j_\partial), l \rangle\, \mu(j_\partial) \end{aligned}$$

hold for any of the variables of $\mathcal{V}$ and $\mathcal{A}$. Again $\mathcal{H}_{\mathcal{A}}(j)$ is assumed to be in the complement of $\mathbf{K}_j$ for any $j \in E(M,N)$. In particular

$$ \mathrm{d} \ln \mathcal{V}(j)(j) = \dim M \qquad \text{and} \qquad \mathrm{d} \ln \mathcal{A}(j)(j) = (\dim M - 1) . $$

Thus $\mathrm{d}\mathcal{V}$ and $\mathrm{d}\mathcal{A}$ are constitutive laws with constitutive fields $\mathcal{H}_{\mathcal{V}}$ given by $\mathcal{H}_{\mathcal{V}}(j) = \bar{j}$ for all $j \in E(M,N)$ and $\mathcal{H}_{\mathcal{A}}$, respectively. •

6. The relation between constitutive fields with (2,0)-stress tensors on M

Here we exhibit te relation of the stress tensor a $(2,0)$-tensor on M to α and $\nabla\mathcal{H}$ in particular if codimension $\dim M = \dim N$. Moreover, we show that working with a stress tensor in the latter case can be more informative. The example however is in case of a bounded medium deforming in an Euclidean space. Let F be a constitutive law on $E(M,N)$ with constitutive field $\mathcal{H}$. As we have seen in (5.5) there are numerous stress forms α representing F, not only $\nabla\mathcal{H}$. They all differ from $\nabla\mathcal{H}$ by some

$$ \beta : E(M,N) \longrightarrow \mathcal{A}^1_E(M,TN) $$

for which $\mathcal{G}(j)\big(\beta(j), \nabla l\big) = 0 \quad \forall l \in C^\infty_j(M,TN)$ and all $j \in E(M,N)$. We say that β has ***vanishing pseudo exact part***. Nevertheless any α representing F is nothing else but the first Piola-Kirchhoff stress tensor (cf. [M,H]) and vice versa any such tensor α determines some F by taking it as a $\mathcal{G}$-density; the constitutive field $\mathcal{H}$ is then given by solving (5.4). This amounts to split off the ***pseudo exact part*** $\nabla\mathcal{H}$ of α. Clearly $\alpha(j)$ depends on j possibly in a non local way.

The tensor $\mathbf{T}(\alpha)$ defined by

$$ \mathbf{T}(\alpha)(j)(X,Y) := \langle \alpha(j)X \, , \, \nabla_Y \bar{j} \rangle \, , $$

for any stress form α of F is called the ***stress tensor*** on M. $\mathbf{T}(\alpha)$ characterizes the medium as well as F or α if $\dim M = \dim N$, (otherwise not, as shown below!) This is easily verified by considering the bundle map

$$ A(j,Tj) : TM \longrightarrow TM \qquad \forall j \in E(M,N) \, , $$

(cf. (3.1)) for which

$$ \mathbf{T}(\alpha)(j)(X,Y) = m(j)\big(A(j,Tj)X \, , \, Y\big) $$

holds for all $X, Y \in \Gamma TM$, and by observing that for any $j \in E(M,N)$ the force densities derived from $T(\alpha)(j)$ are in case of $\dim M = \dim N$

$$ \nabla^* \mathbf{T}(\alpha)(j) := \nabla^* A(j,Tj) = -\sum_s \nabla(j)_{E_s}\big(A(j,Tj)\big)E_s = \nabla^*\alpha(j) = \Delta(j)\mathcal{H}(j) $$

and on ∂M

$$T(\alpha(\mathbf{n}, X) = \langle \alpha(\mathbf{n}), X \rangle$$

for any moving orthonormal frame $E_1, \ldots, E_{\dim M}$ on M.

Conversely if the smooth map $\mathbf{T}(\alpha) : E(M,N) \longrightarrow \Gamma \bigotimes^2 T^*M$ is given, then in case of $\dim M = \dim N$

$$\Delta(j)\mathcal{H}(j) = \nabla^* A(j, Tj) \qquad \text{and} \qquad \nabla_{\mathbf{n}}\mathcal{H}(j) = \alpha(\mathbf{n}) \qquad \forall j \in E(M,N)$$

defines a smooth vector field $\mathcal{H}$ on $E(M,N)$ of which $\nabla\mathcal{H}$ is a stress form representing a well defined constitutive law on $E(M,N)$. Thus we may state:

Proposition 6.1

If $\dim M = \dim N$ then each constitutive field $\mathcal{H}$ determines a stress tensor T and vice versa such that

$$\nabla^* T(j) = \Delta(j)\mathcal{H}(j) \qquad \forall j \in E(M,N) \; .$$

and

$$T(\alpha)(\mathbf{n}, X) = \langle \nabla_{\mathbf{n}}\mathcal{H}(j), \mathbf{n} \rangle$$

•

To see that $T(\alpha)$ does not determine $\nabla^*\alpha$ in general let $\partial M = \emptyset$ and let us consider $d\mathcal{V}$. Then $\nabla\bar{j}$ is the stress form at j of $d\mathcal{V}(j)$. Hence the stress tensor at j is $m(j)$. Obviously $\nabla^* m(j) = 0$ while as $\nabla^*(\nabla\bar{j}) = \Delta(j)\bar{j} \neq 0$ for any $j \in E(M,N)$.

Let F be a constitutive law with stress form α where $\dim M \leq \dim N$. Since $\mathbf{T}(\alpha)(j)$ splits uniquely into a symmetric and skew symmetric part for any $j \in E(M,N)$, we may split $A(\alpha, Tj)$ uniquely into

$$A(\alpha, Tj) = B(\alpha, Tj) + C(\alpha, Tj) \qquad \forall j \in E(M,N)$$

where the right hand side consists of the self-adjoint $B(\alpha, Tj)$ and the skew-adjoint part $C(\alpha, Tj)$ of $A(\alpha, Tj)$, respectively, both formed with respect to $m(j)$. This amounts to split $T(\alpha)$ into the sum of its symmetric respectively skew symmetric parts. The following is rather obvious:

Proposition 6.2

Let $\dim M \leq \dim N$. Given a constitutive field $\mathcal{H} \in \Gamma(TE(M,N))$ then

$$\langle dj\mathcal{B}(\alpha, Tj)X, djY \rangle = \tfrac{1}{2} d m(j)(\mathcal{H}(j))(X,Y) \qquad \forall X, Y \in \Gamma TM$$

•

A much deeper result, however, is the following one: Suppose that we have

$(N, <,>) = (\mathbb{R}^n, <,>_{\mathbb{R}^n})$, M a manifold with boundary satisfying $\dim M = n$ and that F with the stress field $\mathcal{H}$ is invariant under the Euklidean group. An additional (natural) assumption determines a stress form $\beta : E(M,N) \longrightarrow \mathcal{A}^1_E(M,\mathbb{R}^n)$ with vanishing exact part such that

$$\alpha = \nabla\mathcal{H} + \beta$$

yields a symmetric tensor T (cf. [Sch]). This shows that the forms $\beta(j)$ indeed can have a physical significance even though they do not contribute to the force density which is entirely determined by the stress form $\nabla\mathcal{H}$.

7. The area- and volume-sensitive parts of a constitutive law

In this section and the following two sections we present some of the conveniences and advantages of the constitutive field $\mathcal{H}$. They are all in the context of a global description of the medium. In particular we will exhibit here the part of the virtual work which is caused by deforming the total volume and area at each configuration.

Let F be any constitutive law with constitutive field $\mathcal{H}$. First of all we split off $d\mathcal{V}$ from F, based on lemma 5.6: We take the component of $\nabla\mathcal{H}(j)$ along $\nabla\bar{j}$ with respect to $\mathfrak{g}$. This yields for each j

$$\mathcal{H}(j) = \mathbf{p}(j)\cdot\bar{j} + \mathcal{H}_1(j) \tag{7.1}$$

for a well defined $\mathbf{p}(j) \in \mathbb{R}$ and some $\mathcal{H}_1(j) \in C^\infty_j(M,N)$ for which $\nabla\mathcal{H}_1(j)$ is orthogonal to $\nabla\bar{j}$ with respect to $\mathfrak{g}$. Hence $F(j)$ decomposes into

$$F(j)(l) = \mathbf{p}(j)\cdot d\mathcal{V}(j)(l) + \int_M \nabla\mathcal{H}_1(j)\cdot\nabla l\,\mu(j) \qquad \forall l \in C^\infty_j(M,TN)\ . \tag{7.2}$$

$\mathbf{p}\cdot d\mathcal{V}$ is called the ***volume sensitive part*** of F. Due to (4.8) we have in particular

$$F(j)(\bar{j}) = \mathfrak{g}(j)(\nabla\mathcal{H}(j),\nabla\bar{j}) = \mathbf{p}(j)\cdot\|\nabla\bar{j}\|^2_{\mathfrak{g}} = \mathbf{p}(j)\cdot\dim M\cdot\mathcal{V}(j)\ . \tag{7.3}$$

Since both $\|\nabla(j)\bar{j}\|$ and $\bar{j}$ vary smoothly in j, the map $\mathbf{a} : E(M,N) \longrightarrow \mathbb{R}$ is smooth as well. The vector field $\mathcal{H}_{\mathbf{p}} : E(M,N) \longrightarrow C^\infty_E(M,TN)$ assigning to each j the map $\mathbf{a}(j)\cdot\bar{j}$ is called the ***volume sensitive part*** of $\mathcal{H}$. In fact it only depends on F and not on the particular constitutive field $\mathcal{H}$. We proceed accordingly to construct the ***area sensitive part***. By looking at (5.12), (7.1) and (7.3) we obtain the following splitting of constitutive fields.

Theorem 7.1

For each constitutive law F, any constitutive field $\mathcal{H}$ determines uniquely the smooth maps $\mathbf{p},\mathbf{a} : E(M,N) \longrightarrow \mathbb{R}$ given for each $j \in E(M,N)$ by

$$\begin{aligned} \mathbf{p}(j) &:= F(j)(\bar{j})/\|\nabla\bar{j}\|^2_{\mathfrak{g}} = F(j)(\bar{j})/\mathcal{V}(j)\dim M \\ \mathbf{a}(j) &:= F(j)(\mathcal{H}_{\mathcal{A}}(j))/\|\nabla\mathcal{H}_{\mathcal{A}}(j)\|^2_{\mathfrak{g}} = F(j)(\mathcal{H}_{\mathcal{A}}(j))/\mathcal{A}(j)(\dim M - 1) \end{aligned} \tag{7.4}$$

with

$$\|\nabla\bar{j}\|^2_{\mathcal{G}} = \mathcal{V}(j)\dim M \qquad \text{and} \qquad \|\mathcal{H}_{\mathcal{A}}(j)\|^2_{\mathcal{G}} = \mathcal{A}(j)(\dim M - 1) \tag{7.5}$$

and splits uniquely into

$$\mathcal{H}(j) = \mathbf{p}(j)\cdot\bar{j} + \mathbf{a}(j)\cdot\mathcal{H}_{\mathcal{A}}(j) + \mathcal{H}_2(j) \qquad \forall j \in E(M,N) \tag{7.6}$$

where $\nabla\mathcal{H}_2(j)$ is $\mathcal{G}$-orthogonal to the span of $\nabla\bar{j}$ and $\mathcal{H}(j)$. The volume and area sensitive parts of $\mathcal{H}$ and $\mathcal{H}_{\mathcal{A}}$ respectively as well as $\mathcal{H}_2$ vary smoothly in $j \in E(M,N)$.
Hence any constitutive law F with constitutive field $\mathcal{H}$ splits uniquely into

$$F = \mathbf{p}\cdot d\mathcal{V} + \mathbf{a}\cdot d\mathcal{A} + F_2 \tag{7.7}$$

with F_2 the one-form with $\nabla\mathcal{H}_2$ as its $\mathcal{G}$-density. •

Remark 7.2

(i) $\mathbf{p}(j)$ and $\mathbf{a}(j)$ are called the ***pressure*** in $j(M)$ and the ***capillarity*** on $j(\partial M)$, respectively, of the medium at the configuration j (cf. (8.22) for justification).

(ii) If $\partial M = \emptyset$ and $n = 1 + \dim M$ then due to (7.11) and (7.9) $\Delta(j)\cdot\bar{j} = H(j)N(j)$ implying via (7.10)

$$F(j)(H(j)\cdot N(j)) = \mathbf{p}(j)\cdot\|\Delta(j)\bar{j}\|^2_{\mathcal{G}} \qquad \forall j \in E(M,N)$$

where $N(j)$ is the positively oriented unit normal vector field and $H(j)$ denotes the trace of its Weingarten map (the non-normalized mean curvature of j). Moreover $\|\Delta(j)\bar{j}\|^2_{\mathcal{G}} := \int < \Delta(j)\bar{j}, \Delta(j)\bar{j} > \mu(j)$.

(iii) Concerning the generality of $\mathbf{a}(j)\cdot dj$ as an exact one-form on M the following may be pointed out: Given $j \in E(M, I\!R^n)$ and $g \in C^\infty(M, I\!R)$, then $g\cdot dj$ is an exact one-form on M iff $g = const.$ This is seen immediately by computing the exterior differential of $g\cdot dj$ which yields $dg(v)\cdot dj(w) = dg(w)\cdot dj(v)$ for any $v,w \in T_pM$ and any $p \in M$. Hence if $v = (\operatorname{grad} g)(p)$ and $m(j)(v,w) = 0$ then $w \neq 0$ implies $dg(v) = 0$. Thus $dg = 0$.

8. The Ricci-sensitive part of a constitutive law

Next let us study the influence of the Ricci curvature of $m(j)$ and $m(j_\partial)$ to the constitutive field $\mathcal{H}(j)$ for each $j \in E(M,N)$. This is in particular of interest in $\dim M = 3$, since in this case the Ricci tensor determines the Riemann curvature tensor. The metric $m(j)$ defines for each $j \in E(M,N)$ the Levi-Cività connection $\nabla(j)$. In turn $\nabla(j)$ yields the curvature

$$R(j)(X,Y)Z = \nabla(j)_X\nabla(j)_Y Z - \nabla(j)_Y\nabla(j)_X Z - \nabla(j)_{[X,Y]}Z \tag{8.1}$$

for all $X, Y, Z \in \Gamma(TM)$. Taking the trace with respect to X yields the ***Ricci-tensor*** $Ric(j)$ applied to Y, Z, i.e. $Ric(j)(Y, Z)$. There is a unique smooth bundle map $R(j)$ of TM such that

$$Ric(j)(Y,Z) = m(j)(R(j)Y,Z) \qquad \forall Y,Z \in \Gamma TM . \tag{8.2}$$

Hence $TjR(j) : TM \longrightarrow TN$ can be regarded as a stress form at the configuration $j \in E(M, N)$. Its pseudo exact part $\nabla r(j)$ is given as the solution of the Neumann problem

$$\nabla^\star TjR(j) = \Delta(j)r(j) \qquad \text{and} \qquad TjR(j)(\mathbf{n}) = \nabla_{\mathbf{n}} r(j) \qquad \forall j \in E(M,N) . \tag{8.3}$$

A routine computation shows for each $j \in E(M, N)$ an expression for the divergence of TjR, reading as

$$\nabla^\star TjR(j) = -\tfrac{1}{2}\, Tj \operatorname{grad}_j \lambda(j) - \tfrac{1}{2} \operatorname{tr} S(j)(R(j)\ldots,\ldots) , \tag{8.4}$$

where $\lambda(j) := \operatorname{tr} R(j)$ is the scalar curvature and $S(j)(R(j)\ldots,\ldots)$ is given for each $X, Y \in \Gamma(TM)$ by

$$S(j)(R(j)X,Y) = \big(\nabla(TjR(j)Y)\big)^\perp$$

and where tr means the contradiction.

Let us pause to look at the boundary condition a little closer. Based on the equation of Godazzi-Meinardi a straight forward calculation yields the following expression of $R(j)(\mathbf{n})$:

$$R(j)(\mathbf{n}) = \operatorname{grad}_{m(j_\partial)} H(j_\partial) - \operatorname{div}_{j|\partial M} W(j_\partial) \tag{8.5}$$

with $W(j_\partial)$ the Weingarten map of ∂M in M and $H(j_\partial)$ its trace.
Hence

$$TjR(j) = \nabla r(j) + \beta_r(j) \qquad \forall j \in E(M,N) \tag{8.6}$$

for some well defined $\beta_r(j) \in A^1(M, TN)$.

As we computed the volume sensitive part of $\mathcal{H}(j)$, we may split off from $\mathcal{H}(j)$ the component along $\Delta(j)r(j)$ to obtain

$$\mathcal{H}(j) = p_r(j) \cdot r(j) + \mathcal{H}_2(j) \qquad \forall j \in E(M,N) \tag{8.7}$$

with $p_r(j) \in \mathbb{R}$, a physical entity. Clearly we may exhibit the volume sensitive and area sensitive parts of $r(j)$ and get

$$r(j) = s(j) \cdot \bar{j} + b(j) \cdot \mathcal{H}_{\mathcal{A}}(j) + r_1(j) \qquad \forall j \in E(M,N) \tag{8.8}$$

with $s(j) \in \mathbb{R}$ and where $\nabla r_1(j)$ is $\mathcal{G}$-orthogonal both to $\nabla \bar{j}$ and $\nabla \mathcal{H}_{\mathcal{A}}(j)$. The pressure $\mathbf{p}(j)$ decomposes accordingly into

$$\mathbf{p}(j) = \mathbf{p}_1(j) + s(j) \cdot p_r(j) \qquad \forall j \in E(M,N) . \tag{8.9}$$

In analogy to $\mathbf{p}(j)$ we compute $s(j)$ to

$$s(j) = d\mathcal{V}(j)(r(j))/\|\nabla \bar{j}\|^2_{\mathcal{G}} \qquad \text{or} \qquad s(j) = d \ln \mathcal{V}(j)(r(j))/\dim M . \tag{8.10}$$

$s(j)$ is a geometrical constant, in fact it is a topological invariant if $\partial M = \emptyset$ and $\dim M = 2$ (cf. example below), in contrast to $p_r(j)$.

The following lemma shows how these various constitutive fields are related.

Lemma 8.1

In general $r(j)$ and $\beta_r(j)$ in (8.6) are both different from zero and $r(j) \neq \lambda \cdot \bar{j}$ with λ a constant. The $\mathcal{g}$-densities $\nabla \bar{j}, \nabla \mathcal{H}_{\mathcal{A}}(j)$ and $\nabla r_1(j)$ are in general pair wise linearly independent for $j \in E(M,N)$, in fact $\nabla r_1(j)$ is $\mathcal{g}$-orthogonal to both $\nabla \bar{j}$ and $\nabla \mathcal{H}_{\mathcal{A}}(j)$. •

Proof :

The linear independence of $\nabla \bar{j}$ and $\nabla \mathcal{H}_{\mathcal{A}}(j)$ is evident from (5.12) and (7.4). Turning to ∇r_1, let $\dim M = 2$ and $\partial M = \emptyset$. Moreover, let $N = \mathbb{R}^n$ be equipped with a scalar product $\langle \, , \rangle$. Then $dj\, R(j) = \frac{1}{2}\lambda(j) \cdot dj$ with $\lambda(j) := \operatorname{tr} R(j)$ the scalar curvature. Due to 7.2.(iii), the one-form $\beta_r(j) = 0$ only if $\lambda(j) = const$. On an ellipsoid with three different axis (j =inclusion), λ is not a constant. On the ellipsoid mentioned, both $r(j)$ and $s(j)$ are different from zero. Back to full generality, the rest of the lemma follows from the construction. □

What we have done with $Ric(j)$ we can repeat for $Ric(j_\partial)$, the Ricci tensor of $m(j_\partial)$, the metric on ∂M pulled back by $j_\partial := j|\partial M$. The map $r^\partial : \partial M \longrightarrow TN$ can be extended on all of M in an analogous fashion as $(\overline{j_\partial})$ was extended to $\mathcal{H}_{\mathcal{A}}(j)$ in equation (5.12) (the integrability conditions for this Višik problem are satisfied). Let us call this extension by $\mathcal{H}_{r^\partial}(j)$. Due to the equation of Gauss, $\mathcal{H}_{r^\partial}(j)$ and $r(j)$ are linearly independent in general.

Replacing $r(j)$ in (8.9) by $\mathcal{H}_{r^\partial}(j)$ yields accordingly the real numbers $s^\partial(j)$ and $b^\partial(j)$, the components of $\mathcal{H}_{r^\partial}$ along $\mathcal{H}_{\mathcal{A}}$ and $\bar{j}$, respectively. In fact we adopt the notion

$$b^\partial(j) := \int_M \nabla \bar{j} \cdot \nabla \mathcal{H}_{r^\partial}\, \mu(j) = \int_{\partial M} \langle \Delta(j_\partial) r^\partial, \bar{j} \rangle\, \mu(j_\partial) \,. \tag{8.11}$$

Replacing $r(j)$ in (8.8) by $\mathcal{H}_{r^\partial}(j)$ the real $p_r(j)$ turns into $p_r^\partial(j)$. The field $r_1^\partial(j)$ has a covariant derivative $\nabla r_1^\partial(j)$ which is $\mathcal{g}(j)$-orthogonal to both $\nabla \bar{j}$ and $\nabla \mathcal{H}_{\mathcal{A}}(j)$ again. Combining all this we have the following:

Theorem 8.2

A given constitutive field $\mathcal{H}$ uniquely splits into

$$\mathcal{H}(j) = \mathbf{p}(j) \cdot \bar{j} + \mathbf{a}(j)\mathcal{H}_{\mathcal{A}}(j) + p_r(j) \cdot r_1(j) + a_r(j) \cdot r_1^\partial(j) + \mathcal{H}_4(j) \qquad \forall j \in E(M,N) \tag{8.12}$$

where $\nabla \mathcal{H}_4(j)$ is $\mathcal{g}(j)$-orthogonal to the span of $\nabla \bar{j}$, $\nabla \mathcal{H}_{\mathcal{A}}(j)$, $\nabla r_1(j)$ and $\nabla_{r_1^\partial}$.

Moreover pressure and capillarity are influenced, for each $j \in E(M,N)$, by the $Ric(j)$ and $Ric(j_\partial)$ and the respective scalar curvatures in the following fashion:

$$\mathbf{p}(j) = \mathbf{p}_0(j) + s(j) \cdot p_r(j) + b^\partial(j) \cdot a_r(j) \qquad \text{and} \qquad \mathbf{a}(j) = \mathbf{a}_0(j) + s^\partial(j) \cdot a_r(j) + b(j) \cdot p_r(j) \tag{8.13}$$

with

$$
\begin{aligned}
s(j) &= d\ln\mathcal{V}(j)(r(j))/\dim M = \int_M \lambda(j)\,\mu(j)\,, \\
b(j) &= d\ln\mathcal{A}(j)(r(j))/(\dim M-1)\,, \\
s^\partial(j) &= d\ln\mathcal{A}(j)(r^\partial(j))/(\dim M-1) = \int_{\partial M}\lambda_\partial(j)\,\mu(j_\partial) \\
\text{and}\qquad b^\partial(j) &= d\ln\mathcal{V}(j)(\mathcal{H}_{r^\partial}(j))/\dim M\,.
\end{aligned}
\tag{8.14}
$$

If M and ∂M are Ricci-flat (e.g. if $j(M)\subset I\!R^n$ is flat torus), then $\mathbf{p}(j)=\mathbf{p}_0(j)$ and $\mathbf{a}(j)=\mathbf{a}_0(j)$ for any $j\in E(M,N)$. Denoting the virtual works of which the constitutive fields are r_1, r_1^∂ and $\mathcal{H}_4$ by $F_{r_1}, F_{r_1^\partial}$ and F_4, respectively, (8.13) yields

$$
F = \mathbf{p}\cdot d\mathcal{V} + \mathbf{a}\cdot d\mathcal{A} + p_r F_{r_1} + a_r F_{r_1^\partial} + F_4 \tag{8.15}
$$

with

$$
\mathbf{p},\ \mathbf{a},\ p_r,\ a_r : E(M,N)\longrightarrow I\!R
$$

as smooth maps, all physical constituents. •

A refinement of the above theorem is derived by splitting $Tj\,R(j)$ for each $j\in E(M,N)$ into

$$
Tj\,R(j) = \frac{s(j)}{\dim M}\cdot Tj + \frac{\lambda^0(j)}{\dim M}\cdot Tj + Tj\,R^0(j)\,,
$$

where $s(j) := \int\lambda(j)\mu(j)$ and $\lambda^0(j) := \lambda(j)-s(j)$ combined with the condition

$$
g(j)\left(\frac{\lambda(j)}{\dim M}Tj\,,\,Tj\,R^0(j)\right) = 0\,.
$$

Clearly $\mathrm{tr}\,R^0(j)=0$. Then the pseudo exact part $\nabla r(j)$ of $Tj R(j)$ splits accordingly into

$$
r(j) = s(j)\bar{j} + r_\lambda(j) + r_0(j)
$$

where $\nabla r_\lambda(j) + s(j)Tj$ and $\nabla r_0(j)$ are the pseudo exact parts of $\frac{\lambda^0}{\dim M}Tj$ and $Tj\,R^0(j)$, respectively. Proceeding accordingly for $T_{j_\partial}R_\partial(j_\partial)$ and solving the respective Višik problem yields

Lemma 8.3

For any $j\in E(M,N)$ the following splittings hold for any $j\in E(M,N)$:

$$
\begin{aligned}
r(j) &= s(j)\cdot\bar{j} + r_\lambda(j) + r_0(j) \\
\text{and}\qquad r^\partial(j) &= s^\partial(j)\cdot\mathcal{H}_{\mathcal{A}}(j) + r_\lambda^\partial(j) + r_0^\partial(j)\,.
\end{aligned}
$$

•

Remark 8.4

Equation (8.14) shows the part $s(j) \cdot p_r(j) + b^\partial(j) a_r(j)$ of the pressure $\mathbf{p}(j)$ that is used up to bend (Ricci-sensitively) the medium at the configuration j. For the capillarity the analogous statement holds as well.

Let us make an example to illustrate the above procedures: We assume $\dim \partial M = 2$ and $N = I\!R^3$ equipped with a fixed scalar product. Recalling $j|\partial M = j_\partial$, then $R(j) = 0$ while $R_\partial(j) \neq 0$. A a routine computation yields

$$dj_\partial R(j_\partial) = \frac{\lambda(j_\partial)}{2} dj_\partial \qquad \forall j \in E(M,N) \tag{8.16}$$

with $\lambda(j_\partial)$ the scalar curvature of $m(j_\partial)$. As one sees immediately from (8.4) the exact part is not the $\mathcal{G}$-orthogonal projection of $\frac{\lambda(j_\partial)}{2} \cdot dj_\partial$ onto $I\!R \cdot dj_\partial$. By the theorem of Gauss-Bonnet $s^\partial(j)$ is obtained as

$$s^\partial(j) = \tfrac{1}{2} \mathrm{d} \ln \mathcal{A}(j)(r^\partial(j)) = \int_{\partial M} \lambda(j_\partial)\, \mu(j_\partial) = 4\pi \chi_{\partial M} \tag{8.17}$$

with χ_M the Euler characteristic of ∂M. Hence $\mathbf{p}(j)$ and $\mathbf{a}(j)$ decompose for any $j \in E(M,N)$ into

$$\mathbf{p}(j) = \mathbf{p}_0(j) \quad \text{and} \quad \mathbf{a}(j) = \mathbf{a}_0(j) - 4\pi \chi_{\partial M} \cdot a_r(j) \tag{8.18}$$

since

$$\begin{aligned} b^\partial(j) &= \int_{\partial M} \langle j, \Delta(j_\partial) r_\partial \rangle\, \mu(j_\partial) = -\int_M \langle \mathrm{tr}\, S(j), \mathcal{H}_{r^\partial}(j) \rangle\, \mu(j) \\ &+ \int_{\partial M} \langle Tj(\mathbf{n}), r_\partial \rangle\, \mu(j_\partial) = 0 \end{aligned} \tag{8.19}$$

Corollary 8.5

Let $I\!R^n = N$ and $\langle\, , \rangle$ a scalar product, $\dim M = 3$. For any constitutive law F, pressure and capillarity given by the constitutive field $\mathcal{H}$ of F are

$$\mathbf{p} = \mathbf{p}_0 \quad \text{and} \quad \mathbf{a} = \mathbf{a}_0 - 2\pi \chi_{\partial M} a_r(j)\ . \tag{8.20}$$

In case $j(\partial M)$ is a flat torus $\mathbf{a} = \mathbf{a}_0$. •

Let us illustrate (8.13) somewhat in order to justify the terminology of $\mathbf{p}$ being the pressure and $\mathbf{a}$ being the capillarity: Suppose that for some $j' \in E(M, I\!R^n)$

$$\nabla \mathcal{H}' = 0 = \mathbf{p}(j') \cdot \nabla \overline{j}' + \mathbf{a}(j') \cdot \nabla \mathcal{H}_{\mathcal{A}}(j') + p_r(j) \cdot \nabla r_1(j') + p_r^\partial(j) \cdot r_1^\partial(j') + \nabla \mathcal{H}_4(j')\ .$$

This implies the equations

$$\mathbf{p}(j') \cdot \nabla \overline{j}' + \mathbf{a}(j') \cdot \nabla \mathcal{H}_{\mathcal{A}}(j') = 0\ ,\ p_r(j) \cdot \nabla r(j') + p_r^\partial(j) \cdot \nabla r_1^\partial(j') = 0 \ \text{ and } \ \nabla \mathcal{H}_4(j') = 0\ .$$

In particular we find

$$\mathbf{p}(j') \cdot \overline{j}' + \mathbf{a}(j') \cdot \mathcal{H}_{\mathcal{A}}(j') = q_1 \qquad \text{and} \qquad p_r(j') \cdot r(j') + p_r^{\partial}(j') \cdot r_1^{\partial}(j') = q_2 \tag{8.21}$$

with q_1 and q_2 are covariantly constant vector fields along j'. To understand these equations, let $N = I\!R^3$ with $\langle\, ,\, \rangle$ a scalar product and let $\dim M = 3$. Since $\Delta(j')\overline{j}' = 0$ and since $\Delta(j')(j'|\partial M) = H(j'|\partial M) \cdot \mathcal{N}(j)$, with $\mathcal{N}(j) := Tj_{\mathbf{n}}$ we conclude $q_1 = \tau \cdot \mathcal{N}(j)$ with $\tau \in C^\infty(M, I\!R)$. Hence $d\tau(X)\mathcal{N}(j') = -\tau dj'W(j'X)$ holding for all $X \in \Gamma(T\partial M)$, implying $q_1 = 0$. Therefore $H(j'|\partial M) = const.$ Since $j'|\partial M : \partial M \longrightarrow I\!R^3$ is an embedding with constant mean curvature it has to be a sphere (cf. [B,G]). Hence $q_2 = 0$. Thus we may state:

Lemma 8.6

If $(N\, , <\, , >)$ is an Euklidean space and $\dim M = 3$, equation (8.21) implies that $j'(\partial M) \subset I\!R^3$ is a round sphere. The relation between capillarity and pressure reads as

$$\mathbf{p}(j') \,+\, \mathbf{a}(j') \cdot H(j_\partial) \,=\, 0 \tag{8.22}$$

•

Remark 8.7

Formula (8.22) for the pressure, compared with those for a bubble with a thin boundary medium, shows that **a** is twice the capillarity of the boundary medium. The reason is that this boundary medium is in fact three dimensional and has two boundary surfaces, an inner and an outer one.

Finally let us express $\mathbf{p}_0$ and $\mathbf{a}_0$ by F. The equations (8.14) and (8.15) immediately yield

Proposition 8.8

For each $j \in E(M,N)$ the following formulas provide expressions for $\mathbf{p}_0(j)$ and $\mathbf{a}_0(j)$ in terms of special values of $F(j)$.

$$\mathbf{p}_0(j) \,=\, F(j)(\overline{j})/\dim M \,-\, s(j) \cdot F(j)(r(j))/\|r(j)\|_{\mathcal{G}}^2 \,-\, b^{\partial}(j) \cdot F(j)(\mathcal{H}_{r^{\partial}}(j))/\|\mathcal{H}_{r^{\partial}}(j)\|_{\mathcal{G}}^2$$

$$\mathbf{a}_0(j) \,=\, F(j)(\mathcal{H}_{\mathcal{A}}(j))/\dim M \,-\, s^{\partial}(j) F(j)(r^{\partial}(j))/\|r^{\partial}(j)\|^2 \,-\, b(j) \cdot F(j)(r(j))/\|r(j)\|_{\mathcal{G}}^2$$

both holding for all $j \in E(M,N)$. •

9. The Fourier expansion of the force densities in case of $\partial M = \emptyset$ and an Euklidean ambient space

Let $\partial M = \emptyset$ and let $N = \mathbb{R}^n$ equipped with a fixed scalar product $<,>$. One consequence of the characterization of a smoothly deformable medium in terms of a constitutive field $\mathcal{H}$ is that the force density resisting a virtual deformation is $\Delta(j)\mathcal{H}(j)$ at each configuration $j \in E(M, \mathbb{R}^n)$. (In case $\partial M = \emptyset$ we would have to restrict us to those $\mathcal{H}$ for which $\nabla_{\mathbf{n}}\mathcal{H} = 0$ on all of $E(M, \mathbb{R}^n)$.) This allows us to make use of the Fourier expansion of $\mathcal{H}$. This expansion will yield (under additional assumptions) a decomposition of the work into an exact part and a non exact part on a neighbourhood $\mathcal{W}$ of a fixed reference configuration $j_0 \in E(M, \mathbb{R}^n)$.

As it is well known $\Delta(j_0)$ admits a complete system $e_1, e_2, \ldots$ of smooth orthonormed eigenmaps with respective eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$. Since we use Δ at a fixed reference configuration j_0 we need to pull back $\mathcal{H}$ to j_0. This pull back $\widehat{\mathcal{H}}$ is defined by the solution of the following equation

$$\det f(j)\Delta(j)\mathcal{H}(j) = \Delta(j_0)\widehat{\mathcal{H}}(j) \qquad \forall j \in E(M, \mathbb{R}^n) \tag{9.1}$$

where $\mu(j) = \det f(j)\mu(j_0)$ and $m(j)(X,Y) = m(j_0)(f^2(j)X, Y)$ holding for each pair $X, Y \in \Gamma(TM)$.

We may represent $\widehat{\mathcal{H}}(j)$ for any $j \in E(M, \mathbb{R}^n)$ as a uniformly convergent series (cf. [G,H,L])

$$\widehat{\mathcal{H}}(j) = \sum_{i=1}^{\infty} \xi^i(j) e_i \,. \tag{9.2}$$

The reals $\xi^i(j)$ are the ***Fourier coefficients*** of $\widehat{\mathcal{H}}(j)$. As functions of j they are smooth. These coefficients and the eigenvalues of $\Delta(j_0)$ allow us to determine the work caused by a virtual distortion $l \in C^\infty(M, \mathbb{R}^n)$ in terms of the Fourier coefficients. The desired expression is derived as follows:

$$\begin{aligned} F(j)(l) &= \int \langle \Delta(j)\mathcal{H}(j), l\rangle\, \mu(j) = \int \langle \det f(j)\Delta(j)\mathcal{H}(j), l\rangle\, \mu(j_0) \\ &= \int \langle \widehat{\mathcal{H}}(j), \Delta(j_o) l\rangle\, \mu(j_0) = \sum_{i=1}^{\infty} \xi^i(j) \int \langle e_i, \Delta(j_0) l\rangle\, \mu(j_0) \\ &= \sum_{i=1}^{\infty} \xi^i(j)\lambda_i \int \langle e_i, l\rangle\, \mu(j_0) = \sum_{i=1}^{\infty} \lambda_i \xi^i(j)\zeta^i \end{aligned}$$

with $\zeta^1, \zeta^2, \ldots$ the Fourier coefficients of l formed with respect to $e_1, e_2, \ldots$ The Fourier expansion of $\Delta(j_0)\widehat{\mathcal{H}}(j) = \det f(j)\Delta(j)\mathcal{H}(j)$ and the force density $\Delta(j)\mathcal{H}(j)$ hence turn into

$$\Delta(j_0)\widehat{\mathcal{H}}(j) = \sum_{i=1}^{\infty} \lambda_i \xi^i(j) e_i \,. \tag{9.3}$$

and

$$\Delta(j)\mathcal{H}(j) = \sum_{i=1}^{\infty} \xi^i(j) \det f(j)^{-1} e_i \ . \tag{9.4}$$

The eigenvalues λ_i are monotonically growing to infinity, the Fourier coefficients have to fall off accordingly. This shows a characterization of a deformable medium ($\partial M = \emptyset$!) in the Euclidean case which supplements the one given in theorem 5.4.

Proposition 9.1

Let $\partial M = \emptyset$. Given a fixed configuration $j_0 \in E(M, \mathbb{R}^n)$. Any deformable medium without boundary can be characterized by a sequence $\{\xi_1, \xi_2, \dots\}$ of smooth maps $\xi_i : E(M, \mathbb{R}^n) \longrightarrow \mathbb{R}$ such that the series

$$\sum \lambda_i \xi^i(j) \det f(j)^{-1} e_i$$

converges uniformly on M and on $E(M, \mathbb{R}^n)$ with respect to the C^∞-topology (which is metrizable). The map $\mathcal{H}$ is given by 9.4 is then a constitutive map. •

Since measurements of the force densities are possible only up to a certain degree (depending on various physical grounds), say up to ε, we may find some positive integer $\nu(j)$ such that

$$\sum_{i=\nu(j)}^{\infty} \lambda^i \xi_i(j) < \varepsilon \ .$$

Thus only finitely many of the Fourier coefficients, namely $\xi^1(j), \xi^2(j), \dots, \xi^{\nu(j)}(j)$, are relevant for the $\Delta(j)\mathcal{H}(j)$. Hence the assumption

$$\widehat{\mathcal{H}}(j) = \sum_{i=1}^{\nu(j)} \xi^i(j) e_i \qquad \forall j \in E(M, \mathbb{R}^n) \tag{9.5}$$

is reasonable. Clearly $\nu(j)$ may vary with j. We call $\mathcal{H}$ ***finitely determined*** in a neighbourhood $\mathcal{W}$ of j_0 if almost all of its Fourier coefficients vanish on $\mathcal{W}$.

We call $\mathcal{H}$ ***ν-determined*** in $\mathcal{W}$ iff

$$\widehat{\mathcal{H}}(j) = \sum_{i=1}^{\nu} \xi^i(j) e_i \qquad \forall j \in \mathcal{W} \tag{9.6}$$

with ν a fixed positive integer. We will show next that under some additional assumptions on $F|\mathcal{W} \times C_0^\infty(M, \mathbb{R})$ a well determined exact part (depending on a boundary condition) can be split off, even though $\mathcal{W}$ is an infinite dimensional manifold.

Suppose now that $\mathcal{H}$ is ν-determined in some closed neighbourhood $\mathcal{W}$ of some $j_0 \in E(M,N)$ with smooth boundary $\partial\mathcal{W}$. We may therefore regard $\widehat{\mathcal{H}}$ as a map

$$\widehat{\mathcal{H}} : \mathcal{W} \longrightarrow \mathbb{R}^\nu \subset C^\infty(M, \mathbb{R}^n) ,$$

$$\mathbb{R}^\nu := \operatorname{span}(e_1, \ldots, e_\nu) .$$

To get in the technical realm of finite dimensional Hodge theory, we now require that the following splitting holds:

$$\mathcal{W} = Q \times Q^\perp$$

with Q being an orientable ν-dimensional smooth manifold with boundary for which

$$\bigcup_{j' \in Q^\perp} \partial(Q \times \{j'\}) = \partial\mathcal{W} .$$

$\mathcal{H}|\{j\} \times Q^\perp$ is supposed to be constant (the value depends on $j \in Q$). Q is moreover required to be an orientable manifold with (orientable) boundary ∂Q. Clearly, $\dim Q \leq \nu$. The metric $\mathcal{G}$ (cf. sec.2) on $\mathcal{W}$ yields for each $j' \in Q^\perp$ a Riemannian volume on $Q \times \{j'\}$ and a positively oriented unit normal $\mathbf{q}_{j'}$. Given some $j' \in Q^\perp$ we will split off from

$$F|(Q \times \{j'\}) : (Q \times \{j'\}) \times T(Q \times \{j'\}) \longrightarrow \mathbb{R} ,$$

called $F_{j'}$, its smooth exact part $\mathrm{d}\mathcal{U}_{j'}$ (via Hodge decomposition with Neumann boundary condition) to obtain

$$F_{j'} = \mathrm{d}\mathcal{U}_{j'} + \Psi_{j'} \tag{9.7}$$

with $\Psi_{j'}$ a smooth one-form on Q with vanishing exact part. Doing so for all $j' \in Q^\perp$ yields the desired decomposition on $\mathcal{W}$

$$F|\mathcal{W} = \mathrm{d}\mathcal{U} + \Psi , \tag{9.8}$$

since $\mathcal{U}_{j'}$ and $\Psi_{j'}$ will depend smoothly on $j' \in Q^\perp$. The map $\mathcal{U} : \mathcal{W} \longrightarrow \mathbb{R}$ will be smooth, too. This decomposition is obtained by splitting

$$\Delta(j_0)\widehat{\mathcal{H}} : \mathcal{W} \longrightarrow C^\infty(M, \mathbb{R}^n)$$

into a gradient (of the map $\mathcal{U}$) with respect to $\mathcal{G}$ and a divergence free vector field orthogonal to $\mathbf{q}$ on $\partial\overline{\mathcal{W}}$ where $\mathbf{q}$ is defined by $\mathbf{q}|Q \times \{j'\} = \mathbf{q}_{j'}$. In detail we proceed as follows:
$\mathcal{G}$ and the orientation of $Q \times \{j'\}$ define a Laplacian $\boldsymbol{\Delta}_{\mathcal{G}}^{j'}$. Let $\mathcal{H}_{j'} := \mathcal{H}|Q \times \{j'\}$. We may take the divergence $\mathrm{div}_{\mathcal{G}}\Delta(j_0)\widehat{\mathcal{H}}_{j'}$ on Q and solve the Neumann problem

$$\mathrm{div}_{\mathcal{G}}\big(\Delta(j_0)\widehat{\mathcal{H}}_{j'}(j)\big) = \boldsymbol{\Delta}_{\mathcal{G}}\mathcal{U}_{j'}(j) \qquad \text{and} \qquad \mathrm{d}\mathcal{U}_{j'}(j)(\mathbf{q}_{j'}) = F(j)\big(\mathbf{q}_{j'}(j)\big) \tag{9.9}$$

for all $j \in Q \times \{j'\}$. Doing so for each $j' \in Q^\perp$ the map $\Delta(j_0)\widehat{\mathcal{H}}(j)$ admits the splitting

$$\Delta(j_0)\widehat{\mathcal{H}}(j) = \big(\mathrm{Grad}_{\mathcal{G}}\mathcal{U}\big)(j) + \mathbf{Y}(j) \qquad \forall j \in \mathcal{W} \tag{9.10}$$

with $\mathrm{Grad}_{\mathcal{G}}$ the Gradient formed with respect to $\mathcal{G}$ and $\mathcal{G}(j)(\boldsymbol{Y}(j), \mathbf{q}(j)) = 0$ for all $j \in \partial\overline{W}$. Defining

$$\Psi(j)(l) := \int_M \langle \boldsymbol{Y}(j), l \rangle \, \mu(j) \qquad \forall j \in \mathcal{W} \tag{9.11}$$

provides us with the above splitting (9.9). Since the integration on M and the one on $Q \times \{j'\}$ interchange for each $j' \in Q^\perp$ we moreover have

$$(\mathrm{Grad}_{\mathcal{G}} \mathcal{U})(j) = \Delta(j_0)\widehat{\mathcal{H}}_{\mathcal{U}}(j) \qquad \text{and} \qquad \boldsymbol{Y}(j) = \Delta(j_0)\widehat{\mathcal{H}}_{\Psi}(j) \tag{9.12}$$

for well determined (smooth) maps $\widehat{\mathcal{H}}_{\mathcal{U}}, \widehat{\mathcal{H}}_{\Psi} : Q \longrightarrow C^\infty(M, \mathbb{R}^n)$. Solving on M the equations

$$\det f(j)^{-1} \Delta(j_0)\widehat{\mathcal{H}}_{\mathcal{U}}(j) = \Delta(j)\mathcal{H}_{\mathcal{U}}(j)$$

and

$$\det f(j)^{-1} \Delta(j_0)\widehat{\mathcal{H}}_{\Psi}(j) = \Delta(j)\mathcal{H}_{\Psi}(j) \qquad \forall j \in \mathcal{W}$$

and combining it with the theorem 7.3 we have shown the following splitting theorem:

Theorem 9.2

Let $\partial M = \emptyset$ and let F be ν-determined (ν a positive integer) in an closed neighbourhood $\mathcal{W}$ of j_0 in $E(M, \mathbb{R}^n)$ satisfying the assumptions

1) $\mathcal{W}$ is a smooth oriented manifold with boundary ∂W.

2) It splits into

$$\overline{\mathcal{W}} = Q \times Q^\perp$$

with $F|TQ^\perp$ not depending on $j' \in Q^\perp$ and Q being a ν-dimensional oriented manifold with boundary,

and

3) $Q \times \{j'\}$ is a ν-dimensional orientable manifold with boundary such that

$$\partial\mathcal{W} = \bigcup_{j' \in Q^\perp} \partial(Q \times \{j'\}) \,.$$

Then there is a unique splitting on $\mathcal{W}$

$$F = d\mathcal{U} + \Psi \tag{9.13}$$

with

$$\mathcal{U} : \mathcal{W} \longrightarrow \mathbb{R}$$

a smooth map and

$$\Psi : \mathcal{W} \times C^\infty(M, \mathbb{R}^n) \longrightarrow \mathbb{R}$$

a smooth one-form such that

$$F(j)(\mathbf{q}) = d\mathcal{U}(j)(\mathbf{q}) \qquad \text{and} \qquad \Psi(j)(\mathbf{q}) = 0 \,. \tag{9.14}$$

Here $\mathbf{q}$ is the positively oriented $\mathcal{G}$-unit normal vector field along $\partial\mathcal{W}$. The constitutive field $\mathcal{H}$ splits accordingly uniquely into

$$\mathcal{H} = \mathcal{H}_{\mathcal{U}} + \mathcal{H}_{\Psi} , \tag{9.15}$$

where all $\mathcal{H}(j), \mathcal{H}_{\mathcal{U}}(j)$ and $\mathcal{H}_{\Psi}(j)$ are for any $j \in \mathcal{W}$ $\mathcal{G}$-orthogonal to the constant maps in $C^\infty(M, I\!R^n)$. •

If Ψ in (9.13) satisfies $\Psi \wedge d\Psi$ along $Q \times \{j'\}$ for any $j' \in Q^\perp$ then there is an integrating factor (cf. [B,St]) $T : \mathcal{W} \longrightarrow I\!R$ such that for some $G : \mathcal{W} \longrightarrow I\!R$ the one-form on $E(M, N)$ writes as $\Psi = T dG$. This condition, however, is not satisfied in general.

A one-form ω on a finite dimensional manifold admits an integrating factor iff $\omega \wedge d\omega = 0$, i.e. iff ω is of constant rank one.

Corollary 9.3

Let $\partial M = \emptyset$. Under the suppositions of theorem 9.2 and the assumption that $\Psi_{j'}$ is of constant rank one for each $j' \in Q^\perp$. Then

$$F = d\mathcal{U} + T \cdot dG$$

with T and G both real valued maps on $\mathcal{W}$. •

Next let us investigate $d\mathcal{U}$ in (9.13) more closely. Associated with each e_i is the one-form F_i with $i = 1, 2, \ldots$ given for all $j \in E(M, I\!R^n)$ and for all $l \in C^\infty(M, I\!R^n)$ by

$$F_i(j)(l) := \lambda_i \int \langle e_i, l \rangle \, \mu(j) . \tag{9.16}$$

It is j-independent. Hence

$$\mathcal{U}_i : E(M, I\!R^n) \longrightarrow I\!R^n$$

defined by $\mathcal{U}_i(j) := \lambda_i \int_M \langle e_i(j), j \rangle \, \mu_0$ for all $j \in E(M, I\!R^n)$ satisfies

$$F_i(j)(l) = d\mathcal{U}_i(j)(l) \qquad \forall j \in E(M, I\!R^n), \forall l \in C^\infty(M, I\!R^n) ; \tag{9.17}$$

its gradient formed with respect to $\mathcal{G}(j)$ is

$$\mathrm{Grad}_{\mathcal{G}} \mathcal{U}_i(j) = \det f(j)^{-1} e_i \qquad \forall j \in E(M, I\!R^n) .$$

From this we immediately deduce with the help of theorem 7.1:

Lemma 9.4

Let $\partial M = \emptyset$. Any constitutive law F decomposes at each $j \in E(M, I\!R^n)$ uniquely into

$$F(j) = \sum_{i=1}^{\infty} \xi^i(j) \cdot d\mathcal{U}_i(j) , \tag{9.18}$$

where the coefficient functions $\xi^i : E(M, I\!R^n) \longrightarrow I\!R^n$ are smooth. Moreover

$$d\mathcal{U}_i(j)(j) = d\mathcal{V}(j)(\det f(j)^{-1} e_i) = \lambda_i \cdot \mathbf{c}_i(j) \qquad \forall j \in E(M, I\!R^n)$$

where $\mathbf{c}_i(j)$ is the i^{th} Fourier coefficient of j. •

Combining theorem 9.1 with the above lemma immediately yields the following:

Corollary 9.5

Let $\partial M = \emptyset$. If F is ν-determined in a neighbourhood $\mathcal{W}$ of $j_0 \in E(M, I\!R^n)$ and satisfies in addition the supposition of theorem 9.1, then the exact part of F is

$$d\mathcal{U}(j) = \sum_{i=1}^{\nu} \zeta^i(j) d\mathcal{U}_i(j) \qquad \forall j \in E(M, I\!R^n) , \tag{9.19}$$

and the non exact one

$$\Psi(j) = \sum_{i=1}^{\nu} \theta^i(j) d\mathcal{U}_i(j) \qquad \forall j \in E(M, I\!R^n)$$

where the smooth maps

$$\zeta^i : \mathcal{W} \longrightarrow I\!R \qquad \text{and} \qquad \theta^i : \mathcal{W} \longrightarrow I\!R^n \qquad i = 1, 2, \ldots, \nu$$

are such that $\zeta^i d\mathcal{U}_i$ is exact and that $\theta^i d\mathcal{U}_i$ is not exact for any i. Hence

$$\zeta^i(j) = \frac{1}{\lambda_i} d\mathcal{U}(j)(e_i) \qquad \text{and} \qquad \theta^i(j) = \frac{1}{\lambda_i} \Psi(j)(e_i)$$

have to hold for each $i = 1, \ldots, \nu$. •

We may look at $d\mathcal{U}_i$ as a constitutive law. The pressure $\mathbf{p}^i$ involved in $d\mathcal{U}_i$ is for each $i = 1, 2, \ldots$ determined by

$$\mathbf{p}^i(j) = d\mathcal{U}_i(j)/\mathcal{V}(j) \cdot \dim M = \lambda_i \mathbf{c}_i(j)/\mathcal{V}(j) \cdot \dim M = d \ln \mathcal{V}(j)\big(\det f(j)^{-1} e_i\big)/ \dim M \tag{9.20}$$

implying the following expression for the pressure of a constitutive law:

Proposition 9.6

Let $\partial M = \emptyset$. Given any constitutive law with (smooth) coefficient functions $\xi^1, \xi^2, \ldots : E(M, I\!R^n) \longrightarrow I\!R$, the pressure $\mathbf{p}$ determined by F is

$$\mathbf{p}(j) = \frac{1}{\mathcal{V}(j) \cdot \dim M} \cdot \sum_{i=1}^{\infty} \lambda_i \xi^i(j) \cdot \mathbf{c}_i(j) \qquad \forall j \in E(M, I\!R^n) . \tag{9.21}$$

The constitutive field at each $j \in E(M, I\!R^n)$ of $d\mathcal{U}_i$ with i fixed is $\det f(j)^{-1} e_i$ and therefore

$$\mathbf{p}^i(j) = \lambda_i \mathbf{c}_i(j)/\mathcal{A}(j) \cdot \dim M = \mathbf{p}_0^i(j) + s(j) \cdot p_r^i(j) \tag{9.22}$$

holds. Here $p_r^i(j)$ is as in (8.7) and hence is the i^{th} Fourier coefficient of $r(j)$. Hence $\mathbf{p}_0^i(j) \cdot \frac{1}{\lambda_i} \cdot \mathcal{V}(j) \cdot \dim M$ is the i^{th} Fourier coefficient of j if $(M, m(j))$ is Ricci-flat. •

10. The dynamics for boundary less media determined by a constitutive law

Let $\partial M = \emptyset$. As we have mentioned in the previous section, a constitutive law on $E(M,N)$ of a smoothly deformable medium is defined to be a smooth one-form $F : TE(M,N) \longrightarrow \mathbb{R}$ admitting a smooth constitutive vector field $\mathcal{H} \in \Gamma C_E^\infty(M, TN)$.
The ***dynamical form***

$$\mathcal{W}_F : C^\infty(M, T^2N) \longrightarrow \mathbb{R}$$

- the fundamental ingredient of our set up of a dynamics - is given by

$$\mathcal{W}_F(l)(k) := d\mathcal{E}_{kin}(l)(k) - \left(\pi_E^\star F\right)(l)(k) \tag{10.1}$$

for any $l \in C_E^\infty(M, TN)$ and for any $k \in C_E^\infty(M, T^2N)$.

The dynamics determined by F is given by the unique vector field $\mathcal{X}_F$ (if it exists at all) for which

$$\omega_{\mathcal{B}}(\mathcal{X}_F, \mathcal{X}) = \mathcal{W}_F(\mathcal{X}) \qquad \forall \mathcal{X} \in \Gamma T^2 E(M,N)\ . \tag{10.2}$$

The following theorem shows the existence of $\mathcal{X}_F$ and moreover expresses its simple form:

Theorem 10.1

Given a constitutive law F on $E(M,N)$ with constitutive field $\mathcal{H}$ then

$$\mathcal{X}_F(l) = \mathcal{X}_{\mathcal{B}}(l) + \frac{1}{\rho(\pi_N \circ l)} \cdot \left(\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l)\right)^{vert} \qquad \forall l \in C^\infty(M, TN) \tag{10.3}$$

where *vert* denotes the pointwise formed vertical lift of $\Delta(\pi_N \circ l)\mathcal{H}(\pi_N \circ l)$ on N determined by ∇. •

Proof :
If $\mathcal{X}_F$ exists, then again it is unique. Using (A.9) and (10.2) we verify (10.3) for any $l \in C^\infty(M,N)$ by the following calculation:

$$
\begin{aligned}
\omega_{\mathcal{B}}\big(\mathcal{X}_F(l),\mathcal{X}(l)\big) =& \int\limits_M \rho(\pi_N\circ l)\omega\big(\mathcal{X}_{\mathcal{B}}(l),\mathcal{X}(l)\big)\mu(\pi_N\circ l) \\
&+ \int\limits_M \omega\big((\Delta(\pi_N\circ l)\mathcal{H}(\pi_N\circ l))^{vert},\mathcal{X}(l)\big)\mu(\pi_N\circ l) \\
=& \mathrm{d}\mathcal{E}_{kin}(\pi_N\circ l)\big(\mathcal{X}(l)\big) \\
&- \int\limits_M \big\langle\Delta(\pi_N\circ l)\mathcal{H}(\pi_N\circ l), T\pi_N(\mathcal{X}(l))\big\rangle\mu(\pi_N\circ l) \\
&+ \int\limits_M \big\langle\mathcal{X}(l)^{vert}, T\pi_N\circ(\Delta(\pi_N\circ l)\mathcal{H}(\pi_N\circ l)^{vert})\big\rangle\mu(\pi_N\circ l).
\end{aligned} \tag{10.4}
$$

Since the last summand is zero we find for each $l\in C^\infty(M,TN)$

$$
\begin{aligned}
\omega_{\mathcal{B}}\big(\mathcal{X}_F(l),\mathcal{X}(l)\big) =& \mathrm{d}\mathcal{E}_{kin}(\pi_N\circ l)\big(\mathcal{X}(l)\big) \\
&- \int\limits_M \big\langle\Delta(\pi_N\circ l)\mathcal{H}(\pi_N\circ l), T\pi_N\circ\mathcal{X}(l)\big\rangle\mu(\pi_N\circ l) \\
=& \mathrm{d}\mathcal{E}_{kin}(\pi_N\circ l)\big(\mathcal{X}(l)\big) - F(\pi_N\circ l)\big(T\pi_N\circ\mathcal{X}(l)\big)\ ,
\end{aligned}
$$

establishing the claim. □

Definition 10.2

The equation of a motion $\sigma:(-\lambda,\lambda)\longrightarrow E(M,N)$ subjected to F is given by

$$
\ddot{\sigma}(t)=\mathcal{X}_F\left(\dot{\sigma}(t)\right) \qquad \forall t\in(-\lambda,\lambda)\ , \tag{10.5}
$$

combined with initial conditions.

We therefore have :

Theorem 10.3

The equation of a motion $\sigma:(-\lambda,\lambda)\longrightarrow E(M,N)$ subjected to a given constitutive law F with constitutive field $\mathcal{H}\in\Gamma(C_E(M,TN))$ and with the initial data $\sigma(0)=j\in E(M,N)$ as well as $\dot{\sigma}(0)=l_0\in C_j^\infty(M,TN)$ is given by

$$
\ddot{\sigma}(t)=\mathcal{X}_{\mathcal{B}}\left(\dot{\sigma}(t)\right)+\frac{1}{\rho\left(\sigma(t)\right)}\cdot\left(\Delta\left(\sigma(t)\right)\mathcal{H}\left(\sigma(t)\right)\right)^{vert} \qquad \forall t\in(-\lambda,\lambda) \tag{10.6}
$$

or equivalently (with ∇ the covariant derivative of $\mathcal{B}$) by

$$
\nabla_{\frac{d}{dt}}\dot{\sigma}(t)=\frac{1}{\rho(\sigma(t))}\Delta(\sigma(t))\mathcal{H}(\sigma(t))\ . \tag{10.7}
$$

The motion σ is free i.e a geodesic iff $F = 0$. •

Equation (10.6) coincides with the equation of a motion subjected to a constitutive law of [Bi 4]. There the equation was derived by d'Alembert's principle and not on a geometric basis as done here.

The above decomposition theorems and (8.2) yield immediately refinements of the equations of motions subjected to a constitutive law:

Theorem 10.4
Let F be a constitutive law with constitutive field $\mathcal{H}$. Any motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M, N)$ (with any initial condition) is subjected to F iff

$$\begin{aligned}\nabla_{\frac{d}{dt}} \dot{\sigma}(t) &= \frac{\mathbf{p}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\overline{\sigma}(t) \\ &+ \frac{p_r(\sigma(t))}{\rho(\sigma(t))} \cdot (\Delta(\sigma(t))r_1(\sigma(t)) \\ &+ \frac{1}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\mathcal{H}_4(\sigma(t)) \qquad \forall t \in (-\lambda, \lambda)\ .\end{aligned} \tag{10.8}$$

with $r_2(\sigma(t))$ as in (9.12). Moreover the following balance law

$$\begin{aligned}\tfrac{1}{2}\mathrm{d}\mathcal{E}_{kin}(\sigma(t))(\dot{\sigma}(t)) &= \mathbf{p}(\sigma(t)) \cdot \mathrm{d}\mathcal{V}_F(\sigma(t))\dot{\sigma}(t) + \mathbf{a}(\sigma(t))\mathrm{d}\mathcal{A}(\sigma(t))(\dot{\sigma}(t)) \\ &+ \pi_N^\star\Big(p_r(\sigma(t))F_{r_1}(\sigma(t)) + F_4(\sigma(t))\dot{\sigma}(t)\Big)\end{aligned} \tag{10.9}$$

holds true for all $t \in (-\lambda, \lambda)$. F_r and F_4 are the constitutive laws associated with r and $\mathcal{H}_4$ as in (8.13). •

Corollary 10.5
If the constitutive map of F is of the form

$$\mathcal{H}(j) = \mathbf{p}(j) \cdot \overline{j} \qquad \forall j \in E(M, N) \tag{10.10}$$

that is if $F(j) = \mathbf{p}(j) \cdot \mathrm{d}\mathcal{A}(j)$ then the motion σ, subjected to F, satisfies for any $t \in (-\lambda, \lambda)$

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{\mathbf{p}(t)}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\overline{\sigma}(t) \tag{10.11}$$

as well as the balance law

$$\frac{1}{2}\mathrm{d}\mathcal{E}_{kin}(\sigma(t))(\dot{\sigma}(t)) = \mathbf{p}(\sigma(t)) \cdot \mathrm{d}\mathcal{A}(\sigma(t))(\dot{\sigma}(t))\ . \tag{10.12}$$

If N is Euclidean we may let $\bar{j} = j$ for all $j \in E(M,N)$ and conclude

$$\ddot{\sigma}(t) = \frac{\mathbf{p}(\sigma(t))}{\rho(\sigma(t))} \cdot \Delta(\sigma(t))\sigma(t) \qquad \forall t \in (-\lambda, \lambda) . \tag{10.13}$$

In case M is of codimension one then (10.5) rewrites as

$$\nabla_{\frac{d}{dt}} \dot{\sigma}(t) = \frac{\mathbf{p}(\sigma(t))}{\rho(\sigma(t))} \cdot H(\sigma(t)) \cdot N(\sigma(t)) \qquad \forall t \in (-\lambda, \lambda) \tag{10.14}$$

where $N(\sigma(t))$ is the positively oriented unit normal of $j(M)$ in N. •

11. Symmetry groups

Let $\partial M = \emptyset$. Given any density map $\rho : E(M,N) \to C^\infty(M, \mathbf{R}^n)$, the metric $\mathcal{B}$ on $E(M,N)$ (cf. appendix) associated with ρ is invariant under $Diff^+M$, the group of all diffeomorphisms preserving the orientation of M. This is immediate from the solution of the continuity equation (2.5) and the transformation formula of the integral. Equation (A.9) moreover shows immediately the invariance of $\omega_{\mathcal{B}}$ under $Diff^+M$.

Let us suppose that we are given a constitutive law F being invariant under $Diff^+M$, meaning that $(R\psi)^\star F = F$ for all $\psi \in Diff^+M$, where $R\psi$ denotes the right translation by ψ on $E(M,N)$ (cf. sec. 2). More explicitly, $Diff^+M$ invariance of F means

$$F(j \circ \psi)(l \circ \psi) = F(j)(l) \qquad \forall j \in E(M,N) \text{ and } \forall \psi \in Diff^+M \tag{11.1}$$

The dynamical form $\mathcal{W}_F$ is invariant under $Diff^+M$. Differentiating (11.1) with respect to ψ yields for any $j \in E(M,N)$ and any $l \in C^\infty_j(M,TN)$ the equation

$$\nabla_{TjX}(F(j))(l) + F(j)(\nabla_X l) = 0 \qquad \forall X \in \Gamma TM. \tag{11.2}$$

Here $\Gamma(TM)$ is identified with $T_{id}Diff^+M$ and ∇ is the covariant derivative of $\mathcal{B}$ (cf. appendix). The symplectic formalism yields a smooth moment map

$$\mathbf{J} : C^\infty_E(M,TN) \longrightarrow \mathcal{B}^\flat(\Gamma(TM)) \tag{11.3}$$

given for any $l \in C^\infty_j(M,TN)$ and any $j \in E(M,N)$ by the equation

$$\mathbf{J}(l)(X) = \theta_{\mathcal{B}}(l)(TlX) = -\mathcal{B}(\pi \circ l)(l, T\pi_N \circ TlX) = -\mathcal{B}(j)(l, TjX) . \tag{11.4}$$

The relation between $\mathbf{J}$ and integrals of a motion is as follows:

Lemma 11.1

Let F be $Diff^+M$ invariant. For any $X \in \Gamma(TM)$ the map $\mathbf{J}_X : C^\infty_E(M,TN) \longrightarrow I\!R$ defined by

$$\mathbf{J}_X(l) := \mathbf{J}(l)(X) \qquad \forall l \in C^\infty_E(M,TN) \tag{11.5}$$

is constant on any motion subjected to F iff

$$F(j)(TjX) = 0 \qquad \forall j \in E(M,N) \text{ and } \forall X \in \Gamma TM \ .$$

If F is exact and the potential is $Diff^+M$ invariant, then $\mathbf{J}_X$ is a first integral of the motion for each $X \in \Gamma TM$. •

Proof :

Let us compute $d\mathbf{J}_X$: For any $k \in C^\infty_l(M,TN)$ we have

$$\begin{aligned} d\mathbf{J}_X(l)(k) &= d(\theta_{\mathcal{B}}(l)(T(\pi_N \circ l)X))(k) = -d\mathcal{B}(\pi_N \circ l)(l, T(\pi_N \circ l)X)(k) \\ &= -(\mathcal{B}(k^{vert}, T(\pi_N \circ l)X) + \mathcal{B}(l, T(\pi_N \circ k^{vert}X)) = -\mathcal{B}(k^{vert}, T(\pi_N \circ l)X) \ . \end{aligned} \tag{11.6}$$

Because of $\mathcal{X}_{\mathcal{B}}(l)^{vert} = 0$ the choice $k := \mathcal{X}_F(l)$ yields

$$\begin{aligned} dJ_X(l)(\mathcal{X}_F(l)) &= -\ \mathcal{B}(\mathcal{X}_F(l)^{vert}, T(\pi_N \circ l)X) - \mathcal{B}(\mathcal{X}_{\mathcal{B}}(l)^{vert}, T(\pi_N \circ l)X) \\ &= F(\pi_N \circ l)(T(\pi_N \circ l)X) \end{aligned} \tag{11.7}$$

for all $l \in C^\infty_E(M,TN)$. The validity of the assertion is now immediate. □

Next we consider a more general situation coming up rather frequently. If we have differentiable groups $\mathcal{D}$ and I together with the respective smooth representation

$$a : \mathcal{D} \longrightarrow Diff^+M \text{ and } b : I \longrightarrow \mathcal{J}$$

then both a and b yield moment maps. Following the same routine in the proof of Lemma 11.1 we derive the following:

Theorem 11.2

Let F be invariant under both $a(\mathcal{D})$ and $b(I)$. The respective moment maps of a and b yield first integrals of any motion subjected to F for each of the elements in the respective Lie - algebras if for any $j \in E(M,N)$

$$F(j)(Tj\dot{a}X) = 0 \qquad \forall X \in \Gamma(TM) \tag{11.8}$$

as well as

$$F(j)(\dot{b}(c) \cdot \bar{j}) = 0 \qquad \forall c \in T_{id}I \tag{11.9}$$

hold. Here $\dot{a}$ and $\dot{b}$ denote the representation of the respective Lie algebras determined by a and b. •

12. The restriction of a motion subjected to F to a fibre in the principal bundle $E(M,N)$

Again $\partial M = \emptyset$. Each fibre in the principal bundle $E(M,N)$ is of the form $i \circ Diff\ M$ with fixed $i \in E(M,N)$. In this section we impose on the motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M,N)$ subjected to F the constraint that

$$\sigma(t)(M) = i(M) \qquad \forall t \in (-\lambda, \lambda) .$$

To find the equation of such a motion σ we proceed analogous as in the previous sections: We let $\mathcal{B}^i$ be the metric on $i \circ Diff\ M$ obtained by restricting $\mathcal{B}$ to this fibre. This yields immediately the symplectic structure ω^i on $T(j \circ Diff\ M)$, the pullback of $\omega_{\mathcal{B}}$ by the tangent map of the inclusion map $i \circ Diff\ M \hookrightarrow E(M,N)$. Moreover let F^i be the pullback of F by the inclusion map mentioned.

Observing that any tangent vector to $i \circ \psi \in i \circ Diff\ M$ with $\psi \in Diff\ M$ is of the form $T(j \circ \psi)X$ for some $X \in \Gamma(TM)$, the one-form F^i is given by

$$F^i(i \circ \psi)\big(T(i \circ \psi)X\big) = \int_M < \Delta(i \circ \psi)\mathcal{H}(i \circ \psi), T(i \circ \psi)X > \mu(j) . \qquad (12.1)$$

There is a connection on $E(M,N)$ induced by the orthogonal projection of TN to $TiTM$: Given any $l \in C^\infty_{i \circ \psi}(M,TN)$ with $\psi \in Diff\ M$ we let the component $l^\top$ of l in $T_{i \circ \psi} i \circ Diff\ M$ be given by

$$l^\top(p) = T(i \circ \psi)X(l^\top, j) \in T(i \circ \psi)(TM) \qquad \forall p \in M , \qquad (12.2)$$

for a well defined vector field $X(l^\top, j) \in \Gamma(TM)$. Clearly the projection from $TE(M,N)$ to $T(i \circ Diff\ M)$ given by $\top$ is $Diff\ M$ invariant for each $i \in E(M,N)$.

Let $\mathcal{E}^i_{kin}$ denote the kinetic energy on $T(i \circ Diff^+ M)$ given by $\mathcal{B}^i$. Its Euler field on $Ti \circ Diff\ M$ is the spray $\mathcal{X}^i$ of $\mathcal{B}^i$. It is of the form

$$\mathcal{X}^i(T(i \circ g)X) = T^2(i \circ g)S^i(T(i \circ g))^{-1}T(i \circ g)X \qquad (12.3)$$

with S^i the spray of $m(i)$ on TM and $X \in \Gamma(TM)$. We need one more geometric notion to formulate our equations: Let ∇^i denote the covariant derivative of Levi - Cività of $\mathcal{B}^i$ on $i \circ Diff\ M$. Due to general principles in Riemannian geometry and the fact that $\bar{j}$ is normal to $TjTM$ we immediately find for any $j \in E(M,N)$ the following :

Theorem 12.1

The equation of motion $\sigma : (-\lambda, \lambda) \longrightarrow E(M,N)$, subjected on one hand to a given constitutive law F on $E(M;N)$ with constitutive field $\mathcal{H}$ and on the other to the constraint

$$\sigma(t)(M) = i(M) \qquad \forall t \in (-\lambda, \lambda) \qquad (12.4)$$

for a fixed $i \in E(M, \mathbb{R})$, read as

$$\nabla^{i}_{\frac{d}{dt}}\dot{\sigma}(t) = \left(\left(\Delta(\sigma(t))\mathcal{H}(\sigma(t))\right)^{vert}\right)^{\top} \tag{12.5}$$

for all $t \in (-\lambda, \lambda)$, with S_M the spray of $m(i)$ on TM. •

If moreover, we subject the motion to the further constraint, namely that σ maps into $i \circ Diff_{\mu(i)}M$ with $Diff_{\mu(i)}M$, the group of all $\mu(i)$ preserving diffeomorphisms of M, then we arrive at Euler's equation of a perfect fluid on $i(M)$ as in [E,M]), provided $\mathcal{H} = 0$.

Appendix: $\mathcal{B}$, its associated one- and two-forms and its spray

To find the Levi-Cività connection and the one- and two-forms associated with $\mathcal{B}$ we need to differentiate $\mathcal{B}$ which we regard as a map

$$\mathcal{B} : C^\infty_E(M,TN) \underset{\text{E(M,N)}}{\times} C^\infty_E(M,TN) \longrightarrow I\!R.$$

The domain is the fibred product of $C^\infty_E(M,TN)$ with itself over $E(M,N)$. Now let $l(t) \in C^\infty_E(M,TN)$, varying smoothly in $t \in I\!R$ and let $j(t) := \pi_N \circ l(t)$. Setting $j(0) = j, l(0) = l$ and $\dot{l}(0) = k$, then we verify:

$$\begin{aligned} \frac{d}{dt}\mathcal{B}(j(t))(l(t),l(t))|_{t=0} &= \int_M \rho(j)\frac{d}{dt} < l(t), l(t) > |_{t=0}\mu(j) \\ &= 2\mathcal{B}(j)(\nabla_{\frac{d}{dt}} l(t), l) \ = \ 2\mathcal{B}(j)(k^{vert}, l) \end{aligned} \tag{A.1}$$

where *vert* denotes the pointwise formed vertical component of k in T^2N (with respect to the connection given by $< \ , \ >$, cf [G,H,V]). It is regarded at each $p \in M$ as a tangent vector to $l(p) \in T_{\pi_N \circ l(p)}N$ and hence as an element of $T_{\pi_N \circ l(p)}N$. The covariant derivative

$$\nabla : \Gamma C^\infty(M,TN) \longrightarrow \Gamma C^\infty(M,TN)$$

is hence given by

$$\nabla_k \mathcal{L}(p) = \left(T_l\mathcal{L}(k(p))\right)^{vert} \qquad \forall l \in E(M,N) \text{ and } \forall p \in M \tag{A.3}$$

for any choices of $\mathcal{L}$ and $k \in \Gamma C^\infty(M,TN)$. It is metric and obviously torsion free. Here $T_l\mathcal{L}$ denotes the tangent map of $\mathcal{L}$ on $E(M,N)$ at l and *vert* means again the vertical component formed in T^2N. This type of connection is unique for $\mathcal{B}$, as easily seen by following the proof of the analogous statement for finite dimensional manifolds. The curvature of $\mathcal{B}$ is thus inherited from the one om N to the contract of the curvature of $\mathcal{G}$. Equation A.3 yields immediately(cf. [Bi 4]):

Lemma A.1

The covariant derivative ∇ given by (A.3) is the Levi-Cività connection of the metric $\mathcal{B}$. •

We equip the set

$$\mathcal{B}^{\flat}(TE(M,N)) := \{\mathcal{B}(j)(l,\ldots) \mid j \in E(M,N) \text{ and } l \in T_j E(M,N)\} \qquad (A.4)$$

with the C^∞-topology and obtain a Fréchet manifold, the geometric dual of $TE(M,N)$. It is a smooth vector bundle and we use it as a replacement of the cotangent bundle of $E(M,N)$. The one-form $\Theta_{\mathcal{B}}$ associated with $\mathcal{B}$ on this bundle is defined in analogy to the finite dimensional case: It is the pull back by

$$\begin{aligned} \mathcal{B}^{\flat} : TE(M,N) &\longrightarrow \mathcal{B}^{\flat}(TE(M,N)) \\ l &\mapsto \mathcal{B}(\pi_N \circ l)(l,\ldots) \end{aligned} \qquad (A.5)$$

of the canonical one-form on $\mathcal{B}^{\flat}(TE(M,N))$, i.e $\Theta_{\mathcal{B}}$ is given by

$$\begin{aligned} \Theta_{\mathcal{B}}(l)(k) &= -\mathcal{B}(j)\big(l, T\pi_E(k)\big) \\ &= -\mathcal{B}(j)\big(l, T\pi_N \circ k\big) \ . \end{aligned} \qquad (A.6)$$

Here $\pi_E : TE(M,N) \longrightarrow E(M,N)$ and $\pi_N : TN \longrightarrow N$ are the canonical projections. The two - form $\omega_{\mathcal{B}}$ associated with $\mathcal{B}$ is defined by

$$\omega_{\mathcal{B}} := \mathrm{d}\Theta_{\mathcal{B}} \qquad (A.7)$$

where d also denotes the exterior differential for forms on $TE(M,N)$. $\omega_{\mathcal{B}}$ applied for any $j \in E(M,N)$, for any $l \in C^\infty_j(M,N)$ to any two $k_1, k_2 \in C^\infty_l(M, T^2N)$ reads as

$$\begin{aligned} \omega_{\mathcal{B}}(l)(k_1,k_2) &= \mathcal{B}(j)\big(k_2^{vert}, T\pi_N \circ k_1\big) - \mathcal{B}(j)\big(k_1^{vert}, T\pi_N \circ k_2\big) \\ &= \int_M \rho(\pi_N \circ l)\omega^{\flat}(k_1,k_2)\mu(\pi_N \circ l) \end{aligned} \qquad (A.9)$$

where $\omega^{\flat}$ is the pullback of the canonical two - form on the cotangent bundle $T^\star N$ of N by the diffeomorphism

$$\begin{aligned} <\,,\,>^{\flat} : TN &\longrightarrow T^\star N \\ v &\mapsto < v, . > \ . \end{aligned}$$

Fundamental in our setup of a dynamics will be the notion of the spray $\mathcal{X}_{\mathcal{B}}$ of $\mathcal{B}$. It will govern the free motion. It is defined by

$$\omega(\mathcal{X}_{\mathcal{B}}, \mathcal{Y}) = \mathrm{d}\mathcal{E}_{kin}(\mathcal{Y}) \qquad \forall \mathcal{Y} \in \Gamma T^2 E(M,N) \qquad (A.10)$$

with

$$\mathcal{E}_{kin}(l) := \tfrac{1}{2}\mathcal{B}(l,l) \qquad \forall l \in C^\infty(M,TN) \ . \qquad (A.11)$$

Due to the continuity equation and the special form of $\omega_{\mathcal{B}}$ the spray of $\mathcal{B}$ takes the form

$$\mathcal{X}_{\mathcal{B}}(l) = S_N \circ l \qquad \forall l \in C^\infty(M,TN) \; . \tag{A.13}$$

A smooth curve

$$\sigma : (-\lambda,\lambda) \longrightarrow E(M,N) \quad \text{with} \quad \lambda \in I\!R^+$$

is called a ***geodesic*** iff

$$\mathcal{X}_{\mathcal{B}}\big(\dot\sigma(t)\big) = S_N \circ \dot\sigma(t) \qquad \forall t \in (-\lambda,\lambda) \; . \tag{A.14}$$

Since $\nabla_{\frac{d}{dt}}\dot\sigma(t) = T_{\frac{d}{dt}}\dot\sigma(t) - \mathcal{X}_{\mathcal{B}}(\dot\sigma(t))$, where $T_{\frac{d}{dt}}\dot\sigma(t)$ denotes the tangent map $T\dot\sigma : I\!R \times I\!R \longrightarrow TE(M,N)$ evaluated at $(t,1)$, equation (A.14) turns into

$$\nabla_{\frac{d}{dt}}\dot\sigma(t) = 0 \; . \tag{A.15}$$

In summation of the above we state:

Proposition A.2

A smooth curve $\sigma : (-\lambda,\lambda) \longrightarrow E(M,N)$ is a geodesic of $\mathcal{B}$ with the initial conditions

$$\sigma(0) = j \text{ and } \dot\sigma(0) = l.$$

iff

$$\begin{aligned} \sigma_p : (-\lambda,\lambda) &\longrightarrow N \\ t &\mapsto \sigma(t)(p) \end{aligned}$$

is a geodesic in N for any $p \in M$, satisfying the initial conditions $\sigma_p(0) = j(p)$ and $\dot\sigma_p(0) = l(p)$. •

The above proposition implies in particular, that the spray $\mathcal{X}_{\mathcal{B}}$ admits locally a unique flow on $C^\infty_E(M,TN)$.

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