# On the Chebyshev Norm of Polynomial B-Splines

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# On the Chebyshev Norm of Polynomial B-Splines

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**Abstract.** Polynomial B-splines of given order $m$ and with knots of arbitrary multiplicity are investigated with respect to their Chebyshev norm. We present a complete characterization of those B-splines with maximal and with minimal norm, compute these norms explicitly and study their behavior as $m$ tends to infinity.
Furthermore, the norm of the B-spline corresponding to the equidistant distribution of knots is studied.
Finally, we analyse those types of knot distributions, for which the norms of the corresponding B-splines converge to zero as $m \to \infty$.

**Proposed Running Head:** Chebyshev Norm of B-Splines.

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## 1. Introduction

Let be $m \in \mathbb{N}$, $m \geq 1$. Furthermore, for $k \in \mathbb{N}$, $k \leq m$, let us be given a set of knots $x_\varrho \in \mathbb{R}$, $\varrho = 0, 1, \ldots, k$, satisfying

$$x_0 < x_1 < \cdots < x_k .$$

To each knot $x_\varrho$ there is associated a natural number $\nu_\varrho$, called the *multiplicity* of $x_\varrho$ such that

$$\nu_0 + \nu_1 + \cdots + \nu_k = m + 1 .$$

We call a real function $f$ a (polynomial) *B-spline* of order $m$, belonging to the set of knots $\{x_\varrho\}$ with multiplicities $\nu_\varrho$, $\varrho = 0, 1, \ldots, k$, if it possesses the following properties:

1. It is

$$f(x) = 0 \quad \text{for all } x < x_0 \text{ and for all } x > x_k ,$$

2. The restriction of $f$ to the subinterval $[x_\varrho, x_{\varrho+1})$, $\varrho = 0, \ldots, k-2$, and to $[x_{k-1}, x_k]$ belongs to the space $\Pi_{m-1}$ of polynomials of degree at most $m-1$.
3. It is

$$f \in C^{m-1-\nu_\varrho}(U(x_\varrho))$$

for a suitable neighborhood $U(x_\varrho)$ of the knot $x_\varrho$, $\varrho = 0, 1, \ldots, k$,
4. It is

$$\int_{-\infty}^{+\infty} f(x) = \frac{1}{m} .$$

Of course, if $\nu_\varrho = m$ for some $\varrho$, property # 3. only means that $f$ need not even be continuous in $x_\varrho$; in all other cases we deal with a continuous function which implies that we could write down the second property for closed subintervals as well.

It is well-known (cf. [3,5,11]) that there exists one and only one such function $f$. We will denote this B-spline by $B_m$ or, in greater detail, by

$$B_m\left(x \,\middle|\, \begin{matrix} x_0 & x_1 & \cdots\cdots & x_k \\ \nu_0 & \nu_1 & \cdots\cdots & \nu_k \end{matrix}\right) .$$

Furthermore it is not difficult to prove (cf. [3,5,11]) that

$$B_m\left(x \,\middle|\, \begin{matrix} x_0 & x_1 & \cdots\cdots & x_k \\ \nu_0 & \nu_1 & \cdots\cdots & \nu_k \end{matrix}\right) > 0 \quad \text{for } x \in (x_0, x_k) .$$

Hence the supremum of $B_m$ on $[x_0, x_k]$ is identical with the value of the Chebyshev norm

$$\|B_m\| := \sup\{|B_m(x)| \mid x \in \mathbb{R}\} .$$

We now consider, for fixed order $m$, the set $\mathcal{B}_m$ of all B-splines of order $m$ with the normalization

$$x_0 = 0 \quad \text{and} \quad x_k = 1 .$$

In this paper we are interested in the numbers

$$\lambda_m := \sup \{ \|B_m\| \mid B_m \in \mathcal{B}_m \}$$

and

$$\mu_m := \inf \{ \|B_m\| \mid B_m \in \mathcal{B}_m \} .$$

We call a B-spline $B_m \in \mathcal{B}_m$ *maximal*, if its norm is equal to $\lambda_m$, and *minimal*, if it equals $\mu_m$. It will turn out that these numbers are really attained.

In section 4 we will compute these numbers explicitly and present all B-splines of $\mathcal{B}_m$ with norm $\lambda_m$ resp. with norm $\mu_m$. Likewise, in section 5 we will compute the norm of the B-spline with equidistant knots and study the behavior of all these norms as $m$ tends to infinity. The final section 6 is devoted to the question, for which types of knot distributions the norms of the corresponding B-splines tend to zero at all, as $m \to \infty$.

Before that, in the next sections we give some results on B-splines with a small number of knots, and a contour integral representation for B-splines and their derivatives, which will be our essential tool in proving the results.

## 2. B-splines for a small number of knots

In this section we derive, for all $m \in \mathbb{N}$ and $k = 1$ or $2$, explicit representations for the corresponding B-splines as well as for their norms; these results will later turn out to be important.

**Lemma 1:** *For $m \in \mathbb{N}$ we have, with $\nu_0 + \nu_1 = m + 1$,*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right) = \begin{cases} \binom{m-1}{\nu_0-1} x^{m-\nu_0}(1-x)^{m-\nu_1} & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{2.1}$$

*Furthermore*

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right) \right\| = \binom{m-1}{\nu_0 - 1} \frac{(m-\nu_0)^{m-\nu_0}(m-\nu_1)^{m-\nu_1}}{(m-1)^{m-1}} . \tag{2.2}$$

*We always adopt for some special cases the usual definition $0^0 := 1$.*

*Proof.* The representation formula (2.1) is easily verified by checking its properties with respect to the definition of $B_m$ (one could of course as well prove this formula by use of the B-spline recursion formula (see [2,3,11]), but this is not necessary here).

It remains to prove the validity of (2.2). For $\nu_0 = 1$, $\nu_1 = m$, the maximum value of $B_m$ (which in this case just reduces to the monomial $x^{m-1}$ in $[0,1]$) occurs at the point $1$ and is equal to $1$. The analogue is true in the case $\nu_0 = m$, $\nu_1 = 1$. For $\nu_0 > 1$, $\nu_1 > 1$, however, the maximum value of $B_m$ occurs in the open interval $(0,1)$ at the point

$$\xi = \frac{m-\nu_0}{m-1},$$

yieldig the value of the norm given in (2.2). □

**Lemma 2:** *Let $\alpha_m$ denote the minimal norm of all B-splines from $\mathcal{B}_m$ with no inner knots, i.e.*

$$\alpha_m := \min\left\{ \left\|B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right)\right\| \;\middle|\; \nu_0 + \nu_1 = m+1 \right\}.$$

*Then*

$$\alpha_m = \begin{cases} \dfrac{1}{2^{m-1}} \dbinom{m-1}{(m-1)/2} & \text{if } m \text{ is odd,} \\[2ex] \dfrac{1}{2^{m-1}} \dbinom{m-1}{m/2} \dfrac{m^{m/2}(m-2)^{(m-2)/2}}{(m-1)^{m-1}} & \text{if } m \text{ is even.} \end{cases} \tag{2.3}$$

*Furthermore, for $m \geq 2$ this sequence is strongly decreasing, i.e.*

$$\alpha_m > \alpha_{m+1} \qquad \text{for } m \in \mathbb{N}, m \geq 2\,. \tag{2.4}$$

*Proof.* It follows directly from Lemma 1 that

$$\left\|B_1\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_1 & \nu_2 \end{matrix}\right)\right\| = \left\|B_2\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_1 & \nu_2 \end{matrix}\right)\right\| = 1$$

for all possible choices of $\nu_1$ and $\nu_2$. Hence formulae (2.3) are true for $m = 1, 2$, so assume from now on $m \geq 3$.

Let $\nu_0$ increase from $1$ to the largest number less than or equal to $(m-1)/2$, and replace $\nu_1$ in (2.2) always by $m+1-\nu_0$. We investigate the question for which values of $\nu_0$ the expression

$$\Phi(\nu_0) := \left\|B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & m+1-\nu_0 \end{matrix}\right)\right\|$$

is minimal. Obviously $\Phi(1) = 1$. For $1 < \nu_0 \leq (m+1)/2$ we discuss the validity of the inequality

$$\Phi(\nu_0 + 1) < \Phi(\nu_0)\,. \tag{2.5}$$

Using the elementary expression given in (2.2), the inequality

$$\left(\frac{\nu_0}{\nu_0-1}\right)^{\nu_0-1} < \left(\frac{m-\nu_0}{m-\nu_0-1}\right)^{m-\nu_0-1} \tag{2.6}$$

is equivalent to (2.5). The function

$$g(t) := \left(\frac{t}{t-1}\right)^{t-1}, \quad t \in \mathbb{R}, t \geq 2,$$

is strictly increasing with $t$. Hence the inequality (2.6) is valid if and only if

$$\nu_0 < m - \nu_0, \quad \text{i.e.} \quad \nu_0 < m/2$$

holds. It follows that the minimal value in question is attained for $\nu_0 = \nu_1 = (m+1)/2$ if $m$ is odd, i.e. by the norm of the B-spline

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+1}{2} & \frac{m+1}{2} \end{matrix}\right).$$

If $m$ is even, then the minimal value is attained for $\nu_0 = m/2$, $\nu_1 = (m+2)/2$, respectively, by means of symmetry, for $\nu_0 = (m+2)/2$, $\nu_1 = m/2$, and no other cases, i.e. by the norms of the B-splines

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m}{2} & \frac{m+2}{2} \end{matrix}\right) \quad \text{resp.} \quad B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+2}{2} & \frac{m}{2} \end{matrix}\right).$$

The formulae (2.3) now follow immediately.

We still have to prove the inequality (2.4). It is

$$\alpha_2 = 1 > \alpha_3 = \frac{1}{2} > \alpha_4 = \frac{4}{9}.$$

For odd $m$, say $m = 2r+1$ with $r \geq 1$, we get from (2.3) at once

$$\alpha_m - \alpha_{m+1} = \frac{1}{2^{2r}}\binom{2r}{r}\left\{1 - \left(\frac{4r(r+1)}{4r(r+1)+1}\right)^r\right\} > 0.$$

For even $m$, say $m = 2r+2$ with $r \geq 1$, we get

$$\alpha_m - \alpha_{m+1} = \frac{1}{2^{2r+2}}\binom{2r+2}{r+1}\left\{\frac{2r+2}{2r+1}\left(\frac{4r(r+1)}{4r(r+1)+1}\right)^r - 1\right\} > 0,$$

since the inequality

$$\frac{2r+2}{2r+1}\left(\frac{4r(r+1)}{4r(r+1)+1}\right)^r > \frac{2r+2}{2r+1}\left(1 - \frac{r}{4r(r+1)+1}\right) = \frac{8r^3+14r^2+8r+2}{8r^3+12r^2+6r+1} > 1$$

holds. □

**Lemma 3:** *For $m \geq 2$ we have*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-1 & 1 \end{matrix}\right) = \begin{cases} \left(\frac{x}{x_1}\right)^{m-1} & \text{for } 0 \leq x < x_1, \\ \left(\frac{1-x}{1-x_1}\right)^{m-1} & \text{for } x_1 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

*Furthermore*

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-1 & 1 \end{matrix}\right) \right\| = 1 .$$

*Proof.* It is easy to check all properties of this specific B-spline. □

## 3. Contour integral representations of B-splines and their partial derivatives

We will give now some results concerning the representation of the B-spline $B_m$ and of its partial derivatives with respect to the knots in terms of a complex contour integral. These results will also be a major tool for the proof of our Theorem 2. For convenience, we first repeat the well-known contour integral representation of the B-spline itself:

**Lemma 4:** *Let, for $x \in \mathbb{R}$, $C_x$ denote a simply closed and rectifiable curve in the complex plane, such that all the knots $x_\varrho$, $\varrho = 0, \ldots, k$, with $x < x_\varrho$ and no others lie in the interior of that curve.*
*Then, carrying out the integration in the positive sense, we have the representation*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \frac{1}{2\pi i} \int_{C_x} \frac{(z-x)^{m-1}}{\omega(z)} \, dz , \tag{3.1}$$

*where*

$$\omega(z) := z^{\nu_0}(z - x_1)^{\nu_1} \cdots (z - x_{k-1})^{\nu_{k-1}}(z-1)^{\nu_k} . \tag{3.2}$$

*Proof.* This result was given in [8], see also [4]. □

In our subsequent considerations, representation (3.1) will mainly serve as a theoretical tool. However, it should be emphasized that this formula has also important practical implications, a fact which seems to have been underestimated until now, although (3.1) is known since twenty years. Therefore we would like to make a few remarks on this subject first:

**Corollary:** *Let be* $x \in [x_\varrho, x_{\varrho-1})$ *for some* $\varrho$, *and define for all* $\mu$

$$\omega_\mu(z) \;:=\; (z - x_\mu)^{-\nu_\mu} \cdot \omega(z)$$

*with* $\omega$ *from (3.2). Then the following representation holds:*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \sum_{\mu=0}^{\varrho-1} \frac{(-1)^m}{(\nu_\mu - 1)!} \cdot \frac{d^{\nu_\mu - 1}}{dz^{\nu_\mu - 1}} \left(\frac{(x-z)^{m-1}}{\omega_\mu(z)}\right)_{z=x_\mu} \tag{3.3}$$

*Proof.* According to the residue theorem, we obtain from (3.1)

$$\begin{aligned} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) &= \sum_{\mu=\varrho}^{k} \operatorname{Res}_{z=x_\mu} \left(\frac{(z-x)^{m-1}}{\omega(z)}\right) \\ &= -\sum_{\mu=0}^{\varrho-1} \operatorname{Res}_{z=x_\mu} \left(\frac{(z-x)^{m-1}}{\omega(z)}\right). \end{aligned}$$

Since

$$\operatorname{Res}_{z=x_\mu} \left(\frac{(z-x)^{m-1}}{\omega(z)}\right) = \frac{1}{(\nu_\mu - 1)!} \cdot \frac{d^{\nu_\mu - 1}}{dz^{\nu_\mu - 1}} \left(\frac{(z-x)^{m-1}}{\omega_\mu(z)}\right)_{z=x_\mu}$$

(see any textbook on complex analysis), the result follows. □

If we carry out the differentiation in (3.3) explicitly, we see that our B-spline $B_m$ is of the form

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \sum_{\mu=0}^{k} \sum_{j=1}^{\nu_\mu} \beta_{\mu j} (x - x_\mu)_+^{m-j} \tag{3.4}$$

with

$$\beta_{\mu\nu_\mu} \neq 0 \;\text{ for all }\; \mu\,, \tag{3.5}$$

which so far is a well-known result, see e.g. [11, Theorem 4.14]. But in contrast to the usual divided-difference approach, the calculation of the $\beta'_{\mu j}s$ via eqn. (3.3) is - for concrete cases - not very difficult to do. In addition, we have for all $\mu$

$$\beta_{\mu\nu_\mu} = \binom{m-1}{\nu_\mu - 1} \frac{(-1)^{m+\nu_\mu-1}}{\omega_\mu(x_\mu)}\,, \tag{3.6}$$

which sharpens assertion (3.5).

**Theorem 1:** *Let be $k \in \mathbb{N}$, $k \geq 2$. The multiplicities $\nu_\varrho$ of the knots $x_\varrho$, $\varrho = 1,2,\ldots,k-1$ may all satisfy $\nu_\varrho < m-1$.*
*Then the B-spline*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$

*possesses continuous partial derivatives with respect to $x$ and to all the knots $x_1, x_2, \ldots, x_{k-1}$ in the Cartesian product $(0,1) \times D$, where*

$$D := \{x_1, x_2, \ldots, x_{k-1} \mid 0 < x_1 < \cdots < x_{k-1} < 1\}.$$

*Furthermore we have the representations*

$$\frac{\partial}{\partial x} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = -\frac{m-1}{2\pi i} \int_{C_x} \frac{(z-x)^{m-2}}{\omega(z)}\, dz, \tag{3.7}$$

*and, for $\varrho = 1,2,\ldots,k-1$,*

$$\frac{\partial}{\partial x_\varrho} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \frac{\nu_\varrho}{2\pi i} \int_{C_x} \frac{(z-x)^{m-1}}{\omega(z)(z-x_\varrho)}\, dz. \tag{3.8}$$

*Remark.* Formula (3.8) should be compared with Theorem 4.27 in [11].

*Proof of Theorem 1.* The differentiability with repect to $x$ follows from the assumption $\nu_\varrho < m-1$, $\varrho = 1,\ldots,k-1$. Since $C_x$ is rectifiable, formula (3.7) is easily derived.

The right hand side of (3.8) possesses a denominator, in which the multiplicity of each knot is still less than $m$. Hence this right hand side is continuous in the Cartesian product $(0,1) \times D$. It obviously represents the partial derivative of $B_m$ with respect to the knot $x_\varrho$. □

As an immediate consequence of Theorem 1, we can now prove that the partial derivative of $B_m$ with respect to an inner knot $x_\varrho$ can be written in terms of the usual derivative of an $(m+1)^{th}$ order B-spline:

**Corollary:** *Under the assumptions of Theorem 1, the following relation holds:*

$$\frac{\partial}{\partial x_\varrho} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = $$
$$= -\frac{\nu_\varrho}{m} \cdot \frac{\partial}{\partial x} B_{m+1}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_\varrho & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_\varrho + 1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$

*Proof.* This follows easily from equations (3.7) and (3.8). □

## 4. B-splines with largest and smallest Chebyshev norm

In this section we want to compute explicitly the numbers $\lambda_m$ and $\mu_m$, defined in the introduction. Let us first consider the elementary cases $m = 1, 2, 3$, where the last one soon will turn out to be typical also for the general case.

For $m = 1$ and $m = 2$ it follows immediately from Lemma 1 resp. Lemma 3 that

$$\lambda_1 = \mu_1 = 1$$

and

$$\lambda_2 = \mu_2 = 1 .$$

Also the case $m = 3$ can still be treated in an elementary way. For $0 < x_1 < x_2 < 1$ we verify

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & x_1 & x_2 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}\right) = \begin{cases} \dfrac{x^2}{x_1 x_2} & \text{for } 0 \le x < x_1, \\[2mm] \dfrac{-x^2(1 + x_2 - x_1) + 2xx_2 - x_1 x_2}{(1 - x_1)x_2(x_2 - x_1)} & \text{for } x_1 \le x < x_2, \\[2mm] \dfrac{(1-x)^2}{(1-x_1)(1-x_2)} & \text{for } x_2 \le x \le 1, \\[2mm] 0 & \text{elsewhere.} \end{cases}$$

The maximum value of this B-spline is located at the point

$$\tau := \frac{x_2}{1 + x_2 - x_1} ,$$

and the norm turns out to be

$$\left\| B_3\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & x_2 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}\right) \right\| = \frac{1}{1 + x_2 - x_1} .$$

Hence

$$\lambda_3 \ge 1 \quad \text{and} \quad \mu_3 \le 1/2 .$$

The remaining cases of double and triple knots are easily analysed using Lemma 1 and Lemma 3. Here it is also possible to consider the limits for $x_1 \to 0$ or $x_2 \to 1$ or $x_2 \to x_1$ etc. It turns out that $\lambda_3 = 1$, where this value of the norm is only attained by the functions

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 1 & 3 \end{matrix}\right) , \quad B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 3 & 1 \end{matrix}\right)$$

and

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & 2 & 1 \end{matrix}\right) \quad \text{with } \; 0 < x_1 < 1 \, .$$

Analogously we find that $\mu_3 = 1/2$, where this value is attained only by the B-spline

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 2 & 2 \end{matrix}\right) = \begin{cases} 2x(1-x) & \text{for } 0 \le x \le 1 \, , \\ 0 & \text{elsewhere.} \end{cases}$$

We are now dealing with the case of an arbitrary order $m$.

**Theorem 2:** *Let $m$ be any natural number. Then the following assertions are true:*

*1. It is*

$$\lambda_m = 1$$

*and*

$$\mu_m = \alpha_m \, ,$$

*where $\alpha_m$ is given in Lemma 2.*

*2. The maximum value $\lambda_m$ of the norm is attained by the B-splines*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 1 & m \end{matrix}\right) \, , \quad B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ m & 1 \end{matrix}\right) \tag{4.1}$$

*and, if $m \ge 2$,*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-1 & 1 \end{matrix}\right) \quad \textit{with } \; 0 < x_1 < 1 \, , \tag{4.2}$$

*and by no others.*

*The minimum value $\mu_m$ of the norm is attained by the B-spline*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+1}{2} & \frac{m+1}{2} \end{matrix}\right) \, , \tag{4.3}$$

*if $m$ is odd, and by the two B-splines*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m}{2} & \frac{m+2}{2} \end{matrix}\right) \; \textit{and} \; B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+2}{2} & \frac{m}{2} \end{matrix}\right) \, , \tag{4.4}$$

*if $m$ is even, and by no others.*

We remark that the assertion $\lambda_m = 1$ can also be derived easily by using a special case of the Marsden identity (cf. [9]).

*Proof of Theorem 2.* Obviously, we only have to prove the second assertion, since the first one then follows easily by means of Lemma 1 and Lemma 3.
We proceed by induction and remark that the cases $m = 1, 2, 3$ have already been proved. Hence assume $m \geq 4$. Let be $k \in \mathbb{N}$, $k \geq 2$, and consider any fixed B-spline

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) .$$

We claim that under the assumptions of Theorem 1 the gradient vector

$$\operatorname{grad} B_m = \left\{ \frac{\partial B_m}{\partial x}, \frac{\partial B_m}{\partial x_1}, \ldots, \frac{\partial B_m}{\partial x_{k-1}} \right\} \tag{4.5}$$

never vanishes on the Cartesian product $(0,1) \times D$. (Here, if $k = 2$, we assume in addition $\nu_1 < m - 2$, and treat the remaining case $k = 2$, $\nu_1 = m - 2$ separately.) This assertion says that all these B-splines are neither maximal nor minimal. In order to prove it we assume to the contrary that there is a point

$$p := (\tau, y_1, \ldots, y_{k-1}) \in (0,1) \times D$$

such that simultaneously

$$\left(\frac{\partial B_m}{\partial x}\right)_p = 0 \quad \text{and} \quad \left(\frac{\partial B_m}{\partial x_\varrho}\right)_p = 0 \quad \text{for } \varrho = 1, 2, \ldots, k-1$$

hold. According to (3.7) and (3.8), the representations

$$\left(\frac{\partial B_m}{\partial x}\right)_p = -\frac{m-1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-1}}{\omega(z)(z-\tau)}\, dz$$

and, for $\varrho = 1, 2, \ldots, k-1$,

$$\left(\frac{\partial B_m}{\partial x_\varrho}\right)_p = \frac{\nu_\varrho}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-1}}{\omega(z)(z-y_\varrho)}\, dz$$

are valid. Here, of course

$$\omega(z) = z^{\nu_0}(z-y_1)^{\nu_1} \cdots (z-y_{k-1})^{\nu_{k-1}}(z-1)^{\nu_k} .$$

Our indirect assumption yields that, if $\tau$ does not coincide with one of the knots $y_1, \ldots, y_{k-1}$, the linear functional $L_1$, defined on the space of entire functions $h$ by

$$L_1 h := \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-2} h(z)}{\omega^*(z)}\, dz$$

vanishes for all $h \in \Pi_{k-1}$ . If $\tau$ coincides with one of these inner knots, then the linear functional $L_2$ , defined on the space of entire functions $h$ by

$$L_2 h \;:=\; \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-1} h(z)}{\omega^*(z)}\, dz$$

vanishes for all $h \in \Pi_{k-2}$ . Here we have defined

$$\omega^*(z) \;:=\; z^{\nu_0}(z-y_1)^{\nu_1+1} \cdots (z-y_{k-1})^{\nu_{k-1}+1}(z-1)^{\nu_k} .$$

In order to save certain differentiability properties we assume in the second case $k > 2$ ; the remaining case $k = 2$ , $\tau = y_1$ again will be treated separately.

Let us consider the first case now. We choose the entire function $h_1(z) := (z-\tau)^{k-1}$ ; then

$$\begin{aligned} L_1 h_1 &= \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m+k-3}}{\omega^*(z)}\, dz \\ &= \frac{-1}{m+k-2} \left( \frac{\partial}{\partial x} B_{m+k-1}\left(x \,\middle|\, \begin{matrix} 0 & y_1 & \cdots & y_{k-1} & 1 \\ \nu_0 & \nu_1+1 & \cdots & \nu_{k-1}+1 & \nu_k \end{matrix}\right)\right)_{x=\tau} = 0 . \end{aligned} \tag{4.6}$$

Since it is well-known (see [11, Thm. 4.57]) that the first derivative of this B-spline $B_{m+k-1}$ can have only one root in $(0,1)$, the number $\tau$ is uniquely determined by equation (4.6).
Next we choose $h_2(z) := (z-\tau)^{k-2}$ ; then

$$\begin{aligned} L_1 h_2 &= \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m+k-4}}{\omega^*(z)}\, dz \\ &= \frac{1}{(m+k-2)(m+k-3)} \left( \frac{\partial^2}{\partial x^2} B_{m+k-1}\left(x \,\middle|\, \begin{matrix} 0 & y_1 & \cdots & y_{k-1} & 1 \\ \nu_0 & \nu_1+1 & \cdots & \nu_{k-1}+1 & \nu_k \end{matrix}\right)\right)_{x=\tau} \\ &= 0 . \end{aligned}$$

But since the first derivative of our B-spline $B_{m+k-1}$ already vanishes at $x = \tau$ , this cannot hold also for the second derivative, see again [11]. Therefore our assumption leads to a contradiction, and the original assertion is proved in the first case.

In the second case, i.e. if $\tau$ coincides with one of the knots $y_1, \ldots, y_{k-1}$ , we choose $h_3(z) := (z-\tau)^{k-2} (= h_2(z))$ and $h_4(z) := (z-\tau)^{k-3}$ to obtain

$$L_2 h_3 \;=\; 0 \quad \text{and} \quad L_2 h_4 \;=\; 0 ,$$

which again, up to non-zero factors, can be interpreted as

$$\left( \frac{\partial}{\partial x} B_{m+k-1} \right)_{x=\tau} = 0 \quad \text{and} \quad \left( \frac{\partial^2}{\partial x^2} B_{m+k-1} \right)_{x=\tau} = 0$$

for the same B-spline as before. We get the same contradiction.

Now let us consider the case $k = 2$, $\tau = y_1$. Here we have

$$\frac{1}{2\pi i}\int_{C_\tau} \frac{(z-\tau)^{m-\nu_1-2}}{z^{\nu_0}(z-1)^{\nu_2}}\,dz \;=\; 0\,,$$

where $\nu_0 + \nu_2 = m - \nu_1$. Hence, having $\tau = y_1$ in mind again,

$$B_m\left(\tau \,\middle|\, \begin{matrix} 0 & y_1 & 1 \\ \nu_0 & \nu_1 & \nu_2 \end{matrix}\right) = \frac{1}{2\pi i}\int_{C_\tau} \frac{(z-\tau)^{m-\nu_1-1}}{z^{\nu_0}(z-1)^{\nu_2}}\,dz$$

$$= B_{m-\nu_1}\left(\tau \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_2 \end{matrix}\right)$$

and thus

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & y_1 & 1 \\ \nu_0 & \nu_1 & \nu_2 \end{matrix}\right)\right\| = \left\| B_{m-\nu_1}\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_2 \end{matrix}\right)\right\|$$

$$\geq \mu_{m-\nu_1} = \alpha_{m-\nu_1} > \alpha_m \geq \mu_m\,,$$

where we have used inequality (2.4) and the induction hypothesis.

We still have to show that the B-splines

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{matrix}\right) \quad \text{and} \quad B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 2 & m-2 & 1 \end{matrix}\right)$$

are neither maximal nor minimal. Due to symmetry reasons, we may restrict to the first one. It can easily be verified that

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{matrix}\right) = \begin{cases} \dfrac{x^{m-1}}{x_1^{m-2}} & \text{for } 0 \leq x < x_1, \\[2ex] \dfrac{(m-1)(1-x)^{m-2} - (1+\frac{m-2}{1-x_1})(1-x)^{m-1}}{(1-x_1)^{m-2}} & \text{for } x_1 \leq x \leq 1, \\[2ex] 0 & \text{elsewhere.} \end{cases}$$

The maximum value of $B_m$ occurs at the point $\tau$ with

$$1 - \tau \;=\; \frac{(m-2)(1-x_1)}{m-x_1-1}\,,$$

and it follows

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{matrix}\right)\right\| \;=\; \left(\frac{m-2}{m-x_1-1}\right)^{m-2}. \tag{4.7}$$

Obviously, this function is neither maximal nor minimal.

So, the B-splines which are either maximal or minimal have to be of the type (2.1) or (4.2); these cases have been discussed earlier. □

We are now in the position to characterize the asymptotic behaviour of the minimal norms $\mu_m$, as $m$ goes to infinity.

**Theorem 3:** *The sequence* $\{\mu_m\}$ *satisfies the asymptotic relation*

$$\mu_m = \sqrt{\frac{2}{\pi m}} \left(1 + \frac{1}{4m} + O(m^{-2})\right) \quad \textit{for } m \to \infty . \tag{4.8}$$

*Remark.* Relation (4.8) implies in particular that

$$\mu_m = O(m^{-1/2}) \quad \text{for } m \to \infty .$$

This has been conjectured for a long time (see [8]), but could not be proved until now.

*Proof of Theorem 3.* Assume first that $m$ is odd, say $m = 2k+1$ with $k \geq 1$; then, according to Lemma 2,

$$\mu_{2k+1} = \frac{1}{2^{2k}} \binom{2k}{k} . \tag{4.9}$$

This is nothing else but the famous Wallis product, which is known to possess the asymptotic expansion (see [1, # 6.1.49])

$$\frac{1}{2^{2k}} \binom{2k}{k} = \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right) \quad \text{for } k \to \infty . \tag{4.10}$$

Now we replace $k$ by $(m-1)/2$ in equation (4.10). This yields

$$\begin{aligned} \frac{1}{2^{m-1}} \binom{m-1}{(m-1)/2} &= \sqrt{\frac{2}{\pi(m-1)}} \left(1 - \frac{1}{4(m-1)} + O(m^{-2})\right) \\ &= \sqrt{\frac{2}{\pi m}} \left(1 - \frac{1}{m}\right)^{-1/2} \left(1 - \frac{1}{4m} + O(m^{-2})\right) \\ &= \sqrt{\frac{2}{\pi m}} \left(1 + \frac{1}{4m} + O(m^{-2})\right) \end{aligned}$$

for $m \to \infty$.

Now let $m$ be even, $m = 2k$; Lemma 2 implies

$$\begin{aligned} \mu_{2k} &= \frac{1}{2^{2k-1}} \binom{2k-1}{k} \frac{(2k)^k (2k-2)^{k-1}}{(2k-1)^{2k-1}} \\ &= \frac{1}{2^{2k}} \binom{2k}{k} \frac{(2k)^k (2k-2)^{k-1}}{(2k-1)^{2k-1}} \\ &= \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right) \frac{(1-\frac{1}{k})^{k-1}}{(1-\frac{1}{2k})^{2k-1}} \\ &= \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right) \left(1 + \frac{1}{4k} + O(k^{-2})\right) \\ &= \frac{1}{\sqrt{\pi k}} \left(1 + \frac{1}{8k} + O(k^{-2})\right) \quad \text{for } k \to \infty . \end{aligned} \tag{4.11}$$

Here we have used twice the asymptotic relation

$$\begin{aligned}(1-\frac{1}{k})^{k-1} &= \exp\Big((k-1)\log(1-\frac{1}{k})\Big)\\ &= \exp\Big(-1+\frac{1}{2k}+O(k^{-2})\Big)\\ &= \exp(-1)\cdot\Big(1+\frac{1}{2k}+O(k^{-2})\Big)\ .\end{aligned}$$

Putting $k = m/2$ in (4.11) yields the assertion. □

Obviously, more terms of the asymptotic expansion (4.8) can be worked out easily. The numerical values of the first ten numbers $\mu_m$ are given in Table I (see Section 5).

## 5. The equidistant distribution of knots

In many applications, e.g. in the context of Computer Aided Design by spline-curves and -surfaces, B-splines with equally spaced knots are of particular interest. It is therefore natural to ask for the behaviour of their norms; so, let

$$B^e_m(x) := B_m\left(x \,\middle|\, \begin{matrix} 0 & \frac{1}{m} & \cdots & \frac{m-1}{m} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{matrix}\right)$$

and

$$\beta_m := \|B^e_m\|\ .$$

Since $B^e_m(x) = B^e_m(1-x)$ for all $x$, the norm of this function is attained at $x = 1/2$. Hence

$$\beta_m = B^e_m\Big(\frac{1}{2}\Big) = \frac{1}{(m-1)!\,2^{m-1}} \sum_{\mu=0}^{[(m-1)/2]} (-1)^\mu \binom{m}{\mu} (m-2\mu)^{m-1}\ , \tag{5.1}$$

where we have used (3.4) and (3.6). In Table I, we list the first ten values of $\beta_m$ and compare them with the corresponding "optimal" values $\mu_m$ ; furthermore, we present the asymptotic limits (cf. Thms. 3 and 4).

For $m \to \infty$ , we obtain the following asymptotic result:

**Theorem 4:** *The sequence of norms of the equidistant B-splines satisfies the asymptotic relation*

$$\beta_m = \sqrt{\frac{6}{\pi m}}\left(1-\frac{3}{20m}+O(m^{-2})\right) \qquad \textit{for } m\to\infty\ . \tag{5.2}$$

| $m$ | $\mu_m$ | $\sqrt{\frac{2}{\pi m}}$ | $\beta_m$ | $\sqrt{\frac{6}{\pi m}}$ |
|---|---|---|---|---|
| 1 | 1.00000 | 0.79788 | 1.00000 | 1.38197 |
| 2 | 1.00000 | 0.56418 | 1.00000 | 0.97720 |
| 3 | 0.50000 | 0.46065 | 0.75000 | 0.79788 |
| 4 | 0.44444 | 0.39384 | 0.66666 | 0.69098 |
| 5 | 0.37500 | 0.35682 | 0.59895 | 0.61803 |
| 6 | 0.34560 | 0.32573 | 0.55000 | 0.56418 |
| 7 | 0.31250 | 0.30157 | 0.51102 | 0.52233 |
| 8 | 0.29375 | 0.28209 | 0.47936 | 0.48860 |
| 9 | 0.27343 | 0.26596 | 0.45292 | 0.46065 |
| 10 | 0.26018 | 0.25231 | 0.43041 | 0.43701 |

**Table I**

*Proof.* We use Schoenberg's integral representation for cardinal B-splines (cf. [11, Theorem 4.33]), which in our case takes the form

$$B_m^c(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^m e^{imt(2x-1)}\,dt\,,$$

and so

$$\beta_m = B_m^c\left(\frac{1}{2}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^m dt\,. \tag{5.3}$$

This integral is treated in several places in the literature (see [7] or [10, p. 94]), and there one can also find the asymptotic expansion (5.2). □

Interestingly, the values of $\beta_m$ (equidistant case) and $\mu_m$ (minimal case) both tend to zero with the same order of convergence, and, moreover, the asymptotic constants only differ by a factor $\sqrt{3}$. So, the equidistant knot distribution is, from this paper's point of view, a rather good choice.

## 6. For which B-splines does the norm tend to zero at all?

Let us be given, for $m \in I\!N$, an infinite triangular matrix M of knots $x_\mu^{(m)}, \mu = 0, 1, \ldots k_m$, which satisfy

$$0 = x_0^{(m)} < x_1^{(m)} < \cdots < x_{k_m-1}^{(m)} < x_{k_m}^{(m)} = 1\,,$$

where to every knot $x_\mu^{(m)}$ a multiplicity $\nu_\mu^{(m)}$ is prescribed.

We construct to each row of M the corresponding B-spline $B_m(x \mid \mathrm{M})$. We want to analyze the question, for which knot matrices M the sequence of norms

$$\{\|B_m(\cdot \mid \mathrm{M})\|\}_{m=1}^{\infty}$$

tends to zero at all. At first, one could think this is the case for "almost all" of them, i.e. for all B-splines except for the maximal ones given in section 4. But this is not true at all; for example, consider for arbitrary $x \in (0,1)$ and $m \geq 3$ the B-spline $B_m\left(x \,\middle|\, \begin{smallmatrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{smallmatrix}\right)$, whose norm was in (4.7) computed to be

$$\left(\frac{m-2}{m-x_1-1}\right)^{m-2} = \left(1-\frac{x_1-1}{m-2}\right)^{-(m-2)},$$

hence

$$\lim_{m\to\infty} \left\| B_m\left(\cdot \,\middle|\, \begin{smallmatrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{smallmatrix}\right)\right\| = e^{x_1-1} > 0 .$$

Note that $B_m$ is a $C^1$-function! Another counterexample is given by the $C^2$-function

$$B_m\left(x \,\middle|\, \begin{smallmatrix} 0 & \frac{1}{2} & 1 \\ 2 & m-3 & 2 \end{smallmatrix}\right) \qquad (m \geq 3),$$

whose norm equals $1/2$ for all $m$.

Having come so far, one hopes that at least all B-splines with simple knots, i.e. $k_m = m$ for all $m \in I\!N$ converge to zero. But this is wrong too, which can be seen from the following example.

*Example.* Let $m \geq 2$ and $0 < \varepsilon < 1$. We consider the function

$$B_m^*(x) := B_m\left(x \,\middle|\, \begin{smallmatrix} 0 & \xi_1 & \cdots & \xi_{m-1} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{smallmatrix}\right)$$

with the inner knots

$$\xi_\mu := \frac{1+\varepsilon^{2^{m-\mu}}}{2}, \quad \mu = 1, \ldots, m-1 .$$

For $x \in [0, \xi_1)$, $B_m^*$ takes according to (3.4), (3.6) the form

$$B_m^*(x) = \frac{(-1)^m}{\prod_{\mu=1}^{m-1}(-\xi_\mu)(-1)} \cdot x^{m-1} = \frac{(2x)^{m-1}}{\prod_{\mu=1}^{m-1}(1+\varepsilon^{2^\mu})} .$$

Since $\frac{1}{2} \in [0, \xi_1)$, it follows that for all $m$

$$\|B_m^*\| \geq B_m^*(\tfrac{1}{2}) = \frac{1}{\prod_{\mu=1}^{m-1}(1+\varepsilon^{2^\mu})} > \frac{1}{\prod_{\mu=1}^{\infty}(1+\varepsilon^{2^\mu})} = 1 - \varepsilon^2 > 0 .$$

However, we do not like to close this paper with a series of negative examples, and so we give the following sufficient conditions, under which the B-spline's norms tend to zero as $m$ goes to infinity. The first one, stated in Theorem 5, says that the linear convergence of the knot sequence $\{x_\mu^{(m)}\}$, as defined in (6.1), implies zero convergence of the norms.

**Theorem 5:** *Assume that there exist real constants* $0 < K_1 \leq K_2$, *such that for* $\mu = 1, \ldots, m$ *the relation*

$$\frac{K_1}{m} \leq x_\mu^{(m)} - x_{\mu-1}^{(m)} \leq \frac{K_2}{m} \tag{6.1}$$

*holds for all* $m \geq 3$. *Then*

$$\lim_{m\to\infty} \left\| B_m\left( \cdot \left| \begin{matrix} 0 & x_1^{(m)} & \cdots & x_{m-1}^{(m)} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{matrix} \right.\right) \right\| = 0 .$$

*Proof.* First assume that $m$ is even and let, for some $\mu$, $0 \leq \mu \leq m-1$, $\xi$ denote any point in the interval $[x_\mu^{(m)}, x_{\mu+1}^{(m)}]$. Then, using (3.4) and (3.6) again, we obtain

$$\begin{aligned} |B_m(\xi) - B_m(x_\mu^{(m)})| &= \frac{1}{|\omega_\mu(x_\mu^{(m)})|} \cdot (\xi - x_\mu^{(m)})^{m-1} \\ &\leq \frac{1}{|\omega_\mu(x_\mu^{(m)})|} \cdot (x_{\mu+1}^{(m)} - x_\mu^{(m)})^{m-1} \\ &\leq \left(\frac{m}{K_1}\right)^m \cdot \frac{1}{((\frac{m}{2})!)^2} \cdot \left(\frac{K_2}{m}\right)^{m-1} \\ &= \frac{K_2^{m-1}}{K_1^m} \cdot \frac{(2e)^m}{m^{m-1}} = o(m^{-1}) \text{ for } m \to \infty, \end{aligned} \tag{6.2}$$

where we have used that for all $r \in \mathbb{N}$

$$r! > \exp(-r) \cdot r^r$$

holds. Since there are precisely $m$ subintervals of this type, and due to $B_m(x_0^{(m)}) = 0$, inequality (6.2) already proves that

$$\lim_{m\to\infty} B_m(\xi) = 0 \quad \text{for all} \quad \xi \in [0,1],$$

hence the assertion. If $m$ is odd, the arguments are completely analogous. □

Our final Theorem 6 says that for any fixed $m$ the norm of a B-spline corresponding to a *symmetrical ordering* of simple knots *decreases*, if one moves these knots away from the center of the interval. Since we already know that the norm of the B-spline with equidistant knots tends to zero, Theorem 6 implies zero convergence for a rather big class of B-splines (see the corollary).

So, consider now two sets of simple knots

$$0 < \xi_1 < \cdots < \xi_{m-1} < 1 \quad \text{and} \quad 0 < \eta_1 < \cdots < \eta_{m-1} < 1,$$

such that for all $\mu$

$$\xi_\mu = 1 - \xi_{m-\mu} \quad \text{and} \quad \eta_\mu = 1 - \eta_{m-\mu}. \tag{6.3}$$

Let us define $r := [\frac{m-1}{2}]$, and denote the B-splines belonging to the above-defined knot sequences by

$$B_m(x \mid \xi) \quad \text{resp.} \quad B_m(x \mid \eta),$$

where $\xi = (\xi_1, \ldots, \xi_r)$ and $\eta = (\eta_1, \ldots, \eta_r)$. Note that for even $m$, $m = 2k$, we have

$$\xi_k = \frac{1}{2} \quad \text{and} \quad \eta_k = \frac{1}{2}.$$

**Theorem 6:** *Assume, for $m \geq 3$, that there are non-negative real numbers $\varepsilon_1, \ldots, \varepsilon_r$, such that the above-defined knots satisfy*

$$\eta_\mu = \xi_\mu - \varepsilon_\mu, \quad \mu = 1, \ldots, r$$

*(and therefore $\eta_{m-\mu} = \xi_{m-\mu} + \varepsilon_\mu$ for $\mu = 1, \ldots, r$). Then*

$$\|B_m(\cdot \mid \eta)\| \leq \|B_m(\cdot \mid \xi)\|. \tag{6.4}$$

*In addition, if at least one of the numbers $\varepsilon_\mu$ is positive, then strict inequality holds in (6.4).*

*Proof.* For a symmetric distribution of simple knots

$$0 < x_1 < x_2 < \ldots < \tfrac{1}{2} < \ldots < 1 - x_2 < 1 - x_1 < 1$$

let us consider the vector $\hat{x} = (x_1, \ldots, x_r)$ . For the corresponding B-spline we will write

$$B_m(x \mid \hat{x}) \, .$$

Using the representation (3.1) we get

$$\frac{\partial}{\partial x_\nu} B_m(x \mid \hat{x}) = \frac{2x_\nu - 1}{2\pi i} \int_{C_x} \frac{(z-x)^{m-1}}{\omega(z)(z - x_\nu)(z - 1 + x_\nu)} \, dz$$

for $\nu = 1, \ldots, r$ , where

$$\omega(z) = z(z - x_1) \cdots (z - 1 + x_1)(z - 1) \, .$$

Hence

$$\frac{\partial}{\partial x_\nu} B_m(x \mid \hat{x}) = \frac{2x_\nu - 1}{m(m+1)} \cdot \frac{\partial^2}{\partial x^2} B_{m+2}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_\nu & \cdots & 1-x_\nu & \cdots & 1-x_1 & 1 \\ 1 & 1 & \cdots & 2 & \cdots & 2 & \cdots & 1 & 1 \end{matrix}\right) . \tag{6.5}$$

In order to prove Theorem 6 we first remark that, due to the symmetry of the knots we have obviously

$$\|B_m(\cdot \mid \xi)\| = B_m(\tfrac{1}{2} \mid \xi) \quad \text{and} \quad \|B_m(\cdot \mid \eta)\| = B_m(\tfrac{1}{2} \mid \eta) \, .$$

The mean value theorem yields

$$B_m(\tfrac{1}{2} \mid \eta) - B_m(\tfrac{1}{2} \mid \xi) = -\left(\frac{\partial}{\partial x_1} B_m(\tfrac{1}{2} \mid \hat{x}), \ldots, \frac{\partial}{\partial x_r} B_m(\tfrac{1}{2} \mid \hat{x})\right) \cdot (\varepsilon_1, \ldots, \varepsilon_r)^T \tag{6.6}$$

with some vector $\hat{x} = (1-\tau)\xi + \tau\eta$ , $0 < \tau < 1$ .
We claim that the components of the gradient vector are all positive, i.e.

$$\frac{\partial}{\partial x_\nu} B_m(\tfrac{1}{2} \mid \hat{x}) > 0 \quad \text{for } \nu = 1, \ldots, r \, . \tag{6.7}$$

To prove (6.7) we use eqn. (6.5) for $x = \frac{1}{2}$ . Since the B-spline

$$B_{m+2}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_\nu & \cdots & 1-x_\nu & \cdots & 1-x_1 & 1 \\ 1 & 1 & \cdots & 2 & \cdots & 2 & \cdots & 1 & 1 \end{matrix}\right)$$

is invariant under the transform $x \to 1 - x$ , its only maximum value is attained at $x = \frac{1}{2}$ . The first derivative vanishes. The second derivative does not vanish at $x = \frac{1}{2}$ and is hence negative. Because of

$$2x_\nu - 1 < 0$$

we therefore get the desired inequality (6.7) from (6.5). Now equations (6.6) and (6.7) yield the assertion of Theorem 6. □

**Corollary:** *Let the (symmetrically ordered) knots of the B-spline* $B_m(x \mid \eta)$ *satisfy*

$$\eta_\mu \leq \frac{\mu}{m} \quad \textit{for } \mu = 1, \ldots, r \, .$$

*Then*

$$\mu_m \leq \left\| B_m(\cdot \mid \eta) \right\| \leq \beta_m \, ,$$

*i.e. there are two positive numbers* $c_1, c_2$ *, such that*

$$c_1 \, m^{-1/2} \leq \| B_m(\cdot \mid \eta) \| \leq c_2 \, m^{-1/2}$$

*holds.*

*Proof.* Follows directly from the combination of (4.8), (5.2) and Theorem 6. □

So we have finally seen that there is yet a quite big class of B-splines with zero convergence of the norms. For example, this is true for the well-known Perfect splines.

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