Nr. 176/94

TOPOLOGICAL ASPECTS OF THE CHARACTERIZATION OF HIDA DISTRIBUTIONS – A REMARK

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1. Introduction

In the recent years, the dual pair of smooth and generalized random variables on the White Noise space, $(\mathcal{S})$ and $(\mathcal{S})^*$, has found many applications. For example, stochastic (partial) differential equations [LØU 90, LØU 91, Po 92, Po 93], quantum field theory [PS 93] and Feynman integrals [FPS 91, KS 92, LLS 93]. The main advantage of $(\mathcal{S})$ and $(\mathcal{S})^*$ is the $\mathcal{S}$-Transform, which in a nice way characterizes the pair. This transform maps generalized Hida distributions into a space of complex valued functions on $\mathcal{S}(\mathbb{R})$. This space of functions is called the space of U-functionals. Moreover, the $\mathcal{S}$-Transform turns out to be a bijection onto this space [PS 91].

In most applications, one is really working on the space of U-functionals. For this reason, it is natural to topologize the U-functional space. The aim of this paper is to construct the U-functional space using inductive and projective limits of Banach spaces. This construction is in light of the construction of $(\mathcal{S})$ and $(\mathcal{S})^*$ quite natural. With the given topologies, we show our main result: The $\mathcal{S}$-Transform is a homeomorphism.

Related topics have been investigated by other authors: Y.-J. Lee [Le 91] and Y. Yokoi [Yo 93] have characterized $(\mathcal{S})$ and $(\mathcal{S})^*$ using Bargman spaces.

2. Mathematical Preliminaries

Let

$$(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$$

be the white noise probability space. Denote the algebra of polynomials generated by $< x, \xi >, \xi \in \mathcal{S}(\mathbb{R})$, by $\mathcal{P}$. $(\mathcal{S})_p$ denotes the completion of $\mathcal{P}$ in the norm $\|\cdot\|_{2,p}$, where

$$\|\varphi\|_{2,p} = \|\Gamma(H^p)\varphi\|_2, \varphi \in \mathcal{P}$$

Here, $\Gamma(H^p)$ is the second quantization of the harmonic oscillator H. The space of Hida test functions, $(\mathcal{S})$, is defined as the projective limit of $(\mathcal{S})_p$. The dual of $(\mathcal{S})$, the space of Hida distributions is denoted $(\mathcal{S})^*$. We have the following triple:

$$(\mathcal{S}) \subset (L^2) := L^2(\mathcal{S}'(\mathbb{R}), \mu) \subset (\mathcal{S})^*.$$

In the present paper, the $\mathcal{S}$-Transform is investigated. We now give a short review of some important aspects of this transform on the space of generalized Hida distributions.
For $\Phi \in (\mathcal{S})^*$, the $\mathcal{S}$-Transform is defined as

$$\mathcal{S}\Phi(\xi) =< \Phi, : e^{<\cdot,\xi>} :>, \xi \in \mathcal{S}(\mathbb{R}),$$

where $: e^{<\cdot,\xi>} := e^{<\cdot,\xi> - \frac{1}{2}|\xi|_2^2}$. Of course, the outer dual pairing is the one between $(\mathcal{S})$ and $(\mathcal{S})^*$. Using the $\mathcal{S}$-Transform, one can characterize the space $(\mathcal{S})^*$. We need the following two definitions:

Definition 2.1. A complex valued function F on $\mathcal{S}(\mathbb{R})$ is called ray entire at $\eta \in \mathcal{S}(\mathbb{R})$ if and only if for every $\xi \in \mathcal{S}(\mathbb{R})$, the mapping $\lambda \longmapsto F(\eta + \lambda\xi), \lambda \in \mathbb{R}$, has an entire analytic extension. If F is ray entire at every $\eta \in \mathcal{S}(\mathbb{R})$, then F is called ray entire on $\mathcal{S}(\mathbb{R})$. Assume that F is ray entire at 0, and set for $R > 0$ and $\xi \in \mathcal{S}(\mathbb{R})$

$$M(R, \xi) := \sup_{|z|=R} |F(z\xi)|.$$

F is said to be of growth $(\rho, \tau), \rho \geq 0, \tau \geq 0$, on $\mathcal{S}_p(\mathbb{R})$, if and only if there exists a constant $C > 0$, so that for all $\xi \in \mathcal{S}(\mathbb{R})$, and all $R > 0$,

$$M(R, \xi) \leq C e^{\tau R^\rho |\xi|_{2,p}^2}.$$

ρ is called the order of F, τ is called the type of F. F is said to be of growth $(\rho, 0)$ on $\mathcal{S}_p(\mathbb{R})$, if and only if for every $\tau > 0$, there exists a $C > 0$ so that the above bound holds.

Definition 2.2. A complex valued function F on $\mathcal{S}(\mathbb{R})$ is called a U-functional, if and only if F is ray entire on $\mathcal{S}(\mathbb{R})$ and of growth $(2, \tau)$ on $\mathcal{S}_p(\mathbb{R})$, for some $\tau \geq 0$ and some $p \in \mathbb{N}_0$.

We can now formulate the characterization theorem by Potthoff and Streit, [PS 91, KLP 94]:

Theorem 2.3. If $\Phi \in (\mathcal{S})^*$, then $\mathcal{S}\Phi$ is a U-functional. If F is a U-functional, then there exists a unique Φ in $(\mathcal{S})^*$ so that $\mathcal{S}\Phi = F$.

One can also characterize the space $(\mathcal{S})$ in a similar way (cf. [KPS 91]):

Theorem 2.4. If $\varphi \in (\mathcal{S})$, then $\mathcal{S}\varphi$ is ray entire on $\mathcal{S}(\mathbb{R})$ and of growth $(2, 0)$ on $\mathcal{S}_{-p}(\mathbb{R})$, for every $p \in \mathbb{N}_0$. Conversely, if F is ray entire on $\mathcal{S}(\mathbb{R})$ and of growth $(2, 0)$ on $\mathcal{S}_{-p}(\mathbb{R})$ for every $p \in \mathbb{N}_0$, then there exists a unique $\varphi \in (\mathcal{S})$ so that $\mathcal{S}\varphi = F$.

Remark: In [HKP 93] it is shown that the $\mathcal{S}$-Transform of an $(\mathcal{S})^*$-element is always of growth $(2, 1/2)$ on $\mathcal{S}_p(\mathbb{R})$. Traversing the proof of the characterization theorem, it is easy to see that the growth of a U-functional can be chosen to be $(2, 1/2)$.

For a complete account on the White Noise theory, the reader should confer [HKP 93] and the references therein.

3. Spaces of U-Functionals

For each $p \in \mathbb{Z}$ define $\mathcal{U}_p$ to be the space of all mappings

$$F : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C},$$

where F is ray entire, and satisfies the following growth bound:

$$|F(z\xi)| \leq K e^{\frac{1}{2}|z|^2|\xi|^2_{2,-p}}, \forall z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R}).$$

Equip $\mathcal{U}_p$ with the norm

$$\|F\|_p = \sup_{\substack{z\in\mathbb{C} \\ \xi\in\mathcal{S}(\mathbb{R})}} |F(z\xi)|\, e^{-\frac{1}{2}|z|^2|\xi|^2_{2,-p}}$$

Proposition 3.1. $\mathcal{U}_p$ is complete for all $p \in \mathbb{Z}$.

Proof: Let (F_n) be a sequence of elements in $\mathcal{U}_p$ such that

$$\sum_{n=1}^{\infty} \|F_n\|_p < \infty.$$

We show that $\sum F_n$ is an element of $\mathcal{U}_p$. Of course, by the triangle inequality, $\sum F_n$ satisfies the growth bound in $\mathcal{U}_p$. Define

$$F^{(N)}(\xi) = \sum_{n=1}^{N} F_n(\xi).$$

Then $F^{(N)}$ will be a sequence of ray entire functions, and an application of Vitali's theorem gives that $F = \sum F_n$ is ray entire. Hence $\sum F_n$ is an element of $\mathcal{U}_p$. □

For $p < q$ we have that

$$|\cdot|_{2,-p} \geq |\cdot|_{2,-q}.$$

Hence

$$\|F\|_p = \sup_{z,\xi} |F(z\xi)|\, e^{-\frac{1}{2}|z|^2|\xi|^2_{2,-p}} \leq \sup_{z,\xi} |F(z\xi)|\, e^{-\frac{1}{2}|z|^2|\xi|^2_{2,-q}} = \|F\|_q.$$

This means that the norms are comparable in the sense of [GV 68]. The family of norms are also compatible: Let (F_n) be a Cauchy sequence in both of the norms $\|\cdot\|_p$ and $\|\cdot\|_q$, and assume $F_n \longrightarrow 0$ in $\|\cdot\|_p$-norm. We show that $F_n \longrightarrow 0$ in $\|\cdot\|_q$-norm. Note that convergence in $\|\cdot\|_r$-norm implies pointwise convergence. This means that $F_n(z\xi) \longrightarrow 0$ for every z, ξ. Assume $F_n \longrightarrow F$ for some $F \neq 0$ in $\|\cdot\|_q$-norm. Hence, $F(z\xi)$ is different from 0 for at least one pair z, ξ, which is impossible.

Define

$$\mathcal{U} = \bigcap_{p\in\mathbb{Z}} \mathcal{U}_p$$

and equip this space with the projective limit topology. Define

$$\mathcal{U}^* = \bigcup_{p\in\mathbb{Z}} \mathcal{U}_p$$

equipped with the inductive limit topology. We then have the following chain of inclusions

$$\mathcal{U} \subset \ldots \subset \mathcal{U}_2 \subset \mathcal{U}_1 \subset \mathcal{U}_0 \subset \mathcal{U}_{-1} \subset \mathcal{U}_{-2} \subset \ldots \subset \mathcal{U}^*.$$

According to [GV 68], $\mathcal{U}$ is a countably normed space, and hence a Fréchet space.

Remark: The reader should note that $\mathcal{U}^*$ not necessarily is the dual of $\mathcal{U}$. However, we will show later that $\mathcal{U}^*$ can be identified with the dual of $\mathcal{U}$ through the $\mathcal{S}$-Transform.

The two characterization theorems for $(\mathcal{S})$ and $(\mathcal{S})^*$ stated in the preceding section, can now be reformulated as:

Theorem 3.2. The $\mathcal{S}$-Transform is an algebraic bijection from $(\mathcal{S})^*$ onto $\mathcal{U}^*$, and from $(\mathcal{S})$ onto $\mathcal{U}$.

4. Topological Aspects of the $\mathcal{S}$-Transform

Now we want to state our main result:

Theorem 4.1. The $\mathcal{S}$-transform $\mathcal{S}\colon (\mathcal{S})^* \longrightarrow \mathcal{U}^*$ is a homeomorphism if both spaces are equipped with the inductive limit topology (which for $(\mathcal{S})^*$, as well known, coincides with the strong topology, cf. [HKP 93], Chapter 3).

Furthermore also the $\mathcal{S}$-transform $\mathcal{S} : (\mathcal{S}) \longrightarrow \mathcal{U}$ is a homeomorphism if both spaces are equipped with the projective limit topology.

Proof: We will show that for a suitable choice of p and q, $\mathcal{S} : (\mathcal{S})_p \longrightarrow \mathcal{U}_q$ and $\mathcal{S}^{-1} : \mathcal{U}_p \longrightarrow (\mathcal{S})_q$ are continuous, the rest of the proof is done by standard topological arguments.

Proposition 4.2. Let $\Phi \in (\mathcal{S})_p$, $p \in \mathbb{Z}$. Then $\mathcal{S}\Phi \in \mathcal{U}_p$ and $\|\mathcal{S}\Phi\|_p \leq \|\Phi\|_{2,p}$.

Proof: In the proof of the characterization theorem ([PS 91, HKP 93]) it is shown that

$$|\mathcal{S}\Phi(z\xi)| \leq \|\Phi\|_{2,p}\, e^{\frac{1}{2}|z|^2|\xi|^2_{2,-p}}$$

for all $z \in \mathbb{C}$, $\xi \in \mathcal{S}(\mathbb{R})$. Hence

$$\|\mathcal{S}\Phi\|_p = \sup_{\substack{z\in\mathbb{C}\\ \xi\in\mathcal{S}(\mathbb{R})}} |\mathcal{S}\Phi(z\xi)|\, e^{-\frac{1}{2}|z|^2|\xi|^2_{2,-p}} \leq \|\Phi\|_{2,p}.$$

□

Proposition 4.3. Let $F \in \mathcal{U}_p$, $p \in \mathbb{Z}$. Then there exist constants $c_1, c_2 > 0$ independent of p such that for $q \leq p - c_1$, $\mathcal{S}^{-1}F \in (\mathcal{S})_q$ and $\left\|\mathcal{S}^{-1}F\right\|_q \leq c_2 \|F\|_p$.

Proof: Such constants can be found quite easily elaborating the constants in the proof of the characterization theorem [PS 91, HKP 93].

Some particular choices for these constants have been worked out, e.g., in [KS 91, Le 91, Ya 90, KLP 94]. □

As an immediate consequence of Theorem 4.1. we have the following result.

Corollary 4.4. The space $\mathcal{U}^*$ can be identified with the dual space of $\mathcal{U}$.

Proof: Let $\mathcal{U}'$ denote the dual space of $\mathcal{U}$. Then the action of $F \in \mathcal{U}^*$ on $f \in \mathcal{U}$ can be defined by $F(f) :=< \mathcal{S}^{-1}F, \mathcal{S}^{-1}f >$, where the pairing on the right-hand side is the one between $(\mathcal{S})^*$ and $(\mathcal{S})$. Hence $F \in \mathcal{U}'$.

Now for $F \in \mathcal{U}'$ there is a corresponding $(\mathcal{S})^*$-element Φ defined by $< \Phi, \varphi >:= F(\mathcal{S}\varphi)$ for $\varphi \in (\mathcal{S})$. But then $G := \mathcal{S}\Phi \in \mathcal{U}^*$ and for $f \in \mathcal{U}$ $G(f) =< \mathcal{S}^{-1}G, \mathcal{S}^{-1}f >=< \Phi, \mathcal{S}^{-1}f >= F(\mathcal{S}(\mathcal{S}^{-1}f)) = F(f)$, hence $F = G \in \mathcal{U}^*$. □

Another consequence of Theorem 4.1. is the following

Remark: Since $(\mathcal{S})$ and $(\mathcal{S})^*$ are nuclear Fréchet spaces (cf. [HKP 93]), by theorem 4.1. this is also true for $\mathcal{U}$ and $\mathcal{U}^*$. Hence by general theory (e.g. [Sch 71]) $\mathcal{U}$ is the projective limit of a sequence of Hilbert spaces, i.e. a countably Hilbert space in the sense of [GV 68].

Acknowledgements: We would like to thank Jürgen Potthoff for valuable comments and suggestions. The second author is supported by the Norwegian Research Council(NFR), 100549/410.

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