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A Note on Population Growth in a Crowded Stochastic Environment

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abstract

We find an explicit unique solution in the space of Kondratiev distributions, $(\mathcal{S})^{-1}$, to a stochastic differential equation modelling population growth in a crowded stochastic environment.

1. Introduction

In this paper we are going to study a stochastic version of the Verhulst model for population growth,

$$X_t = X_0 + r\int_0^t X_s \diamond (N - X_s)ds + \alpha \cdot \int_0^t X_s \diamond (N - X_s)\delta B_s \tag{1}$$

where r, α, N are constants, N, r positive. δB_s denotes the (generalized) Skorohod integral. A precise meaning of this integral will be given in the next section. We denote by $\diamond$ the Wick product.

(1) was first proposed by Lindstrøm et. al. [LØU] as a modell for population growth in a crowded stochastic environment. For deterministic initial conditions X_0, where $0 \leq X_0 \leq 1$ and $X_0 \neq \frac{1}{2}$, they found an explicit solution to (1) using white noise methods. Their solution is a "true" stochastic variable. The case $X_0 = \frac{1}{2}$ represents some kind of "stochastic bifurcation point", since no stochastic variable exists as a solution for this initial condition *(see Lindstrøm et. al. [LØU] for their remark.)* The main motivation for this paper is to give an explicit solution also for the case $X_0 = \frac{1}{2}$. In section 4 we show that for this initial condition, we do not even have a solution in the space of Hida distributions, $(\mathcal{S})^*$. This suggests that the space of Kondratiev distributions, $(\mathcal{S})^{-1}$, is the natural space for this problem. Moreover, using Wick Calculus on the space of Kondratiev distributions, $(\mathcal{S})^{-1}$, we are able to find an explicit solution of (1) for general initial conditions with positive expectation. Now, however, the solution is no longer a stochastic variable, but a *generalized* stochastic variable living in the abstract space $(\mathcal{S})^{-1}$.

2. Some Preliminaries

We start by recalling some of the basic definitions and features of *the white noise analysis*. For a more complete account, see Hida et. al. [HKPS] and Gjessing et. al. [GHLØUZ].

As usual, let $\mathcal{S}'(\mathbb{R}^d)$ denote the space of tempered distributions on $\mathbb{R}^d$, which is the dual of the well-known Schwartz space $\mathcal{S}(\mathbb{R}^d)$. By the Bochner-Minlos theorem there exists a measure μ on $\mathcal{S}'$ such that

$$\int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}, \phi \in \mathcal{S}$$

where $\|.\|$ is the $L^2(\mathbb{R}^d)$-norm. $\langle\cdot,\cdot\rangle$ is the dual pairing between $\mathcal{S}'$ and $\mathcal{S}$. Let $\mathcal{B}$ denote the Borel sets on $\mathcal{S}'$ (equipped with the weak star topology). Then the triple $(\mathcal{S}', \mathcal{B}, \mu)$ is called *the white noise probability space.*

If we define

$$\tilde{B}_x(\omega) := \tilde{B}_{x_1,\ldots,x_d}(\omega) := \langle\omega, \mathcal{X}_{[0,x_1]\times\ldots\times[0,x_d]}(\cdot)\rangle$$

then $\tilde{B}_x$ has an x-continuous version B_x which becomes a *d-parameter Brownian motion.* The *d-parameter Wiener-Ito integral* of $\phi \in L^2$ is defined by

$$\int_{\mathbb{R}^d} \phi(y) dB_y(\omega) = \langle\omega, \phi\rangle$$

Of special interest will be the space $L^2(\mathcal{S}'(\mathbb{R}^d), \mu)$, or $L^2(\mu)$ for short. *The Wiener-Ito chaos expansion theorem* says that every $F \in L^2(\mu)$ has the form

$$F(\omega) = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n(u) dB_u^{\otimes n}(\omega) \tag{2}$$

where $f_n \in L^2(\mathbb{R}^{nd})$ and f_n is symmetric in its n variables (in the sense that $f_n(u_{\sigma_1}, \ldots, u_{\sigma_n}) = f_n(u_1, \ldots, u_n)$ for all permutations σ, where $u_i \in \mathbb{R}^d$). The right hand side of (2) are the *multiple Ito integrals.*

There is an equivalent expansion of $F \in L^2(\mu)$ in terms of the Hermite polynomials:

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); n = 0, 1, 2, \ldots$$

We explain this more closely: Define the *Hermite function* $\xi_n(x)$ of order n as

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x)$$

where $x \in \mathbb{R}, n = 1, 2, \ldots$. $\{\xi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore the family $\{e_\alpha\}$ of tensor products

$$e_\alpha := e_{\alpha_1,\ldots,\alpha_m} := \xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_d}$$

(where α denotes the multi-index $(\alpha_1, \ldots, \alpha_d)$) forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Assume that the family of all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$ is given a fixed ordering

$$(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n)}, \ldots)$$

where $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_d^{(k)})$. Put

$$e_n = e_{\beta^{(n)}}; n = 1, 2, \ldots$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multi-index. It was shown by Ito that

$$\int_{(\mathbb{R}^d)^n} e_1^{\widehat{\otimes}\alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_m^{\widehat{\otimes}\alpha_m} dB^{\otimes n} = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{3}$$

where $\theta_j(\omega) = \int_{\mathbb{R}^d} e_j(x) dB_x(\omega), n = |\alpha|$ and $\widehat{\otimes}$ denotes the *symmetrized tensor product* (e.g., $f \widehat{\otimes} g(x, y) = \frac{1}{2}[f(x)g(y) + f(y)g(x)]$ if $x, y \in \mathbb{R}$ and similarly for more than two variables). If we define, for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j)$$

then we see that (3) can be written

$$\int_{(\mathbb{R}^d)^n} e^{\widehat{\otimes}\alpha} dB^{\otimes|\alpha|} = H_\alpha(\omega) \tag{4}$$

using multi-index notation: $e^{\widehat{\otimes}\alpha} = e_1^{\widehat{\otimes}\alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_m^{\widehat{\otimes}\alpha_m}$ if $e = (e_1, e_2, \ldots)$. Since the family $\{e^{\widehat{\otimes}\alpha}; |\alpha| = n\}$ forms an orthonormal basis for the symmetric functions in $L^2((\mathbb{R}^d)^n)$, we see by combining (2) and (4) that we have the representation

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{5}$$

(the sum being taken over all multi-indices α of nonnegative integers). Moreover, it can be proved that

$$\|F\|^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2$$

where $\alpha! = \alpha_1! \ldots \alpha_m!$.

The Hida test function space $(\mathcal{S})$ and the Hida distribution space $(\mathcal{S})^*$ can be given the following characterization, due to Zhang [Z].

Theorem 2.1: Let $\psi \in L^2(\mu)$ have the chaos expansion

$$\psi(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

Then ψ is a *Hida test function*, i.e. $\psi \in (\mathcal{S})$, if

$$\sup_{\alpha} c_\alpha^2 \alpha! (2I\!N)^{\alpha k} < \infty, \ \forall \text{ natural numbers } k < \infty$$

where

$$(2I\!N)^\alpha := \prod_{j=1}^{m} (2^d \beta_1^{(j)} \dots \beta_d^{(j)})^{\alpha_j} \text{ for } \alpha = (\alpha_1, \dots, \alpha_m)$$

A *Hida distibution* Ψ, $\Psi \in (\mathcal{S})^*$, is a formal series

$$\Psi = \sum_\alpha b_\alpha H_\alpha \tag{6}$$

where

$$\sup_\alpha b_\alpha^2 \alpha! ((2I\!N)^{-\alpha})^q < \infty \text{ for some } q > 0$$

□

If $\Psi \in (\mathcal{S})^*$ and $\psi \in (\mathcal{S})$ is given as in the theorem, the action of Ψ on ψ is given by

$$\langle\langle \Psi, \psi \rangle\rangle = \sum_\alpha \alpha! b_\alpha c_\alpha \tag{7}$$

Note that no assumptions are made regarding the convergence of the formal series in (6). We can in a natural way regard $L^2(\mu)$ as a subspace of $(\mathcal{S})^*$. In particual, if $F \in L^2(\mu)$ then by (7) the action of F on $\psi \in (\mathcal{S})$ is given by

$$\langle\langle F, \psi \rangle\rangle = E[F \cdot \psi]$$

Since 1 is an element of $(\mathcal{S})$, the expectation function can be extended to $(\mathcal{S})^*$:

$$E[\Psi] = \langle\langle \Psi, 1 \rangle\rangle$$

We will now introduce the spaces $(\mathcal{S})^1$ and $(\mathcal{S})^{-1}$ which were first constructed by Kondratiev [K]. For a complete account on the following results, see Albeverio et. al. [ADKS] and Kondratiev et. al. [KLS]:

Definiton 2.2: Define $(\mathcal{S})^\rho$ and $(\mathcal{S})^{-\rho}$ as follows:

Part a): For $0 \le \rho \le 1$ let $(\mathcal{S})^\rho$ consist of all

$$\psi = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu)$$

such that

$$\|\psi\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2I\!N)^{\alpha k} < \infty \text{ for all } k < \infty$$

Part b): The space $(\mathcal{S})^{-\rho}$ consists of all formal expansions

$$\Psi = \sum_{\alpha} b_\alpha H_\alpha$$

such that

$$\sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty \text{ for some } q < \infty$$

The family of seminorms $\|f\|^2_{\rho,k}; k = 1, 2, \ldots$ defines a topology on $(\mathcal{S})^{\rho}$. □

We remark that $(\mathcal{S}) = (\mathcal{S})^0$ and $(\mathcal{S})^* = (\mathcal{S})^{-0}$ in the above construction. $(\mathcal{S})^{-1}$ will be called the space of Kondratiev distributions.

Definition 2.3: Let $\Phi = \sum_\alpha a_\alpha H_\alpha, \Psi = \sum_\alpha b_\beta H_\beta$ be two elements of $(\mathcal{S})^{-\rho}$. Then the *Wick product* of Φ and Ψ is the element $\Phi \diamond \Psi$ in $(\mathcal{S})^{-\rho}$ given by

$$\Phi \diamond \Psi = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}$$

□

It can be shown that $(\mathcal{S})^1$ is closed under the Wick product.

The *Hermite Transform*, see Lindstrøm et. al. [LØU], has a natural extension to $(\mathcal{S})^{-1}$, the space of Kondratiev distributions:

Definition 2.4: If $\Psi = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-1}$ then the Hermite Transform of $F, \mathcal{H}\Psi = \tilde{\Psi}$, is defined by

$$\tilde{\Psi}(z) = \mathcal{H}\Psi(z) = \sum_{\alpha} b_\alpha z^\alpha$$

where $z = (z_1, z_2, \ldots.) \in \mathbb{C}_0^{\mathbb{N}}$, and

$$z^\alpha = z_1^\alpha z_2^\alpha \ldots z_m^\alpha$$

for $\alpha = (\alpha_1, \ldots, \alpha_m)$. □

The Hermite Transform characterizes $(\mathcal{S})^{-1}$ in the following way:

Lemma 2.5: $\Psi \in (\mathcal{S})^{-1}$ if and only if there exist some $\epsilon > 0, q < \infty$ such that the Hermite transform of Ψ, $\mathcal{H}\Psi$, is a bounded analytic function on $B_q(0, \epsilon)$. □

Convergence of sequences in $(\mathcal{S})^{-1}$ can be characterized in terms of the Hermite Transform as follows:

Lemma 2.6: The following are equivalent

I: $\Psi_n \to \Psi$ in $(\mathcal{S})^{-1}$

II: There exist $\epsilon > 0, q < \infty, M < \infty$ such that

$$\mathcal{H}\Psi_n(z) \to \mathcal{H}\Psi(z) \text{ as } n \to \infty \text{ for } z \in \mathbf{B}_q(0, \epsilon)$$

and

$$|\mathcal{H}\Psi_n(z)| \leq M \text{ for all } n = 1, 2, ...; z \in \mathbf{B}_q(0, \epsilon)$$

where

$$\mathbf{B}_q(0, \epsilon) = \{z = (z_1, z_2, \ldots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_\alpha |z^\alpha|^2 (2\mathbb{N})^{\alpha q} < \epsilon^2\}$$

□

Note that the Hermite Transform transforms the Wick product into an ordinary product.

The Wick product gives a nice relation between functional integration in $(\mathcal{S})^{-1}$ and Skorohod/Ito integration. We define integration in $(\mathcal{S})^{-1}$ as follows:

Definition 2.7: Assume $\Psi_s \in (\mathcal{S})^{-1}$ for each $s \in [0, T]$, where $0 < T \leq \infty$. If

$$\langle\langle \Psi_s, \psi \rangle\rangle \in L^1([0, T], ds)$$

for all $\psi \in (\mathcal{S})^1$, we define the unique $(\mathcal{S})^{-1}$-element $\int_0^T \Psi_s ds$ by

$$\langle\langle \int_0^T \Psi_s ds, \psi \rangle\rangle = \int_0^T \langle\langle \Psi_s, \psi \rangle\rangle ds$$

□

Consider the case $d = 1$, i.e the probability space $\mathcal{S}'(\mathbb{R})$: Define the element

$$W_t = \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon_k}$$

where ϵ_k is the multi-index with zeros except at position k, where it has value 1. It can be shown, see for instance Gjessing et. al. [GHLØUZ], that $W_t \in (\mathcal{S})^*$. Moreover, for a Skorohod integrable element $\Psi_s \in L^2(\mu)$ it can be shown that

$$\int_0^t \Psi_s \diamond W_s ds = \int_0^t \Psi_s \delta B_s$$

where the integral on the right hand side is the Skorohod integral. See Lindstrøm et. al. [LØU], Hida et. al. [HKPS] and Benth [B] for a discussion of this relation. We can say that functional integration in $(\mathcal{S})^{-1}$ involving Wick product with W_t generalizes the Skorohod/Ito integration. This connection motivates the following interpretation of (1): We look for an element X_t in $(\mathcal{S})^{-1}$ which satisfies

$$X_t = X_0 + r \int_0^t X_s \diamond (N - X_s) ds + \alpha \int_0^t X_s \diamond (N - X_s) \diamond W_s ds \tag{8}$$

We end this section with a nice property of the $(\mathcal{S})^{-1}$ space, the so-called Wick Calculus theorem, see theorem 12 in Kondratiev et. al. [KLS]:

Theorem 2.8: Let $\Psi \in (\mathcal{S})^{-1}$. Assume $f : \mathbb{C} \to \mathbb{C}$ is analytic in a neighborhood of $E[\Psi]$. Then

$$f(\tilde{\Psi}(z)) = \tilde{\Phi}(z)$$

is the Hermite Transform of an element Φ of $(\mathcal{S})^{-1}$. □

We remark that the definitions and results presented in the above language can be found in Holden et. al. [HLØUZ].

3. The Solution Of The Population Modell

For simplicity we will assume that $N = 1$ in modell (8). We also assume that $X_0 \in (\mathcal{S})^{-1}$ and that

$$E[X_0] > 0$$

Hermite transforming the stochastic equation (8) into an ordinary complex differential equation, and solving, we obtain the candidate

$$X_t = (1 + \Theta_0 \diamond \exp^\diamond(-rt - \alpha B_t))^{\diamond(-1)} \tag{9}$$

where

$$\Theta_0 = (1 - X_0) \diamond X_0^{\diamond(-1)} = X_0^{\diamond(-1)} - 1 \tag{10}$$

We have written $\exp^\diamond$ for the *Wick exponential*, i.e the element defined by

$$\exp^\diamond \Phi = \sum_{n=1}^{\infty} \frac{1}{n!} \Phi^{\diamond n}$$

(see theorem 2.8 above.) We show that X_t is an element in $(\mathcal{S})^{-1}$:

Define

$$g(z) = z^{-1} - 1$$

Obviously, $g(z)$ is analytic in a neighborhood around $E[X_0] > 0$. Hence, Φ_0 is an element of $(\mathcal{S})^{-1}$. Furthermore, define

$$f(z) = (1 + z)^{-1}$$

We have

$$E[\Phi_0 \diamond \exp^\diamond(-rt - \alpha B_t)] = (E[X_0]^{-1} - 1)e^{-rt}$$

When $0 < E[X_0] \leq 1$ we have

$$0 \leq (E[X_0]^{-1} - 1)e^{-rt} \leq (E[X_0]^{-1} - 1)$$

and, when $E[X_0] > 1$,

$$(E[X_0]^{-1} - 1) \leq (E[X_0]^{-1} - 1)e^{-rt} < 0$$

for all $t \geq 0$. In both cases is the expectation bounded away from -1 for all $t \geq 0$. Hence, there exist constants q, ϵ such that

$$f(\tilde{\Phi}_0(z) \exp(-rt - \alpha \tilde{B}_t(z)))$$

is analytic and bounded for $z \in B_q(0, \epsilon)$, for all $t \geq 0$. This implies by theorem 2.8 that X_t is an element of $(\mathcal{S})^{-1}$ for all $t \geq 0$.

To show that X_t is a solution of equation (8), we must prove that X_t satisfies

$$\frac{dX_t}{dt} = (r + \alpha W_t) \diamond X_t \diamond (1 - X_t)$$

in $(\mathcal{S})^{-1}$. But by lemma 2.6, II this is equivalent with showing that

$$\frac{\tilde{X}_{t+h}(z)-\tilde{X}_t(z)}{h} \to (r+\alpha\tilde{W}_t(z))\tilde{X}_t(z)(1-\tilde{X}_t(z))$$

pointwise boundedly for $z \in B_q(0,\epsilon)$ when $h \to 0$. This can be seen to hold by direct calculation: The Hermite transform of (9) is:

$$\tilde{X}_t(z) = (1+\tilde{\Theta}_0(z)e^{-rt-\alpha\tilde{B}_t(z)})^{-1}$$

Hence, after some manipulation,

$$\frac{\tilde{X}_{t+h}(z)-\tilde{X}_t(z)}{h} = \left(\frac{1-e^{-rh}}{h}e^{-\alpha\tilde{B}_t(z)} + e^{-rh}\frac{(e^{-\alpha\tilde{B}_t(z)} - e^{-\alpha\tilde{B}_{t+h}(z)})}{h}\right)$$

$$\times e^{-rt}\tilde{\Theta}_0(z)\tilde{X}_t(z)\tilde{X}_{t+h}(z)$$

We see that $\tilde{X}_{t+h}(z) \to \tilde{X}_t(z)$ pointwise boundedly for $z \in \mathbf{B}_q(0,\epsilon)$. By definition we have

$$\frac{d}{dt}e^{-\alpha\tilde{B}_t(z)} = \lim_{h\to 0}\frac{e^{-\alpha\tilde{B}_t(z)} - e^{-\alpha\tilde{B}_t(z)}}{h} = \alpha\tilde{W}_t(z)e^{-\alpha\tilde{B}_t(z)}$$

for every $z \in \mathbf{B}_q(0,\epsilon)$. Moreover, we can show that this convergence is bounded on $\mathbf{B}_q(0,\epsilon)$. Hence

$$\frac{\tilde{X}_{t+h}(z)-\tilde{X}_t(z)}{h} \to (r+\alpha\tilde{W}_t(z))\tilde{X}_t(z)(1-\tilde{X}_t(z))$$

pointwise boundedly on $\mathbf{B}_q(0,\epsilon)$.

Since $\tilde{X}_t(z)$ is the unique solution of the Hermite transformed version of equation (8), it follows by injectivity of the Hermite transform that X_t is unique. We have the conclusion:

Theorem 3.1: Assume $X_0 \in (\mathcal{S})^{-1}$ with $E[X_0] > 0$. Then

$$X_t = (1+\Theta_0 \diamond \exp^{\diamond}(-rt-\alpha B_t))^{\diamond(-1)}$$

where

$$\Theta_0 = X_0^{\diamond(-1)} - 1$$

is the unique $(\mathcal{S})^{-1}$ solution of (8) with $N = 1$. □

4. Some Concluding Remarks

As pointed out in the introduction, Lindstrøm et. al. [LØU] did not obtain any solution of (8) for the "stochastic bifurcation point" $X_0 = \frac{1}{2}$. We discuss this special case more in detail: With initial condition $X_0 = \frac{1}{2}$, we obtain $\Theta_0 = 1$ which gives the solution

$$X_t = (1 + \exp^{\diamond}(-rt - \alpha B_t))^{\diamond(-1)} \tag{11}$$

We show that X_t is not an element of the Hida distribution space $(\mathcal{S})^*$: In Hida et. al. [HKPS] the S-transform of an element of $(\mathcal{S})^*$, is defined as

$$SF(\xi) = \langle\langle F, \exp^{\diamond} W_{\xi} \rangle\rangle$$

for $\xi \in \mathcal{S}(\mathbb{R})$. The S-transform of X_t given in (11), is

$$SX_t(\xi) = (1 + \exp(rt + \alpha \int_0^t \xi(s)ds))^{-1}$$

This object is well defined for all $\xi \in \mathcal{S}(\mathbb{R})$, and $v_t = SX_t(\xi)$ is the unique solution of the problem

$$v_t = \frac{1}{2} + r \int_0^t v_s(1 - v_s)ds + \alpha \int_0^t v_s(1 - v_s)\xi(s)ds$$

for each ξ. However, $SX_t(\xi)$ can *not* be extended to an analytic function

$$z \to SX_t(z\xi)$$

on the complex plane $\mathbb{C}$. Hence, X_t is not an element of $(\mathcal{S})^*$. (See Hida et. al. [HKPS] for a characterization of $(\mathcal{S})^*$-elements in terms of the S-transform. We see that for $\xi, \eta \in \mathcal{S}(\mathbb{R})$ the mapping

$$\lambda \to SX_t(\xi + \lambda\eta)$$

can only be analytic in a *neighborhood* of zero in $\mathbb{C}$. This tells us that X_t is not contained in any of the spaces $(\mathcal{S})^{-\rho}$, $\rho \in [0, 1)$. (See Albeverio et. al. [ADKS] for the characterization of these spaces by the S-transform.)

By the uniqueness of the Hermite Transform, the $(\mathcal{S})^{-1}$ element X_t given in (9) and (10) has to coincide with the solution found by Lindstrøm et. al. [LØU] for constant initial conditions $X_0 = x \neq \frac{1}{2}$. As we have seen, the results above are worked out for general initial conditions

$$X_0 \in (\mathcal{S})^{-1}$$

where $E[X_0] > 0$. This means that for stochastic variables as initial conditions we have a solution as well. Note that the case of anticipating initial conditions is also included.

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References.

[ADKS] S.Albeverio, Yu. L. Daletsky, Yu. G. Kondratiev and L. Streit: *Non-Gaussian Infinite Dimensional Analysis*; Preprint, University of Bielefeld (1994)

[B] F. E. Benth: *Integrals in the Hida Distribution Space* $(\mathcal{S})^*$; In T.Lindstrøm, B.Øksendal and A.S.Ustunel (eds): Stochastic Analysis and Related Topics. Vol. 8, Gordon and Breach (1993)

[GHLØUZ] H. Gjessing, H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: *The Wick Product*; In H.Niemi, G.Högnäs, A.N.Shiryaev and A.Melnikov (eds.): Frontiers in pure and applied mathematics Vol 1, TVP Publishers, Moscow (1993)

[HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: *White Noise -An Infinite Dimensional Calculus*; Kluwer Academic Publishers (1993)

[HLØUZ] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T. -S. Zhang: *The Pressure Equation for Fluid Flow in a Stochastic Medium*; Preprint, University of Oslo (1994)

[K] Yu. G. Kondratiev: *Generalized Functions in Problems of Infinite Dimensional Analysis*; Ph. D. Thesis, Kiev University, Ukraine, (1978)

[KLS] Yu. G. Kondratiev, P. Leukert and L. Streit: *Wick Calculus in Gaussian Analysis*; Preprint, University of Bielefeld, (1994)

[LØU] T.Lindstrøm, B.Øksendal, J. Ubøe: *Wick Multiplication And Ito-Skorohod Stochastic Differential Equations*; In S.Albeverio et.al. (eds.): Ideas and Methods in Mathematic Analysis. Cambridge University Press (1992)

[Z] T.-S.Zhang: *Characterizations of White Noise Test Functions and Hida Distributions*; Stochastics and Stochastics Reports **41**, (1992)