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Stability of Solutions of Parabolic PDE's with Random Drift and Viscosity Limit

T. Deck [1,a], *J. Potthoff* [1,a], *G. Våge* [1,b] *and H. Watanabe* [2]

[1] University of Mannheim, D–68131 Mannheim, Germany.
[2] Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Japan.

Abstract Let u_α be the solution of the Itô stochastic parabolic Cauchy problem $\partial u/\partial t - L_\alpha u = \xi \cdot \nabla u,\ u|_{t=0} = f$. We prove that u_α depends continuously on α, when the coefficients in L_α converge to those in L_0. This result is used to study the diffusion limit for the Cauchy problem (in Stratonovich sense): when the coefficients of L_α tends to 0 the corresponding solutions u_α converge to a function u_0 satisfying $\partial u_0/\partial t = \xi \circ \nabla u_0,\ u|_{t=0} = f$. A criterion is provided for the existence of strong limits, e.g. $u_\alpha \to u_0$, in the space of Hida distributions $(\mathcal{S})^*$. As an application we show that weak solutions of the above Cauchy problem are strong solutions.

1. Introduction

Consider an incompressible fluid with velocity field $w(t,x) = (w_1(t,x), w_2(t,x), w_3(t,x))$ at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}^3$. The mass density $u(t,x)$ of particles suspended in this fluid satisfies the equation

$$\frac{\partial u}{\partial t} - Lu = -w \cdot \nabla u,\ \ u|_{t=0} = f, \tag{1.1}$$

where $Lu = \nabla(c\nabla u)$ (derivatives with respect to $x \in \mathbb{R}^3$), $c(t,x)$ is the diffusion coefficient, and $f(x)$ the initial density of the particles. In [Ch], P.-L. Chow proposed the stochastic partial differential equation (1.1) with $w(t,x) = \dot\eta(t,x)$, where $\dot\eta_i(t,x)$ is the formal time derivative of $\eta_i(t,x) = \sum_{j=1}^3 \int_0^t \sigma_{ij}(s,x)\, dB_j(s)$ for $i = 1, 2, 3$, as a model for transport of particles in a turbulent medium. For the one–dimensional case he proved existence and uniqueness of a solution u for this equation. Moreover, he observed that u under natural conditions is a generalized random field. This motivated [P], [DP] to consider the equation using white noise analysis. Applying contraction methods they proved existence and uniqueness of a weak solution of (1.1) for uniformly elliptic operators $L(t,x)$, $x \in \mathbb{R}^d$, of order two when $w(t,x) = \dot\eta(t,x)$ is as above. (1.1) was recently considered again in [PVW]. Applying the Girsanov formula the authors were able to prove existence and uniqueness of a weak solution of (1.1) when w is d-dimensional space–time white noise.

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[a] Partially supported by the DFG.
[b] Supported by the DFG.

In practice the molecular diffusion c is small, i.e., $0 < c \ll 1$. If we assume that $c = \alpha$ is constant, we are led to consider (1.1) with $L_\alpha = \alpha\Delta$, where $0 < \alpha \ll 1$. What happens when α tends to 0? P.-L. Chow answers this question in [Ch], when $w(t,x) = \dot{\eta}(t,x)$, $x \in I\!R$. In this paper we consider the related problem of stability of the solution of (1.1) with respect to perturbations of the differential operator L for more general types of noise. In particular, if L_α for $\alpha \in [0,1]$ is a family of differential operators such that the coefficients of L_α tend uniformly to the coefficients of L_0 as $\alpha \to 0$, then we provide conditions under which the corresponding solutions u_α converge to u_0. Moreover, we show that if (1.1) is interpreted in Stratonovich sense and L_α tends uniformly to 0, the corresponding solutions u_α converge to u_0, where u_0 solves the (degenerate) equation obtained in the limit. If (1.1) is interpreted in Itô sense the same result does not hold; one can actually construct explicit counterexamples.

The paper is organized as follows: In Section 2 we introduce basic notations. Section 3 contains the stability result. In Section 4 we consider the diffusion limit problem for (1.1) when the equation is interpreted in the Stratonovich sense. A crucial lemma needed to prove the stability theorem in Section 3 is given in the appendix. As a corollary to this lemma we obtain conditions under which a weak solution of (1.1) is a strong solution, in the sense that the derivatives exist in the strong sense. Since the solutions constructed in [PVW] satisfy these conditions, it follows that they are strong solutions.

2. Notations and preliminaries

Let $\mathcal{S}(I\!R^{d+1})$ be the space of rapidly decreasing functions and $\big(\mathcal{S}'(I\!R^{d+1}), \mathcal{B}, \mu\big)$ be the white noise probability space. It is well-known that there is a chain of Hilbert spaces $(\mathcal{S})_q$ with inner product $(\cdot,\cdot)_q$, $q \in Z\!\!Z$, such that

$$(\mathcal{S})^* = \bigcup_{q \in I\!N} (\mathcal{S})_{-q} \supset \cdots \supset (\mathcal{S})_{-1} \supset L^2(\mu) \supset (\mathcal{S})_1 \supset \cdots \supset \bigcap_{q \in I\!N} (\mathcal{S})_q = (\mathcal{S}).$$

$(\mathcal{S})$ and $(\mathcal{S})^*$ denote the space of Hida test functions and of Hida distributions, respectively. We equip $(\mathcal{S})^*$ with the strong topology (which coincides with the inductive limit topology), and denote the dual pairing between $(\mathcal{S})$ and $(\mathcal{S})^*$ by $\langle\cdot,\cdot\rangle$. For $T > 0$ and $d \in I\!N$ we set $D_T = \{(t,x) | t \in [0,T], x \in I\!R^d\}$. By a (generalized) random field $u \in C^{1,2}(D_T, \mathcal{S}^*)$ we mean a mapping $u \colon D_T \to (\mathcal{S})^*$ which is continuously differentiable in t and twice continuously differentiable in x, in the strong sense. This means, e.g., that the limit

$$\frac{\partial u}{\partial t}(t,x) := \lim_{\epsilon \to 0} \frac{u(t+\epsilon,x) - u(t,x)}{\epsilon}$$

exists in $(\mathcal{S})^*$ and depends continuously on $(t,x) \in D_T$. In what follows we denote by $\dot{B}(t,x)$, $(t,x) \in I\!R^{d+1}$, space-time white noise in $(\mathcal{S})^*$. From this we obtain a d–vector $\xi(t,x)$ of independent white noise fields for $t \in [0,T]$:

$$\xi(t,x) = \big(\dot{B}(t,x), \dot{B}(t+t_0,x), \ldots, \dot{B}(t+(d-1)t_0,x)\big),$$

where $t_0 > T$ is some fixed constant.

Consider the following Cauchy problem for $u \in C^{1,2}(D_T, \mathcal{S}^*)$, with $\alpha \in [0,1]$ fixed:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - L_\alpha u(t,x) &= \xi(t,x) \cdot \nabla u(t,x), \\ u(0,x) &= f(x). \end{aligned} \tag{2.1}$$

Here f belongs to $C_b^2(I\!R^d)$, the space of twice continuously differentiable functions with bounded derivatives up to the second order. Multiplication with ξ is understood in the sense of Hitsuda-Skorokhod (the natural generalization of Itô's convention).

In [PVW] the authors prove that (2.1) has a unique weak solution $u(t,x)$ in $(\mathcal{S})^*$, when L_α is uniformly elliptic and has coefficients in $C_b^2(I\!R^{d+1})$. A weak $C^{1,2}$–function u is a mapping $u : D_T \to (\mathcal{S})^*$ satisfying $\langle u(\cdot,\cdot), \varphi\rangle \in C^{1,2}(D_T)$ for every test function $\varphi \in (\mathcal{S})$, and the mapping

$$\varphi \mapsto \partial_i \langle u(t,x), \varphi\rangle$$

defines an element in $(\mathcal{S})^*$, the weak x^i–derivative of u, which is also denoted by $\partial_i u(t,x)$ (similarly for higher order derivatives). Clearly, every strong solution of (2.1) is also a weak solution, but the converse need not be true in general.

We apply the S–transformation to (2.1) at $h \in \mathcal{S}(I\!R^{d+1})$, i.e., we consider the dual pairing of (2.1) with the (normalized) test function $\varphi_h = : \exp\langle\cdot, h\rangle : = \exp\{-\frac{1}{2}|h|_2^2\} \exp\langle\cdot, h\rangle$, where $|h|_2$ denotes the norm in $L^2(I\!R^{d+1})$. This yields

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x;h) - L_\alpha v(t,x;h) &= h(t,x) \cdot \nabla v(t,x;h), \\ v(0,x;h) &= f(x), \end{aligned} \tag{2.2}$$

with $v(t,x;h) := S(u(t,x))(h)$, and we denote by the same symbol h the d–vector with components $h_i(t,x) = h(t+(i-1)t_0, x)$. To prove the stability result we will represent the solution v_α of this (non–stochastic) partial differential equation in terms of a stochastic process X_s^α with generator L_α. We remark that this process is not related to the former $\xi(t,x)$; it just serves as a technical device.

Let the elliptic differential operator L_α be defined by

$$L_\alpha u(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} a_\alpha^{ij}(t,x) \frac{\partial^2 u}{\partial x^i \partial x^j}(t,x) + \sum_{i=1}^{d} b_\alpha^i(t,x) \frac{\partial u}{\partial x^i}(t,x), \tag{2.3}$$

where $b_\alpha(t,x)$ is a continuous d–vector and $a_\alpha(t,x)$ is a positive definite, continuous $d \times d$–matrix which has a positive square–root $\sigma_\alpha(t,x)$. We suppose that b_α, σ_α, and the inverse matrix σ_α^{-1} (all denoted by the symbol ρ_α) satisfy

(C1) $$|\rho_\alpha(t,x)| \le K(1+|x|), \quad |\rho_\alpha(t,x) - \rho_\alpha(t,y)| \le K|x-y|,$$

for all $(t,x) \in D_T$, where K is a constant independent of $\alpha \in [0,1]$, and also that

(C2) $$\lim_{\alpha \to 0} \{ \sup_{|x| \le N} |\rho_\alpha(t,x) - \rho_0(t,x)| \} = 0,$$

for all $N > 0$ and $0 \le t \le T$. As is well-known, condition (C1) for b_α and σ_α implies that the ordinary stochastic differential equation (B_s denotes a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$)

$$dX_s^\alpha = \sigma_\alpha(t-s, X_s^\alpha)\, dB_s + b_\alpha(t-s, X_s^\alpha)\, ds, \quad X_0^\alpha = x,$$

has a unique solution $\{X_s^\alpha\}_{0\le s\le t}$. We remark that $t \in [0,T]$ is a fixed parameter in this equation and we often suppress the t- and x-dependence in the notation of X_s^α. As an easy consequence of (C2) and Gronwall's lemma the following uniform continuity property holds, cf. [Fri, p. 119]:

$$\lim_{\alpha\to 0}\{ \sup_{(s,x)\in D_t} \mathbb{E}\big[(X_s^\alpha(x) - X_s^0(x))^2\big]\} = 0, \tag{2.4}$$

where $\mathbb{E}$ denotes the expectation with respect to P.

3. The stability result

We represent the solution v_α of (2.2) in terms of $X_s^\alpha(x)$ (cf. [Fre]):

$$v_\alpha(t,x;h) = \mathbb{E}\big[f(X_t^\alpha)e^{\int_0^t \phi_\alpha(t-s,X_s^\alpha)dB_s - \frac{1}{2}\int_0^t \phi_\alpha^2(t-s,X_s^\alpha)ds}\big],$$

where $\phi_\alpha(t,x) := \sigma_\alpha^{-1}(t,x)h(t,x)$ for all $(t,x) \in D_T$. Because h is rapidly decreasing and σ_α^{-1} satisfies (C1,C2) it is easy to verify:

$$\begin{gathered}\sup\{|\phi_\alpha(t,x)| : (\alpha,t,x) \in [0,1]\times D_T\} =: R < \infty, \text{ and}\\ \lim_{\alpha\to 0}\{ \sup_{x\in\mathbb{R}^d} |\phi_\alpha(t,x) - \phi_0(t,x)|\ \} = 0,\end{gathered} \tag{3.1}$$

for all $t \in [0,T]$. We are now prepared for the main theorem. Conditions (C1) and (C2) will be sufficient to prove stability of u_α. Because these conditions are weaker than those which guarantee existence and uniqueness of u_α we shall assume existence and uniqueness.

Theorem 3.1 *Let L_α, given by (2.3), be such that (C1) and (C2) hold. Let $f \in C_b(\mathbb{R}^d)$ satisfy $|f(x)-f(y)| \le l_0|x-y|$ for all $x,y \in \mathbb{R}^d$ with $l_0 \ge 0$. Assume further that (2.1) has a unique solution $u_\alpha \in C^{1,2}(D_T, S^*)$ for each $\alpha \in [0,1]$. Then $u_\alpha(t,x)$ converges strongly to $u_0(t,x)$ as $\alpha \to 0$, uniformly in D_T.*

Proof: In view of Lemma A.1 from the appendix it suffices to prove:

(*i*) $$|v_\alpha(t,x;zh)| \le K_1 e^{K_2|z|^2\|h\|_\infty^2}, \text{ for all } (\alpha,t,x), \text{ and}$$

(*ii*) $$\sup_{D_T}|v_\alpha(t,x;h) - v_0(t,x;h)| \to 0 \quad \text{as} \quad \alpha\to 0, \text{ for all } h \in \mathcal{S}(\mathbb{R}^{d+1}).$$

Part (i) has already been proved in [PVW], in case there is no parameter α. However, since the constants in (C1) and (C2) are independent of α one obtains – by the same proof – the estimate (i) uniformly in α.

To prove (ii) we set $F_\alpha := \int_0^t \phi_\alpha(t-s, X_s^\alpha)\, dB_s - \frac{1}{2}\int_0^t \phi_\alpha^2(t-s, X_s^\alpha)\, ds$ and estimate:

$$\begin{aligned} |v_\alpha(t,x;h) - v_0(t,x;h)|^2 &= |\mathbb{E}\big[(f(X_t^\alpha) - f(X_t^0))e^{F_0}\big] + \mathbb{E}\big[f(X_t^\alpha)(e^{F_\alpha} - e^{F_0})\big]|^2 \\ &\le 2\mathbb{E}\big[(f(X_t^\alpha) - f(X_t^0))^2\big]\mathbb{E}\big[e^{2F_0}\big] + 2\mathbb{E}\big[f(X_t^\alpha)(e^{F_\alpha} - e^{F_0})\big]^2 \\ &\le 2Ml_0^2\mathbb{E}\big[(X_t^\alpha - X_t^0)^2\big] + 2\|f\|_\infty^2 \mathbb{E}\big[|e^{F_\alpha} - e^{F_0}|\big]^2 \end{aligned} \tag{3.2}$$

The last estimate follows from Lipschitz continuity of f and from the following remark: When $(f_s)_{s\in[0,T]}$ is a continuous, adapted (to Brownian motion) process which is uniformly bounded for all s and ω, we find in [Fri, p. 152], that

$$\mathbb{E}\big[e^{\int_0^t f_s(\omega)dB_s(\omega) - \frac{1}{2}\int_0^t f_s^2(\omega)ds}\big] = 1, \quad \text{for all } t \in [0,T].$$

From this it easily follows that for all $\lambda, \nu \in [0,2]$ there is a constant c, independent of λ, ν, α and t, such that

$$\mathbb{E}\big[e^{\lambda F_\alpha + \nu F_0}\big] \le e^{ct} \le e^{cT} =: M.$$

The first term in (3.2) converges to zero by (2.4), uniformly in D_T. The second term in (3.2) can be estimated as follows:

$$\begin{aligned} \mathbb{E}\big[|e^{F_\alpha} - e^{F_0}|\big]^2 &= \mathbb{E}\big[|\int_0^1 \frac{d}{d\lambda} e^{\lambda F_\alpha + (1-\lambda)F_0}\, d\lambda|\big]^2 \\ &\le \int_0^1 \mathbb{E}\big[|F_\alpha - F_0| e^{\lambda F_\alpha + (1-\lambda)F_0}\big]^2 d\lambda \\ &\le \int_0^1 \mathbb{E}\big[(F_\alpha - F_0)^2\big]\mathbb{E}\big[e^{2\lambda F_\alpha + 2(1-\lambda)F_0}\big]\, d\lambda \\ &\le M\mathbb{E}\big[(F_\alpha - F_0)^2\big]. \end{aligned}$$

Now we use $F_\alpha - F_0 = \int_0^t (\phi_\alpha - \phi_0)\, dB_s - \frac{1}{2}\int_0^t (\phi_\alpha^2 - \phi_0^2)\, ds$, the Itô isometry and (3.1):

$$\begin{aligned} \mathbb{E}\big[(F_\alpha - F_0)^2\big] &\le 2\mathbb{E}\big[(\int_0^t (\phi_\alpha - \phi_0)\, dB_s)^2\big] + 2\mathbb{E}\big[(\frac{1}{2}\int_0^t (\phi_\alpha^2 - \phi_0^2)\, ds)^2\big] \\ &\le 2\mathbb{E}\big[\int_0^t (\phi_\alpha - \phi_0)^2 ds\big] + 2\mathbb{E}\big[\frac{t}{4}\int_0^t (\phi_\alpha - \phi_0)^2(\phi_\alpha + \phi_0)^2 ds\big] \\ &\le 2(1 + R^2T)\mathbb{E}\big[\int_0^t \sup_{x\in\mathbb{R}^d} |\phi_\alpha(t-s,x) - \phi_0(t-s,x)|^2 ds\big]. \end{aligned}$$

In view of (3.1) and the dominated convergence theorem this completes the proof. ∎

4. The zero diffusion limit

In applications of stochastic differential equations one often considers the noise term in the sense of Stratonovich. In order to apply our stability result to this case the noise $\eta(t,x)$ – instead of $\xi(t,x)$ in (2.1) – has to be sufficiently regular with respect to the x–variable. More precisely, let $\eta_i(t,x) \in (\mathcal{S})^*$, $i = 1,\dots,d$, be defined by

$$\langle \eta_i(t,x), \varphi \rangle := \int_{\mathbb{R}^d} \sum_{j=1}^{d} g_{ij}(x-u)\langle \xi_j(t,u), \varphi \rangle du, \quad \varphi \in (\mathcal{S}), \tag{4.1}$$

with non–zero functions $g_{ij} \in \mathcal{S}(\mathbb{R}^d)$.

Corollary 4.1 *Substitute the noise ξ in (2.1) by the smoothed noise (4.1) and assume that the resulting equation satisfies the conditions in Theorem 3.1. Then $u_\alpha(t,x)$ converges strongly to $u_0(t,x)$ as $\alpha \to 0$, uniformly in D_T.*

Proof: As in the proof of Theorem 3.1 we show that (i) and (ii) hold for the modified equation. The S–transform of $\eta_i(t,x)$ reads:

$$\tilde{h}_i(t,x;h) = \int_{\mathbb{R}^d} \sum_{j=1}^{d} g_{ij}(x-u) h_j(t,u)\, du,$$

and we observe: $\tilde{h}_i(\cdot,\cdot;h) \in \mathcal{S}(\mathbb{R}^{d+1})$. Instead of equation (2.2) we get

$$\frac{\partial v}{\partial t}(t,x;h) - L_\alpha v(t,x;h) = \tilde{h}(t,x;h)\cdot \nabla v(t,x;h),$$
$$v(0,x;h) = f(x),$$

and instead of (i) we obtain

$$|v_\alpha(t,x;zh)| \le K_1 e^{K_2|z|^2\|\tilde{h}\|_\infty^2} \quad \text{for all} \quad (\alpha,t,x).$$

Now (i) follows from

$$\|\tilde{h}\|_\infty \le \|h\|_\infty \cdot \sum_{i,j=1}^{d} \|g_{ij}\|_{L^1(\mathbb{R}^d)}.$$

Since $\tilde{h}(\cdot,\cdot;h)$ is rapidly decreasing $\tilde{\phi}_\alpha^{-1}(t,x) = \sigma_\alpha^{-1}(t,x)\tilde{h}(t,x;h)$ also satisfies (3.1). This implies that part (ii) of the proof of Theorem 3.1 holds without any changes when ϕ_α is replaced by $\tilde{\phi}_\alpha$. ■

Remark: The existence and uniqueness proof for solutions of (2.1) given in [PVW] easily extends to the case where ξ is substituted by the (smoothed) noise η. The necessary modifications are similar to those in the above proof.

Informally, the covariance of η reads

$$\mathbb{E}_\mu\big[\eta_i(t,x)\eta_j(s,y)\big] = \delta(t-s)K_{ij}(x,y),$$

where $\mathbb{E}_\mu$ is expectation with respect to the white noise measure μ, and

$$K_{ij}(x,y) = \sum_{r=1}^{d} \int_{\mathbb{R}^d} g_{ir}(x-u)g_{jr}(y-u)\,du.$$

Notice that $K_{ij} := K_{ij}(x,x)$ are constant elements of a symmetric $d \times d$–matrix K. We are now prepared to consider the Cauchy problem in Stratonovich sense (denoted by $\circ$):

$$\frac{\partial u}{\partial t} - L_\alpha u = \eta \circ \nabla u, \quad u(0,x) = f(x). \tag{4.2}$$

Rewritten as an Itô equation this gives (cf. [Ku]):

$$\begin{gathered}\frac{\partial u}{\partial t} - L_\alpha u - \frac{1}{2}\sum_{i,j=1}^{d} K_{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial K_{ij}(x,y)}{\partial x^i}\Big|_{y=x}\frac{\partial u}{\partial x^j} = \eta\nabla u, \\ u(0,x) = f(x).\end{gathered} \tag{4.3}$$

When the coefficients of L_α tend uniformly to zero, the second order terms of (4.3) tend uniformly to $\frac{1}{2}\sum K_{ij}\partial_i\partial_j u$. The combination of this with Corollary 4.1 and the above remark yields:

Corollary 4.2 *Let L_α, $\alpha \in [0,1]$, be uniformly elliptic operators with coefficients in $C_b^2(D_T)$ such that (C1,C2) hold with $b_0 = 0$ and $\sigma_0 = 0$. Assume further that η is given by (4.1) such that K is positive definite, and the inverse of $\sigma_\alpha + K$ satisfies (C1,C2). Then (4.2) has a unique solution $u_\alpha \in C^{1,2}(D_T, \mathcal{S}^*)$ which converges strongly to u_0 as $\alpha \to 0$, uniformly in D_T, and u_0 satisfies*

$$\frac{\partial u}{\partial t} = \eta \circ \nabla u, \quad u(0,x) = f(x).$$

Appendix

The S–transformation is an important tool in white noise analysis. One of its applications is to solve differential equations by solving the corresponding S–transformed equations. A basic question is then how the inverse S–transformation behaves under limits with respect to parameters $x \in \mathbb{R}^n$, like continuity and/or differentiability with respect to x. There are several notions of differentiability in the literature, like weak differentiability, or differentiability of coefficients in chaos or Hermite expansions. In this appendix we derive a sufficient condition for when the limit $\lim_{x\to x_0} S^{-1}F(x)$ exists with respect to the strong topology on $(\mathcal{S})^*$. We remark that the resulting notion of strong differentiability is stronger than each of the before mentioned notions.

Consider the S–transform of $\Phi \in (\mathcal{S})^*$ evaluated at $h \in \mathcal{S}(\mathbb{R}^n)$:

$$F(h) := S\Phi(h) = \langle \Phi, : e^{\langle \cdot, h\rangle} : \rangle.$$

The space of mappings $F : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ obtained in this way is denoted by $\mathcal{U}$. Since S is injective the correspondence between $(\mathcal{S})^*$ and $\mathcal{U}$ is one–to–one.

Lemma A.1 *Let X be a first countable topological space, Y be an arbitrary set, and $F : X \times Y \to \mathcal{U}$ be such that, for every $h \in \mathcal{S}(\mathbb{R}^n)$:*
(i) The function $F(\cdot, y; h) : X \to \mathbb{R}$, $F(x, y; h) := F(x, y)(h)$ is continuous at x_0, uniformly with respect to y, i.e.

$$\sup_{y\in Y} |F(x, y; h) - F(x_0, y; h)| \to 0, \text{ as } x \to x_0.$$

(ii) There exists a continuous seminorm $\|\cdot\|$ on $\mathcal{S}(\mathbb{R}^n)$ and $K > 0$, such that

$$|F(x, y; zh)| \leq K e^{|z|^2 \|h\|^2},$$

for all $(x, y) \in X \times Y$ and all $z \in \mathbb{C}$.
Then there exists $r \in \mathbb{N}$ such that $\Phi_y(x) := S^{-1}(F(x, y)) \in (\mathcal{S})_{-r}$, for all $(x, y) \in X \times Y$. The mapping $x \mapsto \Phi_y(x)$ is continuous at x_0, uniformly with respect to Y:

$$\sup_{y\in Y} \|\Phi_y(x) - \Phi_y(x_0)\|_{-r} \to 0 \quad \text{as} \quad x \to x_0.$$

Consequently Φ_y, considered as a mapping from X into $(\mathcal{S})^*$, is strongly continuous at x_0, uniformly with respect to Y.

Proof: We need two facts from [PS], [KLPSW]: Firstly, (ii) implies there exists $C \geq 0$ and $q \geq 0$ such that $\Phi_y(x) \in (\mathcal{S})_{-q}$ and $\|\Phi_y(x)\|_{-q} \leq C$ for all $(x, y) \in X \times Y$. Secondly, the vector space $\mathcal{E} \subset (\mathcal{S})_{-q}$ generated by $\{\exp\langle \cdot, h\rangle | h \in \mathcal{S}(\mathbb{R}^n)\}$ is dense in $(\mathcal{S})_{-q}$.

Since X is first countable it suffices to consider sequences. Let (x_n) be a sequence in X which converges to x_0. From (i) follows

$$\sup_{y\in Y} |(\Phi_y(x_n), \varphi)_{-q} - (\Phi_y(x_0), \varphi)_{-q}| \to 0 \quad \text{as } n \to \infty, \tag{A.1}$$

for every $\varphi \in \mathcal{E}$. Since $\Phi_y(x_n)$ is bounded in $(\mathcal{S})_{-q}$ we find that (A.1) holds for all $\varphi \in (\mathcal{S})_{-q}$.

Let $\{e_1, e_2, \ldots\}$ be a CONS in $(\mathcal{S})_{-q}$ and consider the orthogonal expansion

$$\Phi_y(x_n) - \Phi_y(x_0) = \sum_{k=1}^{\infty} (a_k^{(n)}(y) - a_k^{(0)}(y)) e_k,$$

where

$$|a_k^{(n)}(y) - a_k^{(0)}(y)| = |(\Phi_y(x_n), e_k)_{-q} - (\Phi_y(x_0), e_k)_{-q}|$$
$$\leq \|\Phi_y(x_n) - \Phi_y(x_0)\|_{-q} \leq 2C.$$

Now (A.1) implies that for each $k \in I\!N$

$$\sup_{y \in Y} |a_k^{(n)}(y) - a_k^{(0)}(y)| \to 0, \text{ as } n \to \infty. \qquad (A.2)$$

The identity mapping $i : (\mathcal{S})_{-q} \hookrightarrow (\mathcal{S})_{-r}$, $r = q+1$, is a Hilbert–Schmidt operator, so that

$$\sum_{k=1}^{\infty} \|e_k\|_{-r}^2 = \sum_{k=1}^{\infty} \|i(e_k)\|_{-r}^2 = \sum_{k=1}^{\infty} N_k^2 =: M < \infty.$$

This implies that for any $\epsilon > 0$ there exists $m = m(\epsilon)$ such that $(2C)^2 \sum_{k=m}^{\infty} N_k^2 < \epsilon/2$. Using this and (A.2) we estimate as follows:

$$\begin{aligned}\|\Phi_y(x_n) - \Phi_y(x_0)\|_{-r}^2 &= \sum_{k=1}^{\infty} |a_k^{(n)}(y) - a_k^{(0)}(y)|^2 N_k^2 \\ &\leq \sum_{k=1}^{m-1} |a_k^{(n)}(y) - a_k^{(0)}(y)|^2 N_k^2 + (2C)^2 \sum_{k=m}^{\infty} N_k^2 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{for all } y \in Y \text{ and all } n \geq n_0.\end{aligned}$$

This shows that the mapping Φ_y from X into $(\mathcal{S})_{-r}$ is continuous, uniformly with respect to Y. Composition of Φ_y with the strongly continuous embedding $(\mathcal{S})_{-r} \hookrightarrow (\mathcal{S})^*$ completes the proof. ■

We can now give a criterion for when a weak solution of (2.1) is a strong solution.

Corollary A.2 *Let u be a weak solution of (2.1), $v(t,x;h) := S(u(t,x))(h)$, and suppose the t–derivative of v and the x–derivatives of v up to the second order satisfy (ii). Then u is a strong solution of (2.1).*

Proof: We first show that the derivatives in (2.1) exist for each fixed $(t,x) \in D_T$ in the strong sense. Let $b_i \in I\!R^d$ be the unit vector in the i–th coordinate direction and consider

$$\Phi_i(\epsilon) := \frac{u(t, x+\epsilon b_i) - u(t,x)}{\epsilon}, \quad \epsilon \neq 0.$$

The S–transform is: $(S\Phi_i(\epsilon))(h) =: F(\epsilon, i; h) = [v(t, x+\epsilon b_i; h) - v(t,x;h)]/\epsilon$. Since $v(\cdot,\cdot;h)$ is differentiable $F(\epsilon,i;h)$ converges to $F(0,i;h) := \partial_i v(t,x;h)$ as $\epsilon \to 0$. Thus, F satisfies (i) in Lemma A.1, with $X := I\!R$ and $Y := \{1, \ldots, d\}$. Also (ii) holds:

$$|F(\epsilon, i; zh)| \leq \frac{1}{|\epsilon|} \int_0^{\epsilon} |\frac{\partial v}{\partial x_i}(t, x+sb_i; zh)| ds \leq \frac{1}{|\epsilon|} \int_0^{\epsilon} K e^{|z|^2 \|h\|^2} ds = K e^{|z|^2 \|h\|^2}.$$

By Lemma A.1 $\Phi_i(\epsilon)$ converges to $\Phi_i(0) := S^{-1}\partial_i v(t,x;\cdot)$ strongly in $(\mathcal{S})^*$ as $\epsilon \to 0$, which means by definition that u is strongly differentiable with respect to x_i. (Of course the same holds for the strong t–derivative of u.) Moreover it follows that the weak derivatives of u coincide with the strong derivatives. But then the same method of proof shows that the strong second order x–derivatives of u exist as well.

By assumption (2.1) holds in the weak sense. Since weak and strong derivatives coincide (2.1) also holds in the strong sense.

Finally u, $\partial_i u$ etc. depend continuously on $(t,x) \in D_T$. This follows from Lemma A.1 for the choice $X := D_T$ (no y–dependence) and the observation that (i) and (ii) hold for $v(t,x;h)$ and for the derivatives up to the second order. ■

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