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Exterior Domain Problems and Decomposition of Tensor Fields in Weighted Sobolev Spaces

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ABSTRACT

The Hodge decomposition is a useful tool for tensor analysis on compact manifolds with boundary. This paper aims at generalising the decomposition to exterior domains $G \subset I\!R^n$. Let $L^2_a\Omega^k(G)$ be the space weighted square integrable differential forms with weight function $(1+|x|^2)^a$, let d_a be the weighted perturbation of the exterior derivative and δ_a its adjoint. Then $L^2_a\Omega^k(G)$ splits into the orthogonal sum of the subspaces of the d_a-exact forms with vanishing tangential component on the boundary, of δ_a-coexact forms with vanishing normal component, and harmonic forms, in the sense of $d_a\lambda = 0$ and $\delta_a\lambda = 0$. For the respective components regularity results are given and corresponding a-priori estimates are shown.

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1. Introduction

The method of Hodge decomposition of differential forms provided a useful tool for the analysis on manifolds with boundary, in particular for solving boundary value problems. For the case of a compact manifold G with boundary it has been shown in [M] that the space $L^2\Omega^k(G)$ of square integrable k-forms splits into

$$L^2\Omega^k(G) = L^2\mathcal{E}^k(G) \oplus L^2\mathcal{C}^k(G) \oplus L^2\mathcal{H}^k(G) \tag{1.1}$$

where

$$L^2\mathcal{E}^k(G) = \{d\alpha \in L^2\Omega^k(G) \mid \mathbf{t}\alpha = 0\} \ , \quad L^2\mathcal{C}^k(G) = \{\delta\beta \in L^2\Omega^k(G) \mid \mathbf{n}\beta = 0\}$$
$$\text{and} \quad L^2\mathcal{H}^k(G) = \{\lambda \in L^2\Omega^k(G) \mid d\lambda = 0 \ , \delta\lambda = 0\} \ . \tag{1.2}$$

Here d is the extension of the exterior derivative $d : \Omega^{k-1}(G) \to \Omega^k(G)$ and $\delta : \Omega^{k+1}(G) \to \Omega^k(G)$ is its adjoint, the co-differential. The conditions $\mathbf{t}\alpha = 0$ and $\mathbf{n}\beta = 0$ indicate that the tangential respectively normal component on the boundary ∂G of the differential forms have to vanish. For precise definitions see Section 2. Identifying the 1-forms $\omega \in \Omega^1(G)$ with vector fields $X_\omega \in \mathcal{X}(G)$ this Hodge-Morrey decomposition (1.1) generalises the well known Helmholtz decomposition, by stating that each vector field uniquely splits into the gradient of $f \in C^\infty(G)$, the generalised curl of a vector field $W \in \mathcal{X}(G)$ and a harmonic (i.e.curl- and divergence-free) field. Here f and W have to satisfy the given boundary conditions.

In the case of G being a non-compact manifold (with boundary), a complete generalisation of that result is missing. A number of partial results have been obtained by several authors, see [B-S], [C], [D] [P], [W1] and [W-W]. This paper aims at filling the gap for arbitrary exterior domains $G \subset \mathbb{R}^n$. Its main purpose is to prove the corresponding Hodge-Morrey decomposition

$$L^2_a\Omega^k(G) = L^2_a\mathcal{E}^k_a(G) \oplus L^2_a\mathcal{C}^k_a(G) \oplus L^2_a\mathcal{H}^k_a(G) \ , \tag{1.3}$$

where $L^2_a\Omega^k(G)$ is the Hilbert space of weighted square integrable differential forms, with the norm

$$\|\omega\|^2_{L^2_a} = \int_G \langle\omega, \omega\rangle \exp(2a\sigma) d^n x \quad \text{where} \quad \sigma = \tfrac{1}{2}\log(1 + |x|^2) \ . \tag{1.4}$$

In order to do so the exterior derivative and the co-differential operator need to be modified by a term corresponding to the choice of the weight. With

$$\begin{aligned} d_a\omega &:= d\omega + a\, d\sigma \wedge \omega \quad \text{mapping} \quad d_a : \Omega^{k-1}(G) \to \Omega^k(G) \quad \text{and} \\ \delta_a\omega &:= \delta\omega - a\left(\mathbf{i}_{\operatorname{grad}\sigma}\omega\right) \quad \text{mapping} \quad \delta_a : \Omega^{k+1}(G) \to \Omega^k(G) \end{aligned} \tag{1.5}$$

for the weighted exterior derivative and its adjoint, the spaces $L^2\mathcal{E}^k(G)$, $L^2\mathcal{C}^k(G)$ and $L^2\mathcal{H}^k(G)$ are replace by

$$L^2_a\mathcal{E}^k_a(G) = \{d_a\alpha \in L^2_a\Omega^k(G) \mid \mathbf{t}\alpha = 0\} \ , \quad L^2_a\mathcal{C}^k_a(G) = \{\delta_a\beta \in L^2_a\Omega^k(G) \mid \mathbf{n}\beta = 0\}$$
$$\text{and} \quad L^2_a\mathcal{H}^k_a(G) = \{\lambda \in L^2_a\Omega^k(G) \mid d_a\lambda = 0 \ , \delta_a\lambda = 0\} \ . \tag{1.6}$$

Let $H^1_{a-1}\Omega^k(G)$ denote the weighted Sobolev space [K] of differential forms normed by $\|\omega\|^2_{H^1_{a-1}} := \|\omega\|^2_{L^2_{a-1}} + \sum_{j=1..n} \|\nabla_j\omega\|^2_{L^2_a}$. The essential object needed to show the decomposition (1.3) is the weighted Dirichlet integral

$$\begin{aligned} &\mathcal{D}_a : H^1_{a-1}\Omega^k(G) \times H^1_{a-1}\Omega^k(G) \longrightarrow \mathbb{R} \\ &\mathcal{D}_a(\omega,\eta) = \ll d_a\omega, d_a\eta \gg_a + \ll \delta_a\omega, \delta_a\eta \gg_a . \end{aligned} \tag{1.7}$$

The aim is to identify a subspace of $H^1_{a-1}\Omega^k(G)$ on which this continuous bilinear form gives an upper bound for the weighted Sobolev norm, that is

$$\|\omega\|^2_{H^1_{a-1}} \leq C(a,G)\, \mathcal{D}_a(\omega,\omega) \tag{1.8}$$

with a constant depending only on a and the geometry of G. We prove that this inequality holds for each $a \neq (1-n/2)$ on the space of all differential forms ω, which have a vanishing tangential component $\mathbf{t}\omega = 0$ and are orthogonal with respect to the L^2_{a-1} norm to the space

$$\mathcal{H}^{k,D}_{a-1}(G) = \{\,\lambda \mid \mathbf{t}\lambda = 0 \text{ and } \mathcal{D}_a(\lambda,\lambda) = 0\,\} . \tag{1.9}$$

Having established this essential estimate, the approach of [S2] towards a proof of the Hodge-Morrey decomposition generalises.

This paper is divided into 8 sections: In Section 2 some basic notations are introduced. The main analytic arguments are found in Section 3 and 4. There a weighted generalisation of the Poincaré inequality [O-K] for differential forms on exterior domains is given. Moreover, it is shown that the weighted Dirichlet integral $\mathcal{D}_a$ satisfies the estimate (1.8) modulo a contribution of order $\|\omega\|^2_{L^2_{a-2}}$. In Section 5 the proof of estimate (1.8) is completed and it is shown how this relates to solving the elliptic boundary value problem

$$\begin{aligned} (\delta_a d_a + d_a\delta_a)\omega &= \eta && \text{on } G \\ \mathbf{t}\omega = 0 \text{ and } \mathbf{t}\delta_a\omega &= 0 && \text{on } \partial G . \end{aligned} \tag{1.10}$$

This allows to prove in Section 6 the Hodge-Morrey decomposition (1.3) for exterior domains, and give corresponding regularity results and estimates for the components. Section 7 is devoted to the decomposition on the subspace of differential forms satisfying boundary conditions. Finally, in Section 8, a short discussion is given about solving boundary value problems for differential forms on exterior domains by means of the Hodge decomposition.

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2. Weighted Sobolev spaces of differential forms

Throughout this paper all differential forms and distributions are defined on an exterior domain $G = \mathbb{R}^n \backslash \widehat{G}$ with a smooth boundary ∂G. Here $\widehat{G} \subset \mathbb{R}^n$ is an open bounded domain so that $\partial G \subset G$ is compact and G is closed. Let $\bigwedge^*(\mathbb{R}^n)$ be the exterior algebra, then the space of smooth differential forms of degree k is $\Omega^k(G) = C^\infty(G; \bigwedge^k(\mathbb{R}^n))$.

By $\Omega^k_c(G)$ the subspace differential forms on G with compact support in $I\!R^n$. Let $\mathcal{F} = (E_1, \dots E_n)$ be a local orthonormal on $U \subset G$. We define a fibrewise product on $\Omega^k(G)$ by

$$\langle \omega, \eta \rangle := \frac{1}{k!} \sum_{j_1=1..n} \cdots \sum_{j_k=1..n} \omega(E_{j_1}, \dots E_{j_k}) \cdot \eta(E_{j_1}, \dots E_{j_k}) , \tag{2.1}$$

where the vector fields E_{j_i} run through $\mathcal{F}$. The product $\langle , \rangle$ is independent of the choice of the frame used for its definition. This give rise to define the Hodge (star) operator, $\star : \Omega^k(G) \to \Omega^{n-k}(G)$, such that $\langle \eta, \omega \rangle d^n x = \eta \wedge (\star\omega)$ for all $\eta \in \Omega^k(G)$. Here $d^n x$ is the standard volume form in $I\!R^n$. The contraction of $\omega \in \Omega^k(G)$ with a vector field Y is defined by

$$(\mathrm{i}_Y \omega)(X_1, \dots X_{k-1}) = \omega(Y, X_1, \dots X_{k-1}) . \tag{2.2}$$

Following the approach of [R] we write for the derivative of a differential form in the direction of a vector field Y

$$(\nabla_Y \omega)(X_1, \dots X_k) := D(\omega((X_1, \dots X_k))(Y) - \sum_{j=1..k} \omega(X_1, \dots \partial_Y X_j, \dots X_k) ; \tag{2.3}$$

if $k = 0$ we identify $D\omega(Y) = \nabla_Y \omega$. Then the exterior derivative reads

$$d\omega(X_0, X_1, \dots X_k) = \sum_{j=0..k} (-1)^j (\nabla_{X_j} \omega)(X_0, \dots \widehat{X}_j, \dots X_k) , \tag{2.4}$$

where $\widehat{X}_j$ means to omit this vector field. For co-differential operator $\delta = \star d \star$ we have

$$\delta\, \omega(X_1, \dots X_{k-1}) := - \sum_{j=1..n} (\nabla_{E_j} \omega)(E_j, X_1, \dots X_{k-1}) , \tag{2.5}$$

where the fields E_j run through an arbitrary orthonormal frame $\mathcal{F}$. The Laplace operator $\Delta = \delta d + d\delta$ on $\Omega^k(G)$ can be written as

$$\Delta\omega = - \sum_{j=1..n} \left(\nabla_{E_j} (\nabla_{E_j} \omega) - \nabla_{\nabla_{E_j} E_j} \omega \right) . \tag{2.6}$$

The space $\Omega^1(G)$ can be identified with the space $\mathcal{X}(G)$ of (smooth) vector fields on G by means of the flat map. That is, each vector field Y on G defines a 1-form $Y^\flat \in \Omega^1(G)$ by demanding $\langle Y^\flat, \omega \rangle = \omega(Y)$ for all $\omega \in \Omega^1(G)$. By direct computation

$$\langle (Y^\flat \wedge \omega), \eta \rangle = \langle \omega, (\mathrm{i}_Y \eta) \rangle , \tag{2.7}$$

and

$$\frac{1}{|Y|^2} \Big(\mathrm{i}_Y (Y^\flat \wedge \omega) + Y^\flat \wedge (\mathrm{i}_Y \omega) \Big) = \omega . \tag{2.8}$$

Moreover, the flat map allows to express the co-differential by the divergence and the exterior derivative of 1-forms by the generalised curl of the corresponding vector field, that is

$$\operatorname{div} Y = \delta Y^\flat \quad \text{and} \quad (\operatorname{curl} Y)^\flat = \star d Y^\flat . \tag{2.9}$$

To describe the boundary behavior let $j : \partial G \to G$ be the inclusion of the boundary. We denote by $\omega|_{\partial G}$ the restriction of $\omega \in \Omega^k(G)$ to ∂G, and by $j^*\omega \in \Omega^k(\partial G)$ its pull back. If N is the outward pointing unit vector field on ∂G, each point $y \in \partial G$ has an open neighborhood $U_y \subset G$ such that

$$\mathcal{F}_N = \{(F_1, F_2, \dots F_n) \mid F_1|_y = N_y \ , \ \langle F_j|_x, F_k|_x\rangle = \delta_{jk} \ \forall x \in U_y\} \tag{2.10}$$

defines a local orthonormal frame. The restriction of $(F_2, \dots F_n)$ to ∂G then is a local orthonormal frame on $U_y \cap \partial G$. In slight abuse of notation we will identify N with its extension F_1. In a neighborhood U of ∂G each $X \in \mathcal{X}(G)$ can be split into $X|_U = X^\perp N + X^\top$, where $X^\top|_{\partial G}$ is a vector field along ∂G. For $\omega \in \Omega^k(G)$ we define the tangential respectively the normal component by

$$\mathbf{t}\omega(X_1, \dots X_k) = \omega|_{\partial G}(X_1^\top, \dots X_k^\top) \quad \text{and} \quad \mathbf{n}\omega = \omega|_{\partial G} - \mathbf{t}\omega \ ; \tag{2.11}$$

if $k = 0$ we set $\mathbf{t}\omega = \omega|_{\partial G}$. The spaces of smooth differential forms with vanishing tangential respectively normal components we denote by

$$\Omega^k_D(G) = \{\omega \in \Omega^k(G) \mid \mathbf{t}\omega = 0\} \quad \text{and} \quad \Omega^k_N(G) = \{\omega \in \Omega^k(G) \mid \mathbf{n}\omega = 0\} \ . \tag{2.12}$$

One easily shows that the Hodge operator $\star$ intertwines the normal and the tangential projection, that is $\star\mathbf{t} = \mathbf{n}\star$. Hence, for each $\omega \in \Omega^k_D(G)$ there is a unique $\eta \in \Omega^{n-k}_N(G)$ such that $\omega = \star\eta$ and vice versa. For 1-forms this can be described in terms of vector fields by means of the flat map, i.e. $\mathbf{t}Y^\flat = (Y^\top)^\flat$ and $\mathbf{n}Y^\flat = Y^\perp N^\flat$.

To get access to the notion of weighted Sobolev spaces, let $r = |x|$ be the radial distance from $x \in G$ to the origin in $\mathbb{R}^n$, and denote by $R_x = \frac{x}{r}$ the radial unit vector. Then

$$\rho^{2a} := \exp(2a\sigma) \quad \text{where} \quad \sigma = \tfrac{1}{2}\log(1 + r^2) \tag{2.13}$$

defines a family of weight functions, and

$$\operatorname{grad}\big(\exp(2a\sigma)\big) = 2a\exp(2a\sigma)(\partial_r\sigma)R_x \ \text{ where } \ \partial_r\sigma = r\exp(-2\sigma) \ . \tag{2.14}$$

Using this, the space $\Omega^k_c(G)$ can be equipped with a family of weighted scalar products defined by

$$\ll \omega, \eta \gg_a := \int_G \exp(2a\sigma)\langle\omega, \eta\rangle d^n x \ . \tag{2.15}$$

The completion of $\Omega^k_c(G)$ with respect to the corresponding norm $\|\omega\|_{L^2_a}$ is denoted by $L^2_a\Omega^k(G)$. If $\mathcal{F}_c = (e_1, \dots e_n)$ is the canonical basis on $G \subset \mathbb{R}^n$, the weighted H^s_a Sobolev norm, inductively defined by

$$\|\omega\|^2_{H^s_a} := \|\omega\|^2_{L^2_a} + \sum_{j=1..n} \|\nabla_{e_j}\omega\|^2_{H^{s-1}_{a+1}} \ . \tag{2.16}$$

The respective completions of $\Omega^k_c(G)$ in these norms, that is the weighted Sobolev spaces of differential forms, are denoted by $H^s_a\Omega^k(G)$. The space $H^0_a\Omega^k(G)$ is identified with

$L_a^2\Omega^k(G)$. Of special interest is the spaces $H_a^1\Omega^k(G)$. The corresponding norm is easily shown to be independent of the choice of the frame used for its definition. That is

$$\|\omega\|_{H_a^1}^2 = \|\omega\|_{L_a^2}^2 + \sum_{j=1..n} \|\nabla_{E_j}\omega\|_{L_{a+1}^2}^2 \tag{2.17}$$

for an arbitrary (local) orthonormal frame $\mathcal{F} = (E_1, \dots E_n)$. In the general case $s > 1$ this frame independence fails. However, any choice of a frame $\mathcal{F}$ on G induces an equivalent topology on $H_a^s\Omega^k(G)$. From the respective definitions it is clear that the exterior derivative and the co-differential extend to bounded linear operators $d : H_a^s\Omega^k(G) \to H_{a+1}^{s-1}\Omega^{k+1}(G)$ and $\delta : H_a^s\Omega^k(G) \to H_{a+1}^{s-1}\Omega^{k-1}(G)$. For corresponding concepts for general (non-compact) Riemannian manifolds see [C], [D] and [E].

To obtain a generalisation of Green's formula for the L_a^2 scalar product we observe that each $y \in G$ has a neighborhood U_y such that

$$\mathcal{F}_R = \{(F_1, F_2, \dots F_n) \mid F_1|_x = R_x \ , \ \langle F_j|_x, F_k|_x\rangle = \delta_{jk} \ \forall\, x \in U_y\} \tag{2.18}$$

defines a local orthonormal frame. Here $R_x = \frac{x}{r}$. For the vector field F_1 we will write also R. By definition of the weight function $\nabla_{F_j} \exp(a\sigma) = 0$ for $j \geq 2$. From (2.4) and (2.5) we then infer that

$$\begin{aligned} d\big(\exp(a\sigma)\eta\big) &= \exp(a\sigma)\, d\eta + \big(\nabla_R \exp(a\sigma)\big) R^\flat \wedge \eta = \exp(a\sigma)\big(d\eta + a(\partial_r\sigma) R^\flat \wedge \eta\big) \\ \delta\big(\exp(a\sigma)\eta\big) &= \exp(a\sigma)\, \delta\eta - \big(\nabla_R \exp(a\sigma)\big)\mathbf{i}_R\eta = \exp(a\sigma)\big(\delta\eta - a(\partial_r\sigma)\mathbf{i}_R\eta\big) \ . \end{aligned} \tag{2.19}$$

In view of this, we define the weighted exterior derivative as

$$d_a\eta := d\eta + a(\partial_r\sigma)R^\flat \wedge \eta \ , \tag{2.20}$$

and the weighted co-differential operator as

$$\delta_a\eta := \delta\eta - a(\partial_r\sigma)\mathbf{i}_R\eta \ . \tag{2.21}$$

These differentials extend to bounded linear operators on the corresponding weighted Sobolev spaces, that is $d_a : H_a^s\Omega^k(G) \to H_{a+1}^{s-1}\Omega^{k+1}(G)$ and $\delta_a : H_a^s\Omega^k(G) \to H_{a+1}^{s-1}\Omega^{k-1}(G)$. Since

$$\exp(a\sigma)d_a d_a\eta = d\big(\exp(a\sigma)d_a\eta\big) = dd\big(\exp(a\sigma)\eta\big) = 0 \ , \tag{2.22}$$

the weighted differentials are nilpotent, that is $d_a^2 = 0$ and $\delta_a^2 = 0$. Moreover,

$$d\Big(\exp(2a\sigma)(\omega \wedge \star\eta)\Big) = \exp(2a\sigma)\Big(d_a\omega \wedge \star\eta - \omega \wedge \star\delta_a\eta\Big) \ , \tag{2.23}$$

so that Stokes theorem yields the weighted generalisation of Green's formula, reading

$$\ll d_a\omega, \eta \gg_a = \ll \omega, \delta_a\eta \gg_a + \int_G \exp(2a\sigma) j^*(\omega \wedge \star\eta) \ . \tag{2.24}$$

For the usual Laplacian $\Delta = \delta d$ acting on scalars $g \in \Omega^0_c(G)$ this implies

$$\begin{aligned}\int_G \exp(2a\sigma)(\Delta g)d^n x = &\int_G (\Delta \exp(2a\sigma))g\, d^n x \\ &+ \int_{\partial G} \exp(2a\sigma) j^* \Big(2ag(\partial_r \sigma) \star R^\flat - \star dg\Big) .\end{aligned} \tag{2.25}$$

Finally we need to introduce the weighted Laplace operator

$$\Delta_a := \delta_a d_a + d_a \delta_a \ : \ \Omega^k(G) \longrightarrow \Omega^k(G) . \tag{2.26}$$

This is an elliptic operator on $\Omega^k(G)$, which is clear by observing that Δ_a differs from the unweighted Laplacian Δ only by lower order terms. Boundary value problems for elliptic operator are called elliptic, if the boundary operator satisfies the Lopatinskiĭ-Šapiro-condition, cf. [H2] ,[R-S]. In the context considered here the following results is relevant:

Lemma 2.1

The boundary value problem

$$\begin{aligned} \Delta_a \omega = \eta \qquad & \text{on } G \\ \mathbf{t}\omega = 0 \ \text{ and } \ \mathbf{t}\delta_a\omega = 0 \qquad & \text{on } \partial G \end{aligned} \tag{2.27}$$

on $\Omega^k(G)$ is elliptic in the sense of Lopatinskiĭ-Šapiro.

ind For the unweighted case a detailed computation can be found in [S2]. Since δ_a differs from the unweighted co-differential δ by lower order terms only, that result generalises to the boundary value problem (2.27) for the weighted operators.

3. A generalised Poincare inequality

The scalar theory of weighted Sobolev spaces is extensively studied in the literature, cf. [K]. Here we need a special generalisation of the weighted Poincaré inequality. We start with a modified version of the Hardy-Littlewood estimate.

Proposition 3.1

Let $\rho > 0$ and $e \neq -1$. Then there exists for $\epsilon' > 0$ a constant $C_{\epsilon'} \geq 0$ such that

$$\int_\rho^\infty |h(t)|^2 t^e dt \leq \left(\frac{2+\epsilon'}{e+1}\right)^2 \int_\rho^\infty |\partial_t h(t)|^2 t^{e+2} dt + C_{\epsilon'} \int_\rho^{\rho+1} |h(t)|^2 dt \tag{3.1}$$

for all compactly supported $h \in C^\infty_c([\rho,\infty))$. For $e > -1$ this holds with $C_{\epsilon'} = 0$. If $e < -1$ the estimate (3.1) also hold for $h \in C^\infty([\rho,\infty))$, which are not compactly supported.

Proof :

For $e < -1$ the classical Hardy-Littlewood inequality reads

$$\int_0^\infty |F(t)|^2 t^e dt \le \left(\frac{2}{e+1}\right)^2 \int_0^\infty |f(t)|^2 t^{e+2} dt \tag{3.2}$$

with $F(t) = \int_0^t |f(s)| ds$, which holds for all piecewise continuous $f : [0,\infty) \to \mathbb{R}$. Given $h \in C^\infty([\rho,\infty))$ let f_h be defined by

$$f_h(t) = \partial_t h(t) \text{ for } t \in [\rho,\infty) \quad \text{and} \quad f_h(t) = 0 \text{ for } t \in [0,\rho) . \tag{3.3}$$

Then $F_h(t) = \int_\rho^t |\partial_s h(s)| ds$, and

$$|h(t)|^2 = \left|h(\rho) + \int_\rho^t \partial_s h(s) ds\right|^2 \le \left(|h(\rho)| + |F_h(t)|\right)^2 \quad \text{for } t \in [\rho,\infty) . \tag{3.4}$$

Since $F_h(t) = 0$ for $t < \rho$, a weighted integration implies by using (3.2)

$$\begin{aligned} \int_\rho^\infty |h(t)|^2 t^e dt &\le (1+\epsilon^2) \int_0^\infty |F_h(t)|^2 t^e dt + (1+\frac{1}{\epsilon^2}) \int_\rho^\infty |h(\rho)|^2 t^e dt \\ &\le \left(\frac{2+2\epsilon}{e+1}\right)^2 \int_0^\infty |f_h(t)|^2 t^{e+2} dt + C_1 |h(\rho)|^2 . \end{aligned} \tag{3.5}$$

To estimate the second term let $I_\rho = [\rho, \rho+1]$. Since the embedding $H^1(I_\rho) \hookrightarrow C^0(I_\rho)$ is compact, there exists by Ehrling's inequality a constant C_2 such that

$$C_1 |h(\rho)|^2 \le C_1 \sup_{t \in I_\rho} |h|^2 \le \epsilon \|h\|^2_{H^1(I_\rho)} + C_2 \|h\|^2_{L^2(I_\rho)} . \tag{3.6}$$

By definition of f_h this proves that

$$\int_\rho^\infty |h(t)|^2 t^e dt \le \left(\frac{2+\epsilon'}{e+1}\right)^2 \int_\rho^\infty |\partial_t h(t)|^2 t^{e+2} dt + C_3 \int_\rho^{\rho+1} |h(t)|^2 dt . \tag{3.7}$$

For $e > -1$ the Hardy-Littlewood inequality (3.2) holds with $F(t) = \int_t^\infty |f(s)| ds$. By assumption $h(t)$ has compact support so that we get by using the same notation as above

$$|h(t)|^2 = \left| \int_t^\infty \partial_s h(s) ds \right|^2 \le |F_h(t)|^2 \quad \text{for } t \in [\rho,\infty) . \tag{3.8}$$

Then (3.2) implies

$$\int_0^\infty |F_h(t)|^2 t^e dt \le \left(\frac{2}{e+1}\right)^2 \int_0^\infty |f_h(t)|^2 t^{e+2} dt = \left(\frac{2}{e+1}\right)^2 \int_\rho^\infty |\partial_t h(t)|^2 t^{e+2} dt . \tag{3.9}$$

Since $F_h(t) = 0$ for $t < \rho$, this proves the result. □

Lemma 3.2

If $a \neq (1-\frac{n}{2})$ there exists for each $\epsilon > 0$ a constant $C_\epsilon \geq 0$ such that

$$\|g\|^2_{L^2_{a-1}} \leq \frac{1+\epsilon}{(a-1+n/2)^2} \sum_{j=1..n} \|\nabla_{E_j} g\|^2_{L^2_a} + C_\epsilon \|g\|^2_{L^2_{a-1}} \quad \forall g \in \Omega^0_c(G) . \tag{3.10}$$

Proof :

If B^r denotes the (open) ball of radius r in $\mathbb{R}^n$, let $\widehat{B}^r$ be its complement and S^r the corresponding sphere. For $g \in \Omega^0_c(G)$ we use polar coordinates and write $g(x) = g(r,\theta) =: h_\theta(r)$. Fixing ρ sufficiently big, such that $\widehat{B}^\rho \subset G$ we have

$$\int_{\widehat{B}^\rho} |g(x)|^2 r^{2a-2} d^n x = \int_{S^r} \Big(\int_\rho^\infty |h_\theta(r)|^2 r^{2a+n-3} dr \Big) d\theta . \tag{3.11}$$

With Proposition 3.1

$$\int_\rho^\infty |h_\theta(r)|^2 r^{2a+n-3} dr \leq \left(\frac{2+\epsilon'}{2a+n-2}\right)^2 \int_\rho^\infty |\partial_r h_\theta(r)|^2 r^{2a+n-1} dr + C_4 \int_\rho^{\rho+1} |h_\theta(r)|^2 r^{n-1} dr. \tag{3.12}$$

Using a radial frame $\mathcal{F}_R$, cf. (2.18), $|\partial_r h_\theta(r)|^2 = |\nabla_R g(x)|^2 \leq \sum_{j=1..n} |\nabla_{E_j} g(x)|^2$. Thus

$$\begin{aligned} &\int_{\widehat{B}^\rho} |g(x)|^2 r^{2a-2} d^n x \\ &\leq \left(\frac{2+\epsilon'}{2a+n-2}\right)^2 \int_{\widehat{B}^\rho} \sum_{j=1..n} |\nabla_{E_j} g(x)|^2 r^{2a} d^n x + C_4 \int_{(\widehat{B}^\rho \cap B^{\rho+1})} |g(x)|^2 d^n x . \end{aligned} \tag{3.13}$$

Moreover, for each power b and each $\rho > 0$ there exist constants c^ρ_b and C^ρ_b such that

$$c^\rho_b r^{2b} \leq \exp(2b\sigma) \leq C^\rho_b r^{2b} \quad \forall r \geq \rho . \tag{3.14}$$

By chosing ρ sufficiently big $C^\rho_{b-1}/c^\rho_b < (1+\epsilon')$ we can estimate

$$\begin{aligned} \|g\|^2_{L^2_{a-1}(G)} &\leq C_5 \|g\|^2_{L^2(B^\rho \cap G)} + C^\rho_{a-1} \int_{\widehat{B}^\rho} |g(x)|^2 r^{2a-2} d^n x \\ &\leq \frac{1+\epsilon}{(a-1+n/2)^2} \sum_{j=1..n} \int_{\widehat{B}^\rho} |\nabla_{E_j} g(x)|^2 \exp(2a\sigma) d^n x + (C_4 + C_5) \|g\|^2_{L^2(B^{\rho+1} \cap G)} . \end{aligned} \tag{3.15}$$

Finally, since $(B^{\rho+1} \cap G)$ is bounded

$$\|g\|^2_{L^2(B^{\rho+1} \cap G)} \leq C_6 \int_{B^{\rho+1} \cap G} |g(x)|^2 \exp(2(a-2)\sigma) d^n x \leq C_6 \|g\|^2_{L^2_{a-2}(G)} , \tag{3.16}$$

which proves the generalised Poincaré inequality (3.10). □

Since differential forms on G also can be considered as vector valued functions on G this estimate generalises to $\Omega^k_c(G)$. By completion in the H^1_{a-1} norm on $\Omega^k_c(G)$ we then get:

Theorem 3.3
If $G \subset I\!R^n$ is an exterior domain, and $a \neq (1-n/2)$, there exists for each $\epsilon > 0$ some C_ϵ such that

$$\|\omega\|^2_{L^2_{a-1}} \leq \frac{1+\epsilon}{(a-1+n/2)^2} \sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} + C_\epsilon \|\omega\|^2_{L^2_{a-2}} \quad \forall\, \omega \in H^1_{a-1}\Omega^k(G). \tag{3.17}$$

4. The weighted Dirichlet integral

The weighted Dirichlet integral we define as the map

$$\begin{aligned} &\mathcal{D}_a : H^1_{a-1}\Omega^k(G) \times H^1_{a-1}\Omega^k(G) \longrightarrow I\!R \\ &\mathcal{D}_a(\omega,\eta) = \ll d_a\omega, d_a\eta \gg_a \; + \; \ll \delta_a\omega, \delta_a\eta \gg_a \;. \end{aligned} \tag{4.1}$$

By construction, $\mathcal{D}_a$ is a symmetric continuous bilinear functional for each $a \in I\!R$. Our aim is to prove the H^1_{a-1} ellipticity of $\mathcal{D}_a$, that is to show that

$$\|\omega\|^2_{H^1_{a-1}} = \|\omega\|^2_{L^2_{a-1}} + \sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} \leq C(a,G)\, \mathcal{D}_a(\omega,\omega) \tag{4.2}$$

on an appropriate subspace of $H^1_{a-1}\Omega^k(G)$. First we show :

Lemma 4.1
(a) If $\omega \in \Omega^k_c(G)$, then

$$\sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} = \mathcal{D}_a(\omega,\omega) + c_1\|\omega\|^2_{L^2_{a-1}} - c_2\|\omega\|^2_{L^2_{a-2}} + \int_{\partial G} \mathcal{B}(\omega) \;, \tag{4.3}$$

where $c_1 = (a^2 + (n-2)a)$, $c_2 = (a^2 - 2a)$, and

$$\mathcal{B}(\omega) = \exp(2a\sigma) j^* \Big(-\tfrac{1}{2} \star d\langle\omega,\omega\rangle + d\omega \wedge \star\omega - \delta\omega \wedge \star\omega + a\langle\omega,\omega\rangle(\partial_r\sigma) \star R^\flat \Big) \,. \tag{4.4}$$

(b) There exists a constant $C > 0$ such that

$$\|\omega\|^2_{H^1_{a-1}} \leq C\Big(\mathcal{D}_a(\omega,\omega) + \|\omega\|^2_{L^2_{a-2}} + \Big|\int_{\partial G} \mathcal{B}(\omega)\Big|\Big) \quad \forall\, \omega \in \Omega^k_c(G)\,. \tag{4.5}$$

Proof :
(a) The identity (2.6) for the Laplace operator implies that

$$\Delta\langle\omega,\omega\rangle = 2\langle\Delta\omega,\omega\rangle - 2\sum_{j=1..n} \langle\nabla_{E_j}\omega, \nabla_{E_j}\omega\rangle \;. \tag{4.6}$$

By weighted integration over G and Eq. (2.25) we get

$$-\frac{1}{2}\Big(\int_G \Delta \exp(2a\sigma)\langle\omega,\omega\rangle d^n x + \int_{\partial G} \mathcal{B}_1(\omega)\Big) + \ll \Delta\omega,\omega \gg_a = \sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} \tag{4.7}$$

$$\text{where}\quad \mathcal{B}_1(\omega) = \exp(2a\sigma) j^*\Big(2a\langle\omega,\omega\rangle(\partial_r\sigma)\star R^\flat - \star d\langle\omega,\omega\rangle\Big)\ .$$

To rewrite $\Delta\omega = (\delta d + d\delta)\omega$ in terms of the weighted differentials d_a and δ_a we observe that

$$\begin{aligned}\delta d\omega &= \delta_a\big(d_a\omega - a(\partial_r\sigma)R^\flat\wedge\omega\big) + a(\partial_r\sigma)\mathbf{i}_R\big(d_a\omega - a(\partial_r\sigma)R^\flat\wedge\omega\big)\\ d\delta\omega &= d_a\big(\delta_a\omega + a(\partial_r\sigma)\mathbf{i}_R\omega\big) - a(\partial_r\sigma)R^\flat\wedge\big(\delta_a\omega - a(\partial_r\sigma)\mathbf{i}_R\omega\big)\ .\end{aligned} \tag{4.8}$$

Using Green's formula (2.24) we get

$$\begin{aligned}\ll \delta d\omega,\omega \gg_a = &\ll d_a\omega, d_a\omega \gg_a - a \ll (\partial_r\sigma)R^\flat\wedge\omega, d_a\omega \gg_a + a \ll (\partial_r\sigma)\mathbf{i}_R d_a\omega,\omega \gg_a\\ &- a^2 \ll (\partial_r\sigma)^2\mathbf{i}_R(R^\flat\wedge\omega),\omega \gg_a + \int_{\partial G}\mathcal{B}_2(\omega)\ ,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\ll d\delta\omega,\omega \gg_a = &\ll \delta_a\omega, \delta_a\omega \gg_a + a \ll (\partial_r\sigma)\mathbf{i}_R\omega, \delta_a\omega \gg_a - a \ll (\partial_r\sigma)R^\flat\wedge\delta_a\omega,\omega \gg_a\\ &- a^2 \ll (\partial_r\sigma)^2 R^\flat\wedge\mathbf{i}_R\omega,\omega \gg_a + \int_{\partial G}\mathcal{B}_3(\omega)\ ,\end{aligned} \tag{4.10}$$

where the boundary terms read

$$\mathcal{B}_2(\omega) = -\exp(2a\sigma)j^*\big(d\omega\wedge\star\omega\big) \quad\text{and}\quad \mathcal{B}_3(\omega) = \exp(2a\sigma)j^*\big(\delta\omega\wedge\star\omega\big)\ . \tag{4.11}$$

With (2.7) and (2.8) we then get

$$\ll \Delta\omega,\omega \gg_a = \mathcal{D}_a(\omega,\omega) - a^2 \ll (\partial_r\sigma)^2\omega,\omega \gg_a + \int_{\partial G}(\mathcal{B}_2(\omega) + \mathcal{B}_3(\omega))\ . \tag{4.12}$$

As far as (4.7) is concerned, we also have to control the contribution of the integral of $\Delta\exp(2a\sigma)\langle\omega,\omega\rangle$. From (2.14) we obtain

$$\begin{aligned}-\frac{1}{2}\Delta\exp(2a\sigma) &= \frac{1}{2}\Big(\partial_r\partial_r + \frac{n-1}{r}\partial_r\Big)\exp(2a\sigma)\\ &= \big(a(\partial_r^2\sigma) + 2a^2(\partial_r\sigma)^2 + a\frac{n-1}{r}(\partial_r\sigma)\big)\exp(2a\sigma)\\ &= \big(2a^2 + (n-2)a\big)\exp(2(a-1)\sigma) - (2a^2 - 2a)\exp(2(a-2)\sigma)\ .\end{aligned} \tag{4.13}$$

With the constants c_1 and c_2 given above this yields

$$-\frac{1}{2}\int_G \Delta\exp(2a\sigma)\langle\omega,\omega\rangle d^n x = \big(c_1 + a^2\big)\|\omega\|^2_{L^2_{a-1}} - \big(c_2 + a^2\big)\|\omega\|^2_{L^2_{a-2}}\ . \tag{4.14}$$

Adding the contributions of (4.12) and (4.14), then Eq. (4.7) implies the identity (4.3).

(b) The contribution of order $\|\omega\|^2_{L^2_{a-1}}$ in (4.3) can be estimated by Poincaré's inequality (3.17) as

$$c_1\|\omega\|^2_{L^2_{a-1}} \leq \gamma_a \sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} + C'\|\omega\|^2_{L^2_{a-2}}$$
$$\text{where} \quad \gamma_a = \frac{(1+\epsilon)(a^2+(n-2)a)}{(a-1+n/2)^2} \; . \tag{4.15}$$

For $n > 2$ and ϵ is sufficiently small one has $(\gamma_a - 1) < 0$ for all $a \in I\!R$, and (4.3) yields

$$0 < (1-\gamma_a) \sum_{j=1..n} \|\nabla_{E_j}\omega\|^2_{L^2_a} \leq \mathcal{D}_a(\omega,\omega) + C'\|\omega\|^2_{L^2_{a-2}} + \Big| \int_{\partial G} \mathcal{B}(\omega) \Big| \; . \tag{4.16}$$

The estimate (4.5) follows by using the Poincaré inequality (3.17) once more. □

Lemma 4.2

The boundary integral of Lemma 4.1 satisfies the estimate

$$\Big| \int_{\partial G} \mathcal{B}(\omega) \Big| \leq C\|\omega\|^2_{L^2(\partial G)} \qquad \forall\, \omega \in \Omega^k_*(G) \; , \tag{4.17}$$

where $\Omega^k_*(G)$ is either of the spaces $\Omega^k_D(G) \cap \Omega^k_c(G)$ or $\Omega^k_N(G) \cap \Omega^k_c(G)$ define in (2.12).

Proof :

If N is the unit normal field on ∂G, and $d^{n-1}_\partial x = \mathbf{i}_N d^n x$ is the associated volume form, the kernel of the boundary integral of (4.4) can be written as

$$\mathcal{B}(\omega) = \exp(2a\sigma)\Big(-\tfrac{1}{2}D\big[\langle\omega,\omega\rangle\big](N) + \langle \mathbf{i}_N d\omega, \omega\rangle - \langle\delta\omega, \mathbf{i}_N\omega\rangle + a\langle\omega,\omega\rangle(\partial_r\sigma)R^\flat(N)\Big)d^{n-1}_\partial x, \tag{4.18}$$

cf. [S2]. Using the frame $\mathcal{F}_N$, cf. (2.10), the boundary condition $\mathbf{t}\omega = 0$ implies that

$$\langle\omega, \mathbf{i}_N d\,\omega\rangle \;=\; 0 \tag{4.19}$$
$$\tfrac{1}{2}D\big[\langle\omega,\omega\rangle\big](N) \;=\; \langle(\nabla_N\omega),\omega\rangle \;=\; \langle\mathbf{i}_N(\nabla_N\omega), \mathbf{i}_N\omega\rangle \; . \tag{4.20}$$

Moreover it follows from (2.5) that

$$\langle\delta\omega, \mathbf{i}_N\omega\rangle \;=\; \langle\delta_\partial\omega, \mathbf{i}_N\omega\rangle - \langle\mathbf{i}_N(\nabla_N\omega), \mathbf{i}_N\omega\rangle \quad , \tag{4.21}$$

with δ_∂ as the co-differential on the boundary manifold ∂G. The second term on the right hand side of Eq. (4.21) cancels with (4.20). Thus we are left with $\langle\delta_\partial\omega, \mathbf{i}_N\omega\rangle$, where

$$\delta_\partial\,\omega(E_{j_2},\dots E_{j_k}) = -\sum_{l=1..(n-1)}\Big(\nabla_{E_l}\big(\omega(E_l,E_{j_2},\dots E_{j_k})\big) \tag{4.22}$$
$$+\,\omega(\partial_{E_l}E_l,E_{j_2},\dots E_{j_k}) + \sum_{i=2..k}\omega(E_l,E_{j_2},\dots,\partial_{E_l}E_{j_i},\dots E_{j_k})\Big) \; .$$

Since $\mathbf{t}\omega = 0$, the first term on the right hand side vanishes, and from the derivatives $\partial_{E_j} E_i$ only the normal components will contribute. These are described by the second fundamental form $\mathcal{K}$ of $\partial G \hookrightarrow G$. Since ∂G is smooth and compact, $\mathcal{K}$ is uniformly bounded, and it follows that

$$\Big| \int_{\partial G} \exp(2a\sigma)\langle \delta_\partial \omega, \mathbf{i}_N \omega \rangle d_\partial^{n-1}x \Big| \leq C_1 \| \omega \|^2_{L^2(\partial G)} . \tag{4.23}$$

For the remaining term of (4.18) we get

$$\Big| a \int_{\partial G} \exp(2a\sigma)\langle \omega, \omega \rangle (\partial_r \sigma) R^\flat(N) d_\partial^{n-1}x \Big| \leq C_2 \|\omega\|^2_{L^2(\partial G)} . \tag{4.24}$$

This proves (4.17) for $\Omega^k_D(G)$. As far as the boundary condition $\mathbf{n}\omega = 0$ is concerned, we observe that $\star$ intertwines the action of $\mathbf{n}$ and $\mathbf{t}$. Thus each $\omega \in \Omega^k_N(G)$ writes as $\omega = \star\eta$ with $\eta \in \Omega^{n-k}_D(G)$. Since $\mathcal{B}(\star\eta) = \mathcal{B}(\eta)$ the estimate (4.17) for $\Omega^k_N(G)$ follows from the corresponding result on $\Omega^{n-k}_D(G)$. □

From this we immediately infer Gaffney's inequality:

Theorem 4.3
If $G \subset \mathbb{R}^n$ is an exterior domain, and $a \neq (1 - n/2)$, there exists a constant $C_a > 0$ such that

$$\| \omega \|^2_{H^1_{a-1}} \leq C_a \big(\mathcal{D}_a(\omega, \omega) + \|\omega\|^2_{L^2_{a-2}} \big) \quad \forall \omega \in H^1_a \Omega^k_*(G) . \tag{4.25}$$

Here $H^1_a\Omega^k_*(G)$ is the completion of either of the two spaces $\Omega^k_D(G)$ or $\Omega^k_N(G)$ in the H^1_{a-1} norm.

Proof :
Since ∂G is compact, the restriction $\omega \mapsto \omega|_{\partial G}$ is a compact map from $H^1_{a-1}\Omega^k(G)$ to $L^2\Omega^k(G)|_{\partial G}$. The Ehrling lemma then implies that for each $\epsilon > 0$ there is a constant C_ϵ such that

$$\| \omega \|^2_{L^2(\partial G)} \leq \epsilon \| \omega \|^2_{H^1_{a-1}(G)} + C_\epsilon \| \omega \|^2_{L^2_{a-2}(G)} \quad \forall \omega \in H^1_{a-1}\Omega^k(G) . \tag{4.26}$$

Chosing ϵ sufficiently small, (4.25) for smooth differential forms follows as a direct consequence of Lemma 4.1 and 4.2. The assertion then follows by a completion in the H^1_{a-1} norm. □

For differential forms on a compact manifold with boundary the estimate (4.25) has first been shown in [G] and hence is referred to as Gaffney's inequality. In the notation of functional analysis [S1] it states in particular that the weighted Dirichlet integral is coercive on the Sobolev $H^1_{a-1}\Omega^k(G)$. For our approach on exterior domains it is essential that $\mathcal{D}_a(\omega, \omega)$ estimates the H^1_{a-1} norm modulo a contribution of order $\|\omega\|^2_{L^2_{a-2}}$. Since the embedding $H^1_{a-1}\Omega^k(G) \hookrightarrow L^2_{a-2}\Omega^k(G)$ is compact, cf. [L], implies also coercivity in the sense of calculus of variation on an appropriate subspace. This is shown in the next section.

5. Potentials of the weighted Dirichlet integral

Harmonic fields in $H^1_{a-1}\Omega^k(G)$ are characterised by the condition $\mathcal{D}_a(\lambda,\lambda)=0$. We write

$$\mathcal{H}^{k,D}_{a-1}(G)=\{\,\lambda\in H^1_{a-1}\Omega^k_D(G)\mid \mathcal{D}_a(\lambda,\lambda)=0\,\} \tag{5.1}$$

for the harmonic fields in $H^1_{a-1}\Omega^k(G)$ which satisfy the boundary condition $\mathbf{t}\lambda=0$. Since $\mathcal{D}_a$ is continuous, this is a closed subspace of $H^1_{a-1}\Omega^k_D(G)$. By Theorem 4.3 it is also a closed subspace of $L^2_{a-1}\Omega^k(G)$. The orthogonal complement of $\mathcal{H}^{k,D}_{a-1}(G)$ in $H^1_{a-1}\Omega^k_D(G)$ with respect to the weighted L^2_{a-1} scalar product $\ll,\gg_{a-1}$, cf. (2.15), we denote by

$$(\mathcal{H}^\perp)^{k,D}_{a-1}(G):=\{\,\omega\in H^1_{a-1}\Omega^k_D G)\mid\ \ll\omega,\kappa\gg_{a-1}=0\ \ \forall\kappa\in\mathcal{H}^{k,D}_{a-1}\,\}\,. \tag{5.2}$$

Then $H^1_{a-1}\Omega^k_D(G)=(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\oplus\mathcal{H}^{k,D}_{a-1}(G)$ and both components are Hilbert spaces.

Lemma 5.1

The Dirichlet integral $\mathcal{D}_a$ is H^1_{a-1}-elliptic on the space $(\mathcal{H}^\perp)^{k,D}_{a-1}(G)$. That is, there are positive constants c and C such that

$$c\,\|\,\omega\,\|^2_{H^1_{a-1}}\le\mathcal{D}_a(\omega,\omega)\le C\,\|\,\omega\,\|^2_{H^1_{a-1}}\qquad\forall\omega\in(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\ . \tag{5.3}$$

Proof :
Let η_i be a minimising sequence for $\mathcal{D}_a(\omega,\omega)$ in the unit sphere

$$S_{\mathcal{H}}:=\{\omega\in(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\mid\|\omega\|_{L^2_{a-2}}=1\}\,. \tag{5.4}$$

By (4.25), the sequence $\|\eta_j\|_{H^1_{a-1}}$ is bounded, and there exists a subsequence η_{j_l} such that $\eta_{j_l}\rightharpoonup\eta$ weakly in $H^1_{a-1}\Omega^k(G)$. By its construction $\eta\in(\mathcal{H}^\perp)^{k,D}_{a-1}(G)$. Since $\mathcal{D}_a(\omega,\omega)$ is weakly lower semicontinuous on $H^1_{a-1}\Omega^k(G)$ we infer that

$$\mathcal{D}_a(\omega,\omega)\ge\mathcal{D}_a(\eta,\eta)\cdot\|\omega\|^2_{L^2_{a-2}}\qquad\forall\omega\in(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\,. \tag{5.5}$$

As shown in [L], the embedding $H^1_{a-1}\Omega^k(G)\hookrightarrow L^2_{a-2}\Omega^k(G)$ is compact. Therefore $\eta_j\to\widehat{\eta}$ (strongly) in $L^2_{a-2}\Omega^k(G)$, up to the selection of a subsequence. The uniqueness of the weak limit then implies that $\eta=\widehat{\eta}\in S_{\mathcal{H}}$, so that $\|\eta\|_{L^2_{a-2}}=1$ and $\mathcal{D}_a(\eta,\eta)>0$. With (5.5) we then get from (4.25)

$$\|\,\omega\,\|^2_{H^1_{a-1}}\le C_a\Big(1+\frac{1}{\mathcal{D}_a(\eta,\eta)}\Big)\,\mathcal{D}_a(\omega,\omega)\quad\forall\omega\in(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\,. \tag{5.6}$$

Since $\mathcal{D}_a$ is continuous on $H^1_{a-1}\Omega^k(G)$ this prove the H^1_{a-1}-ellipticity. □

In the language of calculus of variations this means that $\mathcal{D}_a(\omega,\omega)$ is a coercive quadratic functional on the subspace $(\mathcal{H}^\perp)^{k,D}_{a-1}(G)\subset H^1_{a-1}\Omega^k_D(G)$. Then we have:

Theorem 5.2

If $G \subset I\!R^n$ is an exterior domain and $a \neq (1 - n/2)$, there exists for each $\eta \in L^2_{a+1}\Omega^k(G)$ satisfying the integrability condition

$$\ll \eta, \lambda \gg_a = 0 \qquad \forall \lambda \in \mathcal{H}^{k,D}_{a-1}(G) \ , \tag{5.7}$$

a unique potential $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G) \subset H^1_{a-1}\Omega^k_D(G)$, such that

$$\mathcal{D}_a(\phi_D, \xi) = \ll \eta, \xi \gg_a \quad \forall \xi \in H^1_{a-1}\Omega^k_D(G) \ . \tag{5.8}$$

Proof :
Since $\mathcal{D}_a$ is elliptic on the Hilbert space $(\mathcal{H}^\perp)^{k,D}_{a-1}(G)$, the Lax-Milgram lemma, [S1] guarantees for each bounded linear functional $\mathcal{F} : (\mathcal{H}^\perp)^{k,D}_{a-1}(G) \to I\!R$ the existence of some $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ such that

$$\mathcal{F}(\widetilde{\xi}) = \mathcal{D}_a(\phi_D, \widetilde{\xi}) \qquad \forall \widetilde{\xi} \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G) \ . \tag{5.9}$$

In particular we can choose $\mathcal{F}(.) = \ll \eta, . \gg_a$ with $\eta \in L^2_{a+1}\Omega^k(G)$. Then ϕ_D solves (5.8), but only for $\widetilde{\xi} \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$. An arbitrary $\xi \in H^1_{a-1}\Omega^k_D(G)$ splits into

$$\xi = \widetilde{\xi} + \lambda_\xi \ \text{ where } \ \widetilde{\xi} \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G) \ \text{and} \ \lambda_\xi \in \mathcal{H}^{k,D}_{a-1}(G) \ . \tag{5.10}$$

If $\eta \in L^2_{a+1}\Omega^k(G)$ satisfies the integrability condition (5.7), then

$$\mathcal{D}_a(\phi_D, \xi) = \mathcal{D}_a(\phi_D, \widetilde{\xi}) = \ll \eta, \widetilde{\xi} \gg_a = \ll \eta, \xi \gg_a \ . \tag{5.11}$$

This proves the existence of the potential $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ satisfying (5.8). To show uniqueness let $\phi'_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ be another solution of (5.8). Then $\mathcal{D}_a((\phi_D - \phi'_D), \xi) = 0$ for all $\xi \in H^1_{a-1}\Omega^k_D(G)$. Therefore, $(\phi_D - \phi'_D) \in \mathcal{H}^{k,D}_{a-1}(G)$, which proves that $(\phi_D - \phi'_D) = 0$.
□

By using (formally) Green's formula (2.24) we get from Eq. (5.8)

$$\ll \eta, \xi \gg_a = \ll \Delta_a \phi_D, \xi \gg_a + \int_{\partial G} \exp(2a\sigma) j^*(\xi \wedge \star d_a \phi_D - \delta_a \phi_D \wedge \star \xi) \ , \tag{5.12}$$

holding for all $\xi \in H^1_{a-1}\Omega^k_D(G)$. Here Δ_a is the weighted Laplace operator (2.26). Since $j^*(\xi \wedge \eta) = (j^*\xi) \wedge (j^*\eta)$ the first boundary integral vanishes. Since $H^1_{a-1}\Omega^k_D(G) \subset L^2_{a-1}\Omega^k(G)$ is dense, this shows that $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ is weak solution of the boundary value problem

$$\begin{aligned} \Delta_a \phi_D &= \eta && \text{on } G \\ \mathbf{t}\phi_D = 0 \ &\text{and} \ \mathbf{t}\delta_a \phi_D = 0 && \text{on } \partial G \ . \end{aligned} \tag{5.13}$$

By Lemma 2.1 this is an elliptic problem in the sense of Lopatinskiĭ-Šapiro. Although the standard theory for elliptic systems does not apply to exterior domain problems, cf. [N-W], it has a weighted generalisation, which does. More precise, if $H^s_a(G; V)$ denotes the weighted Sobolev space of distributions on an exterior domain G with values in a vector space V, then [L-M] have shown:

Theorem 5.3
Given a differential operator $A : C^\infty(G;V) \to C^\infty(G;V)$ of order 2 and a boundary operator B such that the system (A, B) is Lopatinskiĭ-Šapiro elliptic. If a distribution χ satisfy the homogeneous boundary condition $B\chi = 0$, then

$$\chi \in H^{s+2}_{a-1}(G;V) \iff A\chi \in H^s_{a+1}(G;V) \tag{5.14}$$

preassumed that the weight parameter a is not exceptional, that is $(a - n/2) \notin \mathbb{Z}$ or $a \in (-\frac{n}{2}+1, \frac{n}{2}-1)$. Moreover, an elliptic a-priori estimate is satisfied, that is

$$\| \chi \|_{H^{s+2}_{a-1}} \leq C \left(\| A\chi \|_{H^s_{a+1}} + \| \chi \|_{L^2_{a+1}} \right) \quad \forall \chi \in H^{s+2}_{a-1}(G;V) \,. \tag{5.15}$$

If $A\chi$ is smooth then $\chi \in C^\infty(G;V)$, too.

One might observe that though for even dimension n the integers $a \in \mathbb{Z}$ are typically exceptional weight parameters. However, $a = 0$ is in any case not exceptional. Since $\Omega^k(G) = C^\infty(G; \bigwedge^k(\mathbb{R}^n))$ we infer from this and the ellipticity of the boundary value problem (5.13):

Corollary 5.4
Let a be not exceptional. For each $\eta \in L^2_{a+1}\Omega^k(G)$ satisfying the integrability condition (5.7) the potential ϕ_D constructed in Theorem 5.2 is a classical solution of the boundary value problem (5.13), i.e. $\phi_D \in H^2_{a-1}\Omega^k(G)$. Moreover, if $\eta \in H^s_{a+1}\Omega^k(G)$, then

$$\|\phi_D\|_{H^{s+2}_{a-1}} \leq C \left(\|\eta\|_{H^s_{a+1}} + \|\phi_D\|_{L^2_{a-1}} \right) \,, \tag{5.16}$$

and if η is smooth, then $\phi_D \in \Omega^k(G)$, too.

The corresponding results can be obtained on $\Omega^k_N(G)$, that is under the boundary condition $\mathbf{n}\omega = 0$. With

$$\mathcal{H}^{k,N}_{a-1}(G) := \{ \lambda \in H^1_{a-1}\Omega^k_N(G) \mid \mathcal{D}_a(\lambda, \lambda) = 0 \} \tag{5.17}$$

and the orthogonal complement $(\mathcal{H}^\perp)^{k,N}_{a-1}(G)$ satisfying

$$H^1_{a-1}\Omega^k_N(G) = (\mathcal{H}^\perp)^{k,N}_{a-1} \oplus \mathcal{H}^{k,N}_{a-1}(G) \,. \tag{5.18}$$

Then all constructions based on Theorem 4.3 can be literally repeated to prove:

Theorem 5.5
If $G \subset I\!R^n$ is an exterior domain and $a \neq (1 - n/2)$, there exists for each $\eta \in L^2_{a+1}\Omega^k(G)$ satisfying the integrability condition

$$\ll \eta, \lambda \gg_a = 0 \qquad \forall \lambda \in \mathcal{H}^{k,N}_{a-1}(G) \ , \tag{5.19}$$

a unique potential $\phi_N \in (\mathcal{H}^\perp)^{k,N}_{a-1}(G)$ such that

$$\mathcal{D}_a(\phi_N, \xi) = \ll \eta, \xi \gg_a \quad \forall \xi \in H^1_{a-1}\Omega^k_N(G) \ . \tag{5.20}$$

Moreover, $\phi_N \in \mathcal{H}^2_{a-1}\Omega^k_N(G)$ and it is a classical solution of the boundary value problem

$$\begin{aligned} \Delta_a \phi_N &= \eta && \text{on } G \\ \mathbf{n}\phi_N = 0 \ \text{ and } \ \mathbf{n} d_a \phi_N &= 0 && \text{on } \partial G \ . \end{aligned} \tag{5.21}$$

If $\eta \in H^s_{a+1}\Omega^k(G)$, and a is not exceptional, then

$$\|\phi_N\|_{H^{s+2}_{a-1}} \leq C\left(\|\eta\|_{H^s_{a+1}} + \|\phi_N\|_{L^2_{a-1}}\right) , \tag{5.22}$$

and if η is smooth, then $\phi_N \in \Omega^k(G)$, too.

We finish this study of potentials corresponding to the Dirichlet integral $\mathcal{D}_a$ with the following observation :

Lemma 5.6
(a) If $\eta = \delta_a \omega$ with $\omega \in H^1_a\Omega^{k+1}(G)$, then η satisfies the integrability condition (5.7) of Theorem 5.2 and the corresponding potential $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ is co-closed, i.e. $\delta_a\phi_D = 0$.
(b) If $\eta = d_a\omega$ with $\omega \in H^1_a\Omega^{k-1}(G)$, then η satisfies the integrability condition (5.19) of Theorem 5.5 and the corresponding potential $\phi_N \in (\mathcal{H}^\perp)^{k,N}_{a-1}(G)$ is closed, i.e. $d_a\phi_N = 0$.

Proof :
From Green's formula (2.24) we infer that

$$\ll \delta_a\omega, \lambda \gg_a = \ll \omega, d_a\lambda \gg_a = 0 \quad \forall \lambda \in \mathcal{H}^{k,D}_{a-1}(G) \ . \tag{5.23}$$

Hence, there exists by Theorem 5.2 a (unique) potential $\phi_D \in H^2_{a-1}\Omega^k(G)$ for $\eta = \delta_a\omega$. Since $d_a\delta_a\phi_D \in L^2_{a+1}\Omega^k(G) \subset L^2_a\Omega^k(G)$ we get from (5.13)

$$\|d_a\delta_a\phi_D\|^2_{L^2_a} = \ll d_a\delta_a\phi_D, \delta_a\big(\omega - d_a\phi_D\big) \gg_a \ . \tag{5.24}$$

With (2.24) and the boundary condition $\mathbf{t}\delta_a\phi_D = 0$ this implies that $d_a\delta_a\phi_D = 0$. Since $\mathbf{t}\phi_D = 0$, then also

$$\|\delta_a\phi_D\|^2_{L^2_a} = \ll d_a\delta_a\phi_D, \phi_D \gg_a = 0 \ . \tag{5.25}$$

This proves the assertion of (a). Part (b) is shown in literally the same way with the roles of d_a and δ_a respectively of $\mathbf{t}$ and $\mathbf{n}$ interchanged. □

6. The Hodge decomposition

To formulate the Hodge decomposition we consider the subspace $\mathcal{E}_a^k(G) \subset \Omega^k(G)$ of smooth k-forms which are the weighted exterior derivative of some $\alpha \in \Omega^{k-1}(G)$ with vanishing tangential component and a finite H^1_{a-1} norm, that is

$$\mathcal{E}_a^k(G) := \{d_a\alpha \mid \alpha \in \Omega^{k-1}(G)\ ,\ \mathbf{t}\alpha = 0\ ,\ \|\alpha\|_{H^1_{a-1}} < \infty\}\ . \tag{6.1}$$

Correspondingly we denote by $\mathcal{C}_a^k(G)$ as the space of smooth δ_a-coexact forms with vanishing normal component, i.e.

$$\mathcal{C}_a^k(G) := \{\, \delta_a\beta \mid \beta \in \Omega^{k+1}(G)\ ,\ \mathbf{n}\beta = 0\ ,\ \|\beta\|_{H^1_{a-1}} < \infty\}\ . \tag{6.2}$$

The space of smooth (d_a, δ_a)-harmonic and weighted square integrable fields we denote by

$$\mathcal{N}_a^k(G) := \{\lambda \in \Omega^k(G) \mid d_a\lambda = 0\ ,\ \delta_a\lambda = 0\ ,\ \|\lambda\|_{L^2_a} < \infty\}\ . \tag{6.3}$$

Proposition 6.1

The spaces $\mathcal{E}_a^k(G)$, $\mathcal{C}_a^k(G)$ and $\mathcal{N}_a^k(G)$, are mutual orthogonal to each other with respect to the weighted scalar product $\ll, \gg_a$. Moreover, $\mathcal{N}_a^k(G)$ is the orthogonal complement of $\mathcal{E}_a^k(G) \oplus \mathcal{C}_a^k(G)$ in the space of smooth weighted square integrable k-forms, that is

$$\mathcal{N}_a^k(G) = \{\kappa \in \Omega^k(G) \mid \|\kappa\|_{L^2_a} < \infty\ ,\ \ll \eta, \kappa \gg_a = 0\ \forall \eta \in \left(\mathcal{E}_a^k(G) \oplus \mathcal{C}_a^k(G)\right)\}\ . \tag{6.4}$$

Proof :
As an immediate consequence of the boundary conditions $\mathbf{t}\alpha = 0$ and $\mathbf{n}\beta = 0$ and the nilpotence of d_a and δ_a we infer from Green's formula (2.24) and the definition of the space $\mathcal{N}_a^k(G)$ that

$$\ll d_a\alpha, \delta_a\beta \gg_a = 0\ ,\ \ll d_a\alpha, \lambda \gg_a = 0 \quad \text{and} \quad \ll \delta_a\beta, \lambda \gg_a = 0 \tag{6.5}$$

for all $d_a\alpha \in \mathcal{E}_a^k(G)$, $\delta_a\beta \in \mathcal{C}_a^k(G)$ and $\lambda \in \mathcal{N}_a^k(G)$. This proves the mutual L^2_a orthogonality of the spaces. In particular $\mathcal{N}_a^k(G)$ is a subset of the L^2_a orthogonal complement of $\mathcal{E}_a^k(G) \oplus \mathcal{C}_a^k(G)$ in the space of smooth k-forms. On the other hand, let κ be an arbitrary smooth square integrable form in that complement. Orthogonality to $\mathcal{E}_a^k(G)$ implies that

$$0 = \ll \kappa, d_a\alpha \gg_a = \ll \delta_a\kappa, \alpha \gg_a \quad \forall \alpha \in \Omega_D^{k-1}(G) \text{ with } \|\alpha\|_{H^1_{a-1}} < \infty\ . \tag{6.6}$$

Since these differential forms α constitute a dense subspace of $L^2_{a-1}\Omega^{k-1}(G)$, this shows that $\delta_a\kappa = 0$. Similarly it follows from $\ll \kappa, \delta_a\beta \gg_a = 0$ that $d_a\kappa = 0$. Therefore $\kappa \in \mathcal{N}_a^k(G)$, which proves Eq. (6.4). $\square$

Theorem 6.2

Let $G \subset \mathbb{R}^n$ be an exterior domain and $L^2_a\Omega^k(G)$ the space of weighted square integrable k-forms, with a non-exceptional, i.e. $(a-n/2) \notin \mathbb{Z}$ or $a \in (-\frac{n}{2}+1, \frac{n}{2}-1)$. Then $L^2_a\Omega^k(G)$ splits into the direct sum

$$L^2_a\Omega^k(G) = L^2_a\mathcal{E}^k_a(G) \oplus L^2_a\mathcal{C}^k_a(G) \oplus L^2_a\mathcal{N}^k_a(G) \tag{6.7}$$

of the L^2_a-closure of the spaces $\mathcal{E}^k_a(G)$, $\mathcal{C}^k_a(G)$ and $\mathcal{N}^k_a(G)$. In particular, each $\omega \in L^2_a\Omega^k(G)$ uniquely decomposes into

$$\omega = d_a\alpha_\omega + \delta_a\beta_\omega + \lambda_\omega \quad \text{with} \quad \alpha_\omega \in H^1_{a-1}\Omega^{k-1}_D(G) \ , \ \beta_\omega \in H^1_{a-1}\Omega^{k+1}_N(G) \ , \tag{6.8}$$

such that $\|\alpha_\omega\|_{H^1_{a-1}} \le C\,\|\omega\|_{L^2_a}$ and $\|\beta_\omega\|_{H^1_{a-1}} \le C\,\|\omega\|_{L^2_a}$.

Proof :
If $\omega \in \Omega^k(G)$ is a smooth differential form with $\|\omega\|_{H^1_a} < \infty$, then

$$\delta_a\omega \in L^2_{a+1}\Omega^{k-1}(G) \quad \text{and} \quad \ll \delta_a\omega, \kappa \gg_a = 0 \quad \forall \kappa \in \mathcal{H}^{k,D}_{a-1}(G) \ . \tag{6.9}$$

Hence $\delta_a\omega$ satisfies the integrability condition (5.7) of Theorem 5.2, and there exists a potential $\phi_D \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G) \subset H^1_{a-1}\Omega^{k-1}_D(G)$ such that

$$\ll \delta_a\omega, \xi \gg_a = \mathcal{D}_a(\phi_D, \xi) \quad \forall \xi \in H^1_{a-1}\Omega^{k-1}_D(G) \ . \tag{6.10}$$

By Corollary 5.4, ϕ_D is smooth. Similarly, by Theorem 5.5, the exterior derivative $d_a\omega$ determines a smooth potential $\phi_N \in (\mathcal{H}^\perp)^{k,N}_{a-1}(G) \subset H^1_{a-1}\Omega^{k+1}_N(G)$ such that

$$\ll d_a\omega, \xi \gg_a = \mathcal{D}_a(\phi_N, \xi) \quad \forall \xi \in H^1_{a-1}\Omega^{k+1}_N(G) \ . \tag{6.11}$$

Choosing $\alpha_\omega := \phi_D$ and $\beta_\omega := \phi_N$, then $d_a\alpha_\omega \in \mathcal{E}^k_a(G)$ and $\delta_a\beta \in \mathcal{C}^k_a(G)$, and we may set

$$\lambda_\omega := \left(\omega - d_a\alpha_\omega - \delta_a\beta_\omega\right) \in \Omega^k(G) \ . \tag{6.12}$$

By construction, $\|\lambda_\omega\|_{L^2_a} < \infty$. With Theorem 5.2 we get for an arbitrary $d_a\widetilde{\alpha} \in \mathcal{E}^k_a(G)$

$$\ll \lambda_\omega, d_a\widetilde{\alpha} \gg_a = \ll \left(\omega - d_a\alpha_\omega\right), d_a\widetilde{\alpha} \gg_a = \mathcal{D}_a(\phi_D, \widetilde{\alpha}) - \ll d_a\alpha_\omega, d_a\widetilde{\alpha} \gg_a \ . \tag{6.13}$$

Since $\alpha_\omega = \phi_D$ and $\delta_a\phi_D = 0$ by Lemma 5.6, this implies that λ_ω is orthogonal to $\mathcal{E}^k_a(G)$. Similarly

$$\ll \lambda_\omega, \delta\widetilde{\beta} \gg_a = 0 \qquad \forall \delta\widetilde{\beta} \in \mathcal{C}^k_a(G) \ . \tag{6.14}$$

From Proposition 6.1 we then infer that $\lambda_\omega \in \mathcal{N}^k_a(G)$, and hence $\omega \in \Omega^k(G)$ splits into

$$\omega = d_a\alpha_\omega + \delta_a\beta_\omega + \lambda_\omega \quad \text{with} \quad d_a\alpha_\omega \in \mathcal{E}^k_a(G) \ \ \delta_a\beta_\omega \in \mathcal{C}^k_a(G) \ \ \lambda_\omega \in \mathcal{N}^k_a(G) \ . \tag{6.15}$$

Now consider $\omega \in L^2_a\Omega^k(G)$. There exists a sequence $\omega_j \in \Omega^k(G)$ with $\|\omega_j\|_{H^1_a} < \infty$ such that $\omega_j \to \omega$ in $L^2_a\Omega^k(G)$. We split $\omega_j = d_a\alpha_j + \delta_a\beta_j + \lambda_j$. By Lemma 5.6, $\alpha_j \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$

and $\delta_a \alpha_j = 0$. From Lemma 5.1 and the orthogonality of the decomposition we infer that

$$\|\alpha_j - \alpha_i\|^2_{H^1_{a-1}} \leq C_1 \mathcal{D}_a\big((\alpha_j - \alpha_i), (\alpha_j - \alpha_i)\big) = C_1 \ll (d_a\alpha_j - d_a\alpha_i), (\omega_j - \omega_i) \gg_a , \tag{6.16}$$

and by the Cauchy-Schwarz inequality

$$\|\alpha_j - \alpha_i\|_{H^1_{a-1}} \leq C_2 \|\omega_j - \omega_i\|_{L^2_a} . \tag{6.17}$$

Therefore α_j is a Cauchy sequence, and hence $\alpha_j \to \alpha_\omega$ in $H^1_{a-1}\Omega^{k-1}_D(G)$ such that $\|\alpha_\omega\|_{H^1_{a-1}} \leq C\,\|\omega\|_{L^2_a}$. Similarly the construction above determines a sequence $\beta_j \to \beta_\omega$ in $H^1_{a-1}\Omega^{k+1}_N(G)$ such that $\delta_a\beta_\omega$ is the $\mathcal{C}^k_a$ component of ω, and satisfies $\|\beta_\omega\|_{H^1_{a-1}} \leq C\,\|\omega\|_{L^2_a}$. Then also the convergence of $\lambda_j \to \lambda_\omega$ in $L^2_a\mathcal{N}^k_a(G)$ is guaranteed. □

As far as the higher order Sobolev spaces $H^s_a\Omega^k(G)$ are concerned we have the following regularity result for the Hodge-Morrey decomposition:

Lemma 6.3

If $\omega \in H^s_a\Omega^k(G)$, then the components of the decomposition (6.8) are determined by differential forms $\alpha_\omega \in H^{s+1}_{a-1}\Omega^{k-1}_D(G)$ and $\beta_\omega \in H^{s+1}_{a-1}\Omega^{k+1}_N(G)$ which satisfy

$$\|\alpha_\omega\|_{H^{s+1}_{a-1}} \leq C\|\omega\|_{H^s_a} \quad \text{and} \quad \|\beta_\omega\|_{H^{s+1}_{a-1}} \leq C\|\omega\|_{H^s_a} . \tag{6.18}$$

Proof :
The differential form $\alpha_\omega \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ which determines the component of ω in $L^2_a\mathcal{E}^k_a(G)$ is by construction the unique solution of Eq. (6.10). For $\omega \in H^s_a\Omega^k(G)$ this is equivalent to the boundary value problem

$$\begin{aligned} &\Delta_a\alpha_\omega = \delta_a\omega && \text{on } G \\ &\mathrm{t}\alpha_\omega = 0 \quad \text{and} \quad \mathrm{t}\delta_a\alpha_\omega = 0 && \text{on } \partial G . \end{aligned} \tag{6.19}$$

The elliptic estimate (5.16) of Corollary 5.4 then implies that

$$\|\alpha_\omega\|_{H^{s+1}_{a-1}} \leq C\big(\|\delta_a\omega\|_{H^{s-1}_{a+1}} + \|\alpha_\omega\|_{L^2_{a-1}}\big) . \tag{6.20}$$

Moreover, by Theorem 6.2, $\|\alpha_\omega\|_{L^2_{a-1}} \leq \|\omega\|_{L^2_a}$, which proves the estimate (6.18) for α_ω. The result for β is shown correspondingly. □

Conversely, the Hodge-Morrey decomposition allows to estimate the differential form ω belonging to certain subspaces of $H^s_a\Omega^k(G)$ uniformly by their exterior derivative and its co-differential. As the first result of this type we have:

Lemma 6.4
If $s \geq 1$ and a is not exceptional, then

$$\|\omega\|_{H^s_a} \leq C\Big(\|d_a\omega\|_{H^{s-1}_{a+1}} + \|\delta_a\omega\|_{H^{s-1}_{a+1}}\Big) \quad \forall \omega \in H^s_a\mathcal{E}^k_a(G) \oplus H^s_a\mathcal{C}^k_a(G), \tag{6.21}$$

with a universal constant C depending on s, a and the geometry of ∂G.

Proof :
Since $\omega \in H^s_a\Omega^k(G)$ is L^2_a-orthogonal to the space $\mathcal{N}^k_a(G)$, the decomposition (6.8) yields

$$\|\omega\|_{H^s_a} \leq \|d_a\alpha_\omega\|_{H^s_a} + \|\delta_a\beta_\omega\|_{H^s_a} \leq \|\alpha_\omega\|_{H^{s+1}_{a-1}} + \|\beta_\omega\|_{H^{s+1}_{a-1}} . \tag{6.22}$$

By construction α_ω is solution of the boundary value problem(6.19), and the corresponding elliptic estimate (6.20) implies that

$$\|\alpha_\omega\|_{H^{s+1}_{a-1}} \leq C_1\big(\|\delta_a\omega\|_{H^{s-1}_{a+1}} + \|\alpha_\omega\|_{L^2_{a-1}}\big) . \tag{6.23}$$

By construction, $\alpha_\omega \in (\mathcal{H}^\perp)^{k,D}_{a-1}(G)$ and $\delta_a\alpha_\omega = 0$. From the estimate (5.3) of Lemma 5.1 and the L^2_a orthogonality of (6.8) we then infer that

$$\|\alpha_\omega\|^2_{L^2_{a-1}} \leq C_2\mathcal{D}_a(\alpha_\omega, \alpha_\omega) = C_2 \ll d_a\alpha_\omega, \omega \gg_a . \tag{6.24}$$

Therefore, by Green's formula (2.24),

$$\|\alpha_\omega\|^2_{L^2_{a-1}} \leq C_3 \ll \alpha_\omega, \delta_a\omega \gg_a \leq C_3\|\alpha_\omega\|_{L^2_{a-1}}\|\delta_a\omega\|_{L^2_{a+1}} , \tag{6.25}$$

which proves that $\|\alpha_\omega\|_{L^2_{a-1}} \leq C_3\|\delta_a\omega\|_{H^{s-1}_{a+1}}$. Similar apply to β_ω, so that

$$\|\beta_\omega\|_{L^2_{a-1}} \leq C_4\|d_a\omega\|_{H^{s-1}_{a+1}} . \tag{6.26}$$

In view of (6.22) this proves the result. □

The estimate (6.21) provides us with a special version of what is called Korn's inequality in continuum mechanics. In the differential form calculus such type of inequalities we first studied by Friedrichs [F], who's own investigations into the Hodge decomposition was motivated by this problem. For exterior domains $G \subset I\!R^3$ a recent result is found in [W2], where the corresponding estimate for the unweighted L^p norm on the spaces $\Omega^1_D(G)$ and $\Omega^k_N(G)$ is given, cf. also the following section.

7. Decomposition results under boundary conditions

Corresponding decomposition results can be established also for the spaces $\Omega^k_D(G)$ and $\Omega^k_N(G)$ of differential forms satisfying the boundary conditions $\mathbf{t}\omega = 0$ and $\mathbf{n}\omega = 0$, respectively. We start with defining the spaces

$$\mathcal{N}^{k,D}_a(G) = \mathcal{N}^k_a(G) \cap \Omega^k_D(G) \quad \text{and} \quad \mathcal{N}^{k,N}_a(G) = \mathcal{N}^k_a(G) \cap \Omega^k_N(G) \tag{7.1}$$

of smooth harmonic fields in $\Omega^k(G)$ which satisfy the respective boundary conditions. Their completions in $H^1_a\Omega^k(G)$ are denoted by $H^1_a\mathcal{N}^{k,D}_a(G)$ and $H^1_a\mathcal{N}^{k,N}_a(G)$. The inclusion $H^1_a\Omega^k(G) \subset H^1_{a-1}\Omega^k(G)$ implies that these spaces are contained in the spaces $\mathcal{H}^{k,D}_{a-1}(G)$ respectively $\mathcal{H}^{k,N}_{a-1}(G)$, discussed in section 4, that is $H^1_a\mathcal{N}^{k,D}_a(G) \subset \mathcal{H}^{k,D}_{a-1}(G)$ and $H^1_a\mathcal{N}^{k,N}_a(G) \subset \mathcal{H}^{k,N}_{a-1}(G)$.

Theorem 7.1

If $G \subset I\!R^n$ is an exterior domain and $a \neq (1 - n/2)$, the spaces $H^1_a\mathcal{N}^{k,D}_a(G)$ and $H^1_a\mathcal{N}^{k,N}_a(G)$ are finite dimensional. Moreover, if a is not exceptional, all their elements are smooth differential forms.

Proof :
Let $D_\mathcal{H} = \{\lambda \in \mathcal{H}^{k,D}_{a-1}(G) \mid \|\lambda\|_{H^1_{a-1}} \leq 1\}$ be the unit disk in $\mathcal{H}^{k,D}_{a-1}(G)$. By Gaffney's inequality (4.25) the H^1_{a-1} norm and the L^2_{a-2} norm are equivalent on $D_\mathcal{H}$, that is

$$\|\lambda\|^2_{H^1_{a-1}} \leq C_a \|\lambda\|^2_{L^2_{a-2}} \quad \forall \lambda \in D_\mathcal{H} . \tag{7.2}$$

Thus $D_\mathcal{H}$ is closed in the L^2_{a-2}topology. Since the embedding $H^1_{a-1}\Omega^k(G) \hookrightarrow L^2_{a-2}\Omega^k(G)$ is compact, this implies that $D_\mathcal{H}$ is compact. Therefore the space $\mathcal{H}^{k,D}_{a-1}(G)$ is finite dimensional, and so is its subspace $H^1_a\mathcal{N}^{k,D}_a(G)$. Moreover, each $\lambda \in H^1_a\mathcal{N}^{k,D}_a(G)$ (weakly) satisfies the elliptic boundary value problem

$$\begin{aligned} &\Delta_a\lambda = 0 && \text{on } G \\ &\mathbf{t}\lambda = 0 \ \text{ and } \ \mathbf{t}\delta_a\lambda = 0 && \text{on } \partial G . \end{aligned} \tag{7.3}$$

By Theorem 5.3 this is also a strong solution, which hence is smooth. For $H^1_a\mathcal{N}^{k,N}_a(G)$ the same argument applies. □

For the basic weight parameter $a = 0$ the spaces of harmonic fields in $\Omega^k_D(G)$ on exterior domains with (possibly non-smooth) boundary have been studies extensively in [D] and [P]. In particular, these authors show, how to relate the dimensions of $\mathcal{N}^{k,D}_a(G)$ and $\mathcal{N}^{k,N}_a(G)$ to the Betti number of the domain G, cf. also [S2].

Turning toward the Hodge decomposition under boundary conditions we need to define the space

$$\widehat{\mathcal{C}}^k_a(G) := \{\, \delta_a\widehat{\beta} \mid \widehat{\beta} \in \Omega^{k+1}(G) \ , \ \mathbf{t}\widehat{\beta} = 0 \ , \ \mathbf{t}\delta_a\widehat{\beta} = 0 \ , \ \|\widehat{\beta}\|_{H^1_{a-1}} < \infty \} . \tag{7.4}$$

The the arguments of Proposition 6.1 apply literally to the case under consideration. That is

$$\mathcal{N}_a^{k,D}(G) = \{\kappa \in \Omega_D^k(G) \mid \|\kappa\|_{L_a^2} < \infty\ ,\ \ll \eta, \kappa \gg_a = 0\ \forall \eta \in \left(\mathcal{E}_a^k(G) \oplus \widehat{\mathcal{C}}_a^k(G)\right)\}\ , \quad (7.5)$$

and the space $\widehat{\mathcal{C}}_a^k(G)$ is orthogonal with respect to the L_a^2 scalar product to $\mathcal{E}_a^k(G)$, as defined by (6.1).

Theorem 7.2

If $G \subset I\!R^n$ is an exterior domain and a is not exceptional, the space $H_a^1\Omega_D^k(G)$ of differential forms satisfying the boundary condition $\mathbf{t}\omega = 0$ splits into the direct sum

$$H_a^1\Omega_D^k(G) = H_a^1\mathcal{E}_a^k(G) \oplus H_a^1\widehat{\mathcal{C}}_a^k(G) \oplus \mathcal{H}_a^1\mathcal{N}_a^{k,D}(G)\ . \quad (7.6)$$

This decomposition is L_a^2 orthogonal. If $\omega \in H_a^s\Omega_D^k(G)$, then $\omega = d_a\alpha_\omega + \delta_a\widehat{\beta}_\omega + \widehat{\lambda}_\omega$, and differential forms α_ω and $\widehat{\beta}_\omega$ satisfy the estimate

$$\|\alpha_\omega\|_{H_{a-1}^{s+1}} \leq C\|\omega\|_{H_a^s} \quad \text{and} \quad \|\widehat{\beta}_\omega\|_{H_{a-1}^{s+1}} \leq C\|\omega\|_{H_a^s}\ . \quad (7.7)$$

Proof :
Given $\omega \in H_a^1\Omega_D^k(G)$ the component $d\alpha_\omega$ is constructed as in Theorem 6.2. From the boundary condition $\mathbf{t}\omega = 0$ we infer that

$$\ll d_a\omega, \kappa \gg_a = \ll \omega, \delta_a\kappa \gg_a = 0 \qquad \forall \kappa \in \mathcal{H}_{a-1}^{k,D}(G)\ . \quad (7.8)$$

Therefore $d_a\omega$ has a potential $\widehat{\phi}_D \in (\mathcal{H}_{a-1}^{k,D})^\perp(G)$ in the sense of Theorem 5.2, which is, by Corollary 5.4, the unique solution of the boundary value problem

$$\begin{aligned} \Delta_a\widehat{\phi}_D &= d_a\omega && \text{on } G \\ \mathbf{t}\widehat{\phi}_D = 0 \ \text{ and } \ \mathbf{t}\delta_a\widehat{\phi}_D &= 0 && \text{on } \partial G\ . \end{aligned} \quad (7.9)$$

Moreover $\ll d_a\delta_a\widehat{\phi}_D, \delta_a d_a\widehat{\phi}_D \gg_a = 0$, which follows from $\mathbf{t}\widehat{\phi}_D = 0$. Arguing as in the proof of Lemma 5.6, this implies that $d_a\widehat{\phi}_D = 0$. Now we set

$$\widehat{\beta}_\omega := \widehat{\phi}_D \quad \text{and} \quad \widehat{\lambda}_\omega := \left(\omega - d_a\alpha_\omega - \delta_a\widehat{\beta}_\omega\right)\ . \quad (7.10)$$

Since, by construction, $d_a\widehat{\beta}_\omega = 0$, the argument of (6.13) applies accordingly, that is

$$\ll \widehat{\lambda}_\omega, \delta_a\widetilde{\beta} \gg_a = \ll \left(\omega - \delta_a\widehat{\beta}_\omega\right), \delta_a\widetilde{\beta} \gg_a = \mathcal{D}_a(\widehat{\phi}_D, \widetilde{\beta}) - \ll \delta_a\widehat{\beta}_\omega, \delta_a\widetilde{\beta} \gg_a = 0 \quad (7.11)$$

for all $\delta_a\widetilde{\beta} \in \widehat{\mathcal{C}}_a^k(G)$. Therefore $\widehat{\lambda}_\omega$ is L_a^2 orthogonal to the space $\widehat{\mathcal{C}}_a^k(G)$. Since also

$$\ll \widehat{\lambda}_\omega, d_a\widetilde{\alpha} \gg_a = 0 \qquad \forall d_a\widetilde{\alpha} \in E_a^k(G)\ , \quad (7.12)$$

this proves that $\widehat{\lambda}_\omega \in \mathcal{N}_a^{k,D}(G)$. Thus the decomposition (7.6) is established. The regularity result and the estimate (7.7) then follow literally as in Lemma 6.3. □

It is obvious that under the boundary condition $\mathbf{n}\omega = 0$ a corresponding result holds true. Defining

$$\widehat{\mathcal{E}}_a^k(G) := \{d_a\widehat{\alpha} \mid \widehat{\alpha} \in \Omega^{k-1}(G)\ ,\ \mathbf{n}\widehat{\alpha} = 0\ ,\ \ \mathbf{n}d_a\widehat{\alpha} = 0\ ,\ \|\widehat{\alpha}\|_{H^1_{a-1}} < \infty\} \tag{7.13}$$

we have:

Theorem 7.3
If $G \subset I\!R^n$ is an exterior domain and a is not exceptional, the space $H^1_a\Omega^k_N(G)$ splits into

$$H^1_a\Omega^k_N(G) = H^1_a\widehat{\mathcal{E}}^k_a(G) \oplus H^1_a\mathcal{C}^k_a(G) \oplus \mathcal{H}^1_a\mathcal{N}^{k,N}_a(G)\ . \tag{7.14}$$

This decomposition is L^2_a orthogonal. If $\omega \in H^s_a\Omega^k_D(G)$, then $\omega = d_a\widehat{\alpha}_\omega + \delta_a\beta_\omega + \widehat{\lambda}_\omega$, where $\widehat{\lambda} \in \mathcal{H}^s_a\mathcal{N}^{k,N}_a(G)$, and differential forms $\widehat{\alpha}_\omega$ and β_ω satisfy the estimate

$$\|\widehat{\alpha}_\omega\|_{H^{s+1}_{a-1}} \le C\|\omega\|_{H^s_a} \quad \text{and} \quad \|\beta_\omega\|_{H^{s+1}_{a-1}} \le C\|\omega\|_{H^s_a}\ . \tag{7.15}$$

On the basis of the decomposition results of Theorem 7.2 and 7.3 the proof of Lemma 6.4 literally generalises to the case of differential forms in $\Omega^k_D(G)$ and $\Omega^k_N(G)$. We have:

Lemma 7.4
If $s \ge 1$ and a is not exceptional, then

$$\|\omega\|_{H^s_a} \le C\Big(\|d_a\omega\|_{H^{s-1}_{a+1}} + \|\delta_a\omega\|_{H^{s-1}_{a+1}}\Big) \quad \forall\, \omega \in H^s_a\mathcal{E}^k_a(G) \oplus H^s_a\widehat{\mathcal{C}}^k_a(G)\ . \tag{7.16}$$

with a universal constant C depending on s, a and the geometry of ∂G. Correspondingly, the estimate (7.16) holds on the spaces $H^s_a\widehat{\mathcal{E}}^k_a(G) \oplus H^s_a\mathcal{C}^k_a(G)$.

This version of Korn's inequality for differential forms satisfying the boundary condition $\mathbf{t}\omega = 0$ or $\mathbf{n}\omega = 0$ is more common in the literature then the result of Lemma 6.4. It is of particular importance that by Theorem 7.1 the spaces $\mathcal{N}^{k,D}_a(G)$ and $\mathcal{N}^{k,N}_a(G)$ are finite dimensional. Consequently Korn's inequality holds true for all differential forms satisfying either of the homogeneous boundary conditions above, modulo a finite dimensional subspace. Moreover, one can show that an estimate of the form (7.16) also hold true on the space

$$H^1_a\Omega_0(G) := \{\omega \in H^1_a\Omega^k(G) \mid \mathbf{t}\omega = 0 \ \text{ and } \ \mathbf{n}\omega = 0\}\ . \tag{7.17}$$

This is due to the fact that $\mathcal{N}^{k,D}_a(G) \cap \mathcal{N}^{k,N}_a(G) = \emptyset$. For a precise argument, see Lemma 2.4.10 in [S2].

8. Boundary value problems for differential forms

It has been discussed in [S1] that the method of Hodge decomposition provides a useful tool to solve boundary value for differential forms. Here will illustrate the Hodge decomposition technique at the example of two special exterior domain problem, and restrict ourselves – for sake of simplicity – to homogeneous boundary conditions.

Lemma 8.1

Let $\rho \in H^{s-1}_{a+1}\Omega^{k-1}(G)$ and $\chi \in H^{s-1}_{a+1}\Omega^{k+1}(G)$ satisfy the integrability conditions

$$\delta_a\rho = 0 \quad , \quad \ll \rho, \widetilde{\kappa} \gg_a = 0 \quad \forall \widetilde{\kappa} \in \mathcal{H}^{k-1,D}_{a-1}(G) \tag{8.1}$$

$$d_a\chi = 0 \quad , \quad \mathbf{t}\chi = 0 \text{ and } \ll \chi, \kappa \gg_{a+1} = 0 \quad \forall \kappa \in \mathcal{N}^{k+1,D}_{a+1}(G) \; . \tag{8.2}$$

Then the boundary value problem

$$\begin{aligned} d_a\omega = \chi \text{ and } \delta_a\omega = \rho & \qquad \text{on } G \\ \mathbf{t}\omega = 0 & \qquad \text{on } \partial G \end{aligned} \tag{8.3}$$

has a unique solution $\omega_0 \in H^s_a\Omega^k(G)$, which is L^2_a-orthogonal to $\mathcal{N}^{k,D}_a(G)$. This solution can be estimated by

$$\|\omega_0\|_{H^s_a} \le C\Big(\|\chi\|_{H^{s-1}_{a+1}} + \|\rho\|_{H^{s-1}_{a+1}}\Big) \; . \tag{8.4}$$

Any other solution of (8.3) differs from ω_0 by an element of $\mathcal{N}^{k,D}_a(G)$.

Proof :
By assumption, ρ is orthogonal to the space $\mathcal{H}^{k-1,D}_{a-1}(G)$ with respect to the pairing $\ll, \gg_a$. Hence there exists a unique potential $\widetilde{\phi}_D \in (\mathcal{H}^{\perp})^{k-1,D}_{a-1}(G)$ in the sense of Theorem 5.2. By Corollary 5.4 it is a strong solution of the boundary value problem

$$\begin{aligned} \Delta_a\widetilde{\phi}_D = \rho & \qquad \text{on } G \\ \mathbf{t}\widetilde{\phi}_D = 0 \text{ and } \mathbf{t}\delta_a\widetilde{\phi}_D = 0 & \qquad \text{on } \partial G \; , \end{aligned} \tag{8.5}$$

which satisfies $\|\widetilde{\phi}_D\|_{H^{s+1}_{a-1}} \le C_1(\|\rho\|_{H^{s-1}_{a+1}} + \|\widetilde{\phi}_D\|_{L^2_{a-1}})$. Using Green's formula (2.24) we then infer from the boundary condition $\mathbf{t}\delta_a\widetilde{\phi}_D = 0$ and the integrability condition $\delta_a\rho = 0$ that

$$\|\rho - \delta_a d_a\widetilde{\phi}_D\|^2_{L^2_{a+1}} = \ll d_a\delta_a\widetilde{\phi}_D, (\rho - \delta_a d_a\widetilde{\phi}_D) \gg_{a+1} = 0 \; . \tag{8.6}$$

Therefore $\omega_\rho := d_a\widetilde{\phi}_D \in H^s_a\Omega^k(G)$ is a solution of the problem

$$\delta_a\omega_\rho = \rho \quad , \quad d_a\omega_\rho = 0 \quad \text{and} \quad \mathbf{t}\omega_\rho = 0 \; . \tag{8.7}$$

Moreover, by (2.24) ω_ρ is orthogonal to $\mathcal{N}^{k,D}_a(G)$ and satisfies the estimate $\|\omega_\rho\|_{H^s_a} \le C_2\|\rho\|_{H^{s-1}_{a+1}}$.

On the other hand, the Hodge-Morrey decomposition (6.8) applied to $\chi \in H^{s-1}_{a+1}\Omega^{k+1}(G)$ yields

$$\chi = d_a\alpha_\chi + \delta_a\beta_\chi + \kappa_\chi \ . \tag{8.8}$$

From the integrability conditions $d_a\chi = 0$ we infer that $\delta_a\beta_\chi = 0$. Hence the condition $\mathbf{t}\chi = 0$ implies that $\kappa_\chi \in \mathcal{N}^{k+1,D}_{a+1}(G)$. By assumption, χ is orthogonal to $\mathcal{N}^{k+1,D}_{a+1}(G)$ so that $\kappa_\chi = 0$ and hence $\chi = d_a\alpha_\chi \in H^{s-1}_{a+1}\mathcal{E}^{k+1}_{a+1}(G)$. From the construction of the Hodge component α_χ, cf. Theorem 6.2, we infer that $\delta_a\alpha_\chi = 0$ and $\ll \alpha_\chi, \widetilde{\lambda} \gg_a = 0$ for all $\widetilde{\lambda} \in \mathcal{N}^{k,D}_a(G)$. Therefore, $\omega := \omega_\rho + \alpha_\chi$ is a solution of the problem

$$\delta_a\omega = \rho \ , \ d_a\omega = \chi \text{ and } \mathbf{t}\omega = 0 \ , \tag{8.9}$$

which is orthogonal to $\mathcal{N}^{k,D}_a(G)$. By the estimate for $\|\omega_\rho\|_{H^s_a}$ and Lemma 6.3 this solution satisfies the demanded inequality (8.4). Finally any other solution of (8.3) has to be a harmonic field with vanishing tangential component, i.e. an element in $\mathcal{N}^{k,D}_a(G)$. □

In the context of the Atiyah-Singer index theorem, the problem (8.3) may be understood as a Dirichlet boundary value problem for the Dirac type operator

$$(d+\delta) \ : \ \bigoplus_{k=0..n} \Omega^k(G) \longrightarrow \Omega^e(G) \oplus \Omega^o(G) \ , \tag{8.10}$$

on the exterior algebra bundle. Here $\Omega^e(G)$ and $\Omega^o(G)$ denote the algebra of differential forms of (arbitrary) even and odd degree, respectively. Hence $\bigoplus \Omega^k(G) = \Omega^e(G) \oplus \Omega^o(G)$ can be considered as the space of sections in a $\mathbb{Z}_2$-graded vector bundle.

Lemma 8.2

Let $\eta \in H^s_a\Omega^k(G)$ and $\mathbf{n}\eta$ be its normal component on ∂G. Then there exists a differential form $\sigma \in H^{s+1}_{a-1}\Omega^{k-1}(G)$ such that

$$\sigma|_{\partial G} = 0 \quad \text{and} \quad \mathbf{n}(d\sigma) = \mathbf{n}\eta \ . \tag{8.11}$$

It satisfies $\|\sigma\|_{H^{s+1}_{a-1}} \le C\|\eta\|_{H^s_a}$, where C only depends on s, a and the geometry of ∂G.

Proof :
Given a normal frame in the sense of (2.10) in a neighborhood of the boundary, that is a local frame of the form $\mathcal{F}_N = (N, F_2, \dots F_n)$ on $U \subset G$. If $\eta \in \Omega^k(G)$ is smooth, the component $\mathbf{n}\eta$ on $\partial G \cap U$ is uniquely determined by the set smooth functions $\eta(N, F_{\varphi(2)}, \dots F_{\varphi(k)})$, where the permutations φ run over all the $\binom{n-1}{k-1}$ shuffles of the fields $(F_2, \dots F_n)$. From $\sigma|_{\partial G} = 0$ and (2.4) we infer that

$$\mathbf{n}(d\sigma)(N, F_{\varphi(2)}, \dots F_{\varphi(k-1)}) = D[\sigma(F_{\varphi(2)}, \dots F_{\varphi(k-1)})](N) \ . \tag{8.12}$$

Hence, the extension problem (8.11) reduces to solve locally on U the system

$$\begin{aligned} \sigma(F_{\varphi(2)}, \dots F_{\varphi(k)})|_{\partial G \cap U} &= 0 \\ \text{and} \quad D[\sigma(F_{\varphi(2)}, \dots F_{\varphi(k)})](N) &= \eta(N, F_{\varphi(2)}, \dots F_{\varphi(k)}) \end{aligned} \tag{8.13}$$

for each permutation φ. Using the tubular neighborhood theorem, [H1], one shows that each of these $\binom{n-1}{k-1}$ scalar problems allow for a smooth extension $\sigma(F_{\varphi(2)}, \ldots F_{\varphi(k)})$ to $U \subset G$, which is compactly supported in U. These extensions can be chosen such that

$$\|\sigma(F_{\varphi(2)}, \ldots F_{\varphi(k)})\|_{H^{s+1}_{a-1}(U)} \leq C_1 \|\eta(N, F_{\varphi(2)}, \ldots F_{\varphi(k)})\|_{H^{s-1/2}(U \cap \partial G)} \tag{8.14}$$

with C_1 depending only on s, a and the geometry of $\partial G \cap U$. Besides (8.13), the boundary condition $\sigma|_{\partial G} = 0$ yields another set of problems, namely

$$\sigma\big(N, F_{\varphi'(2)}, \ldots F_{\varphi'(k-1)}\big)|_{\partial G \cap U} = 0 \ , \tag{8.15}$$

which are solved by the trivial extension $\sigma(N, F_{\varphi'(2)}, \ldots F_{\varphi'(k-1)}) \equiv 0$ on U. Since all the problems of (8.13) and (8.15) are mutually independent of each other, one can construct a smooth compactly supported $\sigma \in \Omega^{k-1}_c(U)$. By compactness of ∂G there is a finite number of neighborhoods U_α covering the boundary such that on which the construction above can be performed on each U_α. By a partition of unity argument the respective (locally defined) differential forms glue together to a global solution $\sigma \in \Omega^{k-1}_c(G)$ of (8.11). By construction,

$$\|\sigma\|_{H^{s+1}_{a-1}(G)} \leq C_2 \|\eta\|_{H^{s-1/2}(\partial G)} \leq C_3 \|\eta\|_{H^s_a(G)} \ , \tag{8.16}$$

where the last inequality follows from the trace theorem. Finally, if $\eta_j \to \eta$ in $H^s_a\Omega^k(G)$ and each η_j is smooth, then there exists a sequence σ_j of smooth compactly supported forms satisfying (8.11), and the statement of the Lemma follows from (8.16). □

Theorem 8.3

Let G be an exterior domain and a not exceptional. If $\chi \in H^{s-1}_{a+1}\Omega^{k+1}(G)$ satisfies the integrability conditions

$$d_a\chi = 0 \ , \quad \mathbf{t}\chi = 0 \ \text{ and } \ \ll \chi, \kappa \gg_{a+1} = 0 \quad \forall \kappa \in \mathcal{N}^{k+1,D}_{a+1}(G) \ , \tag{8.17}$$

then the boundary value problem

$$\begin{aligned} d_a\omega &= \chi && \text{on } G \\ \mathbf{t}\omega = 0 &\text{ and } \mathbf{n}\omega = 0 && \text{on } \partial G \end{aligned} \tag{8.18}$$

has a solution $\omega \in H^s_a\Omega^k(G)$. This solution can be chosen such that $\|\omega\|_{H^s_a} \leq C\|\chi\|_{H^{s-1}_{a+1}}$ with a universal constant C.

Proof :
The Hodge-Morrey decomposition (6.8) of $\chi \in H^{s-1}_{a+1}\Omega^{k+1}(G)$ yields

$$\chi = d_a\alpha_\chi + \delta_a\beta_\chi + \kappa_\chi \ . \tag{8.19}$$

As in the proof of Lemma 8.1 the integrability conditions $d_a\chi = 0$, $\mathbf{t}\chi = 0$ and the orthogonality of χ to $\mathcal{N}^{k+1,D}_{a+1}(G)$ imply that $\chi = d_a\alpha_\chi$ with $\alpha_\chi \in H^s_a\Omega^k_D(G)$. Moreover,

$\|\alpha_\chi\|_{H^s_a} \leq C_1\|\chi\|_{H^{s+1}_{a-1}}$, by Theorem 6.3. Having fixed α_χ, there exists by Lemma 8.2 a differential form $\sigma_\chi \in H^{s+1}_{a-1}\Omega^{k-1}(G)$ such that

$$\sigma_\chi|_{\partial G} = 0 \quad \text{and} \quad \mathbf{n}(d\sigma_\chi) = \mathbf{n}\alpha_\chi \ , \tag{8.20}$$

which can be chosen so that $\|\sigma_\chi\|_{H^{s+1}_{a-1}} \leq C_2\|\alpha_\chi\|_{H^s_a}$. Then $\omega_\chi := \alpha_\chi + d_a\sigma_\chi \in H^s_a\Omega^k(G)$ satisfies the equation $d_a\omega = \chi$ and the boundary condition $\mathbf{n}\omega_\chi = 0$. Moreover, since $\mathbf{t}\sigma_\chi = 0$ implies that $\mathbf{t}d_a\sigma_\chi = 0$, this proves that ω_χ solves the problem (8.18). □

By the $\star$-duality we have:

Corollary 8.4

The boundary value problem

$$\begin{aligned} \delta_a\omega &= \rho \qquad\qquad\qquad \text{on } G \\ \mathbf{t}\omega = 0 &\ \text{ and } \ \mathbf{n}\omega = 0 \quad \text{on } \partial G \end{aligned} \tag{8.21}$$

is solvable in $H^s_a\Omega^k(G)$ for each $\rho \in H^{s-1}_{a+1}\Omega^{k-1}(G)$ satisfying

$$\delta_a\rho = 0 \ , \quad \mathbf{n}\rho = 0 \ \text{ and } \ \ll \rho, \kappa \gg_{a+1} = 0 \quad \forall \kappa \in \mathcal{N}^{k-1,N}_{a+1}(G) \ . \tag{8.22}$$

We note that elliptic techniques are not appropriate to treat the problems (8.18) and (8.21), since the exterior derivative d_a is not an elliptic operator. In fact, these problems are to be considered as an underdetermined system, cf. [R-S]. The study of this particular type of equations is is motivated by its importance for applications.

As a special example the Stokes equation in hydrodynamics should be mentioned. In order to solve the related static problem, a precise knowledge is needed about the range of the divergence operator acting on vector fields $Y \in \mathcal{X}(G)$ subject to the boundary condition $Y|_{\partial G} = 0$. That is, one has to solve the boundary value problem

$$\operatorname{div} Y = \rho \ \text{ and } \ Y|_{\partial G} \equiv 0 \tag{8.23}$$

in a certain Sobolev space and control the norm of this solution. By means of the equivalence between vector analysis and the differential form calculus on $\Omega^1(G)$, discussed in Section 2, the divergence corresponds the co-differential operator δ_a. Hence the problem (8.23) is solved by Corollary 8.4. The same problem has been treated recently in [W3], where quite different techniques are applied.

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