A special Case of the Fundamental Lemma I

(The case $GSp(4,F)$)

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Introduction:

In the following we are concerned with the fundamental lemma for endoscopic groups (see [LS]) in the case, where the underlying group is the group of symplectic similitudes

$$G = GSp(4,F) \ ,$$

i.e. the group of all 4×4-matrices g satisfying

$$g^t \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} .$$

with coefficients in a local field F of positive residue characteristic different from 2. Let q be the number of elements in the residue field.

It was shown in [H], that in principal the endoscopic fundamental lemma holds for the group $GSp(4,F)$. However, the proof given there does not describe the correspondence between functions with matching orbital integrals, which is important for many applications. Often it is enough to know this correspondence for functions in the Hecke algebra. Using the trace formula one can reduce to the study of one particular function, the unit element 1_K of the Hecke algebra. We restrict ourselves to this important case.

There are essentially two types of tori, for which endoscopic comparison lemmas are needed for the group $GSp(4,F)$. In this paper we deal with the first, the other will be considered in another paper.

The torus T:

Consider the subgroup $H \subset GSp(4,F)$ of all matrices

$$(s,s') = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha' & 0 & \beta' \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma' & 0 & \delta' \end{pmatrix} \quad , \quad s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad , \quad s' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

for $s, s' \in Gl(2,F)$, such that $det(s) = det(s')$. Furthermore consider the torus T

$$T = \{(x,x') \in L^* \times L^* | Norm(x/x') = 1\} \ ,$$

for a field extension L/F of degree 2. T can be embedded as maximal torus into H and G. Any embedding is conjugate to one of the following standard form:

Choose $A, A' \in F$ such that $L = F(\sqrt{A}) = F(\sqrt{A'})$. This means $A/A' = \theta^2$ for some $\theta \in F^*$. For simplicity we assume A, A' to be integral. Embed T by the A resp. A'-regular embeddings, which map given elements $x = (x,x')$ of T

$$x = a + b\sqrt{A} \in L \quad , \quad x' = a' + b'\sqrt{A'} \ .$$

to the pair (s,s')

$$s=\phi_A(x)=\begin{pmatrix} a & Ab \\ b & a \end{pmatrix} \quad , \quad s'=\phi_{A'}(x')=\begin{pmatrix} a' & A'b' \\ b' & a' \end{pmatrix} .$$

Two such standard embeddings are conjugate if and only if the quotient of the corresponding θ's is in $Norm(L^*)$.

Problem:

Our aim is to calculate the $GSp(4,F)$ orbital integral

$$O_\eta^G(1_K)=\int_{G_\eta\backslash G} 1_K(g^{-1}\eta g)dg/dg_\eta$$

for the unit element 1_K of the Hecke algebra (= characteristic functions of the unimodular symplectic similitudes). This will be used to calculate the so called κ-orbital integrals in the end:

$$O_\eta^\kappa(1_K)=O_\eta^G(1_K)-O_{\eta'}^G(1_K) ,$$

where

$$\eta'=(\phi_{A_0}(x),\phi_{A_0'\theta^2}(x')) \quad , \quad \eta=(\phi_{A_0}(x),\phi_{A_0'}(x')) ,$$

and where $\theta\in F^*$ is such that $F^*=Norm(L^*)\cup\theta Norm(L^*)$.

We furthermore assume regularity $x,x'\neq 1$ and $x'\neq x,x^\sigma$, where σ denotes the nontrivial automorphism of L/F.

Assumption: Choose A_0,A_0' to be normalized of minimal order 0 or 1. However this does not mean, that $A'=A_0'\theta^2$ is also normalized, because in the unramified case $ord(\theta)=1$.

The endoscopic group M:

As the torus T is the pullback of the maps $Norm : L^*\times L^*\to F^*\times F^*$ and the diagonal embedding $F^*\to F^*\times F^*$, the dual group $\hat{T}$ is the pushout of the the diagonal map $\mathbb{C}^*\times\mathbb{C}^*\to(\mathbb{C}^*)^2\times(\mathbb{C}^*)^2$ and the multiplication map $\mathbb{C}^*\times\mathbb{C}^*\to\mathbb{C}^*$. This is $\hat{T}=(\mathbb{C}^*)^2\times(\mathbb{C}^*)^2/\{(t,t,t^{-1},t^{-1})\}$. The galois group acts by the generator σ of $Gal(L/F)$ permuting the first and second and third and fourth component respectively. The map $(z_1,z_2,z_3,z_4)/(t,t,t^{-1},t^{-1})\mapsto(z_1z_4,z_2/z_1,z_3/z_4)$ induces an isomorphism $\hat{T}\cong(\mathbb{C}^*)^3$. The action of σ is transported to

$$\sigma(x,\nu,\mu)=(x\nu\mu,\nu^{-1},\mu^{-1}) .$$

This defines an embedding of L-groups

$$\psi : {}^LT \to {}^LGSp(4,F) = GSp(4,\mathbb{C}) \times \Gamma$$

by $\psi(x,\nu,\mu) = diag(x, x\nu, x\mu\nu, x\mu)$ and

$$\psi(\sigma) = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \times \sigma \ .$$

Identify $\hat{T}$ with its image. Then $Z(\hat{G})$ is the group of all $(x,\nu,\mu) = (x,1,1)$ and

$$\hat{T}^\Gamma = Z(\hat{G}) \cup Z(\hat{G})\kappa \ ,$$

where $\kappa = (1,-1,-1)$. The centralizer of $\psi(\kappa) = diag(1,-1,1,-1)$ is

$$\hat{M} = \hat{M}^0 = \{ \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \subset GSp(4,\mathbb{C}) \} \ .$$

As $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ is contained in this group, the general construction defines the trivial homomorphism $\rho : \Gamma \to Out(\hat{M})$. We therefore consider ${}^LM = \hat{M} \times \Gamma$ as a subgroup of ${}^LGSp(4,F)$ in the trivial way. The morphism ψ factors in the evident way

$$\psi : {}^LT \to {}^LM \subset {}^LGSp(4,F) \ .$$

The group $M = Gl(2,F)^2/F^*$

$$M = Gl(2,F) \times Gl(2,F)/\{(t,t,t^{-1},t^{-1}) \mid t \in F^*\}$$

has ${}^LM$ as its L-group. Namely $X^*(T_M) = \{n_1x_1^* + n_2x_2^* + n_3x_3^* + n_4x_4^* \mid n_1 + n_2 = n_3 + n_4\}$ and $X_*(T_M) = \mathbb{Z}x_{1*} + .. + \mathbb{Z}x_{4*}/\mathbb{Z}(x_{1*} + x_{2*} - x_{3*} - x_{4*})$ with the obvious roots and coroots $\pm(x_1^* - x_2^*), \pm(x_3^* - x_4^*)$ resp. $\pm(x_{1*} - x_{2*}), \pm(x_{3*} - x_{4*})$. Dualizing gives the root system of the maximal torus in the group $\{(g_1,g_2) \in Gl(2,\mathbb{C}) \mid det(g_1) = det(g_2)\}$.

Consider some torus

$$T_M = (L^* \times L^*)/F^* \subset M = (Gl(2,F)^2)/F^* \ .$$

The isomorphism

$$\rho : (L^* \times L^*)/F^* \cong L^* \times (L^*/F^*) \cong L^* \times N^1(L) \cong T \subset GSp(4,F) \ ,$$

given by the maps

$$(t_1,t_2)/(t^{-1},t) \mapsto (t_1t_2, t_2 \; mod \; t) \mapsto (t_1t_2, t_2^\sigma/t_2) \mapsto (s,s') = (\phi_A(t_1t_2), \phi_{A'}(t_1t_2(t_2^\sigma/t_2))) \ ,$$

defines an admissable embedding of the torus

$$T_M = (L^* \times L^*)/F^* \quad \to T \quad \subset GSp(4,F)$$

$$(t_1,t_2) \; mod \; (t^{-1},t) \mapsto (s,s') = (\phi_A(t_1t_2), \phi_{A'}(t_1(t_2)^\sigma)) \ .$$

Let us record for comparison with [LS]

$$D_G(s,s') = \delta_G(s,s')|1 - x/x'||1 - x/x^\sigma||1 - x'/(x')^\sigma||1 - x'/x^\sigma|$$

and

$$D_H(t_1,t_2) = \delta_H(t_1,t_2)|1 - t_1/t_1^\sigma||1 - t_2/t_2^\sigma| = \delta_H(t_1,t_2)|1 - x/(x')^\sigma||1 - x/x'| \ .$$

Orbital integrals on M:

Observe that the (stable) orbital integrals $SO^M_{(t_1,t_2)}(1_K)$ on M are the same as the orbital integrals on $Gl(2,F) \times Gl(2,F)$, up to some appropriate normalization of measures

$$\int_{T\backslash M} 1_K(g^{-1}(t_1,t_2)g)dg/dt = [o_T : \rho((o_L^*)^2)] \int_{(L^*\backslash Gl(2,F))^2/F^*} 1_K(g^{-1}(t_1,t_2)g)dg/dl \ .$$

Here and in the following 1_K always denotes the unit element of the Hecke algebra of the corresponding group under consideration. We have

$$\int_{Gl(2,F)/L^*} 1_K(g^{-1}t_ig)dg/dl = 1_{o^*(L)}(t_i)(-\frac{1}{q-1} + \frac{q|A_0|^{1/2}}{q-1}|t_i - t_i^\sigma|^{-1})$$

in the case of <u>ramified</u> extensions L/F and

$$\int_{Gl(2,F)/L^*} 1_K(g^{-1}t_ig)dg/dl = 1_{o^*(L)}(t_i)(-\frac{2}{q-1} + \frac{|A_0'|^{1/2}(q+1)}{q-1}|t_i - t_i^\sigma|^{-1})$$

in the case of <u>unramified</u> extensions L/F.

We will show by explicit computations the

Theorem: For residue characteristic different from 2 we have

$$\Delta((x_1,x_2),(t_1,t_2))O_\eta^\kappa(1_K) = SO^M_{(t_1,t_2)}(1_K) \ ,$$

where

$$\Delta((x_1,x_2),(t_1,t_2)) = (\chi_o|\ .\ |)((x-x^\sigma)(x'-(x')^\sigma))$$

is the transfer factor of [LS]. Here χ_0 denotes the quadratic character of F^* attached to the field extension L/F. Furthermore $\eta = (\phi_{A_0}(x), \phi_{A_0}(x')) \in T$ with "image" $(t_1,t_2) \in T_M$ under ρ^{-1}.

Reduction to H :

In the following let denote π and Π prime elements of F respectively L. Using M.Schröder's double coset representatives [S] for $H \setminus G/K$ and appendix 2, we are led to calculate an infinite (actually then finite) sum

$$O_\eta^G(1_K) = \sum_{i=0}^{\infty} \frac{vol_G(K)}{vol_H(K_i)} O_\eta^H(1_{K_i}) \ ,$$

where

$$K_i = \{(h_1,h_2) \in K_H = Gl(2,o) \mid h_1 = h_2^\sigma \ mod\ \pi^i, det(h_1) = det(h_2)\} \ .$$

Here $o = o_F$ and $h^\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, inducing σ on $\phi_{A'}(L^*)$. Finally

$$O_\eta^H(1_{K_i}) = \int_{T\setminus H} 1_{K_i}(h^{-1}\eta h) dh/dt \ .$$

Here integration is over all elements in H

$$h = (h_1,h_2) \quad , \quad det(h_1) = det(h_2) \ .$$

The same formula holds for the κ-orbital integrals.

Remark: Here we used slightly better representatives then in [S], p.106-107.

We can and will use measures on G, H and $G_\eta = T$, such that the volumes of maximal compact subgroups are one, i.e. $vol_G(K) = 1$, $vol(K_H) = 1$ and $vol(o_T) = 1$. This means

$$vol_G(K)/vol_H(K_i) = [K_0 : K_i] \ .$$

Preliminary considerations on (x, x'):

Fix some $(x, x') \in T$. The coordinates a, b and a', b' of x and x' depend on the choice of the parameters A and A'. From now on we fix notation, such that these coordinates will be reserved for some fixed choice of normalized A_0 and A_0'.

Suppose x and x' are units in L^* with $x = x' \ mod \ \Pi$ (i.e. later $\chi > 0$) in case L/F is ramified. Then

$$x = t_1 t_2 \quad , \quad x' = t_1 t_2^\sigma$$

is solvable with units t_1, t_2 in L^*. Namely the image of o_L^* under $t \mapsto t/t^\sigma$ is $N_L^1 \cap (1 + \Pi o_L)$ in the ramified case and N_L^1 in the unramified case. Put

$$\chi = ord(a - a') \quad , \quad f = ord(b) \quad , \quad f' = ord(b') \quad , \quad F = min(f, f') \ .$$

Then $b = \pi^f B, b' = \pi^{f'} B'$ for units B, B' in o^* and

$$q^{f+f'} = |\frac{x - x^\sigma}{\sqrt{A_0}}|^{-1} |\frac{x' - (x')^\sigma}{\sqrt{A_0'}}|^{-1} \ .$$

Furthermore

$$\chi = ord(Tr(x) - Tr(x')) = ord(t_1 t_2 + t_1^\sigma t_2^\sigma - t_1 t_2^\sigma - t_1^\sigma t_2) = ord((t_1 - t_1^\sigma)(t_2 - t_2^\sigma))$$

and

$$|t_2 - (t_2)^\sigma| = |x - x'| = |(a - a')^2 + (b - b')^2 A_0|^{1/2}$$

respectively

$$|t_1 - (t_1)^\sigma| = |x - (x')^\sigma| = |(a - a')^2 + (b + b')^2 A_0'|^{1/2} \ .$$

Lemma 1: Suppose $\chi > 0$ if L/F is unramified. Then the following holds under the assumptions on x, x' made above:

1) $\{|A_0|^{1/2} |t_1 - t_1^\sigma|^{-1}, |A_0'|^{1/2} |t_2 - t_2^\sigma|^{-1}\} = \{q^F, q^{\chi - F - ord(A_0)}\}$

2) $\chi = ord(A_0(b^2 - (b')^2))$, hence

3) $\chi \geq ord(A_0) + 2F$ with equality in case $f \neq f'$.

Proof: Observe that $\chi > 0$ is automatic in the ramified case by our assumptions on x and x'. We have

$$2\chi = ord[(a - a')^2 - (b - b')^2 A_0)] + ord[(a - a')^2 - (b + b')^2 A_0] \ .$$

For $ord(b \pm b') \geq ord(a - a')$ this implies $ord(b \mp b') = 0$ and $ord(b \pm b') = ord(a - a') = \chi > 0$, hence $F = ord(b) = ord(b') = 0$. Claim 1) and 2) follow immediately. Otherwise, if for both signs $ord(b \pm b') < ord(a - a')$, the equality above gives

$$2\chi = ord(b + b')^2 + ord(b - b')^2 = ord(A_0(b^2 - (b')^2)^2 \ ,$$

hence 2). This and $F = min(ord(b), ord(b')) = min(ord(b - b'), ord(b + b'))$ implies 1).

The rings $R = o/\pi^i o$:

Let $R = o/\pi^i o$ denote the residue ring and let A, A' be elements in R. Suppose we are given matrices s, s' in $Gl(2,R)$

$$s = \begin{pmatrix} a & bA \\ b & a \end{pmatrix} \quad , \quad s' = \begin{pmatrix} a' & b'A' \\ b' & a' \end{pmatrix} .$$

(By abuse of language $b \in R$ stands for what later will be residue classes of elements $b/\pi^j \in o$). Let the group of invertible elements $a^2 - b^2 A$ for a, b in R be

$$N_A \subset R^* .$$

Then $N_A = (R^*)^2$ for $A \notin R^*$ and $N_A = R^*$ for $A \in R^*$.

<u>Lemma 2</u>: The centralizer $Gl(2,R)_s$ of s in $Gl(2,R)$ is either $Gl(2,R)$ (if $b = 0$ in R) or the group of all matrices

$$y = \begin{pmatrix} u & vA + Ann(b) \\ v & u + Ann(b) \end{pmatrix} .$$

Furthermore: Such $y \in Gl(2,R)_s$ have unique decompositions into a product of two invertible matrices of the form

$$y = \begin{pmatrix} u & vA \\ v & u \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1+g' \end{pmatrix} \quad , \quad g, g' \in Ann(b) .$$

<u>Warning</u>: This does not define a semidirect product decomposition of $Gl(2,R)_s$!

<u>Proof</u>: One direction is clear from

$$\begin{pmatrix} a & bA \\ b & a \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1+g' \end{pmatrix} = \begin{pmatrix} a & ag + bA \\ b & a + ag' \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1+g' \end{pmatrix} \begin{pmatrix} a & bA \\ b & a \end{pmatrix} .$$

The other follows during the proof (using $y \in Gl(2,R)$) of

<u>Lemma 3</u>: s and s' are conjugate by an element y in $Sl(2,R)$

$$y^{-1} s y = s' \quad \in Gl(2,R) ,$$

if ond only if they are conjugate in $Gl(2,R)$ under some matrix

$$y = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad , \quad \epsilon \in N_A \subset R^* .$$

In other words: $a = a'$, $b' = b/\epsilon$,$b'A' = bA\epsilon$ for some $\epsilon \in N_A$.

Proof: One direction is clear using conjugation with the elements

$$y = g^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad , \quad \epsilon \in R^*$$

in $Sl(2,R)$, for suitable chosen elements $g \in Gl(2,R)_s$ with $det(g) = \epsilon$.

For the other direction we can assume $b'|b$ by symmetry. The case $b' = 0$ is trivial, so we can assume $b' \neq 0$ in R. Then $y^{-1}sy = s'$ implies $Tr(s) = 2a = 2a' = Tr(s')$. By assumption $2 \in R^*$, hence $a = a'$. Let y be $y = \begin{pmatrix} u & v \\ z & w \end{pmatrix}$, then $sy = ys'$ reads $vb' = zAb$,$ub'A' = wbA$, $wb' = ub$, $zb'A' = vb$. Therefore $v = z(Ab/b') + f$ for some $f \in Ann(b')$ and $w = u(b/b') + f'$ for some $f' \in Ann(b')$. This implies

$$y = \begin{pmatrix} u & z(Ab/b') + f \\ z & u(b/b') + f' \end{pmatrix} .$$

Now $b' \neq 0$ implies $Ann(b') \subset \pi R$, hence $det(y) = 1$ gives $(u^2 - Az^2)(b/b') \in (1 + Ann(b')) \subset (R^*)^2$. This proves

$$\epsilon = b/b' \in N_A \subset R^*$$

and $u^2 - Az^2 \in R^*$. This gives for some new $f, f' \in Ann(b')$

$$y = \begin{pmatrix} u & zA + f \\ z & u + f' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} .$$

As the first term is in the centralizer of s we are done.

We get the (unique) product decomposition of the elements

$$\begin{pmatrix} u & zA + f \\ z & u + f' \end{pmatrix} = \begin{pmatrix} u & zA \\ z & u \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 + g' \end{pmatrix}$$

for some $g, g' \in Ann(b')$, as promised in lemma 2, by solving the equation

$$\begin{pmatrix} u & zA \\ z & u \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} f \\ f' \end{pmatrix} .$$

This is possible by $u^2 - Az^2 \in R^*$. Lemma 2 and 3 are proved.

Now consider a field extension $L = F(\sqrt{A})$ for some integer $A \in o$. Let $A \in R$ also denote its residue class. Then we deduce from lemma 2

Corollary: In the situation of lemma 2 we have

$$\#Sl(2,R)_s = \#Sl(2,R) \quad , \quad ord(b) = \nu \geq i$$

if $b=0$ in R or in the case of ramified L/F

$$\#(Sl(2,R)_s) = 2q^{2\nu+i} \quad , \quad ord(b) = \nu < i$$

respectively in the case of unramified L/F

$$\#(Sl(2,R)_s) = \begin{cases} 2q^{2\nu+i} & ord(A) > 0 \\ (q+1)q^{2\nu+i-1} & ord(A) = 0 \end{cases} \quad , \quad ord(b) = \nu < i \ .$$

Proof: For $ord(b) < i$ the number $Sl(2,R)_s$ is

$$\#(Ann(b)) \cdot \#(\{x \in Image(\phi_A(o_L^*)) \mid Norm(x) \in 1 + Ann(b)\})$$

by lemma 2. But $Ann(b) = (\pi^{i-\nu}) \ mod \ \pi^i$ has order q^ν. Furthermore $Norm(o_L^*) = N_A$ and the index of $1 + Ann(b)$ in R^* is $(q-1)q^{i-\nu-1}$. So

$$\#Sl(2,R)_s = \frac{[R^* : N_A]}{(q-1)} q^{2\nu+1-i} \#Image(\phi_A : o_L^* \to Gl(2,R)) \ .$$

But $\phi_A(o_L^*)$ has order $(q-1)q^{2i-1}$ for $\pi|A$ and order $(q^2-1)q^{2i-2}$ for $ord(A) = 0$. Hence the claim follows.

On $T \setminus H$-integration:

In this section assume A_0, A_0' and θ to be normalized of order 0 or 1. Consider regular elements

$$\eta = (\phi_{A_0}(x), \phi_{\theta^2 A_0'}(x'))$$

with centralizer T. We write H as a disjoint union of double cosets

$$\bigcup_{r \in R} T \cdot r \cdot K_0$$

where the representatives r are of the form

$$r \in (\phi_{A_0}(L*), \phi_{\theta^2 A_0'}(L^*))(\begin{pmatrix} 1 & 0 \\ 0 & \pi^j \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \pi^{j'-ord(\theta)}\epsilon_0 \end{pmatrix})$$

for certain $j, j' \in \mathbb{Z}$ and $\epsilon_0 \in o^*$. This implies

$$\int_H f(h)dh = \sum_r [o_T : (T \cap rK_0r^{-1})] \int_T dt \int_{K_0} f(trk)dk \ ,$$

hence

$$\int_{T\backslash H} f(h)dh/dt = \sum_r [o_T/(T\cap rK_0r^{-1})] \int_{K_0} f(rk)dk \ .$$

This allows to calculate the orbital integrals on H by integrations over K_0 and gives

$$O^G_\eta(1_K) = \sum_{i\geq 0}\sum_{r\in R} [o_T/(T\cap rK_0r^{-1})][K_0 : K_i] \int_{K_0} 1_{K_i}(k^{-1}r^{-1}\eta rk)dk \ .$$

Observe $a' + b'\sqrt{A_0'} = a' + (b'/\theta)\sqrt{\theta^2 A_o'}$, hence for $\Theta = \theta/\pi^{ord(\theta)}$

$$\eta_r := r^{-1}\eta r = (\phi_{A_0\pi^{2j}}(x), \phi_{\Theta^2 A_0'\pi^{2j'}\epsilon_0^2}(x'))$$

because of

$$\begin{pmatrix} 1 & 0 \\ 0 & \pi^{ord(\theta)-j'}\epsilon_0^{-1}\end{pmatrix}\begin{pmatrix} a' & b'\theta A_0' \\ b'\theta^{-1} & a'\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \pi^{j'-ord(\theta)}\epsilon_0\end{pmatrix} = \begin{pmatrix} a' & \frac{b'}{\Theta\epsilon_0\pi^{j'}}(\Theta\epsilon_0\pi^{j'})^2 A_0' \\ \frac{b'}{\Theta\epsilon_0\pi^{j'}} & a'\end{pmatrix} \ .$$

Therefore $T\cap rK_0r^{-1} \subset o_T = \{(x,x') \in o_L^* \times o_L^* \mid Norm(x/x') = 1\}$ is the group

$$\{(x,x') = (a+b\sqrt{A_0}, a'+b'\sqrt{A_0'}) \mid Norm(x) = Norm(x') \in o^*; a, a', \pi^{-j}b, \pi^{-j'}b' \in o\} \ .$$

It does not depend on the choice of the (normalized) A_0 resp. A_0' resp. θ.

<u>Notation</u>: Define orders $o_L(j) = o + o\pi^j\sqrt{A_0} \subset o_L$ for $j > 0$, then

<u>Constraints (for fixed r)</u>: $(x,x') \in T\cap rK_0r^{-1}$ means $x \in o_L(j)$ and $x' \in o_L(j')$.

Furthermore the <u>index</u> is

$$[o_T/(T\cap rK_0r^{-1})] = [o_L^* : o_L(j)^*][o_L^* : o_L(j')^*][Norm(o^*)/Norm(o_L(min(j,j'))^*]^{-1} \ .$$

These constraints have also to be satisfied for the nonvanishing of the orbital integral

$$[K_0 : K_i]\int_{K_0} 1_{K_i}(k^{-1}\eta_r k)dk \ .$$

Namely $k^{-1}\eta_r k \in K_i$ implies $\eta_r \in K_0$, hence $(x,x') \in T\cap rK_0r^{-1}$.

Suppose the r-constraints are satisfied. Then evaluating the orbital integral

$$[K_0 : K_i]\int_{K_0} 1_{K_i}(k^{-1}\eta_r k)dk \quad , \quad \eta_r = (s_r, (s_r)')$$

means counting all cosets $kK_i \subset K_0$, for which $k^{-1}\eta_r k \in K_i$. This is the number of all elements $y \in Sl(2, o/\pi^i o)$, such that

$$y^{-1}s_r y \equiv (s_r')^\sigma \ mod\ \pi^i \quad , \quad y \in Sl(2, o/\pi^i o)$$

holds (a problem solved in lemma 3). It is either zero or equal to the order of the <u>centralizer</u> $Sl(2, o/\pi^i o)_{s_r}$.

Double cosets in H:

For normalizated A_0 (resp. A_0') we have

$$Gl(2,F) = \bigcup_{j\geq 0} \phi_{A_0}(L^*) \begin{pmatrix} 1 & 0 \\ 0 & \pi^j \end{pmatrix} Gl(2,o) \ .$$

This gives decompositions (with unique integers $j \geq 0$ and $j' \geq 0$) for elements $(h,h') \in H$

$$h = \phi_{A_0}(l) \begin{pmatrix} 1 & 0 \\ 0 & \pi^j \end{pmatrix} k \qquad k \in Gl(2,o), l \in L^*$$

$$h' = \phi_{\theta^2 A_0'}(l') \begin{pmatrix} 1 & 0 \\ 0 & \pi^{j'-ord(\theta)} \end{pmatrix} k' \qquad k' \in Gl(2,o), l' \in L^* \ .$$

From $det(h) = det(h')$ we conclude

$$det(k)/det(k') \in Norm(L^*)\pi^{j'-j-ord(\theta)} \ .$$

In the ramified case we choose once and for all $\pi = -A_0$, such that $\pi = Norm(\Pi)$. Then $det(k)/det(k') \in Norm(L^*) \cap o^* = (o^*)^2$. Changing l by some unit in o^* allows to assume $det(k) = det(k')$, i.e. $(k,k') \in K_0 \subset H$. We get H as a disjoint union of double cosets

$$\bigcup_{r\in R} T \cdot r \cdot K_0 \quad , \quad R = \mathbb{N}_0 \times \mathbb{N}_0$$

and

$$r = r_{j,j'} = (\phi_{A_0}(\Pi^{-j}), \phi_{A_0'}(\Pi^{-j'}))(\begin{pmatrix} 1 & 0 \\ 0 & \pi^j \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \pi^{j'} \end{pmatrix}) \ .$$

Now consider the unramified case. Here we get first of all $\pi^{j'-j-ord(\theta)} \in Norm(L^*)o^*$ or

$$j' - j - ord(\theta) = 0 \ mod \ 2 \ .$$

Secondly, to achieve $det(k) = det(k')$, we are only allowed to change k or k' by elements in $\phi_{A_0\pi^{2j}}(L^*) \cap Gl(2,o)$ respectively $\phi_{\theta^2 A_0'\pi^{2j'}}(L^*) \cap Gl(2,o)$. So $det(k)/det(k')$ can be changed within the group $Norm(o_L^*(j)o_L^*(j'))$. This is $(o^*)^2$ unless $min(j,j') = 0$. We get H as a disjoint union of double cosets

$$\bigcup_{r\in R} T \cdot r \cdot K_0$$

where $R \subset ((o^*/(o^*)^2) \times \mathbb{N} \times \mathbb{N}) \cup (1 \times \mathbb{N}_0 \times 0) \cup (1 \times 0 \times \mathbb{N})$ consists of all elements $(\epsilon_0, j, j') \in (o^*/(o*)^2) \times \mathbb{N}^2$ with the property

$$j - j' - ord(\theta) = 0 \ mod \ 2 \ .$$

The representatives are

$$r = r_{\epsilon_0,j,j'} = (\phi_{A_0}(\Pi^{-[j/2]}), \phi_{\theta^2 A_0'}(\Pi^{-[j'-ord(\theta)/2]}\epsilon_0'))(\begin{pmatrix} 1 & 0 \\ 0 & \pi^j \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \pi^{j'-ord(\theta)}\epsilon_0 \end{pmatrix}) \ .$$

Here $\epsilon_0' \in o_L^*$ is chosen, such that $Norm(\epsilon_0') = \epsilon_0^{-1}$, where $\epsilon_0 \in o^* \setminus (o^*)^2$.

Summation conditions:

Let us use the notations $F = min(f,f')$, $\nu = f - j$, $\nu' = f' - j'$, $\chi = ord(a - a')$. Then $b = B\pi^{\nu}, b' = B'\pi^{\nu'}$ for some units $B, B' \in o^*$. Put $A_0 = A_0'$ and $\theta = \Theta\pi^{ord(\theta)}$. If L/K is unramified we put $\theta = \pi$. From lemma 3 we get conditions on the summation indices $(i,r) \in \mathbb{N}_0 \times R$, in order to yield a nonzero contribution to the orbital integral. These are:

1) $0 \le i \le \chi$

2) $\nu' \ge i$ iff $\nu \ge i$. If $\nu < i$ then $\nu = \nu'$ has to hold, furthermore also $-\Theta\epsilon B'/B \in N_A = (R^*)^2$ unless L/K is unramified and $\nu = f$.

3) $2f + ord(A_0) - \nu \ge i$ iff $2f' + ord(A_0) - \nu' \ge i$. If $2f + ord(A_0) - \nu < i$, then $f = f'$ and $i \le \chi - \nu$ has to hold.

For condition 3) observe, that the conditions of lemma 3

$$(b/\pi^j)\pi^{2j}A_0 = -(b'/\pi^{j'})\pi^{2j'}A_0'\Theta\epsilon_0\epsilon^{-1} \ mod \ \pi^i$$

and

$$(b/\pi^j) = -(b'/\pi^{j'}\Theta\epsilon_0)\epsilon \ mod \ \pi^i$$

(for some $\epsilon \in N_A$) can be combined and thus simplified. Here ϵ_0 denotes, changing previous notation, the representative of $o^*/(o^*)^2$. Observe $(b/\pi^j)\pi^{2j}A_0 = B\pi^{\nu+(2f-2\nu)}A_0 = BA_0\pi^{2f-\nu}$ and similar in the other case. Therefore $2f + ord(A_0) - \nu < i$ implies $2f' + ord(A_0) - \nu'$, yielding $2f + ord(A_0) - \nu = 2f' + ord(A_0) - \nu'$. But $\nu \le 2f - \nu < i$ and 2) gives equivalently $f = f'$ plus the remaining condition

$$\epsilon = -\Theta\epsilon_0 B'/B \ mod \ \pi^{i-2f-ord(A_0)+\nu} \ .$$

This together with

$$\epsilon^{-1} = -(\Theta\epsilon_0)^{-1}B'/B \ mod \ \pi^{i-\nu}$$

means $(B'/B)^2 = 1 \ mod \ \pi^{i-2f-ord(A_0)+\nu}$ or in other words $i-2f-ord(A_0)+\nu \le ord(B^2-(B')^2)$. Now $2f + ord(A_0) - \nu < i$ forces $i > 0$, which is only possible for $\chi = ord(a - a') > 0$ by 1). From $\chi > 0$ we get $\chi = ord(A_0(b^2 - (b')^2))$ by lemma 1. Hence by lemma 1 either

$$\chi = 2F + ord(A_0) \quad (f \neq f')$$

holds or

$$\chi = 2F + ord(A_0(B^2 - (B')^2)) \quad (f = f') \ .$$

We already know $f = f'$. Therefore it can be further reformulated to:

$$2f + ord(A_0) - \nu < i \quad \text{implies} \quad f = f' \ \text{and} \ i \le \chi - \nu \ .$$

Résumé:

We now express the results of the chapter on $T \setminus H$ integration in terms, which are suitable for summation. We saw, that the domain of possible values i is divided into three parts:

A) Conditions on i (apart from $0 \leq i \leq \chi$):

Range of small i: $0 \leq i \leq \nu$ and $0 \leq i \leq \nu'$

Middle range i: $\nu < i \leq 2f + ord(A_0) - \nu$ and $\nu' < i \leq 2f' + ord(A_0) - \nu'$.

In the middle range automatically $\nu = \nu'$ holds.

Large range i: $2f + ord(A_0) - \nu < i \leq \chi - \nu$ and $2f' + ord(A_0) - \nu' < i \leq \chi - \nu'$.

In the large range automatically $\nu = \nu'$ and $f = f'$ holds.

B) Conditions on ν, ν' and θ:

In the domain of possible values ν, ν' we have, in addition to possible restrictions mentioned above, the conditions:

$$0 \leq \nu \leq f \quad , \quad 0 \leq \nu' \leq f'$$

$$\nu - \nu' = f - f' + ord(\theta) \ mod\ 2 \ .$$

Furthermore in the middle and large range (concerning i) we have contributions only if

$$-\theta\epsilon B'/B \in (o^*)^2$$

is satisfied, at least unless both $\nu = f$ and L/F is unramified!

C) Contributions from the index $index(\nu, \nu')$:

The relevant contributions are

$$\begin{array}{ll} 1 & (\nu, \nu') = (f, f') \\ \frac{q+1}{q} q^{f+f'-\nu-\nu'} & \nu = f, \nu' \neq f' \ or\ \nu' = f', \nu \neq f \\ \frac{1}{2}\frac{(q+1)^2}{q^2} q^{f+f'-\nu-\nu'} & \nu < f, \nu' < f' \ , \end{array}$$

in the unramified case and

$$q^{f+f'-\nu-\nu'}$$

in the ramified case. (See appendix).

D) Contributions from the centralizer $centralizer(\nu, i)$:

$$(q^2-1)q^{3i-2} \text{ for } i \geq 1 \text{ , else } 1 \quad (i \leq \nu \text{ small range})$$

$$2q^{2\nu+i} \quad (i > \nu \text{ middle and large range}) ,$$

unless

$$\frac{q+1}{q} \cdot q^{2\nu+i} \quad (\nu = f, \nu' = f', \ L/F \text{ unramified}) .$$

In principle we now could compute the orbital integrals. We leave this to the reader. We concentrate instead on the κ-orbital integrals.

The summation (ramified case):

Assume, $\eta = (s, s')$ satisfies $x, x' \in o_L^*$ (otherwise our κ-orbital integral is zero). For the κ-orbital integral, we do not get any contribution from the small range of the i-summation in the ramified case, because this contribution is stable (does not depend on θ). The middle and large range contributions remain. The whole summation is empty unless $\chi > 0$, i.e. $x = x' \ mod \ \Pi$ in addition to $x, x' \in o_L^*$. This is true, because $\chi = 0$ implies $i = 0$, which is in the stable range. This allows to apply the preliminary remarks, made earlier.

Let χ_o be the quadratic character of F^* attached to L/F. It is trivial on $(o^*)^2$ with $\chi_0(\theta) = -1$. We get

$$O_\eta^\kappa(1_K) = (2q^{f+f'}\chi_0(-B/B')) \sum_{0 \leq \nu \leq F} \ \sum_{\nu < i \leq \chi - \nu} q^i .$$

or more precisely

$$O_\eta^\kappa(1_K) = 2q^{f+f'}\chi_0(-B/B') \cdot 1_{O_L^* \times (1+\Pi O_L)}(x, x/x') \sum_{0 \leq \nu \leq F} \ \sum_{\nu < i \leq \chi - \nu} q^i .$$

The double sum gives

$$(q-1)^{-1} \sum_{0 \leq \nu \leq F} (q^{\chi+1-\nu} - q^{\nu+1})$$

$$= (q-1)^{-1}[q^{\chi+1}(q^{-F-1}-1)/(q^{-1}-1) - q(q^{F+1}-1)/(q-1)]$$

$$= \frac{q}{(q-1)^2}(1-q^{F+1})(1-q^{\chi-F+1}) = q \int_{Gl(2,F)^2/(L^*)^2} 1_K(g^{-1}(t_1, t_2)g) dg/dl .$$

Hence $O^{\kappa}_{\eta}(1_K)$ is equal to

$$2|\frac{x-x^\sigma}{2\sqrt{A_0}}|^{-1}|\frac{x'-(x')^\sigma}{2\sqrt{A_0}}|^{-1}\chi_0(\frac{x^\sigma-x}{2\sqrt{A_0}})\chi_0(\frac{x'-(x')^\sigma}{2\sqrt{A_0}})\cdot q\int_{Gl(2,F)^2/(L^*)^2} 1_K(g^{-1}(t_1,t_2)g)dg/dl\ .$$

This implies the theorem in the case L/F ramified. The factor 2 disappears, because in the ramified case $[o_T : \rho((o_L^*)^2)] = 2$. Furthermore $|A_0|q = 1$ and $\chi_0(A_0) = \chi_0(-1)$.

The summation (unramified case):

To compute the κ-orbital integral we concentrate on the case $\chi > 0$. The case $\chi = 0$ is very simple, because only the small range summation contributes with the value $(-q)^{f+f'}$. Let us therefore turn to the essential case $\chi > 0$. From the résumé we get the following contributions:

1) Small range: Its contribution will be

$$SRC = \sum_{0\leq\nu\leq f}\sum_{0\leq\nu'\leq f'}\sum_{0\leq i\leq min(\nu,\nu')} index(\nu,\nu')\#centralizer(i)$$

$$= \sum_{0\leq i\leq F}\#centralizer(i)\Big((\sum_{i\leq\nu<f}\sum_{i\leq\nu'<f'}\frac{1}{2}\cdot 2\cdot\frac{(q+1)^2}{q^2}(-q)^{f+f'-\nu-\nu'}) + (\sum_{i\leq\nu'<f'}\frac{q+1}{q}(-q)^{f'-\nu'})$$
$$+(\sum_{i\leq\nu<f}\frac{q+1}{q}(-q)^{f-\nu}) + 1\Big)\ .$$

The representatives ϵ appearing in R give an additional factor 2 in the first sum! So

$$SRC = \sum_{0\leq i\leq F}\#centralizer(i)S(i,f,f')\ ,$$

where $\#centralizer(i) = 1, (q^2-1)q^{3i-2}$ and $S(i,f,f') = (S(i,f)+1)(S(i,f')+1)$ with

$$S(i,f) = \frac{q+1}{q}\sum_{i\leq\nu<f}(-q)^{f-\nu} = (-q)^{f-i} - 1\ .$$

This gives

$$SRC = (-q)^{f+f'}\Big(1+\frac{q^2-1}{q^2}\sum_{0<i\leq F} q^i\Big) = (-q)^{f+f'}\Big(1+\frac{q+1}{q}(q^F-1)\Big)\ .$$

2) Middle range contribution: From $\nu = \nu'$ we get

$$MRC = \sum_{0\leq\nu\leq F}\sum_{\nu<i\leq 2F-\nu}(-1)^{f+f'-2\nu}index(\nu)\cdot\#centralizer(\nu,i)\ .$$

The summation condition forces $\nu < F$, hence

$$MRC = \sum_{0\le\nu<F}\sum_{\nu<i\le 2F-\nu}\left(\frac{1}{2}\frac{(q+1)^2}{q^2}(-q)^{f+f'-2\nu}\right)\cdot\left(2q^{2\nu+i}\right) = (-q)^{f+f'}\frac{(q+1)^2}{q^2}\sum_{0\le\nu<F}\sum_{\nu<i\le 2F-\nu} q^i$$

$$= (-q)^{f+f'}\frac{(q+1)^2}{(q-1)^2 q}\Big(q^{2F+1} - q^{F+1} - q^F + 1\Big) .$$

3) <u>Large range contribution</u>: We have $f = f' = F$ and $\nu = \nu'$. This gives

$$LRC = \sum_{0\le\nu\le F}\sum_{2F-\nu<i\le\chi-\nu} (-1)^{2F+2\nu} index(i)\cdot \#centralizer(\nu,i)$$

$$= \sum_{0\le\nu<F}\sum_{2F-\nu<i\le\chi-\nu}\left(\frac{1}{2}\frac{(q+1)^2}{q^2}q^{2F-2\nu}\right)\cdot\left(2q^{2\nu+i}\right) + \sum_{F<i\le\chi-F} 1\cdot\frac{q+1}{q}q^{2F+i}$$

$$= \frac{(q+1)(-q)^{2F}}{q}\Big(\frac{q+1}{q}\big(\sum_{0\le\nu<F}\frac{q^{\chi-\nu+1}-q^{2F-\nu+1}}{q-1}\big) + \big(\frac{q^{\chi-F+1}-q^{F+1}}{q-1}\big)\Big)$$

$$= (-q)^{f+f'}\frac{q+1}{(q-1)^2 q}\Big((q-1)(q^{\chi-F+1}-q^{F+1}) + (q+1)(q^{\chi+1}+q^{F+1}-q^{\chi-F+1}-q^{2F+1})\Big) .$$

Adding together the contributions from 1),2) and 3) gives

$$SRC + MRC + LRC = \frac{(-q)^{f+f'}}{(q-1)^2}\Big((q+1)q^F - 2\Big)\Big((q+1)q^{\chi-F} - 2\Big) .$$

By lemma 1 the product over the two $Gl(2,F)$-orbital integrals equals

$$\frac{1}{(q-1)^2}\Big((q+1)q^F - 2\Big)\Big((q+1)q^{\chi-F} - 2\Big) .$$

This proves the theorem in the unramified case.

<u>References</u>:

[H] Hales T.C., Shalika germs on $GSp(4)$, asterisque 171-172, p.195 - 256 (1989)

[LS] Langlands R.P. -Shelstad D., On the definition of transfer factors, Math. Ann. 278, p. 219-271 (1987)

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Appendix on orders (ramified case):

Let the situation be as above. Then the orders $o_L(n) \subset o_L$ (where $n > 0$ by assumption) have the following properties:

1) $Norm(o_L(n)^*) = (o^*)^2$

This is clear from $(o^*)^2 \subset Norm(o_L(n)^*) \subset Norm(o_L^*) \subset (o^*)^2$.

2) $[o_L^*(n) : (1 + \pi^n o_L)] = [o^* : (o^* \cap (1 + \pi^n o_L)] = [o^* : (1 + \pi^n o)] = (q-1)q^{n-1}$

3) $[o_L^* : o_L^*(n)] = [o_L^* : (1 + \pi^n o_L)]/[o_L^*(n) : (1 + \pi^n o_L)] = (q-1)q^{2n-1}/(q-1)q^{n-1} = q^n$

4) $[o_T : (T \cap rK_0r^{-1})] = [o_L^* : o_L^*(j)][o_L^* : o_L^*(j')] = q^{j+j'}$

Finally for $R = o/\pi^i o$ and $i > 0$

5) $\#(Sl(2,R)) = (q-1)(q+1)q^{3i-2}$

and

6) $\#(Image(\phi_A : o_L^* \to Gl(2,R)) = q^i(q-1)q^{i-1} = (q-1)q^{2i-1}$ for $i > 0$.

Appendix on orders (unramified case):

Let the situation be as above. Then the orders $o_L(n) \subset o_L$ have the following properties:

1) $Norm(o_L(n)^*) = (o^*)^2$ if $n > 0$ and $Norm(o_L^*) = o^*$

This is clear from $(o^*)^2 \subset Norm(o_L(A)^*) \subset Norm(o_L^*) = (o^*)$.

2) For $n > 0$ we have $[o_L^*(n) : (1 + \pi^n o_L)] = [o^* : (o^* \cap (1 + \pi^n o_L)] = [o^* : (1 + \pi^n o)] = (q-1)q^{n-1}$

3) For $n > 0$ we have $[o_L^* : o_L^*(n)] = [o_L^* : (1 + \pi^n o_L)]/[o_L^*(n) : (1 + \pi^n o_L)] = (q^2 - 1)q^{2(n-1)}/(q-1)q^{n-1} = (q+1)q^{n-1}$.

Using 1) and 3) we get

4) $[o_T : (T \cap rK_0r^{-1})] = [o_L^* : o_L(j)^*][o_L^* : o_L^*(j')][Norm(o^*) : Norm(o_L^*(min(j,j'))]$.
This value is

$$1 \quad , \quad (j = j' = 0)$$

$$(q+1)q^{j+j'-1} \quad , \quad (\text{ exactly one of the } j, j' \text{ is zero})$$

$$\frac{1}{2}(q+1)^2 q^{j+j'-2} \quad , \quad (j,j'>0) \ .$$

Finally for $R = o/\pi^i o$ and $i > 0$

5) $\#(Sl(2,R)) = (q-1)(q+1)q^{3i-2}$

and

6) $\#Image(\phi_A : o_L^* \to Gl(2,R)) = \#\{(u,v) \in R \mid Norm(u+v\sqrt{A_0}) \in R^*\}$. For $i > 0$ this number is

$$(q-1)q^{2i-1} \quad , \quad (\pi|A)$$

$$(q^2-1)q^{2i-2} \quad , \quad (ord(A)=0) \ .$$

Appendix 2:

Let G be a locally compact tdc group, H a closed subgroup and K compact open in G. Let

$$G = \bigcup H \cdot r \cdot K$$

be a disjoint double coset decomposition with denumberable many representatives. Let dg, dh, dk be Haar measures on G, H, K such that $vol_G(K) = 1$ and $vol_K(K) = 1$. Then according to Warner p.478

$$\int_{HxK} f(g)dg = const' \int_{H/H\cap(xKx^{-1})} dh/dh_x \int_K f(hxk)dk$$

$$= const \int_H (\int_K f(hxk)dk)dh$$

The constants depend on the choice of measures dh_x on $H \cap (xKx^{-1})$ and dh on H. Evaluating with the function $f(g) = 1_{xK}(g)$ gives

$$1 = const \int_H 1_{H\cap xKx^{-1}}(h)dh = vol_H(H \cap xKx^{-1}) \ .$$

So if one normalizes dh such that

$$vol_H(K_H) = 1$$

then

$$const = vol_H(H \cap xKx^{-1})^{-1} = [K_H : (H \cap xKx^{-1})] \ .$$