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Abstract. Gel'fand triples of test and generalized functionals in Gaussian spaces are constructed and characterized.

1. Introduction

In recent years there was an increasing interest in white noise analysis, due to its rapid developments in mathematical structure and applications in various domains. Especially, the circle of ideas going under the heading 'characterization theorems' has played quite an important role in the last few years. These results [21], [33], [42], and their variations and refinements (see, e.g., [31], [34], [37], [48], [50], [54], and references quoted there), provide a deep insight into the structure of spaces of smooth and generalized random variables over the white noise space or – more generally – Gaussian spaces. Also, they allow for rather straightforward applications of these notions to a number of fields: for example, Feynman integration [11], [16], [20], [32], representation of quantum field theory [2], [43], stochastic equations [7], [30], [39], [40], [41], intersection local times [10], [49], Dirichlet forms [3], [4], [15], infinite dimensional harmonic analysis [14] and so forth. Moreover, characterization theorems have been at the basis of new methods for the construction of smooth and generalized random variables [24], [34] which seem to be useful in applications untractable by existing methods (e.g., [18], [19]).

The purpose of the present article is four-fould: We wish 1. to clarify and generalize the structure of the existing characterization theorems, and at the same time, 2. to review and unify recent developments in this direction, 3. to establish the connection to rich, related mathematical literature [1], [8], [12], [45], [53], which might be helpful in future developments, and – last but not least – 4. to fill a gap in the article [42]. In this sense,

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the present paper attempts to give known results and some existing 'folklore' around them a general form that can be used as a reference for future research. In the course of doing this, we also establish some new results, for instance an analytic extension property of U-functionals, and the topological invariance of certain spaces of generalized random variables with respect to different construction schemes.

This article is organized as follows. In Section 2 we present some notions and results from complex analysis on topological vector spaces, from Gaussian analysis, and from Fock space theory. In particular, we construct a nuclear rigging

$$(\mathcal{N}) \subset \Gamma(\mathcal{H}) \subset (\mathcal{N})^*$$

of the symmetric Fock space $\Gamma(\mathcal{H})$ over a Hilbert space $\mathcal{H}$. We give a construction of the second quantized space $(\mathcal{N})$ solely in terms of the topology of $\mathcal{N}$, independent of the particular representation as a projective limit. Via the well-known Wiener–Itô–Segal isomorphism this provides a rigging of the L^2-space over a Gaussian measure space by spaces of smooth and generalized random variables. In Section 3 we study the U-functionals associated with the elements in $(\mathcal{N})$ and $(\mathcal{N})^*$. We derive an analytic extension property for U-functionals, and use this to prove theorems which characterize $(\mathcal{N})$ and $(\mathcal{N})^*$ in terms of their $\mathcal{S}$-transforms. In Section 4 we prove two corollaries of the characterization theorem for $(\mathcal{N})^*$ which appear to be useful in applications.

2. Preliminaries

2.1 G–Entire Functions

We provide some well-known facts from complex analysis on topological vector spaces (see, e.g., [8] and [12]) with a view towards applications in the next section.

Let $\mathcal{E}$ be a locally convex complex vector space.

Definition 1. *A mapping P from $\mathcal{E}$ into $\mathbb{C}$ is called an* **n–homogeneous polynomial,** *if it is the composition of the diagonal mapping $\Delta_n : x \mapsto (x, x, \dots, x)$ from $\mathcal{E}$ into $\mathcal{E}^n$ and a symmetric n-linear mapping L from $\mathcal{E}^n$ into $\mathbb{C}$, i.e., $P = L \circ \Delta_n \equiv \widehat{L}$. Let $P_n(\mathcal{E})$ denote the space of all n-homogeneous polynomials.*

Definition 2. *A function F defined on $\mathcal{E}$ with values in $\mathbb{C}$ is said to be* **G–entire** *if for all $\xi, \eta \in \mathcal{E}$ the complex valued function*

$$z \mapsto F(\eta + z\xi), \quad z \in \mathbb{C},$$

is entire. Let $H_G(\mathcal{E})$ denote the set of all G-entire mappings from $\mathcal{E}$ into $\mathbb{C}$.

For $F \in H_G(\mathcal{E})$, $\eta \in \mathcal{E}$, there exists a unique sequence $(\frac{1}{n!}\widehat{d^n F(\eta)}, n \in \mathbb{N}_0)$ of homogeneous polynomials $\frac{1}{n!}\widehat{d^n F(\eta)} \in P_n(\mathcal{E})$, $n \in \mathbb{N}$, such that for all $\xi \in \mathcal{E}$,

$$F(\eta + \xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\eta)}(\xi). \tag{1}$$

Of course, $d^n\widehat{F(\eta)}(\xi)$ is the n–th partial derivative of F at η in the direction ξ. The corresponding n–linear form is denoted by $d^nF(\eta)(\xi_1,\ldots,\xi_n)$, $\xi_1,\ldots,\xi_n \in \mathcal{E}$.

Definition 3. *Let F be a mapping from $\mathcal{E}$ into $\mathbb{C}$. F is called* **entire** *if it is in $H_G(\mathcal{E})$, and if it is continuous. $H(\mathcal{E})$ denotes the space of all entire functions on $\mathcal{E}$.*

Proposition 4. *Let $F \in H_G(\mathcal{E})$. Then $F \in H(\mathcal{E})$ if and only if F is locally bounded.*

We conclude this subsection by stating a result which is related to the celebrated "cross theorem" of Bernstein. For a review of such results we refer the interested reader also to [1]. The following is a special case of a result by Siciak: if we make use of the fact that any segment of the real line in the complex plane has strictly positive transfinite diameter, then Corollary 7.3 in [45] implies

Proposition 5. *Let $n \in \mathbb{N}$, $n \geq 2$, and f be a complex valued function on $\mathbb{R}^n$. Assume that for all $k = 1,2,\ldots,n$, and $(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n) \in \mathbb{R}^{n-1}$, the mapping*

$$x_k \longmapsto f(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n),$$

from $\mathbb{R}$ into $\mathbb{C}$ has an entire extension. Then f has an entire extension to $\mathbb{C}^n$.

2.2 Gaussian Spaces

The primordial object of Gaussian analysis (e.g., [6], [17], [21], [22], [25], [26], [27], [28], [29]) is a real separable Hilbert space $\mathcal{H}$. One then considers a rigging of $\mathcal{H}$, $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}^*$, where $\mathcal{N}$ is a real nuclear space (see below and [13]), densely and continuously embedded into $\mathcal{H}$, and $\mathcal{N}^*$ is its dual ($\mathcal{H}$ being identified with its dual). A typical example (which appears for instance in white noise analysis) is the rigging $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ of $L^2(\mathbb{R})$ (with Lebesgue measure) by the Schwartz spaces of test functions and tempered distributions.

Via Minlos' theorem the canonical Gaussian measure μ on $\mathcal{N}^*$ is introduced by giving its characteristic function

$$C(f) = \int_{\mathcal{N}^*} e^{i\langle\omega,f\rangle}\, d\mu(\omega) = e^{-\frac{1}{2}|f|^2_{\mathcal{H}}}, \quad f \in \mathcal{N}.$$

The space $L^2(\mathcal{N}^*, d\mu) \equiv (L^2)$ of (equivalence classes of) complex valued functions on $\mathcal{N}^*$ which are square-integrable with respect to μ has the well-known Wiener–Itô–Segal chaos decomposition [36], [46], [47], and one has the familiar Segal isomorphism $\mathcal{I}$ between (L^2) and the complex Fock space $\Gamma(\mathcal{H})$ over the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$.

Spaces of smooth functions on $\mathcal{N}^*$ can be constructed by mapping appropriate subspaces of $\Gamma(\mathcal{H})$ into (L^2) via the unitary mapping $\mathcal{I}^{-1} : \Gamma(\mathcal{H}) \rightarrow (L^2)$, see, e.g., the construction using second quantized operators in [6], [17]. In the present context, we prefer to work exclusively in Fock space. At all times a 'translation' into function space language via $\mathcal{I}^{-1}$ is of course equally valid.

2.3 Second Quantized Spaces

Our starting point is a real separable nuclear space $\mathcal{N}$. It is well-known (e.g., [38]) that the topology of $\mathcal{N}$ is equivalent to the projective limit topology of an increasing countable system $(|\cdot|_p,\, p \in \mathbb{N}_0)$ of compatible Hilbertian norms $|\cdot|_p$. In other words, $\mathcal{N}$ is a countably Hilbert space [13],

$$\mathcal{N} \equiv \bigcap_p \mathcal{H}_p,$$

where $\mathcal{H}_p$ is equal to the completion of $\mathcal{N}$ with respect to $|\cdot|_p$. Moreover, the usual Hilbert–Schmidt property for the embeddings holds, i.e., for every $p \in \mathbb{N}_0$ there exists a $p' > p$ so that the embedding ι from $\mathcal{H}_{p'}$ into $\mathcal{H}_p$ is a Hilbert–Schmidt operator. We shall denote the bilinear dual pairing on $\mathcal{N}^* \times \mathcal{N}$ by $\langle \cdot, \cdot \rangle$. A special role – also with a view towards Gaussian spaces – is played by the Hilbert space $\mathcal{H}_0$, which we also denote by $\mathcal{H}$.

Consider the Fock space, e.g., [9], [46], over $\mathcal{H}$

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n},$$

where $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ is the symmetric n–fold tensor product of $\mathcal{H}_{\mathbb{C}}$ with itself. The inner product and norm $\|\cdot\|_0$ of $\Gamma(\mathcal{H})$ are generated by

$$(\varphi^{\otimes n}, \psi^{\otimes n})_{\Gamma(\mathcal{H})} = n!\,(\varphi, \psi)^n_{\mathcal{H}_{\mathbb{C}}}\,.$$

Likewise, for $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $\mathcal{H}_{\mathbb{C},p}^{\widehat{\otimes} n}$ denotes the n–fold symmetric tensor product of $\mathcal{H}_{\mathbb{C},p}$ with itself, and it is considered as a subspace of $\mathcal{H}_{\mathbb{C},p}^{\otimes n}$. The canonical norm of the latter is denoted by $|\cdot|_p$, too (the meaning will be clear from the context). The duals of $\mathcal{H}_{\mathbb{C},p}^{\otimes n}$ and $\mathcal{H}_{\mathbb{C},p}^{\widehat{\otimes} n}$, respectively, are denoted by $\mathcal{H}_{\mathbb{C},-p}^{\otimes n}$ and $\mathcal{H}_{\mathbb{C},-p}^{\widehat{\otimes} n}$, respectively. The Hilbertian norm of $\mathcal{H}_{\mathbb{C},-p}^{\otimes n}$ is denoted by $|\cdot|_{-p}$, and we remark that for $n \in \mathbb{N}$, $p \in \mathbb{Z}$, $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C},p}^{\otimes n}$, $|\Phi^{(n)}|_p$ is equal to the Hilbert–Schmidt norm of $\Phi^{(n)}$ considered as a linear form on $\mathcal{H}_{\mathbb{C},-p}^{\otimes n}$. For $q \in \mathbb{N}_0$, we introduce Hilbert spaces $\Gamma_q(\mathcal{H}_p)$ as the completions of the space of finite direct sums

$$\bigoplus\nolimits_n' \mathcal{H}_{\mathbb{C},p}^{\widehat{\otimes} n}$$

with respect to the inner product determined by

$$(\varphi^{\otimes n}, \psi^{\otimes n})_{\Gamma_q(\mathcal{H}_p)} = 2^{nq} n!\,(\varphi, \psi)^n_{\mathcal{H}_{\mathbb{C},p}}\,, \tag{2}$$

and denote the corresponding norms by $\|\cdot\|_{p,q}$. Finally we set

$$(\mathcal{N}) = \bigcap_{p,q} \Gamma_q(\mathcal{H}_p)\,,$$

equipped with the projective limit topology.

Remarks. Evidently substitution of the value 2 in equation (2) by any other number strictly larger than 1 produces the same space $(\mathcal{N})$. We use the same notation $(\mathcal{N})$ for the nuclear subspace of (L^2) corresponding to $(\mathcal{N})$ under the Wiener–Itô–Segal isomorphism.

Lemma 6. *$(\mathcal{N})$ is nuclear.*

Proof. Nuclearity of $(\mathcal{N})$ follows essentially from that of $\mathcal{N}$. For fixed p, q consider the embedding

$$I : \Gamma_{q'}(\mathcal{H}_{p'}) \to \Gamma_q(\mathcal{H}_p)$$

where p' is chosen such that the embedding

$$\iota : \mathcal{H}_{p'} \longrightarrow \mathcal{H}_p$$

is Hilbert–Schmidt. Then

$$I = \bigoplus_n \iota^{\otimes n}.$$

Its Hilbert–Schmidt norm is easily estimated by using an orthonormal basis (cf., e.g., [17], Appendix A.2) of $\Gamma_{q'}(\mathcal{H}_{p'})$. The result is the bound

$$\|I\|_{\mathrm{HS}}^2 \le \sum_{n=0}^{\infty} 2^{n(q-q')} \|\iota\|_{\mathrm{HS}}^{2n}$$

which is finite for suitably chosen q'. □

Theorem 7. *The topology on $(\mathcal{N})$ is uniquely determined by the topology on $\mathcal{N}$.*

Proof. Let us assume that we are given two different systems of Hilbertian norms $|\cdot|_p$ and $|\cdot|'_k$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|'_{k,l}$ by $\|\cdot\|_{p,q}$ for some p, q (and vice versa which is completely analogous). Since $|\cdot|'_k$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$, there exists p and a constant C such that $|f|'_k \le C\,|f|_p$, for all $f \in \mathcal{N}$, i.e., the injection ι from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by ι also its linear extension from $\mathcal{H}_{\mathbb{C},p}$ into $\mathcal{K}_{\mathbb{C},k}$. It follows from a straightforward modification of the proof of the Proposition on p. 299 in [44], that $\iota^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{\mathbb{C},p}^{\otimes n}$ into $\mathcal{K}_{\mathbb{C},k}^{\otimes n}$. Now we choose q such that $2^{\frac{q-l}{2}} \ge C$. Then

$$\begin{aligned}
\|\cdot\|'^2_{k,l} &= \sum_{n=0}^{\infty} n!\, 2^{nl}\, |\cdot|'^2_k \\
&\le \sum_{n=0}^{\infty} n!\, 2^{nl} C^{2n}\, |\cdot|_p^2 \\
&\le \|\cdot\|_{p,q}^2 \,,
\end{aligned}$$

which had to be proved. □

From general duality theory on nuclear spaces we know that the dual of $(\mathcal{N})$ is given by

$$(\mathcal{N})^* = \bigcup_{p,q} \Gamma_q(\mathcal{H}_p)^*$$

and one verifies that

$$\Gamma_q(\mathcal{H}_p)^* = \Gamma_{-q}(\mathcal{H}_{-p}).$$

We shall denote the bilinear dual pairing on $(\mathcal{N})^* \times (\mathcal{N})$ by $\langle\langle\cdot,\cdot\rangle\rangle$:

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle,$$

where $\Phi \in \Gamma_{-q}(\mathcal{H}_{-p})$ corresponds to the sequence $(\Phi^{(n)},\, n \in \mathbb{N}_0)$ with $\Phi^{(0)} \in \mathbb{C}$, and $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C},-p}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$.

Remark. Consider the particular choice $\mathcal{N} = \mathcal{S}(\mathbb{R})$. Then $(\mathcal{N})^{(*)}$ coincide with (the Fock space equivalents of) the well-known spaces $(\mathcal{S})^{(*)}$ of white noise functionals, see, e.g., [17], [42]. For the norms $\|\varphi\|_p \equiv \|\Gamma(A^p)\varphi\|_0$ introduced there, we have $\|\cdot\|_p = \|\cdot\|_{p,0}$, and $\|\cdot\|_{p,q} \le \|\cdot\|_{p+\frac{q}{2}}$. More generally, if the norms on $\mathcal{N}$ satisfy the additional assumption that for all $p \ge 0$ and all $\varepsilon > 0$ there exists $p' \ge 0$ such that $|\cdot|_p \le \varepsilon|\cdot|_{p'}$, then the construction of Kubo and Takenaka [26] (and other authors) leads to the same space $(\mathcal{N})$. The construction presented here has the advantage of being manifestly independent of the choice of any concrete system of Hilbertian norms topologizing $\mathcal{N}$.

For the exponential vectors

$$\phi_f := \sum_{n=0}^{\infty} \frac{1}{n!} f^{\otimes n}$$

one calculates the norms

$$\|\phi_f\|_{p,q}^2 = e^{2^q |f|_p^2},$$

and hence for all $f \in \mathcal{N}$ they are in $(\mathcal{N})$. This then allows for the following

Definition 8. *Let $\Phi \in (\mathcal{N})^*$. The* **S–transform of Φ** *is the mapping from $\mathcal{N}$ into $\mathbb{C}$ given by*

$$S\Phi(f) := \langle\langle \Phi, \phi_f \rangle\rangle, \quad f \in \mathcal{N}.$$

We note that the exponential vectors $\{\phi_f,\, f \in \mathcal{N}\}$, are a total set $\mathcal{E}$ in $(\mathcal{N})$, and hence elements of $(\mathcal{N})^*$ are characterized by their S–transforms. Furthermore, it is obvious that the S–transform of $\Phi \in (\mathcal{N})^*$ extends to $\mathcal{N}_{\mathbb{C}}$: for $\xi \in \mathcal{N}_{\mathbb{C}}$ set $S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle$, where ϕ_ξ is the complex exponential vector $\sum_n \frac{1}{n!}\xi^{\otimes n} \in (\mathcal{N})$.

3. *U*–Functionals and the Characterization Theorems

We begin with a definition.

Definition 9. *Let $F : \mathcal{N} \longrightarrow \mathbb{C}$ be such that*

C.1 for all $f, g \in \mathcal{N}$, the mapping $\lambda \longmapsto F(g + \lambda f)$ from $\mathbb{R}$ into $\mathbb{C}$ has an entire extension to $\lambda \in \mathbb{C}$,

C.2 for some continuous quadratic form B on $\mathcal{N}$ there exists constants $C, K > 0$ such that for all $f \in \mathcal{N}$, $z \in \mathbb{C}$,

$$|F(zf)| \leq C \exp(K\,|z|^2 |B(f)|).$$

Then F is called a ***U*–functional.**

Remark. Condition C.2 is actually equivalent to the more conventional

C.2′ there exists constants $C, K > 0$ and $p \in \mathbb{N}_0$, so that for all $f \in \mathcal{N}$, $z \in \mathbb{C}$,

$$|F(zf)| \leq C \exp(K\,|z|^2 |f|_p^2). \tag{3}$$

Lemma 10. *Every U-functional F has a unique extension to an entire function on $\mathcal{N}_{\mathbb{C}}$. Moreover, if the bound on F holds in the form (3) then for all $\rho \in (0,1)$,*

$$|F(\xi)| \leq C' \exp(K'\,|\xi|_p^2), \quad \xi \in \mathcal{N}_{\mathbb{C}},$$

with $C' = C(1-\rho)^{-\frac{1}{2}}$, $K' = 2\rho^{-1}e^2 K$.

Proof. First we show that a U–functional F has a G–entire extension. The extension of F (denoted by the same symbol) is given by $F(\eta) = F(g_0 + zg_1)$, $\eta = g_0 + zg_1 \in \mathcal{N}_{\mathbb{C}}$, $g_0, g_1 \in \mathcal{N}$, $z \in \mathbb{C}$. Let $\xi \in \mathcal{N}_{\mathbb{C}}$ be of the form $\xi = g_2 + ig_3$, $g_2, g_3 \in \mathcal{N}$. Consider the mapping

$$(\lambda_1, \lambda_2, \lambda_3) \longmapsto F(g_0 + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3),$$

from $\mathbb{R}^3$ into $\mathbb{C}$. Condition C.1 and Proposition 5 imply that this function has an entire extension to $\mathbb{C}^3$. In particular, F is G–entire on $\mathcal{N}_{\mathbb{C}}$.

Let $\xi \in \mathcal{N}_{\mathbb{C}}$, and consider the Taylor expansion of $F(\xi)$ at the origin (cf. (1)):

$$F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \widehat{d^n F(0)}(\xi). \tag{4}$$

For all $f \in \mathcal{N}$, $n \in \mathbb{N}$, $R > 0$, we obtain from C.2′ and Cauchy's inequality the estimate

$$|\widehat{d^n F(0)}(f)| \leq C\, n!\, R^{-n} e^{R^2 K |f|_p^2}.$$

We choose $R = (\frac{n}{2K})^{\frac{1}{2}}$, and get for $f \in \mathcal{N}$ with $|f|_p = 1$ the inequality

$$|\widehat{d^n F(0)}(f)| \leq C\, n! \left(\frac{2eK}{n}\right)^{n/2}.$$

A standard polarization argument (see, e.g., [35, §3]) and homogeneity of $\widehat{d^n F(0)}$ yield the following bound for the n–linear form $d^n F(0)$:

$$|d^n F(0)(f_1,\ldots,f_n)| \le C(n!\,(2e^2K)^n)^{\frac{1}{2}} \prod_{k=1}^{n} |f_k|_p, \tag{5}$$

where $f_1,\ldots,f_n \in \mathcal{N}$ (and we used $\frac{n^n}{n!} \le e^n$).

Since $d^n F(0)$ is n–linear on $\mathcal{N}_{\mathbb{C}}$, the last inequality gives the estimate

$$|d^n F(0)(\xi_1,\ldots,\xi_n)| \le C\,(n!\,(4e^2K)^n)^{\frac{1}{2}} \prod_{k=1}^{n} |\xi_k|_p, \tag{6}$$

for $\xi_1,\ldots,\xi_n \in \mathcal{N}_{\mathbb{C}}$. In particular, the Taylor coefficients in (4) have absolute value bounded by

$$C\left(\frac{(4e^2K|\xi|_p^2)^n}{n!}\right)^{\frac{1}{2}},$$

and we get (by Schwarz' inequality) the following estimate for all $\rho \in (0,1)$,

$$|F(\xi)| \le C(1-\rho)^{-\frac{1}{2}} e^{2\rho^{-1}e^2K|\xi|_p^2}, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Hence F is locally bounded on $\mathcal{N}_{\mathbb{C}}$, and therefore Proposition 4 implies that F is entire. □

Now we are ready to prove the following generalization of the main result in [42] which characterizes the space $(\mathcal{N})^*$ in terms of its S–transform.

Theorem 11. *A mapping $F : \mathcal{N} \to \mathbb{C}$ is the S-transform of an element in $(\mathcal{N})^*$ if and only if it is a U-functional.*

Proof. Let $\Phi \in (\mathcal{N})^*$. Then $\Phi \in \Gamma_{-q}(\mathcal{H}_{-p})$ for some p, $q \in \mathbb{N}_0$. As we have remarked at the end of Section 2, the S–transform of Φ extends to $\mathcal{N}_{\mathbb{C}}$, and therefore it makes sense to consider the mapping $\xi \mapsto S\Phi(\xi)$ from $\mathcal{N}_{\mathbb{C}}$ into $\mathbb{C}$. We shall show that this mapping is entire. We have

$$S\Phi(\xi) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \xi^{\otimes n}\rangle, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

We estimate as follows:

$$\begin{aligned} |S\Phi(\xi)| &\le \sum_{n=0}^{\infty} |\Phi^{(n)}|_{-p} |\xi|_p^n \\ &\le \Big(\sum_{n=0}^{\infty} n!\,2^{-qn} |\Phi^{(n)}|_{-q}^2\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} 2^{qn} |\xi|_p^{2n}\Big)^{\frac{1}{2}} \\ &= \|\Phi\|_{-p,-q}\, e^{2^{q-1}|\xi|_p^2}. \end{aligned}$$

The last estimation shows that the power series for $S\Phi$ on $\mathcal{N}_{\mathbb{C}}$ converges uniformly on every bounded neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$, and therefore it defines an entire function on this space [12]. In particular, C.1 holds for $S\Phi$. Moreover, the choice $\xi = zf$, $z \in \mathbb{C}$, $f \in \mathcal{N}$, shows that also C.2$'$ is fulfilled. Hence $S\Phi$ is a U–functional.

Conversely let F be a U–functional. We may assume the bound in the form (3). Consider the n–linear form $d^nF(0)$ on $\mathcal{N}_{\mathbb{C}}$ constructed in the proof of Lemma 10. The estimate (6) shows that $d^nF(0)$ is separately continuous on $\mathcal{N}_{\mathbb{C}}$ in its n variables. Hence by the nuclear theorem (e.g., [6], [13]) there exists $\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^*)^{\widehat{\otimes} n}$ so that

$$\langle \Phi^{(n)}, \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n \rangle = \frac{1}{n!}\, d^nF(0)(\xi_1, \ldots, \xi_n), \quad \xi_1, \ldots, \xi_n \in \mathcal{N}_{\mathbb{C}}.$$

Let $p' > p$ be such that the embedding $\iota : \mathcal{H}_{p'} \longrightarrow \mathcal{H}_p$ is Hilbert–Schmidt, and let $(e_k,\ k \in \mathbb{N})$ be an orthonormal basis of $\mathcal{H}_{p'}$ in $\mathcal{N}$. For $n \in \mathbb{N}$, $(e_{k_1} \otimes \cdots \otimes e_{k_n},\ k_i \in \mathbb{N},\ i = 1, \ldots, n)$ is an orthonormal basis of $\mathcal{H}_{\mathbb{C},p'}^{\otimes n}$. Then we can estimate in the following way (cf. (5)):

$$\begin{aligned}
|\Phi^{(n)}|_{-p'}^2 &= \sum_{k_1,\ldots,k_n} |\langle \Phi^{(n)}, e_{k_1} \otimes \cdots \otimes e_{k_n} \rangle|^2 \\
&= \sum_{k_1,\ldots,k_n} (n!)^{-2}\, |d^nF(0)(e_{k_1}, \ldots, e_{k_n})|^2 \\
&\le C^2 (n!)^{-1} (2e^2K)^n \Big(\sum_{k=1}^{\infty} |\iota e_k|_p^2\Big)^n \\
&= C^2 (n!)^{-1} (2e^2K\, \|\iota\|_{\mathrm{HS}}^2)^n
\end{aligned}$$

i.e., $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C},-p'}^{\widehat{\otimes} n}$, and

$$\|\Phi^{(n)}\|_{-p',-q}^2 \le C^2 (2^{1-q} e^2 K\, \|\iota\|_{\mathrm{HS}}^2)^n. \tag{7}$$

For Φ given by the sequence $(\Phi^{(n)},\ n \in \mathbb{N}_0)$ $(\Phi^{(0)} \equiv F(0))$ we have

$$\begin{aligned}
\|\Phi\|_{-p',-q}^2 &\le C^2 \sum_{n=0}^{\infty} (2^{1-q} e^2 K\, \|\iota\|_{\mathrm{HS}}^2)^n \\
&= C^2 (1 - 2^{1-q} e^2 K\, \|\iota\|_{\mathrm{HS}}^2)^{-1} \\
&< +\infty,
\end{aligned}$$

if we choose q large enough so that $2^{1-q} e^2 K\, \|\iota\|_{\mathrm{HS}}^2 < 1$. In particular, $\Phi \in (\mathcal{N})^*$, and for $f \in \mathcal{N}$ we have by (4),

$$\begin{aligned}
S\Phi(f) &= \sum_{n=0}^{\infty} \langle \Phi^{(n)}, f^{\otimes n} \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\, \widehat{d^nF(0)(f)} \\
&= F(f).
\end{aligned}$$

Uniqueness of $\Phi = S^{-1}F$ follows from the fact that the exponential vectors are total in $(\mathcal{N})$. $\square$

As a by-product of the above proof we obtain the following localization result for generalized functionals.

Corollary 12. *Given a U-functional F satisfying C.2'. Let $p' > p$ be such that the embedding $\iota : \mathcal{H}_{p'} \to \mathcal{H}_p$ is Hilbert-Schmidt, and $q \in \mathbb{N}_0$ so that $\rho := 2^{1-q}e^2 K \, \|\iota\|_{\mathrm{HS}}^2 < 1$. Then $\Phi := S^{-1}F \in \Gamma_{-q}(\mathcal{H}_{-p'})$, and*

$$\|\Phi\|_{-p',-q} \leq C(1-\rho)^{-1/2}. \tag{8}$$

For analogous results in white noise analysis see, e.g., [23], [37], [50].

Within the framework established here one can treat the following and numerous other examples in a unified way.

Example 13. We choose the triplet

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

and equip $\mathcal{S}'(\mathbb{R}^n)$ with the Gaussian measure with characteristic functional

$$C(f) = e^{-\frac{1}{2}\int f^2(t)\, d^n t}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Then the framework allows to discuss functionals of white noise with n–dimensional time parameter [48].

Example 14. If we choose a finite direct sum of identical copies of $\mathcal{S}(\mathbb{R})$ as the basic real nuclear space we obtain the characterization of the space of Hida distributions of the noise of an n-dimensional Brownian motion [48].

We close this section by the corresponding characterization theorem for $(\mathcal{N})$. This result is independently due to [21], [31], [33], and has been generalized and modified in various ways, e.g., [37], [50], [54].

Theorem 15. *A mapping $F : \mathcal{N} \to \mathbb{C}$ is the S–transform of an element in $(\mathcal{N})$ if and only if it admits C.1 and the following condition*

C.3 there exists a system of norms $(|\cdot|_{-p}, p \in \mathbb{N}_0)$, which yields the inductive limit topology on $\mathcal{N}^*$, and such that for all $p \geq 0$ and $\epsilon > 0$ there exists $C_{p,\epsilon} > 0$ so that

$$|F(zf)| \leq C_{p,\epsilon} \exp\left(\epsilon |z|^2 \, |f|_{-p}^2\right), \quad f \in \mathcal{N},\ z \in \mathbb{C}. \tag{9}$$

If for F conditions C.1 and C.3 are satisfied we say that F is of order 2 and minimal type.

Proof. If $\varphi \in (\mathcal{N})$ then condition C.1 is satisfied as a consequence of Theorem 11. For any $p, q \geq 0$ we estimate as follows

$$\begin{aligned}
|S\varphi(zf)| &= |\sum_{n=0}^{\infty} \langle \varphi^{(n)}, (zf)^{\otimes n} \rangle| \\
&\leq \sum_{n=0}^{\infty} |z|^n |\varphi^{(n)}|_p \, |f|_{-p}^n \\
&\leq (\sum_{n=0}^{\infty} n! \, 2^{nq} \, |\varphi^{(n)}|_p^2)^{1/2} (\sum_{n=0}^{\infty} \frac{1}{n!} \, (2^{-q}|z|^2|f|_{-p}^2)^n)^{1/2} \\
&= \|\varphi\|_{p,q} \exp(2^{1-q}|z|^2|f|_{-p}^2).
\end{aligned}$$

Hence condition C.3, too, is necessary.

Conversely, let F be a U–functional of order 2 and minimal type. From F, construct a sequence $\varphi = (\varphi^{(n)}, \, n \in \mathbb{N}_0)$ of continuous linear forms $\varphi^{(n)}$ on $\mathcal{N}^{\hat{\otimes} n}$ as in the proof of Theorem 10. We have to show that φ belongs to $\Gamma_q(\mathcal{H}_r)$ for all $r, q \in \mathbb{N}_0$. Let $r, q \in \mathbb{N}_0$ be given. Choose $p > r$ such that the injection $\iota : \mathcal{H}_p \to \mathcal{H}_r$ is Hilbert–Schmidt. Then so is the injection $\iota^* : \mathcal{H}_{-r} \to \mathcal{H}_{-p}$. $\varepsilon > 0$ in (9) is chosen so that $\rho := \varepsilon 2^{1+q} e^2 \, \|\iota^*\|_{\mathrm{HS}}^2 < 1$. Then the analogue of (7) reads

$$\begin{aligned}
\|\varphi^{(n)}\|_{r,q}^2 &\leq C_{p,\varepsilon}^2 (2^{q+1} e^2 \varepsilon \, \|\iota^*\|_{\mathrm{HS}}^2)^n \\
&= C_{p,\varepsilon}^2 \rho^n,
\end{aligned}$$

and we get

$$\begin{aligned}
\|\varphi\|_{r,q} &= (\sum_{n=0}^{\infty} \|\varphi^{(n)}\|_{r,q}^2)^{\frac{1}{2}} \\
&\leq C_{p,\varepsilon}(1-\rho)^{-\frac{1}{2}}.
\end{aligned}$$

Thus $\varphi \in (\mathcal{N})$, and the proof is complete. □

4. Corollaries

One useful application of Theorem 11 is the discussion of convergence of a sequence of generalized functionals. A first version of this theorem is worked out in [42]. Here we use our more general setting to state

Theorem 16. *Let $(F_n, \, n \in \mathbb{N})$ denote a sequence of U–functionals such that*

1. *$(F_n(f), \, n \in \mathbb{N})$ is a Cauchy sequence for all $f \in \mathcal{N}$,*
2. *there exists a continuous norm $|\cdot|$ on $\mathcal{N}$ and $C, K > 0$ such that $|F_n(zf)| \leq C e^{K|z|^2|f|^2}$ for all $f \in \mathcal{N}$, $z \in \mathbb{C}$, and for almost all $n \in \mathbb{N}$.*

Then $(S^{-1}F_n, \, n \in \mathbb{N})$ converges strongly in $(\mathcal{N})^*$.

Proof. The assumptions and inequality (8) imply that there exist $p, q \geq 0$ and $\rho \in (0,1)$ such that for all $n \in \mathbb{N}$,

$$\|\Phi_n\|_{-p,-q} \leq C(1-\rho)^{-\frac{1}{2}}$$

where $\Phi_n = S^{-1}F_n$. Since $\mathcal{E}$ is total in $\Gamma_{-q}(\mathcal{H}_{-p})$, assumption 1 implies that $(\langle\langle \Phi_n, \varphi\rangle\rangle,\ n \in \mathbb{N})$ is a Cauchy sequence for all $\varphi \in (\mathcal{N})$. Since $(\mathcal{N})^*$ is the dual of the countably Hilbert space $(\mathcal{N})$, which is in particular Fréchet, it follows from the Banach–Steinhaus theorem that $(\mathcal{N})^*$ is weakly sequentially complete. Thus there exists $\Phi \in (\mathcal{N})^*$ such that Φ is the weak limit of $(\Phi_n,\ n \in \mathbb{N})$. The proof is concluded by the remark that weak and strong convergence of sequences coincide in the duals of nuclear spaces (e.g., [13]). □

As a second application we consider a theorem which concerns the integration of a family of generalized functionals.

Theorem 17. *Let $(\Lambda, \mathcal{A}, \nu)$ be a measure space, and $\lambda \mapsto \Phi_\lambda$ a mapping from Λ to $(\mathcal{N})^*$. We assume that the S-transform $F_\lambda = S\Phi_\lambda$ satisfies the following conditions:*

1. *for every $f \in \mathcal{N}$ the mapping $\lambda \mapsto F_\lambda(f)$ is measurable,*
2. *there exists a continuous norm $|\cdot|$ on $\mathcal{N}$ so that for all $\lambda \in \Lambda$, F_λ satisfies the bound $|F_\lambda(zf)| \leq C_\lambda e^{K_\lambda |z|^2 |f|^2}$, and such that $\lambda \mapsto K_\lambda$ is bounded ν-a.e., and $\lambda \mapsto C_\lambda$ is integrable with respect to ν.*

Then there are $q, p \geq 0$ such that $\Phi.$ is Bochner integrable on $\Gamma_{-q}(\mathcal{H}_{-p})$. Thus in particular,

$$\int_\Lambda \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{N})^*,$$

and

$$S\left(\int_\Lambda \Phi_\lambda \, d\nu(\lambda)\right)(f) = \int_\Lambda S\Phi_\lambda(f)\, d\nu(\lambda), \quad f \in \mathcal{N}.$$

Proof. In inequality (3) for $F_\lambda(zf)$ we can replace K_λ by its bound. With this modified estimate and Corollary 12 we can find $p, q \geq 0$ and $\rho \in (0,1)$ such that for all $\lambda \in \Lambda$,

$$\|\Phi_\lambda\|_{-p,-q} \leq C_\lambda (1-\rho)^{-\frac{1}{2}}. \tag{10}$$

Since the right hand side of (10) is integrable with respect to ν, we only need to show the weak measurability of $\lambda \mapsto \Phi_\lambda$ (see [52]). But this is obvious because $\lambda \mapsto \langle\langle \Phi_\lambda, \varphi\rangle\rangle$ is measurable for all $\varphi \in \mathcal{E}$ which is total in $\Gamma_q(\mathcal{H}_p)$. □

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