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Guido Walz
Fakultät für Mathematik und Informatik
Universität Mannheim
D-68131 Mannheim
Germany
walz@math.uni-mannheim.de

Abstract. In this paper we investigate some properties of trigonometric B-splines, which form a finitely-supported basis of the space of trigonometric spline functions. We establish a complex integral representation for trigonometric B-splines, which is in certain analogy to the polynomial case, but the proof of which has to be done in a different and more complicated way. Using this integral representation, we can prove some identities concerning the evaluation of a trigonometric B-spline, its derivative and its partial derivative w.r.t. the knots. As a corollary of the last mentioned identity, we obtain a result on the tangent space of a trigonometric spline function. Finally we show that – in the case of equidistant knots – the trigonometric B-splines of odd order form a partition of a constant, and therefore the corresponding B-spline curve possesses the convex-hull property. This is also illustrated by a numerical example.

Keywords. Trigonometric Splines, Trigonometric B-Splines, Partition of Unity, Convex-Hull Property, Integral Representation, Recursion Formula.

AMS Subject Classification. 41 A 15 , 42 A 10

1. Introduction and Preliminaries

Trigonometric spline functions have been considered the first time by Schoenberg [8]. Originally they were introduced as functions which are piecewise in the space

$$\mathcal{T}_m \;:=\; \mathrm{span}\{1, \cos(x), \sin(x), \ldots, \cos(kx), \sin(kx)\} \tag{1.1}$$

of dimension $m = 2k+1$. Among other things, Schoenberg already proved the existence of locally supported trigonometric splines, the so-called trigonometric B-splines. However, maybe due to the fact that it makes no sense to consider the space $\mathcal{T}_m$ for even values of m, the question for a recurrence relation for these functions was open for quite a long time. In 1979, T. Lyche and R. Winther [4] had the great idea to introduce the "intermediate" spaces

$$\mathcal{T}_m \;:=\; \mathrm{span}\{\cos(\frac{x}{2}), \sin(\frac{x}{2}), \ldots, \cos(\frac{kx}{2}), \sin(\frac{kx}{2})\} \tag{1.2}$$

for $m = 2k$, $k \in \mathbb{N}$, which enabled them to establish a recurrence relation for trigonometric B-splines of arbitrary order, i.e., regardless if m is even or odd (see (1.4) and (1.5) below).

A nice compendium of the fundamental properties of trigonometric splines can be found in L.L. Schumaker's book [9]; moreover, quite recently the interest in these functions has increased significantly, see for example [7] or, in particular, [3] and the many references therein.

The objective of the present paper is the following: In the first part (Section 2) we will establish a contour integral representation for trigonometric B-splines in analogy to the one which is well-known for the polynomial case (see [5,6,10]). This will enable us to prove in Section 3 quite general recursions for trigonometric B-splines themselves, their derivatives with respect to the variable x, as well as for the partial derivative with respect to the knots; for the (easier-to-handle) case of polynomial B-splines, similar results were proved in [11]. Moreover, we will see how the tangent space for trigonometric splines looks like.

Furthermore, a convex-hull property in the usual sense will be proved, for trigonometric splines with equidistant knots (Section 4) as well as for the subspace of cosine splines with arbitrary knots (Section 5). In a numerical exmaple we will apply this to a curve design problem. The question of classical chebychev approximation will be considered in a forthcoming paper [12].

In the rest of this introductory section, we will define the trigonometric B-splines and repeat some basic results. With some natural number m, consider a knot sequence $\Delta = \{x_j\}_{j\in\mathbb{Z}}$, such that

$$\cdots \le x_{-1} \le x_0 \le x_1 \le \cdots \le x_j \le \cdots \tag{1.3}$$

and $x_j < x_{j+m} < x_j + 2\pi$ for all $j \in \mathbb{Z}$.

For arbitrary fixed index ν, we define the following recursion formula: For $j = \nu, \ldots, \nu + m - 1$, put

$$T_{1,j}(x) \ := \ \begin{cases} 1, & \text{for } x_j \le x < x_{j+1}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.4a}$$

Then compute for $\tau = 2, \ldots, m$ and $j = \nu, \ldots, \nu + m - \tau$ the functions

$$T_{\tau,j}(x) \ := \ \sin\Big(\frac{x - x_j}{2}\Big)\, B_{\tau-1,j}(x) \ + \ \sin\Big(\frac{x_{j+\tau} - x}{2}\Big)\, B_{\tau-1,j+1}(x) , \tag{1.4b}$$

where

$$B_{\tau-1,\mu}(x) \ := \ \begin{cases} \dfrac{T_{\tau-1,\mu}(x)}{\sin\Big(\frac{x_{\mu+\tau-1} - x_\mu}{2}\Big)} & x_\mu < x_{\mu+\tau-1} \\ 0, & x_\mu = x_{\mu+\tau-1}, \end{cases} \tag{1.4c}$$

for $\mu = j$ or $\mu = j+1$. Then the final result of this recursion, i.e. the function $T_{m,\nu}$, is the *trigonometric B-spline* of order m associated with the knots $x_\nu, \ldots, x_{\nu+m}$.

It is well-known (see [3] or [4]) that the B-spline $T_{m,\nu}$ is piecewise in the space $\mathcal{T}_m$; more precisely, the functions $\{T_{m,\nu}\}_{\nu \in \mathbb{Z}}$ span the space of trigonometric splines of order m, but we will not go into details about this.

The support of the B-spline $T_{m,\nu}$ is the interval $[x_\nu, x_{\nu+m}]$, in the interior of which it is strictly positive; furthermore it is easy to see that this support is minimal, i.e., there is no non-trivial trigonometric spline of order m, which is zero outside a smaller knot interval than $[x_\nu, x_{\nu+m}]$.

Remarks.

1) What we have defined are the so-called *normalized* trigonometric B-splines, as they were called in [3] or [7]; we will discuss later (see Theorem 4.1) another type of normalization, which leads to trigonometric B-splines which share the convex-hull property.

2) If $x_\mu = x_{\mu+\tau-1}$ for some μ, then the corresponding denominator in the first line of (1.4c) vanishes; but then also $T_{\tau-1,\mu}(x) \equiv 0$, and thus putting $B_{\tau-1,\mu}(x) = 0$ in this case is well-motivated.

3) Of course, for each $\tau = 1, \ldots, m$, the function $T_{\tau,j}$ defined in (1.4b) is also a B-spline (of order τ); but for all $\tau < m$, it may happen that $T_{\tau,j}$ vanishes identically.

4) If, for some τ, $x_\mu < x_{\mu+\tau-1}$ for $\mu = j$ and $\mu = j+1$, then recursion formula (1.4) can be written in the more instructive form

$$T_{\tau,j}(x) \ := \ \frac{\sin\Big(\frac{x - x_j}{2}\Big)}{\sin\Big(\frac{x_{j+\tau-1} - x_j}{2}\Big)} T_{\tau-1,j}(x) \ + \ \frac{\sin\Big(\frac{x_{j+\tau} - x}{2}\Big)}{\sin\Big(\frac{x_{j+\tau} - x_{j+1}}{2}\Big)} T_{\tau-1,j+1}(x) . \tag{1.5}$$

This is in particular true for all τ, if the knots are all distinct, or, in other words, of multiplicity one.

5) If, for some ν,

$$x_{\nu+1} \ = \ x_{\nu+2} \ = \ \cdots \ = \ x_{\nu+m} ,$$

which implies that

$$x_\nu < x_{\nu+1} = x_{\nu+m} < x_{\nu+m+1} ,$$

then, according to (1.4), $T_{m,\nu}$ takes the simple form

$$T_{m,\nu}(x) = \begin{cases} \dfrac{\sin^{m-1}\left(\frac{x-x_\nu}{2}\right)}{\sin^{m-1}\left(\frac{x_{\nu+1}-x_\nu}{2}\right)}, & \text{for } x_\nu \le x < x_{\nu+1} , \\ 0 , & \text{otherwise} . \end{cases} \tag{1.6}$$

Throughout the paper we will make use of some addition theorems for trigonometric functions; for convenience, we collect these in the following proposition.

Proposition 1.1: *For all $\alpha, \beta \in \mathbb{C}$, the following relations hold:*

$$2\sin(\alpha)\cdot\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta); \tag{1.7}$$

$$2\sin(\alpha)\cdot\cos(\beta) = \sin(\alpha-\beta) + \sin(\alpha+\beta); \tag{1.8}$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta); \tag{1.9}$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta). \tag{1.10}$$

2. An Integral Representation for Trigonometric B-Splines

A fundamental tool for most of our proofs will be the following contour integral representation of the functions $T_{m,\nu}$. A similar result for polynomial B-splines is known since many years (see [5,6,10]), but the proofs given there do not go through for trigonometric splines.

Theorem 2.1: *For given $m, \nu \in \mathbb{N}$ and $x \in \mathbb{R}$ let $C = C_x$ be a circle in the complex plane which does not go through a knot and has all knots x_μ with $x < x_\mu \le x_{\nu+m}$ and no others, nor one of the points $x_\mu + 2k\pi$, $k \in \mathbb{Z} \setminus \{0\}$, in its interior. Then*

$$T_{m,\nu}(x) = \frac{\sin\left(\frac{x_{\nu+m}-x_\nu}{2}\right)}{4\pi i} \cdot \int_C \frac{\sin^{m-1}\left(\frac{z-x}{2}\right) dz}{\sin\left(\frac{z-x_\nu}{2}\right)\sin\left(\frac{z-x_{\nu+1}}{2}\right)\cdots\sin\left(\frac{z-x_{\nu+m}}{2}\right)} . \tag{2.1}$$

Proof. Let us denote the right hand side of (2.1) by $I_{m,\nu}(x)$. We prove the equality $I_{m,\nu} = T_{m,\nu}$ by induction w.r.t. m, so let $m = 1$. In this case,

$$I_{1,\nu}(x) \;=\; \frac{\sin\left(\frac{x_{\nu+1}-x_\nu}{2}\right)}{4\pi i} \cdot \int_C \frac{dz}{\omega(z)} \tag{2.2a}$$

with

$$\omega(z) \;:=\; \sin\left(\frac{z-x_\nu}{2}\right) \cdot \sin\left(\frac{z-x_{\nu+1}}{2}\right). \tag{2.2b}$$

If $x \geq x_{\nu+1}$, the integrand in (2.2a) has no poles in the interior of C, and so $I_{m,\nu}(x) = 0$.

If $x_\nu \leq x < x_{\nu+1}$, there is one pole of the integrand in the interior of C, namely the knot $x_{\nu+1}$. According to the Residue Theorem, we obtain

$$\int_C \frac{dz}{\omega(z)} \;=\; \frac{2\pi i}{\omega'(x_{\nu+1})} \;=\; \frac{4\pi i}{\sin\left(\frac{x_{\nu+1}-x_\nu}{2}\right)},$$

since

$$\omega'(z) \;=\; \frac{1}{2}\left(\sin\left(\frac{z-x_\nu}{2}\right)\cos\left(\frac{z-x_{\nu+1}}{2}\right) + \cos\left(\frac{z-x_\nu}{2}\right)\sin\left(\frac{z-x_{\nu+1}}{2}\right)\right),$$

and so $I_{m,\nu}(x) = 1$.

Finally, if $x < x_\nu$, we again use the Residue Theorem and find that

$$\int_C \frac{dz}{\omega(z)} \;=\; 2\pi i\left(\frac{1}{\omega'(x_\nu)} + \frac{1}{\omega'(x_{\nu+1})}\right) \;=\; 0\,; \tag{2.3}$$

due to continuity, (2.3) still holds true for $x_{\nu+1} \to x_\nu$, and so the theorem is proved for $m = 1$.

We assume now that, for some $m \geq 2$, statement (2.1) is true for $m-1$, and prove its validity for m itself.

Case 1. It is $x_\nu < x_{\nu+m-1}$ and $x_{\nu+1} < x_{\nu+m}$. In this case, we have to verify that the relation

$$I_{m,\nu}(x) \;=\; \frac{\sin\left(\frac{x-x_\nu}{2}\right)}{\sin\left(\frac{x_{\nu+m-1}-x_\nu}{2}\right)} I_{m-1,\nu}(x) + \frac{\sin\left(\frac{x_{\nu+m}-x}{2}\right)}{\sin\left(\frac{x_{\nu+m}-x_{\nu+1}}{2}\right)} I_{m-1,\nu+1}(x). \tag{2.4}$$

holds. To do this, we use the linearity of the integral and find that the right hand side of (2.4) equals

$$\frac{1}{4\pi i} \cdot \int_C \frac{\left\{\sin\left(\frac{x-x_\nu}{2}\right)\sin\left(\frac{z-x_{\nu+m}}{2}\right) + \sin\left(\frac{x_{\nu+m}-x}{2}\right)\sin\left(\frac{z-x_\nu}{2}\right)\right\} \cdot \sin^{m-2}\left(\frac{z-x}{2}\right)dz}{\sin\left(\frac{z-x_\nu}{2}\right)\sin\left(\frac{z-x_{\nu+1}}{2}\right)\cdots\sin\left(\frac{z-x_{\nu+m}}{2}\right)}. \tag{2.5}$$

Now we apply (1.7) to the bracketed term in the numerator of (2.5) and obtain the equality

$$\begin{aligned}
&\sin\Big(\frac{x-x_\nu}{2}\Big)\sin\Big(\frac{z-x_{\nu+m}}{2}\Big)+\sin\Big(\frac{x_{\nu+m}-x}{2}\Big)\sin\Big(\frac{z-x_\nu}{2}\Big)\\
&\quad = \frac{1}{2}\cos\Big(\frac{x-x_\nu-z+x_{\nu+m}}{2}\Big)-\frac{1}{2}\cos\Big(\frac{x-x_\nu+z-x_{\nu+m}}{2}\Big)\\
&\qquad +\frac{1}{2}\cos\Big(\frac{x_{\nu+m}-x-z+x_\nu}{2}\Big)-\frac{1}{2}\cos\Big(\frac{x_{\nu+m}-x+z-x_\nu}{2}\Big)\\
&\quad = \frac{1}{2}\cos\Big(\frac{x_{\nu+m}-x_\nu-z+x}{2}\Big)-\frac{1}{2}\cos\Big(\frac{x_{\nu+m}-x_\nu+z-x}{2}\Big)\\
&\quad = \sin\Big(\frac{x_{\nu+m}-x_\nu}{2}\Big)\cdot\sin\Big(\frac{z-x}{2}\Big)\,,
\end{aligned}$$

which, together with (2.5), completes the proof of Theorem 2.1 in Case 1.

Case 2. It is $x_\nu < x_{\nu+1} = x_{\nu+m}$. In this case we verify directly that $I_{m,\nu}$, which reads in this case

$$I_{m,\nu}(x) = \frac{\sin\Big(\frac{x_{\nu+m}-x_\nu}{2}\Big)}{4\pi i}\cdot\int_C \frac{\sin^{m-1}\Big(\frac{z-x}{2}\Big)\,dz}{\sin\Big(\frac{z-x_\nu}{2}\Big)\cdot\sin^m\Big(\frac{z-x_{\nu+1}}{2}\Big)}\,, \tag{2.6}$$

coincides with (1.6). Obviously, $I_{m,\nu}(x) = 0$ for $x \geq x_{\nu+m}$. Moreover, using the result of Case 1 we obtain immediately that

$$I_{m,\nu}(x) = 0 \quad \text{for} \quad x < x_\nu\,, \tag{2.7}$$

since the integral depends continuously on the parameters $x_{\nu+1},\ldots,x_{\nu+m}$.

Now let $x_\nu \leq x < x_{\nu+1}$. Here, according to the Residue Theorem and using (1.7), we have

$$\begin{aligned}
I_{m,\nu}(x) &= \frac{1}{2}\sin\Big(\frac{x_{\nu+1}-x_\nu}{2}\Big)\cdot\operatorname{Res}\left(\frac{\sin^{m-1}\Big(\frac{z-x}{2}\Big)}{\sin\Big(\frac{z-x_\nu}{2}\Big)\cdot\sin^m\Big(\frac{z-x_{\nu+1}}{2}\Big)};z=x_{\nu+1}\right)\\
&= -\frac{1}{2}\sin\Big(\frac{x_{\nu+1}-x_\nu}{2}\Big)\cdot\operatorname{Res}\left(\frac{\sin^{m-1}\Big(\frac{z-x}{2}\Big)}{\sin\Big(\frac{z-x_\nu}{2}\Big)\cdot\sin^m\Big(\frac{z-x_{\nu+1}}{2}\Big)};z=x_\nu\right)\\
&= -\frac{1}{2}\sin\Big(\frac{x_{\nu+1}-x_\nu}{2}\Big)\cdot\frac{\sin^{m-1}\Big(\frac{x_\nu-x}{2}\Big)}{\frac{1}{2}\cdot\sin^m\Big(\frac{x_\nu-x_{\nu+1}}{2}\Big)}\\
&= \frac{\sin^{m-1}\Big(\frac{x-x_\nu}{2}\Big)}{\sin^{m-1}\Big(\frac{x_{\nu+1}-x_\nu}{2}\Big)}\,.
\end{aligned}$$

Case 3. It is $x_\nu = x_{\nu+m-1} < x_{\nu+m}$. In this case, the statement can be proved in complete analogy to Case 2. $\square$

In the next section, the representation (2.1) will be used for proving some new identities for trigonometric B-splines, which cover as special cases the recursion (1.4) resp. the differential recursion (3.16), proved in [4].

3. Identities for Trigonometric B-Splines

The first theorem in this section can be viewed as a generalization of the recursion formula (1.4). Our aim is to express the B-spline $T_{m,\nu}$ by some others of order $m-k$, $k \in \{1,\dots,m-1\}$. Since the index ν is not important here, and since we shall have indices enough to struggle with, we set throughout this section $\nu = 0$ and use the notation $T_m := T_{m,\nu}$.

We need some more notation: For fixed $k \in \{1,\dots,m-1\}$, let N denote the following set of k-tuples:

$$N := \{(j_1,\dots,j_k);\ 0 \le j_1 < j_2 < \dots < j_k \le m\}. \tag{3.1}$$

Futhermore, for each $(j_1,\dots,j_k) \in N$, we denote by

$$T_{m-k}(x \,|\, j_1,\dots,j_k) \tag{3.2}$$

the B-spline of order $m-k$ with knots $\{x_0,\dots,x_m\} \setminus \{x_{j_1},\dots,x_{j_k}\}$. Finally we define, again for each $(j_1,\dots,j_k) \in N$, two indices r and s by

$$r := \min\{\, i\ ;\ 0 \le i \le k+1,\ j_i > i-1\}, \quad \text{and} \tag{3.3a}$$

$$s := \max\{\, i\ ;\ 0 \le i \le k+1,\ j_i < m-k+i\}, \tag{3.3b}$$

where we have set formally $j_0 := -1$ and $j_{k+1} := m+1$. Note that these indices count how many of the first resp. last subsequent knots are deleted in (3.2).

Now we are ready to formulate the following result.

Theorem 3.1 (Generalized Recursion Formula): *For each* $(j_1,\dots,j_k) \in N$, *set*

$$\mu_{(j_1,\dots,j_k)} := \frac{\sin\left(\frac{x_{m-k+s}-x_{r-1}}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}. \tag{3.4}$$

Furthermore, let there be given functions $\lambda_{(j_1,\dots,j_k)}(x)$ *such that for each* $z \in \mathbb{C}$ *the relation*

$$\sum_{(j_1,\dots,j_k)\in N} \lambda_{(j_1,\dots,j_k)}(x) \cdot \mu_{(j_1,\dots,j_k)} \cdot \prod_{i=1}^{k} \sin\left(\frac{z-x_{j_i}}{2}\right) = \sin^k\left(\frac{z-x}{2}\right) \tag{3.5}$$

holds, with $\lambda_{(j_1,\ldots,j_k)}(x)$ *independent of* z .

Then the following identity is true:

$$T_m(x) \;=\; \sum_{(j_1,\ldots,j_k)\in N} \lambda_{(j_1,\ldots,j_k)}(x)\cdot T_{m-k}(x\,|\,j_1,\ldots,j_k)\ . \tag{3.6}$$

Before proving this result, we would like to illustrate it by considering some aspects of the special case $k=1$. Here we obtain with

$$\mu_0 \;=\; \frac{\sin\left(\frac{x_m-x_1}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}\ , \qquad \mu_m \;=\; \frac{\sin\left(\frac{x_{m-1}-x_0}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}\ , \tag{3.7}$$

and $\mu_j = 1$ for $j=1,\ldots,m-1$, the identity

$$\sum_{j=0}^{m} \lambda_j(x)\cdot T_{m-1}(x\,|\,j) \;=\; T_m(x)\ , \tag{3.8}$$

whenever the λ_j 's satisfy

$$\sum_{j=0}^{m} \lambda_j(x)\cdot \mu_j\cdot \sin\left(\frac{z-x_j}{2}\right) \;=\; \sin\left(\frac{z-x}{2}\right) \tag{3.9}$$

for $z \in \mathbb{C}$.

For example, if $x_0 < x_{m-1}$ and $x_1 < x_m$, we may choose

$$\lambda_0(x) \;=\; \frac{\sin\left(\frac{x_m-x}{2}\right)}{\sin\left(\frac{x_m-x_1}{2}\right)}\ , \qquad \lambda_m(x) \;=\; \frac{\sin\left(\frac{x-x_0}{2}\right)}{\sin\left(\frac{x_{m-1}-x_0}{2}\right)}\ , \tag{3.10}$$

and $\lambda_j(x) = 0$ for $j=1,\ldots,m-1$, which leads us back to the recursion (1.5) (with τ replaced by m).

But there are also other possibilities: For two indices $\alpha,\beta \in \{1,\ldots,m-1\}$ with $x_\alpha \neq x_\beta$, set

$$\lambda_\alpha(x) \;=\; \frac{\sin\left(\frac{x-x_\beta}{2}\right)}{\sin\left(\frac{x_\alpha-x_\beta}{2}\right)}\ , \qquad \lambda_\beta(x) \;=\; \frac{\sin\left(\frac{x_\alpha-x}{2}\right)}{\sin\left(\frac{x_\alpha-x_\beta}{2}\right)}\ ,$$

and $\lambda_j(x) = 0$ otherwise. Then, (3.9) is satisfied, and we obtain the recursion

$$\lambda_\alpha(x)\cdot T_{m-1}(x\,|\,\alpha) \;+\; \lambda_\beta(x)\cdot T_{m-1}(x\,|\,\beta) \;=\; T_m(x)\ . \tag{3.11}$$

Proof of Theorem 3.1. We first observe that, according to the definition of r and s, the support of the B-spline $T_{m-k}(x\,|\,j_1,\ldots,j_k)$ is precisely the interval $[x_{r-1}, x_{m-k+s}]$; in particular, $T_{m-k}(x\,|\,j_1,\ldots,j_k)$ is identically zero, if $x_{r-1} = x_{m-k+s}$.

Hence, using (2.1) we may write

$$\sum_{(j_1,\ldots,j_k)\in N} \lambda_{(j_1,\ldots,j_k)}(x)\cdot T_{m-k}(x\,|\,j_1,\ldots,j_k)$$

$$= \sum_{\substack{(j_1,\ldots,j_k)\in N\\ x_{r-1}<x_{m-k+s}}} \lambda_{(j_1,\ldots,j_k)}(x)\cdot T_{m-k}(x\,|\,j_1,\ldots,j_k)$$

$$= \sum_{\substack{(j_1,\ldots,j_k)\in N\\ x_{r-1}<x_{m-k+s}}} \lambda_{(j_1,\ldots,j_k)}(x)\cdot \frac{\sin\left(\frac{x_{m-k+s}-x_{r-1}}{2}\right)}{4\pi i}\cdot \int_C \frac{\sin^{m-k-1}\left(\frac{z-x}{2}\right)dz}{\prod\limits_{\substack{\mu=0\\ \mu\notin\{j_1,\ldots,j_k\}}}^{m} \sin\left(\frac{z-x_\mu}{2}\right)}$$

$$= \sum_{\substack{(j_1,\ldots,j_k)\in N\\ x_{r-1}<x_{m-k+s}}} \lambda_{(j_1,\ldots,j_k)}(x)\cdot \mu_{(j_1,\ldots,j_k)}\cdot \frac{\sin\left(\frac{x_m-x_0}{2}\right)}{4\pi i}\cdot \int_C \frac{\sin^{m-k-1}\left(\frac{z-x}{2}\right)dz}{\prod\limits_{\substack{\mu=0\\ \mu\notin\{j_1,\ldots,j_k\}}}^{m} \sin\left(\frac{z-x_\mu}{2}\right)}$$

$$= \sum_{(j_1,\ldots,j_k)\in N} \lambda_{(j_1,\ldots,j_k)}(x)\cdot \mu_{(j_1,\ldots,j_k)}\cdot \frac{\sin\left(\frac{x_m-x_0}{2}\right)}{4\pi i}\cdot \int_C \frac{\sin^{m-k-1}\left(\frac{z-x}{2}\right)dz}{\prod\limits_{\substack{\mu=0\\ \mu\notin\{j_1,\ldots,j_k\}}}^{m} \sin\left(\frac{z-x_\mu}{2}\right)}.$$

Since all $\lambda_{(j_1,\ldots,j_k)}(x)$ are independent of z, we can use the linearity of the integral and simplify the last expression further to

$$\frac{\sin\left(\frac{x_m-x_0}{2}\right)}{4\pi i}\cdot \int_C \sum_{(j_1,\ldots,j_k)\in N} \lambda_{(j_1,\ldots,j_k)}(x)\cdot \mu_{(j_1,\ldots,j_k)}\cdot \frac{\prod\limits_{i=1}^{k}\sin\left(\frac{z-x_{j_i}}{2}\right)\cdot \sin^{m-k-1}\left(\frac{z-x}{2}\right)dz}{\prod\limits_{\mu=0}^{m}\sin\left(\frac{z-x_\mu}{2}\right)}$$

$$= \frac{\sin\left(\frac{x_m-x_0}{2}\right)}{4\pi i}\cdot \int_C \frac{\sin^{k}\left(\frac{z-x}{2}\right)\cdot \sin^{m-k-1}\left(\frac{z-x}{2}\right)dz}{\prod\limits_{\mu=0}^{m}\sin\left(\frac{z-x_\mu}{2}\right)},$$

$$= T_m(x)\,,$$

where we have used (3.5). This completes the proof of Theorem 3.1. □

Our next result provides a generalization of the differential recursion fomula for trigonometric B-splines, originally proved in [4].

Theorem 3.2 (Generalized Differential Recursion Formula): *With the numbers μ_j defined in (3.4) resp. (3.7), let $\lambda_j(x)$ be real functions, such that for all $z \in \mathbb{C}$ the relation*

$$\sum_{j=0}^{m} \lambda_j(x) \cdot \mu_j \cdot \sin\Big(\frac{z-x_j}{2}\Big) \;=\; \frac{1-m}{2} \cdot \cos\Big(\frac{z-x}{2}\Big) \tag{3.12}$$

holds.

Then for each point x, where $\frac{d}{dx}T_m(x)$ exists, the following identity is true:

$$\frac{d}{dx}\,T_m(x) \;=\; \sum_{j=0}^{m} \lambda_j(x) \cdot T_{m-1}(x\,|\,j)\,. \tag{3.13}$$

Proof. Differentiating the Representation (2.1) w.r.t x, we obtain

$$\frac{d}{dx}\,T_m(x) \;=\; \frac{(1-m)\sin\Big(\frac{x_m-x_0}{2}\Big)}{8\pi i} \cdot \int_C \frac{\cos\Big(\frac{z-x}{2}\Big)\cdot \sin^{m-2}\Big(\frac{z-x}{2}\Big)\,dz}{\sin\Big(\frac{z-x_0}{2}\Big)\sin\Big(\frac{z-x_1}{2}\Big)\cdots\sin\Big(\frac{z-x_m}{2}\Big)}\,. \tag{3.14}$$

From this point on, the proof runs along the same lines as the one of Theorem 3.1 and is therefore omitted here. □

Using relations (1.8) and (1.9), it is easily verified that (3.12) holds in particular for

$$\lambda_0(x) \;=\; \frac{1-m}{2}\cdot\frac{\cos\Big(\frac{x_m-x}{2}\Big)}{\sin\Big(\frac{x_m-x_1}{2}\Big)}\,, \qquad \lambda_m(x) \;=\; \frac{m-1}{2}\cdot\frac{\cos\Big(\frac{x-x_0}{2}\Big)}{\sin\Big(\frac{x_{m-1}-x_0}{2}\Big)}\,, \tag{3.15}$$

and $\lambda_j(x)=0$ for $j=1,\ldots,m-1$; with these choices for the λ_j 's, Relation (3.13) reads

$$\frac{d}{dx}\,T_m(x) \;=\; \frac{m-1}{2}\cdot\frac{\cos\Big(\frac{x-x_0}{2}\Big)}{\sin\Big(\frac{x_{m-1}-x_0}{2}\Big)}\cdot T_{m-1}(x\,|\,m) + \frac{1-m}{2}\cdot\frac{\cos\Big(\frac{x_m-x}{2}\Big)}{\sin\Big(\frac{x_m-x_1}{2}\Big)}\cdot T_{m-1}(x\,|\,0)\;, \tag{3.16}$$

a formula, which was originally proved in [4].

Remark. Of course it is no problem to differentiate the representation (3.14) further, and thus to derive identities like (3.13) for higher derivatives of T_m. But then, the right hand sides of (3.12) become quite unpleasant, and so I think that such identities will not be of great practical importance.

Now that we have found an identity which represents the derivative of T_m w.r.t. x, in the next theorem we establish a corresponding result for the partial derivative of T_m w.r.t. one of the knots.

For its precise formulation, we need to extend our previous notation: Let

$$T_m^{(i)}(x\,|\,j) \tag{3.17}$$

denote the trigonometric B-spline of order m, which arises from T_m by deleting the knot x_j, and adding the knot x_i one more time. Note that, according to this notation,

$$T_m^{(j)}(x\,|\,j) \;=\; T_m(x)\;. \tag{3.18}$$

Furthermore, let for $j \in \{0,\ldots,m\}$, ρ_j denote the multiplicity of the knot x_j, i.e., the number of times x_j appears in the set $\{x_0,\ldots,x_m\}$. We assume from now on that $\rho_j < m$ for all j, and that at least one inner knot exists.

Theorem 3.3 (Partial Derivative): *Choose some index $i \in \{0,\ldots,m\}$, and let again μ_j, $j = 0,\ldots,m$, denote the numbers*

$$\mu_0 \;=\; \frac{\sin\left(\frac{x_m-x_1}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}\,, \qquad \mu_m \;=\; \frac{\sin\left(\frac{x_{m-1}-x_0}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}\,, \qquad \textit{and } \mu_j = 1 \textit{ for } j = 1,\ldots,m-1\,. \tag{3.19}$$

Furthermore, let there be given real numbers $\lambda_0,\ldots,\lambda_m$, such that for all $z \in \mathbb{C}$ the relation

$$\sum_{j=0}^{m} \lambda_j \mu_j \sin\left(\frac{z-x_j}{2}\right) \;=\; \frac{\rho_i}{2}\cdot\cos\left(\frac{z-x_i}{2}\right) \tag{3.20}$$

holds. Then for each x, where $\frac{d}{dx_i}T_m(x)$ exists, the following identity is true:

$$\frac{d}{dx_i}T_m(x) \;=\; \sum_{j=0}^{m} \lambda_j T_m^{(i)}(x\,|\,j)\;. \tag{3.21}$$

In particular, we have

$$\frac{d}{dx_i}T_m(x) \;=\; \frac{\rho_i}{2}\cdot\begin{cases} \left(\dfrac{\cos\left(\frac{x_m-x_i}{2}\right)}{\sin\left(\frac{x_m-x_1}{2}\right)}T_m^{(i)}(x\,|\,0) \;-\; \dfrac{\cos\left(\frac{x_0-x_i}{2}\right)}{\sin\left(\frac{x_{m-1}-x_0}{2}\right)}T_m^{(i)}(x\,|\,m)\right), & 1 \le i \le m-1, \\[2ex] \left(\dfrac{\cos\left(\frac{x_m-x_0}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}T_m(x) \;-\; \dfrac{1}{\sin\left(\frac{x_{m-1}-x_0}{2}\right)}T_m^{(0)}(x\,|\,m)\right), & i = 0, \\[2ex] \left(\dfrac{1}{\sin\left(\frac{x_m-x_1}{2}\right)}T_m^{(m)}(x\,|\,0) \;-\; \dfrac{\cos\left(\frac{x_m-x_0}{2}\right)}{\sin\left(\frac{x_m-x_0}{2}\right)}T_m(x)\right), & i = m, \end{cases} \tag{3.22}$$

i.e., for each i, $\frac{d}{dx_i}T_m$ can be written as a linear combination of two m^{th} order B-splines.

Proof. Differentiating the Representation (2.1) w.r.t. x_i , we obtain

$$\frac{d}{dx_i}T_m(x) = \frac{\sin\left(\frac{x_m-x_0}{2}\right)}{4\pi i}\cdot\int\limits_C \frac{\sin^{m-1}\left(\frac{z-x}{2}\right)\cdot\frac{\rho_i}{2}\cdot\cos\left(\frac{z-x_i}{2}\right)dz}{\sin\left(\frac{z-x_0}{2}\right)\cdots\sin\left(\frac{z-x_m}{2}\right)\cdot\sin\left(\frac{z-x_i}{2}\right)}. \tag{3.23}$$

Using (2.1) also for the B-splines $T_m^{(i)}(x\,|\,j)$, this leads us directly to (3.21).

To verify (3.22) we have to show that

$$\cos\left(\frac{x_m-x_i}{2}\right)\sin\left(\frac{z-x_0}{2}\right)-\cos\left(\frac{x_0-x_i}{2}\right)\sin\left(\frac{z-x_m}{2}\right)=\sin\left(\frac{x_m-x_0}{2}\right)\cos\left(\frac{z-x_i}{2}\right) \tag{3.24}$$

for all i . But this can easily be done, using some of the identities listed in Proposition 1.1, and regarding that $1=\cos\left(\frac{x_0-x_0}{2}\right)=\cos\left(\frac{x_m-x_m}{2}\right)$. □

For some $n\in\mathbb{N}$, $n\geq 2$, let

$$\mathcal{S}_m := \mathcal{S}_m\begin{pmatrix} x_1 & \cdots & \cdots & x_{n-1} \\ \rho_1 & \cdots & \cdots & \rho_{n-1}\end{pmatrix}$$

denote the space of trigonometric splines with (inner) knots $x_1,\ldots x_{n-1}$, of multiplicities $\rho_1,\ldots\rho_{n-1}$. It is well-known (cf. [3, 9]) that – very similar to the polynomial case – this linear space possesses a B-spline basis, and that

$$\dim\mathcal{S}_m = m+\sum_{j=1}^{n-1}\rho_j\,. \tag{3.25}$$

Using the result of Theorem 3.3, the next corollary follows in complete analogy to the polynomial case.

Corollary 3.4 (Tangent Space): *Consider some arbitrary, but fixed $s\in\mathcal{S}_m$. Then the tangent space $T(s)$ of s is the space*

$$T(s)=\mathcal{S}_m\begin{pmatrix} x_1 & \cdots & \cdots & x_{n-1} \\ \rho_1+1 & \cdots & \cdots & \rho_{n-1}+1\end{pmatrix}$$

of dimension

$$\dim T(s) = m+n-1+\sum_{j=1}^{n-1}\rho_j\,.$$

4. A Convex-Hull Property

In contrast to the polynomial ones (see e.g. [1,2]), trigonometric B-splines in general do *not* establish a partition of unity, and therefore the trigonometric B-spline curve

$$t(x) := \sum_{\nu} b_\nu T_{m,\nu}(x) \tag{4.1}$$

cannot be expected to lie in the convex hull of the control points b_ν.

To overcome this difficulty, P.E.Koch, T.Lyche, M.Neamtu & L.L.Schumaker [3] introduced quite recently the concept of trigonometric convexity and control curves, thus obtaining nice results concerning curve design properties of trigonometric B-splines.

In the present paper, we go a bit more the "classical" way and show that – for equidistant knots and m odd, which corresponds to the space $\mathcal{T}_m$ as defined in (1.1) – the $T_{m,\nu}$'s indeed add up to a non-zero constant and so, after a suitable re-normalization, the trigonometric B-splines form a partition of unity. In the remainder of this section, we work out and prove now these roughly sketched ideas.

Theorem 4.1 (Partition of Unity): *Let $m \in \mathbb{N}$ be odd, and assume that*

$$x_{\nu+1} - x_\nu = h > 0 \qquad \text{for all } \nu \in \mathbb{Z}\,. \tag{4.2}$$

Then there is a positive constant C_m, which depends on h, but not on $\{x_\nu\}$, such that

$$\sum_{\nu \in \mathbb{Z}} N_{m,\nu}(x) \equiv 1\,, \tag{4.3a}$$

where

$$N_{m,\nu}(x) := C_m \cdot T_{m,\nu}(x) \tag{4.3b}$$

for all $\nu \in \mathbb{Z}$.

The constants C_m can be given explicitly, see the proof. For example, it is

$$\begin{aligned} &C_1 = 1\,, \qquad C_3 = \cos\left(\frac{h}{2}\right), \\ &C_5 = \frac{2}{3}\left(\cos\left(\frac{h}{2}\right)\cdot\cos\left(\frac{3h}{2}\right) + \frac{1}{2}\right), \quad \text{and} \\ &C_7 = \frac{2}{5}\left(\cos\left(\frac{h}{2}\right)\cdot\cos\left(\frac{3h}{2}\right)\cdot\cos\left(\frac{5h}{2}\right) + \frac{1}{2}\left(\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)+\cos\left(\frac{5h}{2}\right)\right)\right). \end{aligned} \tag{4.4}$$

Before proving Theorem 4.1, we state an easy, but for applications important corollary.

Corollary 4.2 (Convex-Hull Property): *For each* $x \in \mathbb{R}$, *the point*

$$t(x) := \sum_{\nu \in \mathbb{Z}} b_\nu N_{m,\nu}(x) \tag{4.5}$$

lies in the convex hull of the control points b_j, $j \in J_x$. *Here,* J_x *denotes the finite set*

$$J_x = \{j \in \mathbb{Z}\,;\, N_{m,j}(x) \neq 0\}\,.$$

Proof. Since $C_m > 0$, and since $T_{m,\nu} \geq 0$ for all $x \in \mathbb{R}$ (cf. [3, 9]), the statement follows immediately from Theorem 4.1. □

Proof of Theorem 4.1. For $m = 1$, the result is obvious, so let $m \geq 3$. We make use of the Marsden type identity

$$\sin^{m-1}\Big(\frac{y-x}{2}\Big) = \sum_{\nu \in \mathbb{Z}} T_{m,\nu}(x) \cdot \prod_{i=1}^{m-1} \sin\Big(\frac{y - x_{\nu+i}}{2}\Big)\,, \tag{4.6}$$

which is due to T.Lyche & R.Winther [4], see also [9]. Here, $y \in \mathbb{R}$ is some dummy variable, and (4.6) is a trigonometric polynomial in y from the space $\mathcal{T}_m$, as defined in (1.1). The basic idea of our proof is now to compare the constant terms of this polynomial on both sides of the identity (4.6).

From the well-known relation

$$\sin^{m-1}(\varphi) = \frac{1}{2^{m-1}}\left(2\sum_{j=0}^{\frac{m-3}{2}}(-1)^{\frac{m-1}{2}-j}\binom{m-1}{j}\cos((m-1-2j)\varphi) + \binom{m-1}{\frac{m-1}{2}}\right), \tag{4.7}$$

valid for all odd integers m and $\varphi \in \mathbb{R}$, we deduce that the constant term on the left hand side of (4.6), denoted as l_m, equals

$$l_m = \frac{1}{2^{m-1}} \cdot \binom{m-1}{\frac{m-1}{2}} \tag{4.8}$$

and is obviously positive. Here we have used that, for $k \in \mathbb{N}_0$,

$$\cos\Big(2k\Big(\frac{y-x}{2}\Big)\Big) = \cos(ky)\cos(kx) + \sin(ky)\sin(kx)\,, \tag{4.9}$$

due to (1.10), which implies that no other term in (4.7) contributes to the constant.

Now we consider the product on the right hand side of (4.6). Since $m-1$ is even, we may group the factors in pairs and find, again using (1.7) that

$$\prod_{i=1}^{m-1} \sin\Big(\frac{y - x_{\nu+i}}{2}\Big) = \prod_{j=1}^{\frac{m-1}{2}} \left(\sin\Big(\frac{y - x_{\nu+j}}{2}\Big) \cdot \sin\Big(\frac{y - x_{\nu+m-j}}{2}\Big)\right)$$

$$= \prod_{j=1}^{\frac{m-1}{2}} \left(\frac{1}{2} \left(\cos\Big(\frac{x_{\nu+m-j} - x_{\nu+j}}{2} \Big) - \cos(y - \gamma_m) \right) \right)$$

$$= \left(\frac{1}{2}\right)^{\frac{m-1}{2}} \prod_{j=1}^{\frac{m-1}{2}} \left(\cos\Big(\frac{2j-1}{2} h \Big) - \cos(y - \gamma_m) \right), \tag{4.10}$$

where we have defined

$$\gamma_m := \frac{x_{\nu+m-1} + x_{\nu+1}}{2} .$$

Now we have to recall the definition of the *elementary symmetric functions* $\sigma_k^{(N)}$ of order N, which can be done through the polynomial identity

$$(\alpha_1 - z)(\alpha_2 - z) \cdots (\alpha_N - z) = \sum_{k=0}^{N} (-z)^{N-k} \sigma_k^{(N)} ,$$

or explicitly

$$\sigma_k^{(N)} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \prod_{l=1}^{k} \alpha_{i_l} \tag{4.11}$$

with $\sigma_0^{(N)} = 1$. For example, we have

$$\sigma_1^{(N)} = \sum_{l=1}^{N} \alpha_l \quad \text{and} \quad \sigma_N^{(N)} = \prod_{l=1}^{N} \alpha_l .$$

Using this notation, we can re-write (4.10) as

$$\left(\frac{1}{2}\right)^{\frac{m-1}{2}} \sum_{k=0}^{\frac{m-1}{2}} (-1)^k \cos^k(y - \gamma_m)\, \sigma_{\frac{m-1}{2}-k}^{(\frac{m-1}{2})} \tag{4.12}$$

where the σ's are defined through (4.11) with

$$\alpha_j = \cos\Big(\frac{2j-1}{2} h \Big) .$$

Obviously, they depend only on h. Now we have to filter out the constant terms (w.r.t. y) in the representation (4.12). To do this, we apply the elementary identities (cf. (4.7))

$$\cos^{2n}(\varphi) = \frac{1}{2^{2n}} \left(2 \sum_{j=0}^{n-1} \binom{2n}{j} \cos(2(n-j)\varphi) + \binom{2n}{n} \right) \tag{4.13a}$$

and

$$\cos^{2n-1}(\varphi) = \frac{1}{2^{2n-2}} \sum_{j=0}^{n-1} \binom{2n-1}{j} \cos((2(n-j)-1)\varphi) , \tag{4.13b}$$

valid for all $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

From (4.13) we recognize that in (4.12) only the terms with k even, say $k = 2n$, contribute to the constant, and that this contribution equals

$$\left(\frac{1}{2}\right)^{\frac{m-1}{2}} \frac{1}{2^{2n}} \binom{2n}{n} \sigma^{(\frac{m-1}{2})}_{\frac{m-1}{2}-2n} \tag{4.14}$$

Summing up all these contributions now gives us the constant term r_m on the right hand side of (4.6); it is

$$r_m = \left(\frac{1}{2}\right)^{\frac{m-1}{2}} \sum_{n=0}^{[\frac{m-1}{4}]} \frac{1}{2^{2n}} \binom{2n}{n} \sigma^{(\frac{m-1}{2})}_{\frac{m-1}{2}-2n} . \tag{4.15}$$

So we have proved that

$$l_m = r_m \sum_{\nu \in \mathbb{Z}} T_{m,\nu}(x) \tag{4.16}$$

with l_m and r_m defined in (4.8) resp. (4.15). Since l_m is positive, and since each $T_{m,\nu}$ is nonnegative everywhere and positive within its support, we conclude that $r_m > 0$. Therefore, setting

$$C_m := \frac{r_m}{l_m} = 2^{\frac{m-1}{2}} \cdot \left(\binom{m-1}{\frac{m-1}{2}}\right)^{-1} \cdot \sum_{n=0}^{[\frac{m-1}{4}]} \frac{1}{2^{2n}} \binom{2n}{n} \sigma^{(\frac{m-1}{2})}_{\frac{m-1}{2}-2n} , \tag{4.17}$$

Theorem 4.1 is proved. □

Let us now work out the application of this result to a curve design problem, and illustrate it by a numerical example. Since we are forced to work with B-splines with simple knots, and due to the fact that we are dealing with trigonometic functions (which are surely the prototype of periodic functions), it is natural to consider problems for *closed* spline curves.

We first have to make our normed B-splines $N_{m,\nu}$ periodic: To do this, we choose some $n \in \mathbb{N}$, $n \geq 2$, and define for $j = 0, \ldots, n-1$, the periodic trigonometric B-splines $P_{m,j}$ through

$$P_{m,j}(x) := \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \equiv j \bmod n}} N_{m,\nu}(x) . \tag{4.18}$$

Obviously, these functions are periodic on the interval $[x_0, x_n]$, and their restrictions to this interval establish a basis of the corresponding periodic spline space. Note that for each $x \in [x_0, x_n]$ only a finite number of terms on the right hand side of (4.18) is different from zero; in particular, if $n \geq m$, what we want to assume from now on, there are at most two non-zero summands.

Consider now a set of *control points* $b_0, \ldots, b_n \in \mathbb{R}^d$ with $b_0 = b_n$. We define the closed trigonometric spline curve associated with these control points as the mapping

$$t : [x_0, x_n] \rightarrow \mathbb{R}^d ,$$
$$t(x) = \sum_{j=0}^{n} b_j P_{m,j}(x) ,$$

in complete analogy to the well-known polynomial case (cf. [1,2,9]). Due to Corollary 4.2, each point $t(x)$, $x \in [x_0, x_n]$, lies in the convex hull of the control points $b_0, \ldots, b_n$.

For $d = 2$, $n = 5$ and $m = 3$, this behavior is illustrated quite nicely in Figure 1 (here, the control polygon concides with the boundary of the convex hull).

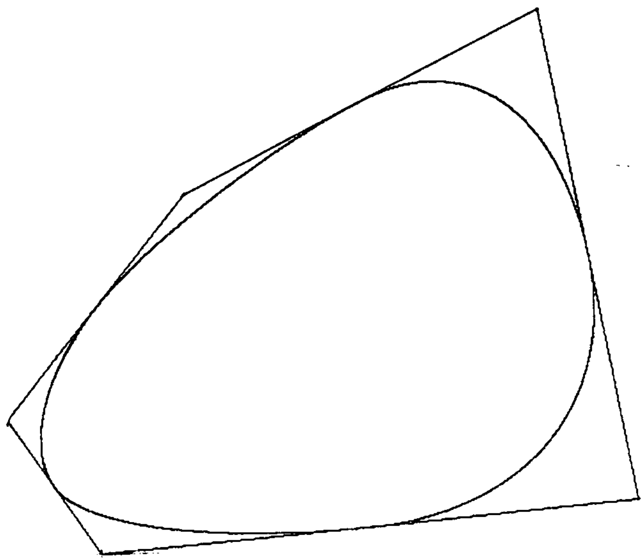

Figure 1. A Trigonometric Spline Curve and its Control Polygon

5. Remarks on Cosine-Splines

In some situations it maybe useful to consider trigonometric splines, which are piecewise in the space

$$\mathcal{C}_m := \text{span}\{1, \cos(x), \ldots, \cos((m-1)x)\} , \tag{5.1}$$

the so-called cosine-splines. Here, we have to impose the condition that $x_j < x_{j+m} < x_j + \pi$ for all $j \in \mathbb{Z}$, since otherwise $\mathcal{C}_m$ is no ECT space. For distinct knots, the B-splines $C_{m,\nu}$ for this trigonometric spline space are defined through the recursion

$$C_{m,\nu}(x) = \frac{\cos(x) - \cos(x_\nu)}{\cos(x_{\nu+m-1}) - \cos(x_\nu)} C_{m-1,\nu}(x) + \frac{\cos(x_{\nu+m}) - \cos(x)}{\cos(x_{\nu+m}) - \cos(x_{\nu+1})} C_{m-1,\nu+1}(x) , \tag{5.2a}$$

where for all j

$$C_{1,j}(x) := \begin{cases} 1, & \text{for } x_j \le x < x_{j+1}\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{5.2b}$$

see e.g. [4].

We only want to mention here that most of the results presented in our paper also hold for these B-splines $C_{m,\nu}$: An integral representation similar to (2.1) exists, and therefore identities like those presented in Section 3 can be derived. Moreover, the B-splines $C_{m,\nu}$ might be of particular interest for curve design techniques, since they form a partition of unity for all m and for arbitrary knot vectors, as follows easily from (5.2), using induction on m.

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Priv.-Doz. Dr. Guido Walz
Fakultät für Mathematik und Informatik
Universität Mannheim
D-68131 Mannheim
Germany
walz@math.uni-mannheim.de