A special Case of the Fundamental Lemma II

(The case $GSp(4,F)$)

R.Weissauer, March 1993

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Introduction:

In the following we are concerned with the endoscopic fundamental lemma (see [LS] for the general conjecture) in the case, where

$$G = GSp(4,F)$$

is the group of all 4×4-matrices, such that

$$g^t \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} .$$

with coefficients in a local field F of positive residue characteristic different from 2.

It was shown in [H], that the endoscopic fundamental lemma holds for the group $GSp(4,F)$. However, this proof does not describe the correspondence between functions with matching orbital integrals, which is important for applications. Often it is enough to know this correspondence for functions in the spherical Hecke algebra. Using the trace formula one can reduce to the study of one particular function, the unit element 1_K of the Hecke algebra. We restrict ourselves to this important case.

There are essentially two types of tori, for which endoscopic comparison lemmas are needed for the case of the group $GSp(4,F)$. In this paper we deal with the generic type, the degenerate one was considered in a previous paper [W].

The torus T:

Consider some field extension E/F of degree 4 with an involution σ, which has fixed field E^+ such that $[E:E^+]=2$. Let q be the number of elements in the residue field of E^+. Then $E^+ = F(\sqrt{A})$ for some element $A \in F^*$. We assume $A = A_0$ to be normalized, i.e. to be integral of minimal possible order. Consider the subgroup $H \subset GSp(4,F)$ of all matrices

$$\eta = \begin{pmatrix} \alpha_1 & \alpha_2 A^{-1} & \beta_1 & \beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 A \\ \gamma_1 & \gamma_2 A^{-1} & \delta_1 & \delta_2 \\ \gamma_2 A^{-1} & \gamma_1 A^{-1} & \delta_2 A^{-1} & \delta_1 \end{pmatrix} .$$

This group is isomorphic to the group

$$\phi : Gl(2,E^+)^0 = \{ g \in Gl(2,E^+) \mid det(g) \in F^* \} .$$

The isomorphism ϕ^{-1} maps the matrix above to the matrix

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where $a = \alpha_1 + \alpha_2\sqrt{A^{-1}}$, $b = \beta_1 + \beta_2\sqrt{A^{-1}}$, $c = \gamma_1 + \gamma_2\sqrt{A^{-1}}$, $d = \delta_1 + \delta_2\sqrt{A^{-1}}$.

Consider the torus T

$$T = \{x \in E^* | Norm_{E/E^+}(x) = xx^\sigma \in F^*\} \ .$$

T can be embedded as maximal torus into H and G. Any embedding is conjugate to one of the following standard form:

Choose $D \in E^+$ such that $E = E^+(\sqrt{D})$. For simplicity we assume D to be integral. Embed T by the D-regular embeddings, mapping elements

$$x = a + b\sqrt{D} \quad a, b \in E^+$$

of T to

$$s = \phi_D(x) = \begin{pmatrix} a & Db \\ b & a \end{pmatrix} \in Gl(2, E^+)^0 \ .$$

Via the isomorphism above, we can view T as a maximal subtorus of H and G. Two such standard embeddings are conjugate (in G or H) if and only if the quotient of the corresponding D's is of the form θ^2 for some θ in $F^* \cdot Norm_{E/E^*}(E^*)$.

Problem:

Our aim is to calculate the $GSp(4, F)$ orbital integral

$$O_\eta^G(1_K) = \int_{G_\eta \backslash G} 1_K(g^{-1}\eta g) dg/dg_\eta$$

for the unit element 1_K of the Hecke algebra (= characteristic functions of the unimodular symplectic similitudes). This will be used to calculate the so called κ-orbital integrals in the end:

$$O_\eta^\kappa(1_K) = O_\eta^G(1_K) - O_{\eta'}^G(1_K) \ ,$$

where

$$\eta = \phi_{D_0}(x) \quad , \quad \eta' = \phi_{D_0\theta^2}(x) \quad , \quad (x \in T) \ ,$$

and where $\theta \in (E^+)^*$ is such that $E^* = F^* Norm_{E/E^+}(E^*) \cup \theta F^* Norm_{E/E^+}(E^*)$. Here we assume the $G - M$ regularity in the sense that $E = F(x)$.

For this we can restrict ourselves to the case, where E/F is a noncyclic galois extension. In all other cases $(E^+)^* = F^* \cdot Norm_{E/E^+}(E^*)$ and the κ-orbital integrals vanish. For the noncyclic galois extensions E/F exactly one of the two extensions E/E^+, E^+/F is ramified respectively unramified.

Assumption: Choose $D = D_0$ to be normalized of minimal order 0 or 1. We can assume, that $D_0 \in F^*$. However this does not mean, that $D = D_0\theta^2$ is also normalized or in F^*, because in case E/E^+ is unramified $ord(\theta) = 1$. Using this convention we us as prime elements $\pi_{E^+} = -D_0$ resp. $\pi_{E^+} = \sqrt{A_0}$ according to whether E/E^+ is ramified or not.

The endoscopic group M:

As the torus T is the pullback of the maps $Norm : E^* \to (E^+)^*$ and the diagonal embedding $K^* \to (E^+)^*$, the dual group $\hat{T}$ is the pushout of the the diagonal map $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ and the multiplication map $(\mathbb{C}^*)^2 \to \mathbb{C}^*$. This is $\hat{T} = (\mathbb{C}^*)^4/\{(t,t,t^{-1},t^{-1})\}$. The galois group $G(E/F)$ acts by the generator σ of $Gal(E/E^+)$ permuting, say the first and second and third and fourth component respectively. Let $\tau \neq \sigma$ be a second involution in $G(E/F)$ with fixed field $L = F(\sqrt{D_0})$, where we assume $D_0 \in F^*$ to be normalized, i.e. integral of minimal possible order. Then τ acts by permuting, say the first and third and second and fourth component. The map $(z_1,z_2,z_3,z_4)/(t,t,t^{-1},t^{-1}) \mapsto (z_1z_4, z_2z_4, z_3/z_4)$ induces an isomorphism $\hat{T} \cong (\mathbb{C}^*)^3$. The action of σ is transported to

$$\sigma(t_1,t_2,t_3) = (t_2t_3, t_1t_3, t_3^{-1}) \ .$$

The action of τ is transported to

$$\tau(t_1,t_2,t_3) = (t_2t_3, t_2, t_1/t_2) \ .$$

This defines an embedding of L-groups

$$\psi : {}^LT \to {}^LGSp(4,F) = GSp(4,\mathbb{C}) \times \Gamma$$

by $\psi(t_1,t_2,t_3) = diag(t_1,t_2,t_2t_3,t_1t_3)$ and

$$\psi(\sigma) = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \times \sigma$$

$$\psi(\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

Identify $\hat{T}$ with its image under ψ. Then $Z(\hat{G})$ is the group of all $(t_1,t_2,t_3) = (x,x,1)$ and $\hat{T}^\Gamma = Z(\hat{G}) \cup Z(\hat{G})\kappa$, where $\kappa = (1,-1,-1)$. The centralizer of $\psi(\kappa) = diag(1,-1,1,-1)$ is

$$\hat{M} = \hat{M}^0 = \{ \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \subset GSp(4,\mathbb{C})\} \ .$$

As $\psi(\sigma), \psi(\tau)$ are contained in this group, the general construction defines the trivial homomorphism $\rho : \Gamma \to Out(\hat{M})$. We therefore consider ${}^L M = \hat{M} \times \Gamma$ as a subgroup of ${}^L GSp(4,F)$ in the trivial way. The morphism ψ factors in the evident way $\psi : {}^L T \to {}^L M \subset {}^L GSp(4,F)$. The group

$$M = Gl(2,F)^2/F^* \ ,$$

more precisely $M = Gl(2,F) \times Gl(2,F)/\{(t,t,t^{-1},t^{-1}) \mid t \in F^*\}$ has ${}^L M$ as its L-group. Namely $X^*(T_M) = \{n_1 x_1^* + n_2 x_2^* + n_3 x_3^* + n_4 x_4^* \mid n_1 + n_2 = n_3 + n_4\}$ and $X_*(T_M) = \mathbb{Z}x_{1*} + .. + \mathbb{Z}x_{4*}/\mathbb{Z}(x_{1*} + x_{2*} - x_{3*} - x_{4*})$ with the obvious roots and coroots $\pm(x_1^* - x_2^*), \pm(x_3^* - x_4^*)$ resp. $\pm(x_{1*} - x_{2*}), \pm(x_{3*} - x_{4*})$. Dualizing gives the root system of the maximal torus in the group $\{(g_1, g_2) \in Gl(2,\mathbb{C}) | \ det(g_1) = det(g_2)\}$.

Now let $L = E^{\langle\tau\rangle} = F(\sqrt{D_0})$, $L' = E^{\langle\tau\sigma\rangle} = F(\sqrt{A_0 D_0})$ and $E^+ = E^{\langle\sigma\rangle}$ be the three fixed fields of E. Consider some torus $T_M = (L^* \times (L')^*)/K^*$ in $M = (Gl(2,F)^2)/K^*$. The isomorphism

$$\rho : (L^* \times (L')^*)/K^* \cong T \ ,$$

given by the map $(t_1,t_2)/(t^{-1},t) \mapsto t_1 t_2 \mapsto \phi_D(t_1 t_2)$, defines an admissable embedding of the torus

$$\rho : \ T_M = (L^* \times (L')^*)/K^* \ \to \ T \ \subset GSp(4,F) \ .$$

Let us record for comparison with [LS] for $x = t_1 t_2 \in T$

$$D_G(x) = \delta_G(x)|1 - x/x^\tau||1 - x/x^\sigma||1 - x^\tau/x^{\tau\sigma}||1 - x^\tau/x^\sigma|$$

$$D_H(t_1,t_2) = \delta_H(t_1,t_2)|1 - t_1/t_1^\sigma||1 - t_2/t_2^\sigma| = \delta_H(t_1,t_2)|1 - x/x^{\sigma\tau}||1 - x/x^\tau| \ .$$

The valuation is chosen, such that $|\pi_F| = q_0^{-1}$, where q_0 is the cardinality of the residue field of F.

Orbital integrals on M:

The orbital integrals on M are stable

$$SO^M_{(t_1,t_2)}(1_K) = \int_{T\backslash M} 1_K(g^{-1}(t_1,t_2)g)dg/dt$$

Here by some abuse of notation 1_K will denote the unit elements of Hecke algebras whatsoever.

We will show by explicit computations

Theorem: For residue characteristic different from 2 we have

$$\Delta(x,(t_1,t_2))O^\kappa_\eta(1_K) = SO^M_{(t_1,t_2)}(1_K) \ ,$$

where

$$\Delta(x,(t_1,t_2)) = (\chi_{L/F}|\ .\ |)((x-x^\sigma)(x^\tau - x^{\sigma\tau}))$$

is the transfer factor of [LS]. Here $\chi_{L/F}$ denotes the quadratic character of $(F)^*$ attached to the field extension L/F. Furthermore $\eta = \phi_{D_0}(x) \in \phi_{D_0}(T)$ is assumed to be $G-M$ regular with image $(t_1,t_2) \in T_M$ under ρ^{-1}.

Remark: Observe that the (stable) orbital integrals on M are the same as the orbital integrals on $Gl(2,F) \times Gl(2,F)$, up to some appropriate normalization of measures

$$SO^M_{(t_1,t_2)}(1_K) = [o_T : \rho((o_L^* \times o_{L'}^*)] \int_{(L^* \times (L')^*/F^*)\backslash (Gl(2,F)^2/F^*)} 1_K(g^{-1}(t_1,t_2)g)dg/dl\ .$$

We have

$$\int_{L^*\backslash Gl(2,F)} 1_K(g^{-1}t_1 g)dg/dl = 1_{o^*(L)}(t_1)\Big(-\frac{f(L:K)}{q_0-1} + \frac{|D_0|^{1/2}(q_0-1+f(L:K))}{q_0-1}|t_1-t_1^\sigma|^{-1}\Big)$$

for L/F and similar for $\int_{(L')^*\backslash Gl(2,F)} 1_K(g^{-1}t_2 g)dg/dl$

$$1_{o^*(L')}(t_2)\Big(-\frac{f(L':K)}{q_0-1} + \frac{|D_0A_0|^{1/2}(q_0-1+f(L':K))}{q_0-1}|t_2-t_2^\sigma|^{-1}\Big)$$

in case of L'/F. Here $f(L/F)$ resp. $f(L'/F)$ denote the degrees of the residue field extension. The valuations are normalized, such that $|\pi_F| = q_0^{-1}$.

Reduction to H :

Using a double coset representatives for $H \backslash G/K$ and [W] appendix 2, we are led to calculate an infinite (actually then finite) sum

$$O^G_\eta(1_K) = \sum_{i=0}^{\infty} \frac{vol_G(K)}{vol_H(K_i)} \int_{T\backslash H} 1_{K_i}(h^{-1}\eta h)dh/dt\ .$$

where

$$K_i = Gl(2, o_{E^+}(i))^0 \subset Gl(2, o_{E^+})^0 = K_0 = K_H\ .$$

Here $o_{E^+}(i)$ denotes the order $o_F + \pi_F^i o_{E^+}$ in o_{E^+} for $i > 0$. The index ^{0} indicates, that determinants should be in $o^* = o_F^*$.

The analogous formula holds for the κ-orbital integrals.

Proof of formula: We use appendix 2 and the double coset decomposition

$$G = \bigcup_{i=0}^{\infty} H \cdot z(i) \cdot GSp(4,o)\ ,$$

where

$$z(i) = \begin{pmatrix} 1 & 0 & -\mu^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $\mu = A\pi_F^i$. See [S], A.19.3(2).

$$z(i)^{-1}\eta z(i) = \begin{pmatrix} \alpha_1+\gamma_1 & \frac{\alpha_2+\gamma_2}{A} & \frac{\beta_1-\alpha_1+\delta_1-\gamma_1}{\mu} & \frac{\beta_2+\delta_2}{\mu} \\ \alpha_2 & \alpha_1 & \frac{\beta_2-\alpha_2}{\mu} & \frac{\beta_1 A}{\mu} \\ \mu\gamma_1 & \frac{\mu\gamma_2}{A} & \delta_1-\gamma_1 & \delta_2 \\ \frac{\mu\gamma_2}{A} & \frac{\mu\gamma_1}{A} & \frac{\delta_2-\gamma_2}{A} & \delta_1 \end{pmatrix}$$

for

$$\phi^{-1}(\eta) = \begin{pmatrix} a & \frac{b}{\mu} \\ \mu c & d \end{pmatrix} .$$

If this matrix is in $GSp(4,o)$, then $\alpha_1, \alpha_2, \delta_1, \delta_2 \in o$. Furthermore then $\gamma_1 \in o$ and $\alpha_2 = -\gamma_2 = -\delta_2 \ mod\ (A)$, hence especially $\gamma_2 \in o$. Furthermore

$$\beta_1 \in (\mu/A)$$

$$\beta_2 = \alpha_2 = -\delta_2 \ mod\ (\mu)$$

$$\beta_1 - \alpha_1 + \delta_1 - \gamma_1 = 0 \ mod\ (\mu) .$$

Especially $\beta_1, \beta_2 \in o$. Put

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & b \\ a+c-b-d & b+d \end{pmatrix} .$$

Then these conditions are equivalent to $g \in Gl(2, o_{E^+})^0$ together with $\alpha_2, \delta_2 \in (\mu)$, $\gamma_1 \in (\mu)$ and $\beta_1 \in (\gamma/A)$ and $\gamma_2 \in (A)$. If we replace g by

$$\breve{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{A} \end{pmatrix}^{-1} g \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{A} \end{pmatrix} ,$$

then these conditions are equivalent to

$$\breve{g} \in Gl(2, o_{E^+}(\frac{\mu}{A}))$$

and $det(g) \in o_F^*$. In other words for $\mu = A\pi_F^i$, the isomorphism

$$\breve{\phi} : Gl(2, E^+)^0 \to H \subset GSp(4,F)$$

$$g \mapsto \phi\Big(Ad_{h(\mu)}(g)\Big) \quad , \quad h(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{A} \end{pmatrix} \in H$$

identifies $(\check{\phi})^{-1}(T \cap z(i)GSp(4,o)z(i)^{-1})$ with the subgroup $K_i \subset Gl(2, o_{E^+})^0$

$$K_i = Gl(2, o_{E^+}(i))^0 \ .$$

Warning: The isomorphism allows to work still with the same $x \in E^*$, but $\check{\phi}$ does not preserve the galois action. We will have to take care of this at the end by making the substitutions $\sigma \mapsto \sigma$, $\tau \mapsto \sigma\tau$ and $\sigma\tau \mapsto \tau$ in the final formulas.

We can and will use measures on G, H and $G_\eta = T$, such that the volumes of maximal compact subgroups are one, i.e. $vol_G(K) = 1$, $vol(K_H) = 1$ and $vol(o_T) = 1$. This means

$$vol_G(K)/vol_H(K_i) = [K_0 : K_i] \ .$$

Preliminary considerations on x:

Fix some $x \in T$ and assume $\tau(\sqrt{D_0}) = \sqrt{D_0}$, i.e. $L = F(\sqrt{D_0})$. The coordinates a, b and $a' = a^\tau, b' = b^\tau$ of

$$x = a + b\sqrt{D_0}$$

and

$$x^\tau = a' + b'\sqrt{D_0}$$

depend on the choice of the parameter $D = D_0$. From now on we fix notation, such that these coordinates will be reserved for some fixed choice of normalized $D_0 \in F^*$.

Observe, that E/E^+ is ramified iff L'/F and L/F are ramified. Furthermore if E/E^+ is unramified, then L/F is unramified and E^+/F and L'/F are ramified. Especially L'/F is always ramified.

Assumption: Suppose $x \in T$ is a unit in E^* with $x = x^\tau \ mod \ \Pi_E$ (in later notation $\chi > 0$) in case E/E^+ is ramified.

Then

$$x = t_1 t_2$$

is solvable with units $t_1 \in o_L^*$ and $t_2 \in o_{L'}^*$.

Proof: $x \in T \cap o_E^*$ implies $x/x^{\sigma\tau} \in N_L^1 \cap o_L^*$. Namely $x \in T$ implies $xx^\sigma = x^\tau x^{\sigma\tau}$, which gives $x/x^{\sigma\tau} \in E^{\langle\tau\rangle} = L$. Furthermore $N_{L/F}(x/x^{\sigma\tau}) = xx^\sigma/x^{\sigma\tau}x^\tau = 1$. We claim, that there is a unit $t_1 \in o_L^*$, such that $x/x^{\sigma\tau} = t_1/t_1^\sigma$. The image of o_L^* under $t \mapsto t/t^\sigma$ is $N_L^1 \cap (1 + \pi_L o_L)$, if L/F ramifies, and N_L^1 otherwise. But L/F ramifies iff E/E^+ ramifies. The assumption $\chi > 0$ in the case, when E/E^+ ramifies, shows $x^\tau = x(\pi_E)$, hence

$$x/x^{\sigma\tau}(\pi_L) = x/x^{\sigma\tau}(\Pi_E) = x/x^\tau(x^\tau/x^{\sigma\tau})(\Pi_E) = x/x^\tau(\Pi_E) = 1(\Pi_E) = 1(\pi_L) \ .$$

This garanties the existance of a unit $t_1 \in o_L^*$ such that $x/x^{\sigma\tau} = t_1/t_1^\sigma = t_1/t_1^{\sigma\tau}$. But then $t_2 = x/t_1$ is a unit, and $t_2^{\tau\sigma}/t_2 = 1$, hence $t_2 \in o_L^*$ and $x = t_1 t_2 \in o_L^* o_{L'}^*$.

Because of $E = F(x)$ we have $a \neq a'$. Let the valuation on E^+ be normalized with minimal value 1. Put

$$\chi = ord(a - a') \quad , \quad F = ord(b) = ord(b') \ .$$

Then $\chi < \infty$. Furthermore $b = \pi_{E^+}^F B, b' = \pi_{E^+}^F B'$ for units B, B' in $o_{E^+}^*$. For the absolute values, normalized such that $|\pi_F| = q_0^{-1}$, we get

$$q^F = |\frac{x - x^\sigma}{\sqrt{D_0}}|^{-2} \ .$$

Furthermore

$$\chi = ord(Tr_{E/E^+}(x) - Tr_{E/E^+}(x^\tau)) = ord(t_1 t_2 + t_1^\sigma t_2^\sigma - t_1 t_2^\sigma - t_1^\sigma t_2) = ord((t_1 - t_1^\sigma)(t_2 - t_2^\sigma))$$

and

$$|t_2 - t_2^\sigma| = |t_2 - t_2^\sigma||t_1| = |x - x^\tau| = |(a - a')^2 + (b - b')^2 D_0|^{1/2}$$

respectively

$$|t_1 - t_1^\sigma| = |t_1 - t_1^\sigma||t_2| = |x - x^{\tau\sigma}| = |(a - a')^2 + (b + b')^2 D_0|^{1/2} \ .$$

Lemma 1: Suppose $\chi > 0$ (if E/E^+ is unramified). Then the following holds under the assumptions on x made above:

1) $\{|D_0|^{1/2}|t_1 - t_1^\sigma|^{-1}, |D_0|^{1/2}|t_2 - t_2^\sigma|^{-1}\} = \{q^{F/2}, q^{(\chi - F - ord(D_0))/2}\}$

2) $\chi = ord(D_0(b^2 - (b')^2))$, hence

3) $\chi \geq ord(D_0) + 2F$.

Proof: Observe that $\chi > 0$ is automatic for E/E^+ ramified by our assumptions on x. We have

$$2\chi = ord[(a - a')^2 - (b - b')^2 D_0)] + ord[(a - a')^2 - (b + b')^2 D_0] \ .$$

For $ord(b \pm b') \geq ord(a-a')$ this implies $ord(b \mp b') = 0$ and $ord(b \pm b') = ord(a-a') = \chi > 0$, hence $F = ord(b) = ord(b') = 0$. Claim 1) and 2) follow immediately. Otherwise, if for both signs $ord(b \pm b') < ord(a-a')$, the equality above gives

$$2\chi = ord(D_0(b+b')^2) + ord(D_0(b-b')^2) = ord(D_0(b^2-(b')^2)^2 \ ,$$

hence 2). This and $F = min(ord(b-b'), ord(b+b'))$ implies 1).

Finally one has the following

Parity rule: If E/E^+ is unramified, then $ord(D_0) = 0$ and χ is odd (because $a - a' \in E^+ \setminus F$ and E^+/F ramifies). On the other hand $ord((t_1 - t_1^\sigma)/2\sqrt{D_0})$ is even (L/F is unramified) and $ord((t_2 - t_2^\sigma)/2\sqrt{D_0})$ is odd (by lemma 1).

The residue rings $S = o_{E^+}/\pi_F^i$:

Let S denote some residue ring o_{E^+}/π_F^i and R the corresponding ring $R = o_F/\pi_F^i$. Observe, that the image of the order $o_{E^+}(i)$ in S is exactly the subring R of S. Obviously for $A = A_0$

$$S = R[X]/(X^2 - A) = R + R\sqrt{A} \ ,$$

where $\sqrt{A} = X \ mod \ (X^2 - A)$. Let $Gl(2,S)^0$ be the group of matrices with determinant in R^*. The image of K_i in $Gl(2,S)^0$ is $Gl(2,R)$.

The integrals

$$[K_0 : K_i] \int_{K_0} 1_{K_i}(k^{-1}sk)dk \ ,$$

for $\sigma \in T$ are zero unless $s \in T \cap K_0$. Let us assume this therefore. Then the value of this integral is the number of left cosets $kK_i \subset K_0$ for which $k^{-1}sk \in K_i$ or equivalently the number of left cosets $y \cdot Gl(2,R) \subset Gl(2,S)^0$, such that

$$y^{-1}sy \in Gl(2,R) \ .$$

Assumption: Suppose s is of the form

$$s = \begin{pmatrix} a & bD \\ b & a \end{pmatrix} ,$$

where $a, b \in S$ and $D_0|D$. We will show

Lemma 2: Let $e = e(E/E^+)$ be the ramification index. We will show, that the following conditions are equivalent:

1) It exists an $y \in Gl(2,S)^0$ such that $y^{-1}sy \in Gl(2,R)$.

2) It exists a unit $\epsilon \in (S^*)^e$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon}{\epsilon^\tau} \end{pmatrix} s \begin{pmatrix} 1 & 0 \\ 0 & \frac{\epsilon}{\epsilon^\tau} \end{pmatrix}^{-1} = s^\tau \ .$$

3) It exists a unit $\epsilon \in (S^*)^e$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} s \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^{-1} \in Gl(2,R)$$

in other words: $a \in R, \epsilon b \in R, \epsilon^{-1}bD \in R$.

Proof: The statement are trivial for $b = 0$. Hence let us assume $b \neq 0$. Consider a solution $y \in Gl(2,S)^0$ of 1) or also a "weak" solution $y \in Gl(2,S)$. Remark: It is clear, that any weak solution y with

$$det(y) = r \cdot det(z) \in R^* \cdot det(Gl(2,S)_s) = R^* \cdot N_D$$

can be made into a solution by replacing y by

$$z^{-1}y \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}^{-1} \ .$$

If $e = 1$, E/E^+ is unramified and E^+/F is ramified, then $R^*(S^*)^2 = S^* \subset R^* det(Gl(2,S)_s)$. If $e = 2$, E/E^+ is ramified and E^+/F is unramified, then $R^* det(Gl(2,S)_s) = (S^*)^2$. Hence 3) obviously implies 1). Furthermore 2) and 3) are trivially equivalent. So let us discuss the remaining implication 1) implies 2):

Of course $y^{-1}sy \in Gl(2,R)$ is equivalent to $y^{-1}sy = y^{-\tau}s^\tau y^\tau$, which is equivalent to

$$xsx^{-1} = s^\tau$$

for

$$x = y^\tau y^{-1} \quad , \quad x^\tau = x^{-1} \ .$$

But x uniquely determines the left coset $yGl(2,R)$ of the (weak) solution y.

If y is a solution, then $det(x) = 1$ and

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1^\tau & x_2^\tau \\ x_3^\tau & x_4^\tau \end{pmatrix} = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} ,$$

hence

$$x = \begin{pmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & x_1^\tau \end{pmatrix} \quad r_2, r_3 \in R \ .$$

This gives, because of $a = a^\tau$, the conditions $x_1^\tau b = b^\tau x_1$, $x_1 bD = x_1^\tau b^\tau D^\tau$, $\sqrt{A}r_2 b = \sqrt{A}r_3 b^\tau D^\tau$. For b/b^τ (because of $b \neq 0$ and $ord(b) = ord(b^\tau)$) one gets

$$\frac{b}{b^\tau} x_1^\tau = x_1 + Ann_S(b)$$

$$\frac{b^\tau}{b} x_1^\tau D^\tau = x_1 D + Ann_S(b)$$

$$\frac{b}{b^\tau} r_2\sqrt{A} = r_3\sqrt{A}D^\tau + Ann_S(b) \ .$$

But $det(x) = 1$. Hence

$$x_1 x_1^\tau - r_2 r_3 \sqrt{A}^2 = 1 \ ,$$

hence

$$\frac{b}{b^\tau} = (\frac{b}{b^\tau} x_1^\tau) x_1 - (r_2\sqrt{A}\frac{b}{b^\tau}) r_3\sqrt{A}$$
$$= x_1^2 + Ann_S(b) - r_3^2 A D^\tau + Ann_S(b) \ .$$

But $Ann_S(b) \subset \pi_{E^+} S$ because $b \neq 0$ and $\pi_F | A_0 D_0 | A D^\tau$, because E/F is always ramified! Hence

$$b/b^\tau = x_1^2 \ mod \ \pi_{E^+} S \ .$$

Therefore $x_1 \in S^*$ and

$$b/b^\tau = x_1/x_1^\tau + Ann_S(b) \ .$$

One finds therefore

$$x_1 x_1^\tau = 1 \ mod \ \pi_{E^+}$$

and

$$x \in \begin{pmatrix} x_1 & r_3\sqrt{A}D^\tau \frac{x_1^\tau}{x_1} + Ann_S(b) \\ r_3\sqrt{A} & x_1^\tau \end{pmatrix} \ .$$

However $r_2\sqrt{A} = r_3\sqrt{A}\frac{b^\tau}{b}D^\tau = \frac{x_1}{x_1^\tau} D \ mod \ Ann_S(b)$. Therefore

$$x \in \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1^\tau}{x_1} \end{pmatrix} \begin{pmatrix} x_1 & \frac{x_1}{x_1^\tau} r_3\sqrt{A}D + Ann_S(b) \\ \frac{x_1}{x_1^\tau} r_3\sqrt{A} & x_1 \end{pmatrix}$$

$$\subset \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1^\tau}{x_1} \end{pmatrix} \cdot Gl(2,S)_s \ .$$

We get (2) (weak form)

$$xsx^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1^\tau}{x_1} \end{pmatrix} s \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1^\tau}{x_1} \end{pmatrix}^{-1} = s^\tau \ .$$

Here $x_1 \in S^*$. Hence 3) (weak form)

$$xsx^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix}^{-1} s \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \in Gl(2,R) \ .$$

It only remains to show

$$x_1 \in (S^*)^e$$

in the case $e(E/E^+) = 2$. But if E/E^+ is ramified (E^+/F is unramified) we have $N_D = (S^*)^2$ and $R^* \subset (S^*)^2$. But we have shown $x_1 x_1^\tau = 1 \ mod \ \pi_{E^+}$. The norm map of the residue field extension of E^+/F is surjective. Therefore the kernel is contained in a subgroup, whose quotient has a index-2 subgroup, thus it is contained in the group of squares. This implies

$$x_1 \in (S^*)^2$$

if $e = 2$. Lemma 2 is proved.

In the situation of lemma 2 consider the set of solutions $\mathcal{S}$

$$\{y \in Gl(2,S)^0/Gl(2,R) \quad | \quad y^{-1}sy \in Gl(2,R)\}$$

and the set of solutions $\mathcal{C}$

$$\{x \in Sl(2,S) \ | \quad x^\tau = x^{-1}, \quad xsx^{-1} = s^\tau\} \ .$$

The map $y \mapsto x = y^\tau y^{-1}$ defines an injection

$$i : \mathcal{S} \ \hookrightarrow \ \mathcal{C} \ .$$

This map is not surjective in general. However, if $e(E/E^+) = 2$, then $R \to S$ is etale and $Sl(2)$ is smooth and connected. Therefore the map i is surjective, therefore bijective. If $e(E/E^+) = 1$, then E^+/F is ramified. Therefore

$$y^\tau y^{-1} = id \ mod \ \pi_{E^+} \ ,$$

whereas $x^\tau = x^{-1}$ implies $x^2 = id \ mod \ \pi_{E^+}$, hence only

$$x = \pm id \ mod \pi_{E^+} \ .$$

Indeed if $x \in \mathcal{C}$, then also $-x \in \mathcal{C}$. But if $x = id \ mod \ \pi_{E+}$ is in $\mathcal{C}$, then one shows easily $x = y^\tau y^{-1}$ for $y \in Gl(2,S)^0$ (by recursion mod powers of π_{E+}). But then it is clear, that $x \in \mathcal{C}$ implies $y \in \mathcal{S}$. Therefore $\mathcal{C} = i(\mathcal{S}) \cup -i(\mathcal{S})$. We get

$$\#(\mathcal{S}) = \frac{1}{2} e(E^+/E) \#(\mathcal{C}) \ .$$

So in our counting problem we are reduced to count all $x \in Sl(2,S)$ with $x = x^* = (x^\tau)^{-1}$ and $xsx^{-1} = s^\tau$.

Nonvanishing of $\mathcal{C}$:

Suppose we are given

$$s = \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \ .$$

Consider the set $\mathcal{C}$

$$\mathcal{C} = \{ x \in Sl(2,S) \mid x^\tau = x^{-1}; \ xsx^{-1} = s^\tau \} \ .$$

Put $b = B\pi_{E+}^\nu$ and

$$inv(b,D) = (b/b^\tau) D \in S \ .$$

Actually this is only welldefined modulo $Ann_S(b)$, but we later choose some fixed representative. Suppose

$$\mathcal{C} \neq \emptyset \ .$$

This implies

$$inv(b,D)^\tau = inv(b,D) \quad mod \ Ann_S(b) \ .$$

Namely one has $(b^\tau/b)D^\tau = (x_1/x_1^\tau)D \ mod \ Ann_S(b)$ and $(x_1/x_1^\tau)D = (b/b^\tau)D \ mod \ Ann_S(b)$. So we can and will fix some

$$inv(b,D) \in R$$

depending only on s.

Description of $\mathcal{C}$:

Then we have already shown, that

$$\mathcal{C} = \{ x = \begin{pmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & x_1^\tau \end{pmatrix} \mid det(x) = 1; x_1^\tau = x_1(b^\tau/b) \ mod \ Ann_S(b);$$

$$r_2\sqrt{A} = inv(b,D) r_3 \sqrt{A} + a \text{ for } a \in Ann_S(b) \} \ .$$

where of course

$$a \in Ann_S(b) \cap \sqrt{A} R \ .$$

We get from $b = B\pi_{E+}^{\nu}$

$$b^{\tau}/b = (B^{\tau}/B)(-1)^{e\nu} .$$

This gives, modulo $Ann_S(b)$, the relation $(x_1/B)^{\tau} = (-1)^{e\nu}(x_1/B)$. But $x_1/B \in S^*$ and therefore $(-1)^{e\nu} = 1$ automatically has to hold. Only the case $e = 1$, where E^+/F is ramified, has to be considered. But then $S^* \cap \sqrt{A}R = \emptyset$. Hence

$$(x_1/B)^{\tau} = (x_1/B) \quad \text{or} \quad x_1 \in B \cdot R .$$

We get for $\mathcal{C}$

$$\{x = \begin{pmatrix} r_1 B(1+\delta) & inv(b,D)r_3\sqrt{A} + a \\ r_3\sqrt{A} & r_1 B^{\tau}(1+\delta)^{\tau} \end{pmatrix} \mid det(x) = 1; a, \delta \in Ann_S(b); a^{\tau} = -a\} \ / \ \cong ,$$

where we consider solutions $x = (r_1, r_3, \delta, a)$ and $x' = (r_1', r_3', \delta', a')$ equivalent $x \cong x'$, if $r_3 = r_3', a = a'$ and $r_1(1+\delta) = r_1'(1+\delta')$. So we have a description of $\mathcal{C}$ in terms of equivalence classes of elements $(r_1, r_3, \delta, a) \in R^2 \times Ann_S(b)^2$ fullfilling equations

$$a^{\tau} = -a$$

and

$$r_1^2 BB^{\tau}(1+\delta)(1+\delta)^{\tau} - r_3^2 A \cdot inv(b,D) - ar_3\sqrt{A} = 1 .$$

We can now choose $\delta \in Ann_S(b), \alpha \in Ann_S(b\sqrt{A}) \cap R$ and $r_3 \in R$ arbitrarily. Put $a = \sqrt{A}\alpha$. Then there are exactly two solutions $r_1 \in R$ of the equation

$$r_1^2 BB^{\tau}(1+\delta)(1+\delta)^{\tau} - r_3^2 A \cdot inv(b,D) - \alpha r_3 = 1 .$$

This follows, because the coefficient BB^{τ} is a square in R^*. Otherwise $\mathcal{C} = \emptyset$. Furthermore $A \cdot inv(b,D)$ and αA are in the maximal ideal. Extracting square roots from units congruent one is always possible and gives exactly two solutions. The equivalence relations identifies $1 + Ann_S(b) \cap R$ solutions. Therefore

$$\#\mathcal{C} = 2\#R\#Ann_S(b)[(Ann_S(b) \cap R\sqrt{A}) : (Ann_S(b) \cap R)] .$$

This is

$$2\#R\#Ann_S(b)$$

in the case where E/E^+ is ramified and also in the case E/E^+ unramified, if $ord(b) = \nu$ is even. However this is satisfied if $b \neq 0$ and $\mathcal{C} \neq \emptyset$.

We get

Lemma 3: Suppose that the set of solutions $\mathcal{S}$ is nonempty. Then the number of solution $\#\mathcal{S}$ is either $Sl(2,S)/Sl(2,R)$ if $b = 0$ in S or otherwise

$$\#\mathcal{S} = e(E/E^+)\#R\#Ann_S(b) .$$

On $T \setminus H$-integration:

In this section assume $D_0 \in E^+$ and $\theta \in E^+$ to be normalized of order 0 or 1. Consider regular elements

$$\eta = \phi_{\theta^2 D_0}(x) \in H$$

with centralizer $\phi_{\theta^2 D_0}(T)$. We write H as a disjoint union of double cosets

$$\bigcup_{r \in R} \phi_{\theta^2 D_0}(T) \cdot r \cdot K_0 \ ,$$

where the representatives $r \in Gl(2, E^+)^0$ are of the form

$$r \ \in \ \phi_{\theta^2 D_0}(T) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E+}^{j-ord(\theta)} \epsilon_0 \end{pmatrix})$$

for certain $j \in \mathbb{N}$ and $\epsilon_0 \in o_{E+}^*$. This implies

$$\int_H f(h) dh = \sum_r [o_T : (T \cap r K_0 r^{-1})] \int_T dt \int_{K_0} f(trk) dk \ ,$$

hence

$$\int_{T \setminus H} f(h) dh/dt = \sum_r [o_T/(T \cap r K_0 r^{-1})] \int_{K_0} f(rk) dk \ .$$

It allows to calculate the orbital integrals on H by integrations over K_0 and gives

$$O_\eta^G(1_K) = \sum_{i \geq 0} \sum_{r \in R} [o_T/(T \cap r K_0 r^{-1})][K_0 : K_i] \int_{K_0} 1_{K_i}(k^{-1} r^{-1} \eta r k) dk \ .$$

Observe $x = a + b\sqrt{A_0} = a + (b/\theta)\sqrt{\theta^2 D_0}$, hence for $\Theta = \theta / \pi_{E+}^{ord(\theta)}$

$$\eta_r := r^{-1} \eta r = \phi_{\Theta^2 D_0 \pi_{E+}^{2j} \epsilon_0^2}(x))$$

because η_r is

$$\begin{pmatrix} 1 & 0 \\ 0 & \pi_{E+}^{ord(\theta)-j} \epsilon_0^{-1} \end{pmatrix} \begin{pmatrix} a & b\theta D_0 \\ b\theta^{-1} & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E+}^{j-ord(\theta)} \epsilon_0 \end{pmatrix} = \begin{pmatrix} a & \frac{b}{\Theta \epsilon_0 \pi_{E+}^j} (\Theta \epsilon_0 \pi_{E+}^j)^2 D_0 \\ \frac{b}{\Theta \epsilon_0 \pi_{E+}^j} & a' \end{pmatrix} \ .$$

Therefore $T \cap r K_0 r^{-1} \subset o_T = \{x \in o_E^* \mid Norm(x) \in o^*\}$ is the group

$$\{x = a + b\sqrt{D_0} \mid Norm(x) \in o^*; a, \pi_{E+}^{-j} b \in o_{E+}\} \ .$$

It does not depend on the choice of the (normalized) D_0 resp. θ.

Constraints (for fixed r and corresponding j): $x \in T \cap rK_0r^{-1}$ means $x \in o_E(j) \cap T$.

Furthermore the index is

$$[o_T/(T \cap rK_0r^{-1})] = [o_E^* : o_E(j)^*][(Norm_{E/E^+}(o_E^*) \cap o^*) : (Norm_{E/E^+}(o_E(j)^*) \cap o^*)]^{-1} .$$

Certain constraints have to be satisfied, if the orbital integral

$$[K_0 : K_i] \int_{K_0} 1_{K_i}(k^{-1}\eta_r k)dk$$

should not vanish. Namely $k^{-1}\eta_r k \in K_i$ implies $\eta_r \in K_0$, hence $x \in T \cap rK_0r^{-1}$.

Suppose these r-constraints are satisfied. Then evaluating the orbital integral

$$[K_0 : K_i] \int_{K_0} 1_{K_i}(k^{-1}\eta_r k)dk$$

means counting all cosets $kK_i \subset K_0$, for which $k^{-1}\eta_r k \in K_i$. This is the number of all cosets $y \in Gl(2, o_{E^+}/\pi_{E^+}^i o_{E^+})^0/Gl(2, o_{E^+}(i))$ in $Gl(2, o_{E^+}/\pi_F^i o_{E^+})$, such that

$$y^{-1}\eta_r y \in Gl(2, o_{E^+}(i)) \ mod\ \pi_{E^+}^i \quad ,$$

holds (a number computed in lemmas 2 and 3).

Double cosets in H:

For normalized D_0 we have

$$Gl(2, E^+) = \bigcup_{j \geq 0} \phi_{D_0}(E^*) \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^j \end{pmatrix} Gl(2, o_{E^+}) .$$

This gives a decomposition (with unique integers $j \geq 0$) for elements $h \in H$

$$h = \phi_{\theta^2 D_0}(l) \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^{j-ord(\theta)} \end{pmatrix} k \qquad k \in Gl(2, o_{E^+}), l \in E^* .$$

In order to adjust the data to the group H we are only allowed to change k to $k_0 k$ for some $k_0 \in Gl(2, o_{E^+})$ with the property

$$k_0 \in \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^{j-ord(\theta)}\epsilon_0 \end{pmatrix}^{-1} \phi_{\theta^2 D_0}(E^*) \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^{j-ord(\theta)}\epsilon_0 \end{pmatrix} ,$$

which means

$$k_0 \in o_E^*(j) \ .$$

Hence k_0 can be assumed to lie in K_0, whenever $Norm(o_E^*(j))o^* = o_{E^+}^*$. This is the case only if $j = 0$ and E/E^+ is unramified. In all other cases $Norm(o_E^*(j))o^* = (o_{E^+}^*)^2$ has index two in $o_{E^+}^*$.

On the other hand $h \in H$ imposes conditions on $det(k)$. Namely $det(h) \in F^*$ implies

$$(*) \qquad det(k)\pi_{E^+}^{j-ord(\theta)} = 1 \ \text{ in } \ (E^+)^*/(F^* \cdot Norm_{E/E^+}(E^*)) \cong H^1(F,T) \ .$$

In the case, where E/E^+ is <u>ramified</u>, we choose $\pi_{E^+} = Norm_{E/E^+}(\Pi_E) = -D_0$ to be in $Norm_{E/E^+}(E^*)$. Then (*) implies $det(k) \in Norm(L^*) \cap o_{E^+}^* = (o_{E^+}^*)^2$. Changing k by some k_0 with determinant in $(o_{E^+}^*)^2$ allows to assume $det(k) \in o^*$, hence $k \in K_0 \subset H$.

We therefore get H as a disjoint union of double cosets

$$\bigcup_{r\in R} \phi_{\theta^2 D_0}(T) \cdot r \cdot K_0 \quad , \quad R = \mathbb{N}_0$$

and for $j \in \mathbb{N}_0$

$$r = r_j = \phi_{\theta^2 D_0}(\Pi_E^{-j+ord(\theta)}) \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^{j-ord(\theta)} \end{pmatrix} \quad (\pi_{E^+} \in F^* \cap Norm_{E/E^+}(E^*)) \ .$$

Now consider the case, where E/E^+ is <u>unramified</u>. Then $det(k)\pi_{E^+}^{j-ord(\theta)} \in F^* Norm_{E/E^+}(E^*)$ implies

$$j - ord(\theta) = 0 \ mod \ 2 \ .$$

Secondly, to achieve $det(k) \in o^*$, we are only allowed to change $det(k)$ by elements in the group $Norm_{E/E^+}(o_E^*(j))$. This is $(o^*)^2$ unless $j = 0$. Let ϵ_0 be in $o_{E^+}^* \setminus (o_{E^+}^*)^2$. Then we get H as a disjoint union of double cosets

$$\bigcup_{r\in R} \phi_{\theta^2 D_0}(T) \cdot r \cdot K_0$$

where $R \subset ((o^*/(o^*)^2) \times \mathbb{N} \ \cup \ (1 \times 0)$ consists of all elements $(\epsilon_0, j) \in (o^*/(o*)^2) \times \mathbb{N}_0$ with the property

$$j = ord(\theta) \ mod \ 2 \ .$$

The representatives are

$$r = r_{\epsilon_0,j} = \phi_{\theta^2 D_0}(\Pi_E^{(-j+ord(\theta))/2}\epsilon_0') \begin{pmatrix} 1 & 0 \\ 0 & \pi_{E^+}^{j-ord(\theta)}\epsilon_0 \end{pmatrix} \ .$$

Here $\epsilon_0' \in o_E^*$ is chosen, such that $Norm_{E/E^+}(\epsilon_0') = \epsilon_0^{-1}$ and where $\epsilon_0 \in o_{E^+}^*/(o_{E^+}^*)^2$. Furthermore $\pi_{E^+} = \sqrt{A_0}$.

Summation conditions:

Let us use the notations F, $\nu = F - j$, $\chi = ord(a - a')$. Then $b/\pi_{E+}^j = B\pi_{E+}^\nu, b'/\pi_{E+}^j = B'\pi^\nu$ for some units $B, B' \in o_{E+}^*$. Put $\theta = \Theta\pi_{E+}^{ord(\theta)}$. If E/E^+ is unramified we put $\theta = \pi_{E+}$. From lemma 2 we get conditions on the summation indices $(i,r) \in \mathbb{N}_0 \times R$ for $r = (\epsilon_0, j)$, in order to yield a nonzero contribution to the orbital integral. These are:

1) $0 \leq i \leq \chi$ meaning $a = a' \ mod \ \pi_{E+}^i$

2) $\nu < i$ then $(\pi_{E+}^\nu B\Theta^{-1}\epsilon_0^{-1})/(\pi_{E+}^\nu B\Theta^{-1}\epsilon_0^{-1})^\tau = \epsilon/\epsilon^\tau \ mod \ \pi_{E+}^{i-\nu}$ has to hold for some $\epsilon \in (o_{E+}^*)^{e(E/E^+)}$.

3) If $2F + ord(D_0) - \nu < i$, then $i \leq \chi - \nu$ has to hold.

For condition 3) observe, that the conditions of lemma 2

$$(\pi_{E+}^\nu B\Theta^{-1}\epsilon_0^{-1})/(\pi_{E+}^\nu B\Theta^{-1}\epsilon_0^{-1})^\tau = \epsilon/\epsilon^\tau \ mod \ \pi_{E+}^{i-\nu}$$

and

$$(\pi_{E+}^{2F-\nu} B\Theta\epsilon_0 D_0)/(\pi_{E+}^{2F-\nu} B\Theta\epsilon_0 D_0)^\tau = \epsilon^\tau/\epsilon \ mod \ \pi_{E+}^{i-2F-ord(D_0)+\nu}$$

(for some $\epsilon \in (o_{E+}^*)^{e(E/E^+)}$) can be combined and thus simplified. For that we assume $\pi_{E+}^2 \in F$ because either $\pi_{E+} = Norm_{E/E^+}\sqrt{D_0} = -D_0$ or $\pi_{E+} = \sqrt{A_0}$. It means $(B^2 D_0)/(B^2 D_0)^\tau = 1 \ mod \ \pi_{E+}^{i-2F-ord(D_0)+\nu}$ or in other words $i - 2F - ord(D_0) + \nu \leq ord(D_0 B^2 - (D_0 B^2)^\tau)$. Now $2F + ord(D_0) - \nu < i$ forces $i > 0$, which is only possible for $\chi = ord(a - a') > 0$ by 1). From $\chi > 0$ we get $\chi = ord(D_0(b^2 - (b')^2))$ by lemma 1. Hence by lemma 1

$$\chi = 2F + ord(D_0 B^2 - (D_0 B^2)^\tau) \ .$$

Therefore the condition can be reformulated to

$$2F + ord(D_0) - \nu < i \quad \text{implies} \quad i \leq \chi - \nu \ .$$

Résumé:

We now express the results of the chapter on $T \setminus H$ integration in terms, which are suitable for summation. We saw, that the domain of possible values i is divided into three parts:

A) Conditions on i (apart from $0 \leq i \leq \chi$ relevant for $\chi = 0$):

Range of small i: $0 \leq i \leq \nu$

Large range i: $\nu < i \leq \chi - \nu$

B) Conditions on ν and θ:

In the domain of possible values ν we have, in addition to possible restrictions mentioned above, the conditions:

$$0 \leq \nu \leq F$$

$$\nu = F + ord(\theta) \ mod \ 2 \qquad (\text{ if } e(E/E^+) = 1) \ .$$

Furthermore in the large range (concerning i) we have contributions only if

$$\pi_{E^+}^{\nu}(B\Theta^{-1}\epsilon_0^{-1}) \in (o_{E^+}^*)^{e(E/E^+)}o \ mod \ \pi_{E^+}^i$$

is satisfied. This forces $\nu = 0 \ mod \ 2$ in the large range for $e(E/E^+) = 1$.

C) Contributions from the index $index(\nu)$:

The relevant contributions are

$$1 \qquad (\nu = F)$$

$$\frac{(q+1)}{2q} q^{F-\nu} \qquad (\nu < F) \ ,$$

in the case where E/E^+ is unramified and

$$q^{F-\nu}$$

in the case, where E/E^+ is ramified. (See appendix).

D) Contributions from counting solutions $sol(\nu, i)$:

Put $S = o_{E^+}/\pi_F^i$ and $R = o/\pi_F^i$. Then $sol(\nu, i)$ is

$$\#Sl(2,S)/\#Sl(2,R) \qquad (i \leq \nu \text{ small range})$$

$$e(E/E^+)\#R\#Ann_S(\pi_{E^+}^{\nu}) \qquad (i > \nu \text{ large range}) \ .$$

In principle we now could compute the orbital integrals. We leave this to the reader. We concentrate on the κ-orbital integrals instead.

The summation (E/E^+ ramified):

For η corresponding to $x = t_1 t_2$, both $t_1 \in L^*$ and $t_2 \in L'$ can be assumed to be units. Otherwise our κ-orbital integral is zero. This follows from the r-constraints and the following fact: For the κ- orbital integral we do not get any contribution from the small range of the i-summation in the ramified case, because this contribution is stable (does not depend on θ). The middle and large range contributions remain. The whole summation is therefore empty unless $\chi > 0$, i.e. $x = x' \ mod\ \Pi$ in addition to $x, x' \in o_L^*$. This is true, because $\chi = 0$ implies $i = 0$, which is in the stable range. This allows to apply the preliminary remarks, made earlier.

Let χ_o be the quadratic character of $(E^+)^*$ attached to E/E^+. It is trivial on $(o_{E^+}^*)^2$ with $\chi_0(\theta) = -1$. We get from $q^{F-\nu} \cdot (2q^\nu q_0^i) = 2q^F q_0^i$ with $q_0 = q^{1/2}$

$$O_\eta^\kappa(1_K) = 2q^F \chi_0(B) \cdot 1_{o_L^* \times o_{L'}}(t_1, t_2) \sum_{0 \le \nu \le F} \ \sum_{\nu < i \le \chi - \nu} q_0^i \ .$$

The double sum gives

$$(q_0 - 1)^{-1} \sum_{0 \le \nu \le F} (q_0^{\chi+1-\nu} - q_0^{\nu+1})$$

$$= (q_0 - 1)^{-1}[q_0^{\chi+1}(q_0^{-F-1} - 1)/(q_0^{-1} - 1) - q_0(q_0^{F+1} - 1)/(q_0 - 1)]$$

$$= \frac{q_0}{(q_0 - 1)^2}(1 - q_0^{F+1})(1 - q_0^{\chi - F}) = q_0 \int_{Gl(2,F)^2/(L^*)^2} 1_K(g^{-1}(t_1, t_2)g)dg/dl \ .$$

Hence $O_\eta^\kappa(1_K)$ is equal to

$$2|\frac{x - x^\sigma}{2\sqrt{D_0}}|^{-2} \chi_0(\frac{x^\sigma - x}{2\sqrt{D_0}}) \cdot q_0 \int_{Gl(2,F)^2/(L^*)^2} 1_K(g^{-1}(t_1, t_2)g)dg/dl \ .$$

The factor 2 disappears, because in the ramified case $[o_T : \rho((o_L^*)^2)] = 2$. Furthermore q_0 cancels $|2\sqrt{D_0}|^2$. Finally $\chi_0(\frac{x - x^\sigma}{2\sqrt{D_0}}) = \chi_{L/F}((x - x^\sigma)(x^\tau - x^{\sigma\tau}))\chi_{L/F}(2^{-2}D_0^{-1})$ and $\chi_{L/F}(2^{-2}D_0^{-1}) = \chi_{L/F}(-1)$. If we now make the substitutions $\sigma \mapsto \sigma$, $\tau \mapsto \sigma\tau$ and $\sigma\tau \mapsto \tau$, in order to compensate the effect of the isomorphism $\check{\phi}$, we get the desired formula. The theorem in the case, where E/E^+ is ramified, is proved.

The summation (E/E^+ unramified):

To compute the κ-orbital integral we concentrate on the case $\chi > 0$. The case $\chi = 0$ is excluded by the parity rule, χ is always odd. From the résumé we get the following contributions:

1) Small term range: Its contribution will be

$$SRC = \sum_{0\le\nu\le F}\sum_{0\le i\le\nu}(-1)^{F-\nu}\#(\epsilon_0 - reprs.)\cdot index(\nu)\cdot sol(i)$$

$$= \sum_{0\le i\le F} sol(i)\Big(\big(\sum_{i\le\nu<f}(-1)^{F-\nu}\cdot 2\cdot\frac{(q+1)}{2q}(-q)^{F-\nu}\big)+1\Big)\ .$$

The representatives ϵ_0 appearing in R give an additional factor 2 in the first sum! So

$$SRC = \sum_{0\le i\le F} sol(i)(-q)^{F-i}\ .$$

But $sol(i) = q^{3i/2}$ or 0 depending on the parity of i (because we use the primes π^i_{E+}). Thus

$$SRC = (-q)^F\sum_{0\le j\le F/2} q^j = \frac{(-q)^F}{(q-1)^2}[(q-1)q^{1+[F/2]}-(q-1)]\ .$$

Here $[x]$ is the largest integer $n\le x$.

2) Large range contribution: We get

$$LRC = \sum_{\nu<i\le\chi-\nu}\sum_{0\le\nu\le F}(-1)^{F-\nu}\cdot\#(\epsilon_0 - reprs.)\cdot index(\nu)\cdot sol(\nu,i)\ .$$

The number $sol(\nu,i)=0$ unless ν and i are even. Hence

$$LRC = \sum_{\nu<i\le\chi-\nu,i\ even}\Big(\sum_{0\le\nu<F,\nu\ even}(-1)^F\cdot 2\cdot q^{i/2+\nu}\cdot\frac{q+1}{2q}q^{F-\nu}\Big)$$

$$+\Big(\sum_{F<i\le\chi-F,i,F\ even}(-1)^F\cdot 1\cdot q^{i/2+F}\cdot 1\Big)\ .$$

The summations over i and then ν give

$$\frac{(-q)^F}{(q-1)^2}\Big((q+1)q^{\frac{\chi+1}{2}}-(q+1)q^{\frac{\chi-F}{2}}-(q+1)q^{\frac{F+1}{2}}+(q+1)\Big)$$

in the case F odd and

$$\frac{(-q)^F}{(q-1)^2}\Big((q+1)q^{\frac{\chi+1}{2}}-2q^{\frac{\chi-F+1}{2}}-(q^2+1)q^{\frac{F}{2}}+(q+1)\Big)$$

in case F is even.

3) <u>Total contribution</u>: Adding together the contributions from 1),2) gives

$$SRC + LRC = \frac{(-q)^F}{(q-1)^2}\left(-2 + (q+1)q^{\frac{ev}{2}}\right)\left(-1 + qq^{-1/2}q^{\frac{odd}{2}}\right) ,$$

where ev is the even number among $F, \chi - F$ and odd is the odd one (parity rule). But $|D_0| = 1$, $|A_0| = q^{-1/2}$, $q = q_0$, $(-1)^F = \chi_0((x - x^\sigma)/2\sqrt{D_0}) = \chi_{L/F}((x - x^\sigma)(x^{\sigma\tau} - x^\tau))$. In the present case the first of the two $Gl(2,F)$-orbital integrals (for M) corresponds to the unramified extension L/F, the second to the ramified extension L'/F. This proves the theorem in the unramified case, once we apply the substitutions $\tau \mapsto \sigma\tau$ and vice verca.

References:

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Appendix on orders (E/E^+ ramified):

Let the situation be as above. Then the orders $o_E(j) = o_{E^+} + \pi^j_{E^+} o_E \subset o_E$ (where $j > 0$ by assumption) have the following properties:

1) $Norm(o_E(n)^*) = (o^*_{E^+})^2$

This is clear from $(o^*_{E^+})^2 \subset Norm(o_E(n)^*) \subset Norm(o^*_E) \subset (o^*_{E^+})^2$.

2) $[o^*_E(n) : (1+\pi^n_{E^+} o_E)] = [o^*_{E^+} : (o^*_{E^+} \cap (1+\pi^n_{E^+} o_E)] = [o^*_{E^+} : (1+\pi^n_{E^+} o_{E^+})] = (q-1)q^{n-1}$

3) $[o^*_E : o^*_E(n)] = [o^*_E : (1+\pi^n_{E^+} o_E)]/[o^*_E(n) : (1+\pi^n_{E^+} o_E)] = (q-1)q^{2n-1}/(q-1)q^{n-1} = q^n$

4) $[o_T : (T \cap rK_0 r^{-1})] = [o^*_L : o^*_L(j)] = q^j$ because $(o^*_E)^{(\sigma+1)(\tau-1)} = (o_E(j)^*)^{(\sigma+1)(\tau-1)}$ follows from 1).

Finally for $S = o_{E^+}/\pi^i_{E^+} o_{E^+}$ and $i > 0$

5) $\#(Sl(2,S)) = (q-1)(q+1)q^{3i-2}$

and

6) $\#(Image(\phi_D : o^*_L \to Gl(2,S)) = q^i(q-1)q^{i-1} = (q-1)q^{2i-1}$ for $i > 0$.

Appendix on orders (E/E^+ unramified):

Let the situation be as above. Let $o_E(j) = o_{E^+} + \pi^j_{E^+} o_E \subset o_E$ for $j > 0$.

1) $Norm(o_E(j)^*) = (o^*_{E^+})^2$ if $j > 0$ and $Norm(o^*_E) = o^*_{E^+}$

This is clear from $(o^*_{E^+})^2 \subset Norm(o^*_E(j)) \subset Norm(o^*_E) = (o^*_{E^+})$ and $[o^*_{E^+} : (o^*_{E^+})^2] = 2$ and $o^*_E(j) \subset o^*_{E^+}(1+\pi_E o_E)$.

2) For $n > 0$ we have $[o^*_E(n) : (1+\pi^n_{E^+} o_E)] = [o^*_{E^+} : (o^*_{E^+} \cap (1+\pi^n_{E^+} o_E)] = [o^*_{E^+} : (1+\pi^n_{E^+} o_{E^+})] = (q-1)q^{n-1}$

3) For $n > 0$ we have $[o^*_E : o^*_E(n)] = [o^*_E : (1+\pi^n_{E^+} o_E)]/[o^*_E(n) : (1+\pi^n_{E^+} o_E)] = (q^2-1)q^{2(n-1)}/(q-1)q^{n-1} = (q+1)q^{n-1}$.

Using 1) and 3) we get

5) $[o_T : (T \cap rK_0 r^{-1})]$ is 1 or $\frac{1}{2}(q+1)q^{j-1}$. This follows because $Norm_{E/E^+}(o^*_E) \cap o^* = Norm_{E/E^+}(o_E(j)^*) \cap o^*$ for $j = 0$ and has index two for $j > 0$ (namely $(o^*_{E^+})^2 \cap o^* = (o^*)^2$ in this case has index two in $o^*_{E^+} \cap o^* = o^*$.

This value is

$$\begin{array}{cc} 1 \quad , & (j=0) \\ \frac{1}{2}(q+1)q^{j-1} \quad , & (j>0) \ . \end{array}$$