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Abstract

We give a necessary alternation condition for unique local best approximation from $S_{m,k}$, the set of splines of degree m with k free knots. This result is related to a conjecture of L.L. Schumaker. Moreover, we give a characterization of functions from the interior of the strong unicity set for $S^1_{m,k}$, the set of splines of degree m with k free simple knots, and show that this set is dense in the unicity set. Then we give a general characterization of suns for strong unicity and show that $S^1_{m,k}$ is a set of this type, although it is not a sun.

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Introduction

Best uniform approximation by $S_{m,k}$, the set of splines of degree m with k free knots, is a highly nonlinear problem. In this paper we investigate uniqueness problems in connection with the structure of $S_{m,k}$ and of general nonlinear families.

In Section 1 we give a necessary alternation condition for unique local best approximations from $S_{m,k}$ (Theorem 1.1). Schumaker [21] conjectured that for each function f in $C[a,b]$, there exists a best approximation s_f from $S_{m,k}$ such that $f - s_f$ has at least $m + 2k + 2$ alternating extreme points. Later in [22], he gave a counterexample. Our result shows that Schumaker's conjecture holds for the case when f has a unique (local) best approximation with k simple knots (Corollary 1.2).

In [15] we gave an alternation characterization of strongly unique best approximations in $S^1_{m,k}$, the set of splines of degree m with k free knots. By applying this result and Theorem 1.1 we show that the strong unicity set is a dense subset of the unicity set for $S^1_{m,k}$ (Theorem 1.5). Moreover, we give an alternation characterization of the functions from the interior of the strong unicity set for $S^1_{m,k}$ (Theorem 1.4).

In Section 2 we investigate the structure of $S^1_{m,k}$. The known results on the structure of $S_{m,k}$ and $S^1_{m,k}$ are negative. These sets are not suns; local and global best approximations are not the same; and best approximations cannot be characterized by alternation properties for the linear tangent space — in contrast e.g. to rational approximation (see e.g. Schumaker [21], Braess [3], [4]). We show that all these properties hold if we consider the class of strongly unique best approximations from $S^1_{m,k}$ ($m \geq 2$). In particular, $S^1_{m,k}$ is a sun for strong unicity, although it is not a sun. In this context we give a result on arbitrary nonlinear families: A nonlinear family is a sun for strong unicity if and only if local and global strongly unique best approximations coincide, respectively strongly unique best approximations can be characterized by alternation properties for the linear tangent space (Theorem 2.2). We finally note that the set $S^1_{m,k}$ which plays an important role in our investigations is a dense open subset of $S_{m,k}$. Some of the results in this paper were announced in [14].

1. Splines with Free Knots

The space of all r-times continously differentable real-valued functions on an interval $[a,b]$ is denoted by $C^r[a,b]$. Let points $a = x_0 < x_1 < \cdots < x_r < x_{r+1} = b$ and integers $m \geq 1$, $1 \leq m_i \leq m+1$, $i = 1,\ldots,r$, be given. We denote by$S_m\binom{x_1,\ldots,x_k}{m_1,\ldots,m_r}$ the space of **polynomial splines** of degree m with r fixed knots $x_1,\ldots,x_r$ of multiplicities $m_1,\ldots,m_r$, and by $S_{m,k}$ the set of **polynomial splines** of degree m with k **free (multiple) knots**, where $k \geq 1$ (see e.g. Braess [4], Nürnberger [16], Schumaker [23]). Here we use the convention that a spline has a knot of multiplicity $m+1$ if for this spline no continuity is required at the knot. We investigate best approximation in the uniform norm $\|h\| = \sup_{t\in[a,b]} |h(t)|$ $(h \in C[a,b])$. A function $s_f \in S_{m,k}$ is called **unique local best approximation** of $f \in C[a,b]$ from $S_{m,k}$ if there exists an $\varepsilon > 0$ such that for all $s \in U(s_f,\varepsilon) = \{s \in S_{m,k} \mid \|s - s_f\| \leq \varepsilon\}$, $s \neq s_f$, we have

$$\|f - s\| > \|f - s_f\|\,.$$

We call points $a \leq t_1 < \cdots < t_p \leq b$ alternating extreme points of a function $h \in C[a,b]$ if there exists a sign $\sigma \in \{-1,1\}$ such that

$$\sigma(-1)^i h(t_i) = \|h\| \quad,\ i = 1,\ldots,p\,.$$

The number of alternating points of h on a subinterval $[c,d]$ of $[a,b]$ is denoted by $A(h)|_{[c,d]}$.

First, we give a necessary alternation condition for unique local best approximations from $S_{m,k}$. We note that tangent methods which are applied in the description of best and strongly unique best approximations do not work in this case (see e.g. Braess [4], Mulansky [11], Nürnberger [15]).

Theorem 1.1 *Let a function $f \in C[a,b]$ and a spline $s_f \in S_m\binom{x_1,\ldots,x_r}{m_1,\ldots,m_r} \cap C^1[a,b] \setminus S_{m,k-1}$ be given, where $k = \sum_{i=1}^{r} m_i$. Moreover, let $a < y_1 \leq \cdots \leq y_{r+k} < b$ be the knots of s_f counting each knot x_i (m_i+1)-times, $i = 1,\ldots,r$, and let $y_{-m} < \cdots < y_{-1} < y_0 = a$, $b = y_{r+k+1} < y_{r+k+2} < \cdots < y_{r+k+m+1}$ be arbitrary points. If s_f is a unique local best approximation of f from $S_{m,k}$, then for every interval $[y_i, y_{i+m+j}] \subset [y_{-m}, y_{r+k+m+1}]$, $j \geq 1$, we have*

$$A(f - s_f)|_{[y_i,y_{i+m+j}]} \geq j + 1\,. \tag{1}$$

Proof: Suppose that there exists an interval $[y_i, y_{i+m+j}]$ such that

$$A(f - s_f)|_{[y_i,y_{i+m+j}]} = d \leq j\,. \tag{2}$$

We will show that there exists a sequence $(s_n) \subset S_{m,k}$ such that $s_n \neq s_f$ for all n, $s_n \to s_f$ and $\|f - s_n\| \leq \|f - s_f\|$, i.e. s_f is not a unique local best approximation of f from $S_{m,k}$. This is done as follows. There exist unique sets $T_1, \ldots, T_d$ such that

$$E(f - s_f) \cap [y_i, y_{i+m+j}] = \bigcup_{\mu=1}^{d} T_\mu$$

and for all $\mu \in \{1, \ldots, d-1\}$, $\widetilde{t}_\mu \in T_\mu$ and $\widetilde{t}_{\mu+1} \in T_{\mu+1}$, we have $\widetilde{t}_\mu < \widetilde{t}_{\mu+1}$ and

$$\left(f(\widetilde{t}_\mu) - s_f(\widetilde{t}_\mu)\right) \cdot \left(f(\widetilde{t}_{\mu+1}) - s_f(\widetilde{t}_{\mu+1})\right) < 0 .$$

We choose arbitrary points $t_1 < \cdots < t_{d-1}$ such that for all $\mu \in \{1, \ldots, d-1\}$, $\widetilde{t}_\mu \in T_\mu$ and $\widetilde{t}_{\mu+1} \in T_{\mu+1}$, we have $\widetilde{t}_\mu < t_\mu < \widetilde{t}_{\mu+1}$. First, we show that we may assume that

$$\left.\begin{array}{l} j \geq 2\,, \quad A(f - s_f)|_{[y_i, y_{i+m+j}]} = j \text{ and every interval} \\ (y_\mu, y_{\mu+m+\nu}) \subsetneqq (y_i, y_{i+m+j})\,, \quad \nu \geq 1\,, \\ \text{contains at least } \nu \text{ points from } \{t_1, \ldots, t_{j-1}\}\,. \end{array}\right\} \qquad (3)$$

We only consider the case $j \geq 2$, since the case $j = 1$ is similar and even simpler. Now, if there exists an interval $(y_\mu, y_{\mu+m+\nu})$ as in (3) with less than ν points from $\{t_1, \ldots, t_{d-1}\}$, then by definition of the sets T_μ and the points t_μ we have

$$A(f - s_f)|_{[y_\mu, y_{\mu+m+\nu}]} \leq \nu\,.$$

Then we consider $[y_\mu, y_{\mu+m+\nu}]$ instead of $[y_i, y_{i+m+j}]$ and go to further subintervals, if necessary. This process ends after finitely many steps. Therefore, we may assume that (3) holds, except that we may have

$$A(f - s_f)|_{[y_i, y_{i+m+j}]} \leq j\,,$$

i.e. $d \leq j$. But, since $(y_i, y_{i+m+j-1})$ contains at least $j - 1$ points from $\{t_1, \ldots, t_{d-1}\}$, it follows that $d \geq j$ which implies $d = j$. This shows that we may assume that (3) holds.

Next, we set

$$S = \left\{ s \in S_m \begin{pmatrix} x_1, & \ldots & , x_r \\ m_1+1, & \ldots, & m_r+1 \end{pmatrix} \mid s = 0 \text{ on } [a,b] \backslash (y_i, y_{i+m+j}) \right\}.$$

Moreover, we define

$$K_m(z,t) = (t - z)_+^m \quad , \quad (z,t) \in [a,b] \times [a,b]$$

and denote by $K_m[z_1, \ldots, z_{l+1}, t]$ the divided difference of order l of the function $z \to K_m(z,t)$ with respect to the points $z_1, \ldots, z_{l+1}$. It is well known that

$$S = \text{span}\{B_i^m, \ldots, B_{i+j-1}^m\},$$

where for all $\mu \in \{i, \ldots, i+j-1\}$,

$$B_\mu^m(t) = (-1)^{m+1} K_m[y_\mu, \ldots, y_{\mu+m+1}, t]$$

is the B-spline of degree m associated with the knots $y_\mu \leq \cdots \leq y_{\mu+m+1}$, and that S is a j-dimensional weak Chebyshev subspace of $C[a,b]$ (see e.g. Nürnberger [16], Schumaker [23]). Now, it follows from Jones & Karlovitz [9] (see also Deutsch, Nürnberger & Singer [6]) that there exists a nontrivial function $s = \sum_{\mu=i}^{i+j-1} a_\mu B_\mu^m \in S$ such that

$$(-1)^{\mu+1} s(t) \geq 0 \quad , \quad t \in [t_{\mu-1}, t_\mu] \cap [a,b] \,, \quad \mu = 1, \ldots, j \,, \tag{4}$$

where $t_0 = y_i$ and $t_j = y_{i+m+j}$. Obviously, we have

$$s(t_\mu) = 0 \,, \quad \mu = 1, \ldots, j-1 \,.$$

Let any point $t \in (y_i, y_{i+m+j}) \backslash \{t_1, \ldots, t_{j-1}\}$ be given. It follows from (3) that the points $t_1, \ldots, t_{j-1}, t$ satisfy the Schoenberg-Whitney-condition with respect to the basis $B_i^m, \ldots, B_{i+j-1}^m$, i.e. the determinant $D\begin{pmatrix} B_i^m, \ldots, B_{i+j-1}^m \\ t_1, \ldots, t_{j-1}, t \end{pmatrix} \neq 0$ (see e.g. Nürnberger [16]). Since $s \neq 0$, it follows that $s(t) \neq 0$. Therefore, it follows from (4) that

$$(-1)^{\mu+1} s(t) > 0 \quad , \quad t \in (t_{\mu-1}, t_\mu) \cap [a,b] \,, \quad \mu = 1, \ldots, j \,, \tag{5}$$

i.e. s changes sign at the points $t_1, \ldots, t_{j-1}$.

We now perturb the points from $\{y_i, \ldots, y_{i+m+j}\} \cap (a,b)$ and denote the resulting points by $y_{i,\varepsilon}, \ldots, y_{i+m+j,\varepsilon}$. This is done as follows. Let

$$\{x_p, \ldots, x_{p+q}\} = \{y_i, \ldots, y_{i+m+j}\} \cap (a,b)$$

such that $x_p < \cdots < x_{p+q}$. Let $\varepsilon = (\varepsilon_p, \ldots, \varepsilon_{p+q})$ be given, where $\varepsilon_p, \ldots, \varepsilon_{p+q} > 0$. We associate to each number ε_μ a sign $\sigma_\mu \in \{-1, 1\}$, $\mu = p, \ldots, p+q$. For $\mu \in \{p, \ldots, p+q\}$, we count each point x_μ m_μ-times and add a further point $x_\mu + \sigma_\mu \varepsilon_\mu$. Moreover, we add the points $y_i, \ldots, y_{i+m+j} \notin (a,b)$ and denote the resulting points by $y_{i,\varepsilon} \leq \cdots \leq y_{i+m+j,\varepsilon}$.

Then we set

$$s_\varepsilon = \sum_{\mu=i}^{i+j-1} a_\mu B^m_{\mu,\varepsilon} ,$$

where for all $\mu \in \{i, \ldots, i+j-1\}$,

$$B^m_{\mu,\varepsilon}(t) = (-1)^{m+1} K_m[y_{\mu,\varepsilon}, \ldots, y_{\mu+m+1,\varepsilon}, t]$$

is the B-spline of degree m associated with knots $y_{\mu,\varepsilon} \leq \cdots \leq y_{\mu+m+1,\varepsilon}$. Since

$$\lim_{\varepsilon\to 0}(y_{i,\varepsilon}, \ldots, y_{i+m+j,\varepsilon}) = (y_i, \ldots, y_{i+m+j})$$

and since the B-splines $B^m_i, \ldots, B^m_{i+j-1}$ are continuous, it follows that for all $\mu \in \{i, \ldots, i+m+j\}$,

$$\lim_{\varepsilon\to 0} \|B^m_\mu - B^m_{\mu,\varepsilon}\| = 0$$

which implies

$$\lim_{\varepsilon\to 0} \|s - s_\varepsilon\| = 0 . \tag{6}$$

Let a sufficiently large natural number n be given. We will show that there exists a sufficiently small $\varepsilon^{(n)}$ such that

$$s_n = s_f + \frac{1}{n} s_{\varepsilon^{(n)}} \in S_{m,k} .$$

First, we set

$$z_{\mu,\nu} = x_\mu \quad , \quad \nu = 1, \ldots, m_\mu + 1 , \quad \mu = 1, \ldots, r .$$

Then s_f can be written as

$$s_f(t) = \sum_{\mu=0}^{m} b_\mu t^\mu + \sum_{\mu=1}^{r} \sum_{\nu=1}^{m_\mu} b_{\mu,\nu} K_m[z_{\mu,1}, \ldots, z_{\mu,\nu}, t] .$$

Moreover, s can be written as

$$s(t) = \sum_{\mu=p}^{p+q} \sum_{\nu=1}^{m_\mu+1} c_{\mu,\nu} K_m[z_{\mu,1}, \ldots, z_{\mu,\nu}, t]$$

and s_ε can be written as

$$s_\varepsilon(t) = \sum_{\mu=p}^{p+q} \left(\sum_{\nu=1}^{m_\mu} c_{\mu,\nu,\varepsilon} K_m[z_{\mu,1}, \ldots, z_{\mu,\nu}, t] + c_{\mu,m_\mu+1,\varepsilon} K_m[z_{\mu,1}, \ldots, z_{\mu,m_\mu}, z_{\mu,\varepsilon}, t] \right) ,$$

where

$$z_{\mu,\varepsilon} = x_\mu + \sigma\varepsilon_\mu \quad , \quad \mu = p, \ldots, p+q .$$

Note, that $c_{\mu,\nu,\varepsilon} \to c_{\mu,\nu}$ and

$$K_m[z_{\mu,1},\ldots,z_{\mu,m_\mu},z_{\mu,\varepsilon},t] \to K_m[z_{\mu,1},\ldots,z_{\mu,m_\mu},z_{\mu,m_\mu+1},t]$$

for $\varepsilon \to 0$. Let $\mu \in \{p,\ldots,p+q\}$ be given. By definition we have

$$\begin{aligned} K_m[z_{\mu,1},\ldots,z_{\mu,m_\mu},z_{\mu,\varepsilon},t] &= \\ &= \frac{K_m[z_{\mu,1},\ldots,z_{\mu,m_\mu},t] - K_m[z_{\mu,2},\ldots,z_{\mu,m_\mu},z_{\mu,\varepsilon},t]}{z_{\mu,m_\mu} - z_{\mu,\varepsilon}} \\ &= \frac{K_m[z_{\mu,1},\ldots,z_{\mu,m_\mu},t] - K_m[z_{\mu,2},\ldots,z_{\mu,m_\mu},z_{\mu,\varepsilon},t]}{-\sigma_\mu\varepsilon_\mu}. \end{aligned}$$

Now, let a sufficiently large n be given. We may choose $\varepsilon^{(n)} = (\varepsilon_p^{(n)},\ldots,\varepsilon_{p+q}^{(n)})$ sufficiently small and appropriate signs $\sigma_p^{(n)},\ldots,\sigma_{p+q}^{(n)}$ such that for all $\mu \in \{p,\ldots,p+q\}$,

$$\frac{1}{n}\left(c_{\mu,m_\mu,\varepsilon^{(n)}} - \frac{1}{\sigma_\mu\varepsilon_\mu^{(n)}}c_{\mu,m_\mu+1,\varepsilon^{(n)}}\right) = -b_{\mu,m_\mu}. \tag{7}$$

We note that for all $\mu \in \{p,\ldots,p+q\}$, $c_{\mu,m_\mu+1,\varepsilon^{(n)}} \neq 0$. This follows from the fact $s_{\varepsilon^{(n)}}$ is from the j-dimensional weak Chebyshev space

$$S_{\varepsilon^{(n)}} = \operatorname{span}\{B^m_{i,\varepsilon^{(n)}},\ldots,B^m_{i+j-1,\varepsilon^{(n)}}\}$$

and has $j-1$ sign changes, since s has $j-1$ sign changes. If $c_{\mu,m_\mu+1,\varepsilon^{(n)}} = 0$ for some $\mu \in \{p,\ldots,p+q\}$, then $s_{\varepsilon^{(n)}}$ would be from a weak Chebyshev space of dimension less than j and would have at most $j-2$ sign changes. Finally, it follows from (7) that for all $\mu \in \{p,\ldots,p+q\}$, the spline s_n has a knot of multiplicity $m_\mu - 1$ at x_μ and a knot of multiplicity one at $x_\mu + \sigma_\mu\varepsilon_\mu$. This implies that $s_n \in S_{m,k}$ for all n. Since s changes sign at the points $t_1,\ldots,t_{j-1}$, for sufficiently small ε, the spline s_ε changes sign at points $t_{1,\varepsilon},\ldots,t_{j-1,\varepsilon}$ such that

$$\lim_{\varepsilon\to 0}|t_\mu - t_{\mu,\varepsilon}| = 0 \quad , \quad \mu = 1,\ldots,j-1. \tag{8}$$

Therefore, $s_\varepsilon \in S_\varepsilon$ has no further sign change which implies knot

$$(-1)^{\mu+1}s_\varepsilon(t) \geq 0 \quad , \quad t \in [t_{\mu-1,\varepsilon},t_{\mu,\varepsilon}] \cap [a,b] \quad , \quad \mu = 1,\ldots,j. \tag{9}$$

Moreover, we have

$$s_\varepsilon(t) = 0 \quad , \quad t \in [a,b]\backslash(y_{i,\varepsilon},y_{i+m+j,\varepsilon}). \tag{10}$$

By the choice of the points $t_1,\ldots,t_{j-1}$ and by (8), there exists a constant $c > 0$ and a sign $\sigma \in \{-1,1\}$ such that

$$\sigma(-1)^\mu(f(t) - s_f(t)) \le \|f - s_f\| - c \quad , \quad t \in [t_{\mu-1,\varepsilon}, t_{\mu,\varepsilon}] \cap [a,b] \quad , \quad \mu = 1,\ldots,j\,. \tag{11}$$

We may assume that $\sigma = 1$. Then it follows from (9) and (11) that for sufficiently large n, $\mu \in \{1,\ldots,j\}$ and $t \in [t_{\mu-1,\varepsilon(n)}, t_{\mu,\varepsilon(n)}] \cap [a,b]$,

$$\begin{aligned}
-\|f - s_f\| \le (-1)^\mu(f(t) - s_f(t)) &\le (-1)^\mu(f(t) - s_f(t)) - (-1)^\mu \cdot \frac{1}{n} s_{\varepsilon(n)}(t) \\
&= (-1)^\mu(f(t) - s_n(t)) \\
&\le \|f - s_f\| - c + \frac{1}{n}\|s_{\varepsilon(n)}\| \\
&\le \|f - s_f\|\,.
\end{aligned}$$

Then it follows from (10) that $\|f - s_n\| \le \|f - s_f\|$. Finally, it follows from (6) that

$$\lim_{n\to\infty} \|s_f - s_n\| = \lim_{n\to\infty} \frac{1}{n}\|s_{\varepsilon(n)}\| = 0\,.$$

This proves Theorem 1.1.

Schumaker [21] conjectured that for each $f \in C[a,b]$, there exists a best approximation $s_f \in S_{m,k}$ such that $A(f - s_f)|_{[a,b]} \ge m + 2k + 2$. Later in [22], he gave a counterexample. The following result which follows directly from Theorem 1.1 shows that Schumaker's conjecture is true for unique (local) best approximations in

$$S^1_{m,k} = \{s \in S_{m,k} \backslash S_{m,k-1} \mid s \text{ has } k \text{ simple knots}\}\,.$$

Corollary 1.2 *If $f \in C[a,b]$ has a unique (local) best approximation s_f from $S_{m,k}$ with $s_f \in S^1_{m,k}$, then*

$$A(f - s_f)|_{[a,b]} \ge m + 2k + 2\,.$$

We can compare Theorem 1.1 with known results on strong unicity. A function $s_f \in S_{m,k}$ is called **strongly unique best approximation** of $f \in C[a,b]$ from $S_{m,k}$ if there exists a constant $K_f > 0$ such that for all $s \in S_{m,k}$,

$$\|f - s\| \ge \|f - s_f\| + K_f\|s - s_f\|\,.$$

It was proved in Nürnberger [15] that a necessary condition for strongly unique local best approximations is given by replacing the closed interval in (1) by the open interval.

Moreover, condition (1) for open intervals is a sufficient condition for strongly unique best approximations if the knots of the best spline approximation are counted with double multiplicity. Therefore, in the case of simple knots, the following alternation characterization holds which we use in the subsequent investigations.

Theorem 1.3 *Let $f \in C[a,b]$, $m \geq 2$ and $s_f \in S_m\binom{x_1,\dots,x_k}{1,\dots,1} \backslash S_{m,k-1}$ be given. Moreover, let $a < y_1 \leq \cdots \leq y_{2k} < b$ be the knots of s_f, counting each knot twice, and let $y_{-m} < \cdots < y_{-1} < y_0 = a$, $b = y_{2k+1} < y_{2k+2} < \cdots < y_{2k+m+1}$. Then the following statements are equivalent:*

(i) *s_f is a strongly unique best approximation of f from $S_{m,k}$.*

(ii) *For every interval $(y_i, y_{m+i+j}) \subset (y_{-m}, y_{2k+m+1})$, $j \geq 1$, we have*

$$A(f - s_f)|_{(y_i, y_{i+m+j})} \geq j + 1 .$$

It was shown by Braess [3] that best approximations from $S_{m,k}$ cannot be characterized by alternation properties. A first alternation characterization of a subclass of best approximations is given by Theorem 1.3. Next, we give a further result of this type. We define the set

$$SU^1(S_{m,k}) = \{f \in C[a,b] \ \mid \ f \text{ has a strongly unique best approximation } s_f \text{ from } S_{m,k} \text{ and } s_f \in S^1_{m,k}\} .$$

The interior of a set is briefly denoted by int.

Theorem 1.4 *For $m \geq 2$ the following statements are equivalent:*

(i) *$f \in int\, SU^1(S_{m,k})$.*

(ii) *There exists a function $s_f \in S_m\binom{x_1,\dots,x_k}{1,\dots,1}$ such that $A(f - s_f)|_{[a,b]} \geq 2k + m + 2$ and $A(f - s_f)|_{[x_p, x_{p+q+1}]} < 2q + m + 2$ for every interval $[x_p, x_{p+q+1}] \subsetneq [a,b]$, $q \geq 0$.*

Proof. (ii)$\Rightarrow$(i). Suppose that (ii) holds, but (i) fails. Since (ii) holds, a simple computation shows that the condition in Theorem 1.3 is satisfied, i.e. s_f is a strongly unique best approximation of f from $S_{m,k}$. Since (i) fails, there exists a sequence (f_n) in $C[a,b]$ such that for all n, $f_n \notin SU^1(S_{m,k})$. It follows from Schumaker [21] that for each n, there exists a spline $s_n \in P_{S_{m,k}}(f_n) \cap C[a,b]$. Since by Nürnberger [17] the metric projection $P_{S_{m,k}}$ is upper semicontinuous at f and $P_{S_{m,k}}(f) = \{s_f\}$, we have $s_n \to s_f$. We denote by $x_1 < \cdots < x_k$ (respectively $x_{1,n} \leq \cdots \leq x_{k,n}$) the knots of s_f (respectively s_n).

Since $s_n \to s_f$, it follows that

$$x_{i,n} \to x_i \quad , \quad i = 1, \ldots, k.$$

Let a natural number n be given. Since $s_n \in P_{S_{m,k}}(f_n)$, it follows from Braess [3] that there exists an interval$[x_{p_n,n}, x_{p_n+q_n+1,n}] \subset [a,b]$, $q_n \geq 0$, such that

$$A(f_n - s_n)|_{[x_{p_n,n}, x_{p_n+q_n+1,n}]} \geq m + 2q_n + 2 \, , \tag{12}$$

where $x_{0,n} = a$ and $x_{k+1,n} = b$.

By going to a subsequence, we may assume that for all n, $p_n = p$ and $q_n = q$.

Case 1. $[x_{p,n}, x_{p+q+1,n}] \neq [a,b]$

By taking limits, it follows from (12) that

$$A(f - s_f)|_{[x_p, x_{p+q+1}]} \geq m + 2q + 2$$

which contradicts (ii).

Case 2. $[x_{p,n}, x_{p+q+1,n}] = [a,b]$

In this case,

$$A(f_n - s_n)|_{[a,b]} \geq m + 2k + 2 \, . \tag{13}$$

We denote by

$$a < y_{1,n} \leq \cdots \leq y_{2k,n} < b$$

the knots of s_n counting each knot twice and choose arbitrary points

$$y_{-m,n} < \cdots < y_{-1,n} < y_{0,n} = a$$

and

$$b = y_{2k+1,n} < y_{2k+2,n} < \cdots < y_{2k+m+1,n} \, .$$

Since s_n is **not** a strongly unique best approximation of f_n from $S_{m,k}$, it follows from Theorem 1.3 that there exists $(y_{i_n,n}, y_{i_n+m+j_n,n}) \subset (y_{-m,n}, y_{2k+m+1,n})$, $j \geq 1$, such that

$$A(f_n - s_n)|_{(y_{i_n,n}, y_{i_n+m+j_n,n})} < j + 1. \tag{14}$$

Again by going to a subsequence, we may assume that for all n, $i_n = i$ and $j_n = j$. A simple computation shows that by (13) and (14),

$$A(f_n - s_n)|_{[a,y_{i,n}]} \geq m + (i-1) + 2 \tag{15}$$

or

$$A(f_n - s_n)|_{[y_{i+m+j,n},b]} \geq m + 2k - (i + m + j) + 2\,. \tag{16}$$

There exist indices μ and ν such that $y_{i,n} = x_{\mu,n}$ and $y_{i+m+j,n} = x_{\nu,n}$. Since the knots $y_{i,n}$ are double knots, it follows from (15) and (16) that

$$A(f_n - s_n)|_{[a,x_{\mu,n}]} \geq m + 2(\mu - 1) + 2$$

or

$$A(f_n - s_n)|_{[x_{\nu,n},b]} \geq m + 2(k - \nu) + 2\,.$$

By taking limits, it follows that

$$A(f - s_f)|_{[a,x_\mu]} \geq m + 2(\mu - 1) + 2$$

or

$$A(f - s_f)|_{[x_\nu,b]} \geq m + 2(k - \nu) + 2$$

which contradicts (ii).

(i)$\Rightarrow$(ii). Suppose that (i) hold, but (ii) fails. Since $f \in SU^1(S_{m,k})$, it follows from Theorem 1.3 that for the strongly unique approximation $s_f \in S_{m,k}$ of f we have

$$A(f - s_f)|_{[a,b]} \geq m + 2k + 2\,. \tag{17}$$

Since (ii) fails, there exists an interval $[x_p, x_{p+q+1}] \subsetneq [a,b]$ such that

$$A(f - s_f)|_{[x_p,x_{p+q+1}]} \geq 2q + m + 2\,. \tag{18}$$

Then we may choose $m + 2k + 2$ alternating extreme points $a \leq t_1 < \cdots < t_{m+2k+2} \leq b$ of $f - s_f$ such that $[x_p, x_{p+q+1}]$ contains at least $m + 2q + 2$ points from $M := \{t_1, \ldots, t_{m+2k+2}\}$. We will show that there exists a sequence (f_n) in $C[a,b]$ converging to f such that for all n,

$$E(f_n - s_f) = M\,.$$

Let (V_n) be a neighborhood basis of M. For each n, there exists a function $h_n \in C[a,b]$ such that

$$\begin{aligned} h_n(t) &= 1 \quad , \text{ if } t \in M \\ h_n(t) &= 1 - \frac{1}{n} \quad , \text{ if } t \in [a,b] \backslash V_n \end{aligned}$$

and

$$1 - \frac{1}{n} < h_n(t) < 1 \quad , \text{ if } t \in V_n \backslash M .$$

Moreover, for each n, we set $f_n = h_n(f - s_f) + s_f$. Then the sequence (f_n) has the desired property. Now, it follows from Schumaker [21] that $s_f \in P_{S_{m,k}}(f_n)$ for all n. Moreover by the choice of the points $t_1, \ldots, t_{m+2k+2}$ we have

$$A(f_n - s_f)|_{[a,x_p)} < 2p + 1$$

or

$$A(f_n - s_f)|_{(x_{p+q+1},b]} < 2(k - p - q) + 1 .$$

Therefore, it follows from Theorem 1.3 that for all n, s_f is not a strongly unique best approximation of f_n from $S_{m,k}$. This proves Theorem 1.4.

Condition (ii) in Theorem 1.3 was developed by Schumaker [21], and it was shown by several authors (Schumaker [21], Arndt [1], Schaback [20]) that this condition implies that s_f is a unique respectively strongly unique best approximation of f from $S_{m,k}$. A class of functions which satisfy this condition are functions $f \in C^{m+1}[a,b]$ with $f^{(m+1)}(t) \neq 0$ for all $t \in (a,b)$. (see Johnson [8], Schumaker [21], Braess [3], Schaback [20]).

By applying Theorem 1.1 and Theorem 1.3 we obtain a density result on the relationship of unique and strongly unique best approximation. For this we define the set

$$\begin{aligned} U^1(S_{m,k}) = \{ & f \in C[a,b] \mid f \text{ has a unique} \\ & \text{best approximation } s_f \text{ from } S_{m,k} \text{ and } s_f \in S^1_{m,k} \} . \end{aligned}$$

Theorem 1.5 *For $m \geq 2$, $SU^1(S_{m,k})$ is a dense subset of $U^1(S_{m,k})$.*

Proof: Let a function $f \in U^1(S_{m,k})$ be given. We will show that there exists a sequence (f_n) in $SU^1(S_{m,k})$ which converges to f. Let $s_f \in S_m\binom{x_1,\ldots,x_k}{1,\ldots,1} \backslash S_{m,k-1}$ be the unique best approximation of f from $S_{m,k}$. By using the notation of Theorem 1.1, it follows from this result that for every interval $[y_i, y_{i+m+j}] \subset [y_{-m}, y_{2k+m+1}]$, $j \geq 1$, we have

$$A(f - s_f)|_{[y_i, y_{i+m+j}]} \geq j + 1 . \tag{19}$$

We set

$$Z = \{z_1, \ldots, z_p\} = \{x_1, \ldots, x_k\} \cap E(f - s_f) .$$

Let (V_n) be an open neighborhood basis of Z. For each n, we define

$$h_n(t) = \begin{cases} \operatorname{sgn}(f(t) - s_f(t)) \cdot \|f - s_f\| & , \quad \text{if } t \in \overline{V}_n \\ f(t) - s_f(t) & , \quad \text{if } t \in [a,b] \backslash V_n \end{cases}$$

and extend h_n linearly on $V_{n-1} \backslash \overline{V}_n$. Then the sequence $(h_n) \subset C[a,b]$ converges to $f - s_f$ which implies that $f_n = h_n + s_f$ converges to f. Moreover, it follows from (19) and the definition of $f_n - s_f = h_n$ that for every interval $(y_i, y_{i+m+j}) \subset (y_{-m}, y_{2k+m+1})$, $j \geq 1$, we have

$$A(f_n - s_f)|_{(y_i, y_{i+m+j})} \geq j + 1 .$$

Then by Theorem 1.3, $(f_n) \subset SU^1(S_{m,k})$. This proves Theorem 1.5.

2. The Structure of Nonlinear Sets

In this section we investigate the structure of nonlinear sets in connection with strong unicity and give applications to splines with free knots and rational functions.

We denote by $C(T)$ the space of all continuous real-valued functions on a compact space T, endowed with the uniform norm $\|h\| = \sup_{t \in T} |h(t)|$. Let V be an open subset of $I\!R^N$. Moreover, let $A : V \to C(T)$ be given such that the mapping

$$(t, x) \longrightarrow \left(\frac{\partial A(x)(t)}{\partial x_1}, \ldots, \frac{\partial A(x)(t)}{\partial x_N} \right), \quad (t,x) \in T \times V ,$$

is continuous, and let $G = A(V)$. For $\overline{g} = A(\overline{x}) \in G$, we denote by

$$TG(\overline{x}) = \operatorname{span} \left\{ \frac{\partial A(\overline{x})}{\partial x_1}, \ldots, \frac{\partial A(\overline{x})}{\partial x_N} \right\}$$

the tangent space of $\overline{g}$. We consider best uniform approximation of functions in $C(T)$ by elements of G. A function $g_f \in G$ is called **strongly unique best approximation** of $f \in C(T)$ from G if there exists a constant $K_f > 0$ such that for all $g \in G$,

$$\|f - g\| \geq \|f - g_f\| + K_f \|g - g_f\| .$$

Moreover, g_f is called **strongly unique local best approximation** of f from G if there exists $\varepsilon > 0$ and $K_{f,\varepsilon} > 0$ such that for all

$$g \in U(g_f, \varepsilon) = \{g \in G \mid \|g - g_f\| \leq \varepsilon\},$$

$$\|f - g\| \geq \|f - g_f\| + K_{f,\varepsilon}\|g - g_f\|.$$

We now introduce the notation of a sun for strong unicity.

Definition 2.1 *A subset $\widetilde{G}$ of G is called a sun for to strong unicity if each $f \in C(T)$ satisfies the following property: if $g_f \in \widetilde{G}$ is a strongly unique best approximation of f from G, then g_f is a strongly unique best approximation of $f_\mu = g_f + \mu(f - g_f)$ from G for all $\mu \geq 0$.*

Next, we give a characterization of suns for strong unicity.

Theorem 2.2 *Let $\widetilde{G}$ be a subset of G such that $\dim TG(x_f) = N$, if $g_f = A(x_f) \in \widetilde{G}$ is a strongly unique best approximation of $f \in C(T)\backslash G$. Then the following statements are equivalent:*

(i) *$\widetilde{G}$ is a sun for strong unicity.*

(ii) *For every $f \in C(T)$, $g_f \in \widetilde{G}$ is a strongly unique best approximation of f from G, if g_f is a strongly unique local best approximation of f from G.*

(iii) *For every $f \in C(T)$, $g_f = A(x_f) \in \widetilde{G}$ is a strongly unique best approximation of f from G if and only if 0 is a strongly unique local best approximation of $f - g_f$ from $TG(x_f)$ and $\dim TG(x_f) = N$.*

Proof. (i)$\Rightarrow$(ii). Suppose that (i) holds and let $g_f \in \widetilde{G}$ be a strongly unique local best approximation of $f \in C(T)$ from G, i.e. there exist $\varepsilon > 0$ and $K_{f,\varepsilon} > 0$ such that for all $g \in U(g_f, \varepsilon)$,

$$\|f - g\| \geq \|f - g_f\| + K_{f,\varepsilon}\|g - g_f\|. \tag{20}$$

First, we shows that for all μ with $0 \leq \mu \leq 1$, g_f is a strongly unique local best approximation of $f_\mu = g_f + \mu(f - g_f)$ from G.

Let $g \in U(g_f, \varepsilon)$ be given. Then by (20) and $0 \leq \mu \leq 1$, it follows that

$$\|f_\mu - g\| \;\; \geq \;\; \|f - g\| - \|f - f_\mu\|$$

$$
\begin{aligned}
&\geq \|f - g_f\| + K_{f,\varepsilon} \cdot \|g - g_f\| - \|f - f_\mu\| \\
&= \|f - g_f\| - (1-\mu)\|f - g_f\| + K_{f,\varepsilon} \cdot \|g - g_f\| \\
&= \|f_\mu - g_f\| + K_{f,\varepsilon} \cdot \|g - g_f\| . \qquad (21)
\end{aligned}
$$

Next, we show that for all μ with $0 \leq \mu \leq \min\left\{\frac{\varepsilon}{4\|f-g_f\|}, 1\right\}$, g_f is a strongly unique best approximation of f_μ from G

Let μ as above be given. If $g \in G \backslash U(g_f, \varepsilon)$, then

$$
\begin{aligned}
\|f_\mu - g\| &\geq \|g - g_f\| - \|f_\mu - g_f\| \\
&\geq \frac{1}{2} \cdot \varepsilon + \frac{1}{2} \cdot \|g - g_f\| - \mu\|f - g_f\| \\
&\geq 2\mu\|f - g_f\| + \frac{1}{2} \cdot \|g - g_f\| - \mu\|f - g_f\| \\
&= \mu\|f - g_f\| + \frac{1}{2} \cdot \|g - g_f\| \\
&= \|f_\mu - g_f\| + \frac{1}{2} \cdot \|g - g_f\| . \qquad (22)
\end{aligned}
$$

If we set $K_f = \min\left\{K_{f,\varepsilon}, \frac{1}{2}\right\}$, then it follows from (21) and (22) that for all $g \in G$,

$$
\|f_\mu - g\| \geq \|f_\mu - g_f\| + K_f\|g - g_f\| . \qquad (23)
$$

It follows from (23) and (i) that g_f is a strongly unique best approximation of $f = g_f + \frac{1}{\mu}(f_\mu - g_f)$ from G. This shows that (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii). Suppose that (ii) holds. We first note that by Satz 4.2.2 in Hettich & Zencke [7], $g_f = A(x_f) \in \widetilde{G}$ is a strongly unique local best approximation with respect to the parameter of $f \in C(T)$ (i.e. there exist $\varepsilon > 0$ and $\widetilde{K}_{f,\varepsilon} > 0$ such that for all $x \in U(x_f, \varepsilon)$, $\|f - A(x)\| \geq \|f - A(x_f)\| + \widetilde{K}_{f,\varepsilon}\|x - x_f\|$) if and only if $\dim TG(x_f) = N$ and 0 is a strongly unique best approximation of $f - g_f$ from $TG(x_f)$.

Moreover, since $A : V \rightarrow C(T)$ is Fréchet differentable, it follows from the mean value theorem and the inverse function theorem that if $\dim TG(x_f) = N$, then g_f is strongly unique local best approximation of f from G if and only if g_f is a strongly unique local best approximation with respect to the parameter of f from G. By using these statements and our assumption, it is easy to see that (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i). Suppose that (iii) holds and let $g_f = A(x_f) \in \widetilde{G}$ be a strongly unique best approximation of $f \in C(T)$ from G. Then it follows from (iii) that 0 is a strongly unique best approximation of $f_\mu - g_f = \mu(f - g_f)$ from $TG(x_f)$ for all $\mu \geq 0$ and $\dim TG(x_f) = N$.

This implies that g_f is a strongly unique best approximation of f_μ from G. This proves Theorem 2.2.

We note that in [14], statement (iii) in Theorem 2.2 was formulated incorrectly, since the condition that $\dim TG(x_f) = N$ was omitted.

Remark 2.3 The proof of Theorem 2.2 shows that the implication (i)⇒(ii) of Theorem 2.2 holds for arbitrary sets G in a normed linear space.

By using Theorem 1.11 on p. 55 in Braess [4], it follows from the proof of Theorem 2.2 that Theorem 2.2 holds for C^1-manifolds G in a normed linear space by replacing the tangent space by the tangent cone and omitting the assumptions on the dimension of the tangent spaces.

It is well known that if we replace in the statements of Theorem 2.2 strongly unique best approximations by best approximations and if we omit the assumptions on the dimension of the tangent spaces, then (i) and (ii) are equivalent, but (i) and (iii) are not equivalent (see Braess [4]).

Examples 2.4 It follows from Theorem 1.3 in this paper and the characterization of strongly unique best spline approximations with fixed knots in Nürnberger [12] (see also Nürnberger et al. [18]) that (iii) in Theorem 2.2 holds for $\widetilde{G} = S^1_{m,k}$ and $G = S_{m,k}$. (The tangent space of a spline in $S^1_{m,k}$ is $S_m\binom{x_1,\ldots,x_k}{2,\ldots,2}$.) Therefore, by Theorem 2.2, $S^1_{m,k}$ is a sun for strong unicity. On the other hand, it is well known that $S^1_{m,k}$ is *not* a sun (see e.g. Braess [4]).

A further prototype of a sun for strong unicity is the set $G = R_{p,q}$ of rational functions. This follows from the well-known alternation characterization of strongly unique best rational approximations. Moreover, $G = R_{p,q}$ is also a sun (see e.g. Braess [3] and Meinardus [10]).

In addition, it follows from Theorem 3 in Barrar & Loeb [2] that the subset $\widetilde{G}$ of normal functions (i.e. functions whose tangent space has maximal dimension) of a set G satisfying the local and global Haar condition is a sun for strong unicity. Moreover, the set G is also a sun (see e.g. Braess [3] and Meinardus [10]).

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