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Abstract

In dealing with denotational semantics of programming languages partial orders resp. metric spaces have been used with great benefit in order to provide a meaning to recursive and repetitive constructs. This paper presents two methods to define a metric on a subset M of a cpo D such that M is a complete metric spaces and the metric semantics on M coincides with the cpo semantics on D when the same semantic operators are used. The first method is to add a 'length' on a cpo which means a function $\rho : D \to I\!N_0 \cup \{\infty\}$ of increasing power. The second is based on the ideas of [9] and uses pseudo rank orderings, i.e. monotone sequences of monotone functions $\pi_n : D \to D$. We show that *SFP* domains can be characterized as special kinds of rank orderded cpo's. We also discuss the connection between the Lawson topology and the topology induced by the metric.

Contents

1 Introduction

In dealing with semantics of programming languages partial orders resp. metric spaces have been used with great benefit in order to provide a meaning to recursive and repetitive constructs. There have been various attempts to reconcile or to relate the two approaches [10, 19, 27, 29]. The observation that the Scott topology associated with a cpo D is not Hausdorff and consequently cannot be obtained by a metric on D led some authors to search for weaker notions of metric as the partial metric [19] and the quasi uniformities [27]. The most promising approach is based on generalized metric spaces as proposed by Lawvere [15]. Both partial order and metric can be considered as special cases of generalized metric. Generalized fixed point theorems that specialize to the classical Banach theorem and Tarski theorem can be proved [24].

The Lawson topology is metrizable under certain conditions, e.g. if D is a compact algebraic cpo with a countable basis, but compactness is too strong a condition in many applications. On the other hand, looking at concrete mathematical stuctures one may observe that

- many structures (e.g. strings, Mazurkiewicz traces, pomsets, event structures and various kinds of trees) allow for both a metric and a partial order setting
- there are certain features of languages that give rise to problems in the partial order setting, e.g. the sequential operator which can easily be handled in the metric setting [3], and viceversa, e.g. unguarded recursion.

In this paper we establish two concepts by which we may obtain a metric from a partial order. First we consider partial orders with a length, i.e. a function ρ which assigns to each element x of D a length $\rho(x) \in \mathbb{N}_0 \cup \{\infty\}$. If the elements of D are interpreted as processes the length $\rho(x)$ is the maximal number of atomic steps which are needed for the execution of x. E.g. the length of a string is its usual length, the length of a tree is its height. We show that a length on a partial order D induces an ultrametric on a subset M of D. In order to ensure the completeness of M we introduce the concept of continuous weight. A weight means a length which ensures the existence of finite cuts where the n-cut $x[n]$ of an element $x \in D$ represents a process whose behaviour is given by the first n steps of x. Continuity of a weight means that the function $x \mapsto x[n]$ is continuous.

The second concept is based on the idea of [9] introducing a pseudo rank ordering on a partial order D (i.e. a monotone sequence of monotone functions $\pi_n : D \to D$). We show that a pseudo rank ordering induces an ultrametric on a subset M of D which is complete if D is a cpo and that continuous weights can be considered as special cases of pseudo rank orderings.

The paper is organized as follows: In section 2 we introduce the concepts of a length, weight, continuous weight resp. pseudo rank ordering and study the properties of the induced metric space. Section 3 treats semantic operators and establishes conditions that guarantee that these operators display the necessary contraction properties for the induced metric space. In section 4 we sketch some examples that show that existing settings fit in our framework. Section 5 shows that Plotkins *SFP* domains [22] can be characterized as special kinds of rank ordered cpo's. In section 6 we discuss the relationship between the Lawson topology and the topology induced by the metric on a weighted resp. rank ordered cpo. In section 7 we briefly discuss the connection to related work.

2 From partial order to metric

In this section we introduce three types of 'measuring' functions of increasing power on a partial order: the length (section 2.1) , the weight (section 2.2) and continuous weight (section 2.3). In section 2.4 we investigate the connection to the work of Bruce and Mitchell [9]. We show that a continuous weight induces a ranking in the sense of [9].

In the following a poset means a pair $(D, \sqsubseteq)$ consisting of a set D and a partial order $\sqsubseteq$ on D. We often write D instead of $(D, \sqsubseteq)$. By a pointed poset we mean a poset D which has a bottom element (denoted by $\perp_D$ or $\perp$). If D is a poset and $x \in D$ then

$$x \downarrow = \{y \in D : y \sqsubseteq x\}, \quad x \uparrow = \{y \in D : x \sqsubseteq y\}$$

and for each subset X of D:

$$X \downarrow = \bigcup_{x \in X} x \downarrow, \quad X \uparrow = \bigcup_{x \in X} x \uparrow$$

X is called leftclosed iff X is nonempty and $X \downarrow = X$. X is called directed iff X is leftclosed and each pair of elements in X has an upper bound in X. A dcpo means a pointed poset in which each directed subset X has a least upper bound (which is denoted by $\bigsqcup X$ or $\mathrm{lub}(X)$). We use the notion cpo to denote a pointed poset in which each monotone sequence (x_n) has a least upper bound (denoted by $\bigsqcup x_n$). Then each dcpo is a cpo. If D, D' are pointed posets then we say a function $f : D \to D'$ is continuous iff f is monotone and for each monotone sequence (x_n) in D for which $\bigsqcup x_n$ exists then $\bigsqcup f(x_n)$ exists and

$$f\left(\bigsqcup x_n\right) = \bigsqcup f(x_n).$$

2.1 Pointed posets with a length

Definition 2.1 *A* length *on a pointed poset* $(D, \sqsubseteq)$ *is a function* $\rho : D \to I\!N_0 \cup \{\infty\}$ *such that for all* $x, y \in D$*:*

(i) $\rho(x) = 0 \iff x = \perp_D$

(ii) $x \sqsubseteq y \implies \rho(x) \leq \rho(y)$

Fin(D, ρ) *or shortly Fin*(D) *denotes the collection of all* $y \in D$ *such that* $\rho(y) < \infty$*. For all* $x \in D$ *we define:*

$$\downarrow_\rho^n(x) = \{y \in D : y \sqsubseteq x, \ \rho(y) \leq n\}$$

$$\downarrow_\rho^{\text{fin}}(x) = \bigcup_{n \geq 0} \downarrow_\rho^n(x) = Fin(D) \cap x\downarrow$$

An element $x \in D$ *is called* approximable *(w.r.t.* ρ*) iff* x *is the least upper bound of* $\downarrow^{\text{fin}}(x)$*.* x *is called* finitely approximable *iff* x *is approximable and* $\downarrow_\rho^n(x)$ *is finite for all* $n \geq 0$*.*

$\mathcal{M}(D, \sqsubseteq, \rho)$ *or shortly* $\mathcal{M}(D)$ *denotes the set of approximable elements,* $\mathcal{M}_{\text{fin}}(D, \sqsubseteq, \rho)$ *or shortly* $\mathcal{M}_{\text{fin}}(D)$ *the set of finitely approximable elements.*

In the following we often omit the index ρ and write $\downarrow^n(x)$ or $\downarrow^{\text{fin}}(x)$.

Theorem 2.2 *Let* $(D, \sqsubseteq)$ *be a pointed poset and* ρ *a length on* $(D, \sqsubseteq)$*. Then*

$$d[\rho](x, y) = \inf \left\{ \frac{1}{2^n} : \downarrow^n(x) = \downarrow^n(y) \right\}$$

is a pseudo ultrametric on D *and an ultrametric on* $\mathcal{M}(D)$*.* $\mathcal{M}_{\text{fin}}(D)$ *is a closed subspace of* $\mathcal{M}(D)$*.*

Proof: It is clear that $0 \leq d[\rho](x, y) = d[\rho](y, x) \leq 1$. The strong triangle inequality can easily be verified. Now we assume that x and y are approximable and we show that

$$d[\rho](x, y) = 0 \iff x = y.$$

$\Longleftarrow$ is clear. Now we assume that x, y are approximable and $d[\rho](x, y) = 0$. Then $\downarrow^n(x) = \downarrow^n(y)$ for all $n \geq 0$. Hence $\downarrow^{\text{fin}}(x) = \downarrow^{\text{fin}}(y)$. Since x and y are approximable:

$$x = \text{lub}\left(\downarrow^{\text{fin}}(x)\right) = \text{lub}\left(\downarrow^{\text{fin}}(y)\right) = y.$$

Next we show that $\mathcal{M}_{\text{fin}}(D)$ is a closed subspace. If $x = \lim x_n$ where (x_n) is a Cauchy sequence in $\mathcal{M}_{\text{fin}}(D)$ then we have to show that x is finitely approximable, i.e. that $\downarrow^m(x)$ is finite for all $m \geq 0$. Let $m \geq 0$. There exists $n \geq 0$ such that $d[\rho](x, x_n) \leq 1/2^m$. Then

$$\downarrow^m(x) = \downarrow^m(x_n)$$

is finite. □

Remark 2.3 Let $(D,\sqsubseteq)$ be a pointed poset. Then $\rho : D \to \mathbb{N}_0 \cup \{\infty\}$,

$$\rho(x) = \begin{cases} 0 & : \text{ if } x = \bot_D \\ 1 & : \text{ otherwise} \end{cases}$$

is a length on $(D,\sqsubseteq)$, called the `discrete length`. The induced metric space $\mathcal{M}(D)$ is D with the discrete metric.

Lemma 2.4 *Let ρ be a length on a pointed poset $(D,\sqsubseteq)$ and (x_n) a monotone sequence in $\mathcal{M}(D)$ such that* $\lim x_n$ *exists. Then* $\bigsqcup x_n$ *exists and*

$$\lim_{n\to\infty} x_n = \bigsqcup_{n\geq 0} x_n.$$

Proof: Let $x = \lim x_n$. First we show that $x_n \sqsubseteq x$ for all $n \geq 0$.

Claim 1: $\downarrow^{\text{fin}}(x_n) \subseteq \downarrow^{\text{fin}}(x)$ for all $n \geq 0$

Proof: Let $n \geq 0$ and $y \in \downarrow^{\text{fin}}(x_n)$. Then $y \in \downarrow^m(x_n)$ for some $m \geq 0$. Since $x = \lim x_n$ there exists $k \geq n$ with $d[\rho](x_k, x) \leq 1/2^m$. Since $x_n \sqsubseteq x_k$ we have $\downarrow^m(x_n) \subseteq \downarrow^m(x_k)$. Hence

$$y \in \downarrow^m(x_n) \subseteq \downarrow^m(x_k) = \downarrow^m(x).$$

By Claim 1 we get that x is an upper bound of $\downarrow^{\text{fin}}(x_n)$. Since x_n is approximable:

$$x_n = \text{lub}\left(\downarrow^{\text{fin}}(x_n)\right) \sqsubseteq x.$$

Claim 2: $x = \bigsqcup x_n$

Proof: Let $y \in D$ with $x_n \sqsubseteq y$ for all $n \geq 0$. First we show that $\downarrow^{\text{fin}}(x) \subseteq \downarrow^{\text{fin}}(y)$.

Let $z \in \downarrow^{\text{fin}}(x)$. Then $z \in \downarrow^m(x)$ for some $m \geq 0$. Let $k \geq 0$ such that $d[\rho](x_k, x) \leq 1/2^m$. Then (since $x_k \sqsubseteq y$):

$$z \in \downarrow^m(x) = \downarrow^m(x_k) \subseteq \downarrow^m(y) \subseteq \downarrow^{\text{fin}}(y).$$

We conclude that y is an upper bound of $\downarrow^{\text{fin}}(x)$. Since x is approximable:

$$x = \text{lub}\left(\downarrow^{\text{fin}}(x)\right) \sqsubseteq y.$$

□

Theorem 2.5 *Let ρ be a length on a cpo $(D,\sqsubseteq)$ such that $(\mathcal{M}(D), d[\rho])$ is a complete metric space. Then*

$$\bigsqcup_{n\geq 0} x_n = \lim_{n\to\infty} x_n$$

for each Cauchy sequence (x_n) in $\mathcal{M}(D)$ which is monotone in D.

Proof: follows immediately by Lemma 2.4. □

Definition 2.6 *Let ρ be a length on a pointed poset $(D,\sqsubseteq)$. A* `tower` *in $(D,\sqsubseteq,\rho)$ is a sequence (x_n) in $\mathcal{M}(D)$ with $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ and*

$$\downarrow^n(x_n) = \downarrow^n(x_{n+1})$$

for all $n \geq 0$.

(x_n) is a tower in $(D, \sqsubseteq, \rho)$ if and only if $d[\rho](x_n, x_m) \leq 1/2^n$ for all $m \geq n \geq 0$. In particular, each tower is a Cauchy sequence. On the other hand, a sequence (x_n) in $\mathcal{M}(D)$ is a Cauchy sequence if and only if there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is a tower and

$$\downarrow^k (x_n) = \downarrow^k (x_{n_k})$$

for all $n \geq n_k$ and $k \geq 0$. We obtain:

Lemma 2.7 *Let ρ be a length on a pointed poset $(D, \sqsubseteq)$. Then $(\mathcal{M}(D), d[\rho])$ is a complete metric space if and only if for each tower (x_n) in $(D, \sqsubset, \rho)$ there exists $x \in \mathcal{M}(D)$ with $\downarrow^n (x_n) = \downarrow^n (x)$ for all $n \geq 0$.*

2.2 Pointed posets with a weight

In this section we consider a special kind of a length on a pointed poset, called a weight. A weight on a pointed poset means a length which ensures the existence of 'n-cuts', i.e. a greatest element in $\downarrow^n (x)$.

Definition 2.8 *Let $(D, \sqsubseteq)$ be a pointed poset. A* `weight` *on $(D, \sqsubseteq)$ is a length ρ on $(D, \sqsubseteq)$ such that*

$$x[n] = \mathrm{lub}(\downarrow^n (x))$$

exists for all $x \in D$, $n \geq 0$ and

$$\rho(x[n]) \leq n.$$

The tripel $(D, \sqsubseteq, \rho)$ is called a `weighted poset`. *$x[n]$ is called the n-*`cut` *of x w.r.t. ρ. We define:*

$$\mu_n^\rho : D \to D, \quad \mu_n^\rho(x) = x[n].$$

If $(D, \sqsubseteq)$ is a cpo and ρ a weight on $(D, \sqsubseteq)$ then we put

$$\mu^\rho = \bigsqcup_{n \geq 0} \mu_n^\rho : D \to D, \quad \text{i.e. } \mu^\rho(x) = \bigsqcup_{n \geq 0} x[n].$$

Remark 2.9 In general, a length is not a weight: Let $D = \{\bot, x_1, x_2, \top\}$ and $\sqsubseteq$ be given by

$$\bot \sqsubseteq x_1 \sqsubseteq \top, \quad \bot \sqsubseteq x_2 \sqsubseteq \top.$$

Then $\rho(\bot) = 0$, $\rho(x_1) = \rho(x_2) = 1$ and $\rho(\top) = 2$ is a length on $(D, \sqsubseteq)$ but not a weight since $\downarrow_\rho^1 (\top) = \{\bot, x_1, x_2\}$ does not have a greatest element. □

If ρ is a length on a cpo then in general $\mathcal{M}(D)$ is not a cpo. In the case of a weight we have:

Lemma 2.10 *Let ρ be a weight on a cpo $(D, \sqsubseteq)$ and let (x_n) be a monotone sequence in $\mathcal{M}(D)$. Then $\bigsqcup x_n \in \mathcal{M}(D)$.*

In particular: $\mathcal{M}(D)$ endowed with the restriction of $\sqsubseteq$ is a cpo and the inclusion $\mathcal{M}(D) \to D$ is continuous.

Proof: Let $x = \bigsqcup x_n$ and $x' = \mu^\rho(x) = \bigsqcup x[k]$. Then $x' \sqsubseteq x$ and $x' \in \mathcal{M}(D)$. We have to show that $x' = x$.

Since $x_n \sqsubseteq x$ we have $x_n[k] \sqsubseteq x[k]$ for all n, $k \geq 0$. Therefore

$$\bigsqcup_{n\geq 0} x_n[k] \sqsubseteq x[k]$$

for all $k \geq 0$. Since $x_n \in \mathcal{M}(D)$:

$$x = \bigsqcup_{n\geq 0} x_n = \bigsqcup_{n\geq 0}\bigsqcup_{k\geq 0} x_n[k] = \bigsqcup_{k\geq 0}\bigsqcup_{n\geq 0} x_n[k] \sqsubseteq \bigsqcup_{k\geq 0} x[k] = x'.$$

We conclude $x = x' \in \mathcal{M}(D)$. □

2.3 Continuous weights

In general the metric space $\mathcal{M}(D)$ induced by a weight is not complete even if D is a dcpo. In this section we present a condition which ensures that the induced metric space $\mathcal{M}(D)$ of a weighted poset $(D, \sqsubseteq, \rho)$ is complete. We start with a characterisation of those weighted posets whose induced metric space is complete: We show that if $\mathcal{M}(D)$ is complete then the functions μ_n^ρ are in some sense 'continuous'.

Lemma 2.11 *Let ρ be weight on a cpo $(D, \sqsubseteq)$. Then the following are equivalent:*

(i) $(\mathcal{M}(D), d[\rho])$ is a complete metric space.

(ii) $\mu_n^\rho(\bigsqcup x_m) = \bigsqcup_{m\geq 0} \mu_n^\rho(x_m)$ for each tower (x_m) in $(D, \sqsubseteq, \rho)$.

If ρ is a weight on a pointed poset $(D, \sqsubseteq)$ then we have the implication (i) $\Longrightarrow$ (ii).

Proof: Let $(D, \sqsubseteq)$ be a pointed poset and ρ a weight on $(D, \sqsubseteq)$. For simplicity $M = \mathcal{M}(D)$, $d = d[\rho]$, $\mu_n = \mu_n^\rho$, $\mu = \mu^\rho$.

(i) $\Longrightarrow$ (ii): If (x_n) is a tower in $(D, \sqsubseteq, \rho)$ then (x_n) is a monotone Cauchy sequence. Since M is complete $\lim x_n$ exists and and $\lim x_n = \bigsqcup x_n$ (Lemma 2.4). Since $d(x_n, x_m) \leq 1/2^n$ for all $m \geq 0$ we have: $d(x_n, x) \leq 1/2^n$, i.e.

$$\downarrow^n (x) = \downarrow^n (x_n) = \downarrow^n (x_m)$$

for all $m \geq n \geq 0$. Therefore $\mu_n(x) = \mu_n(x_n) = \mu_n(x_m)$ for all $m \geq n$. Hence

$$\mu_n(x) = \bigsqcup \mu_n(x_m).$$

(ii) $\Longrightarrow$ (i): Let $(D, \sqsubseteq)$ be a cpo, (x_m) a tower and $x = \bigsqcup x_m$. Then by (ii):

$$x[n] = \bigsqcup_{m\geq 0} x_m[n] = x_n[n].$$

Therefore $\downarrow^n (x) = \downarrow^n (x_n)$. By Lemma 2.7 we get the completeness of $\mathcal{M}(D)$. □

Definition 2.12 *Let $(D, \sqsubseteq)$ be a pointed poset. A* `continuous weight` *on $(D, \sqsubseteq)$ is a weight ρ on $(D, \sqsubseteq)$ such that the functions μ_n^ρ are continuous.*

Theorem 2.13 *Let ρ be a continuous weight on a cpo $(D, \sqsubseteq)$. Then $(\mathcal{M}(D), d[\rho])$ and $(\mathcal{M}_{\mathrm{fin}}(D), d[\rho])$ are complete metric spaces with*

$$\lim_{n \to \infty} x_n = \bigsqcup_{n \geq 0} x_n$$

for each Cauchy sequence in $\mathcal{M}(D)$ resp. $\mathcal{M}_{\mathrm{fin}}(D)$ which is monotone in D.

Proof: follows by Theorem 2.5 and Lemma 2.11. □

Definition 2.14 *Let ρ be a weight on a pointed poset $(D, \sqsubseteq)$. $(D, \sqsubseteq)$ is called ρ-*`complete` *iff for each tower (x_n) in $(D, \sqsubseteq, \rho)$ the least upper bound $\bigsqcup x_n$ exists.*

By definition, each tower is monotone. Hence each weighted cpo is ρ-complete.

Lemma 2.15 *Let ρ be a continuous weight on a pointed poset $(D, \sqsubseteq)$. Then the following are equivalent:*

(i) $(\mathcal{M}(D), d[\rho])$ is a complete metric space.

(ii) $(D, \sqsubseteq)$ is ρ-complete.

If ρ is a weight then we have the implication (i) $\Longrightarrow$ (ii).

Proof:

(i) $\Longrightarrow$ (ii): Let ρ be a weight and (x_n) a tower. Then (x_n) is a Cauchy sequence. Since $\mathcal{M}(D)$ is complete $x = \lim x_n$ exists. By Lemma 2.4: $x = \bigsqcup x_n$. Hence $(D, \sqsubseteq)$ is ρ-complete.

(ii) $\Longrightarrow$ (i): By Lemma 2.7 we have to show that for each tower (x_n) there exists $x \in \mathcal{M}(D)$ with $\downarrow^m (x) = \downarrow^m (x_m)$ for all $m \geq 0$.

Let (x_n) be a tower and let $x = \bigsqcup x_n$. Since ρ is a continuous weight we have: μ_m^ρ is continuous and therefore (since (x_n) is a tower):

$$x[m] = \bigsqcup_{n \geq 0} x_n[m] = x_m[m].$$

Hence $\downarrow^m (x) = \downarrow^m (x_m)$. □

Lemma 2.16 *Let ρ be a continuous weight on a cpo $(D, \sqsubseteq)$ and $M_{\mathrm{fin}} = \mathcal{M}_{\mathrm{fin}}(D, \sqsubseteq, \rho)$. Let (x_n) be a monotone sequence in M_{fin}.*

(a) If (x_n) is a tower and $x = \bigsqcup x_n$ exists in D then $x \in M_{\mathrm{fin}}$.

(b) If (x_n) has an upper bound y in M_{fin} and $x = \bigsqcup x_n$ then $x \in M_{\mathrm{fin}}$ and there exists a subsequence $(x_{n_k})_{k \geq 0}$ of (x_n) such that

$$x[k] = x_{n_k}[k] = x_n[k]$$

for all $n \geq n_k$, $k \geq 0$.

In particular: If x is the least upper bound of (x_n) in M_{fin} then x is the least upper bound of (x_n) in D.

Proof: ad (a): Let $x = \bigsqcup x_n$ where (x_n) is a tower in M_{fin}. Then $x \in \mathcal{M}(D)$ by Lemma 2.10 and $x_n[k] = x_k[k]$ for all $n \geq k \geq 0$. Since ρ is continuous:

$$x[k] \;=\; \bigsqcup_{n \geq 0} x_n[k] \;=\; x_k[k].$$

Since $x_k \in M_{\text{fin}}$ we get: $x_k[k]\downarrow = x[k]\downarrow = \downarrow^k(x)$ is finite. I.e. $x \in M_{\text{fin}}$.

Now we show (b): We assume that $x_n \sqsubseteq y$ where (x_n) is a monotone sequence in M_{fin} and $y \in M_{\text{fin}}$, $x = \bigsqcup x_n$. For all $k \geq 0$ we have:

$$x_0[k]\downarrow \subseteq x_1[k]\downarrow \subseteq x_2[k]\downarrow \subseteq \ldots \subseteq y[k]\downarrow$$

Since $y[k]\downarrow$ is finite there exists $N_k \geq 0$ with

$$x_{N_k}[k]\downarrow = x_n[k]\downarrow$$

for all $n \geq N_k$. Hence $x_{N_k}[k] = x_n[k]$ for all $n \geq N_k$. Let

$$n_0 = 0, \quad n_{k+1} \;=\; \max\,\{N_{k+1},\; n_k + 1\}.$$

Then $x_{n_k}[k] = x_n[k]$ for all $n \geq n_k$. In particular (x_{n_k}) is a tower in M_{fin} and $x = \bigsqcup x_{n_k}$. By (a) we get: $x \in M_{\text{fin}}$ and as we saw above:

$$x[k] \;=\; x_{n_k}[k] \;=\; x_n[k]$$

for all $n \geq n_k$, $k \geq 0$. □

Remark 2.17 Let ρ be a length on a pointed poset $(D, \sqsubseteq)$. Then ρ can be considered as a length on the pointed posets $M = \mathcal{M}(D)$ and on $M_{\text{fin}} = \mathcal{M}_{\text{fin}}(D)$. All elements of M are approximable. The finite approximable elements of M are the finitely approximable elements of D. In M_{fin} all elements are finitely approximbale.

If ρ is a weight on D then also on M and M_{fin}. In this case the n-cut of $x \in M$ or $x \in M_{\text{fin}}$ in M resp. M_{fin} is the n-cut of x in D.

If ρ is a continuous weight and D is a cpo then ρ is also a continuous weight on M and on M_{fin}. Here we use the fact that the least upper bounds in M resp. M_{fin} (if they exist) are the least upper bounds in D (Lemma 2.10 and Lemma 2.16). In this case M is ρ-complete if and only if the induced metric space is complete if and only if D is ρ-complete (by Lemma 2.15). Since M_{fin} is a closed subspace of M we get: If D is ρ-complete then M and then also M_{fin} are complete metric spaces. By Lemma 2.15 M_{fin} is also ρ-complete.

Lemma 2.18 *Let $(D, \sqsubseteq)$ be a cpo and ρ a continuous weight on D such that*

$$D[n] \;=\; \{\; x \in D \;:\; \rho(x) \leq n \;\}$$

is finite for all $n \geq 0$. Then $\mathcal{M}(D) = \mathcal{M}_{\text{fin}}(D)$ is a compact metric space.

Proof: It is clear that $\mathcal{M}(D) = \mathcal{M}_{\mathrm{fin}}(D)$. Now we show the compactness of $\mathcal{M}(D)$: Let (x_m) be a sequence in $\mathcal{M}(D)$. We define by induction on n a subsequence (x_{m_n}) and an infinite subset I_n of $I\!N_0$ such that $x_{m_n}[n] = x_m[n]$ for all $m \in I_n$.

In the case $n = 0$ we may define $m_0 = 0$ and $I_0 = I\!N_0$. Now we assume that $n \geq 0$ and that x_{m_n} and I_n are defined. For all $m \in I_n$ we have: $x_m[n+1]$ is an element of the finite set $D[n+1]$. Hence there exists an infinite subset I_{n+1} of I_n and $m_{n+1} \in I_n$, $m_{n+1} > m_n$ with:

$$x_m[n+1] = x_{m_{n+1}}[n+1] \quad \forall\, m \in I_{n+1}$$

Then $(x_{m_n}[n])_{n\geq 0}$ is a monotone sequence in D. Since D is a cpo $x = \bigsqcup x_{m_n}[n]$ exists. By Lemma 2.10: $x \in \mathcal{M}(D)$. Since ρ is a continuous weight we get:

$$x[n] = x_{m_n}[n] = x_{m_k}[n]$$

for all $k \geq n \geq 0$. We conclude $\lim x_{m_n} = x$. □

2.4 Pseudo rank ordered cpo's

Rank ordered sets were introduced in [9]. They are special kinds of ultrametric spaces. Here we extend the notion of rank orderings on cpo's. We show that the induced ultrametric space of a rank ordered cpo D is complete. In addition we show that the concept of continuous weights can be considered as a special case of rank orderings.

Definition 2.19 *Let M be a nonempty set. A* `pseudo rank ordering` *on M is a family $\tilde{\pi} = (\pi_n)_{n\geq 0}$ of functions $\pi_n : M \to M$ such that:*

(i) π_0 *is constant*

(ii) $\pi_n \circ \pi_m = \pi_m \circ \pi_n = \pi_n$ *for all* $0 \leq n \leq m$.

$\tilde{\pi}$ *is called a* `rank ordering` *on M iff in addition*

(iii) If $x, y \in M$ and $\pi_n(x) = \pi_n(y)$ for all $n \geq 0$ then $x = y$.

A `(pseudo) rank ordered set` *is a pair $(M, \tilde{\pi})$ consisting of a set M and a (pseudo) rank ordering $\tilde{\pi}$ on M.*

Lemma 2.20 *If $(M, \tilde{\pi})$ is a pseudo rank ordered set then*

$$d[\tilde{\pi}](x_1, x_2) = \inf \left\{ \frac{1}{2^n} : \pi_n(x_1) = \pi_n(x_2) \right\}$$

is a pseudo ultrametric on M. If $\tilde{\pi}$ is a rank ordering on M then $d[\tilde{\pi}]$ is an ultrametric on M.

Definition 2.21 *Let $(D, \sqsubseteq)$ be a pointed poset. A* `(pseudo) rank ordering` *on $(D, \sqsubseteq)$ is a pseudo rank ordering $\tilde{\pi} = (\pi_n)_{n\geq 0}$ on D such that:*

(i) $\pi_0 \;=\; \lambda x.\bot_D$

(ii) π_n *is continuous*

(iii) $\pi_n \;\sqsubseteq\; id_D$

A `(pseudo) rank ordered poset` *is a tripel* $(D, \sqsubseteq, \tilde{\pi})$ *consisting of a pointed poset* $(D, \sqsubseteq)$ *and a (pseudo) rank ordering* $\tilde{\pi}$ *on* $(D, \sqsubseteq)$. *A* `(pseudo) rank ordered cpo` *is a (pseudo) rank ordered poset* $(D, \sqsubseteq, \tilde{\pi})$ *where* $(D, \sqsubseteq)$ *is a cpo.*

Lemma 2.22 *Let* $(D, \sqsubseteq, \tilde{\pi})$ *be a pseudo rank ordered poset. Then:*

(a) $(\pi_n)_{n\geq 0}$ *is monotone.*

(b) If $(D, \sqsubseteq)$ *is a cpo and* $\tilde{\pi}$ *is a rank ordering on* $(D, \sqsubseteq)$ *then* $\bigsqcup \pi_n \;=\; id_D$.

(c) If $\bigsqcup \pi_n \;=\; id_D$ *then* $\tilde{\pi}$ *is a rank ordering on* $(D, \sqsubseteq)$.

Proof:

(a) Let $x \in D$. Since $\pi_n \;=\; \pi_{n+1} \circ \pi_n$ and $\pi_n(x) \sqsubseteq x$ we get by the monotony of π_{n+1}:

$$\pi_n(x) \;=\; \pi_{n+1}(\pi_n(x)) \;\sqsubseteq\; \pi_{n+1}(x).$$

(b) Let $(D, \sqsubseteq)$ be a cpo and $\tilde{\pi}$ a rank ordering on $(D, \sqsubseteq)$. We have to show that $x \;=\; \bigsqcup \pi_n(x)$. Let $y \;=\; \bigsqcup \pi_n(x)$. Since π_m is continuous we get for all $m \geq 0$:

$$\pi_m(y) \;=\; \bigsqcup_{n\geq 0} \pi_m(\pi_n(x)) \;=\; \bigsqcup_{n\geq m} \pi_m(\pi_n(x)) \;=\; \bigsqcup_{n\geq m} \pi_m(x) \;=\; \pi_m(x).$$

By condition (iii) of rank ordered sets: $x = y$.

(c) If $x, y \in D$, $\pi_n(x) = \pi_n(y)$ for all $n \geq 0$ then $\;x \;=\; \bigsqcup \pi_n(x) \;=\; \bigsqcup \pi_n(y) \;=\; y$. $\square$

Lemma 2.23 *Let* $(D, \sqsubseteq, \tilde{\pi})$ *be a rank ordered cpo. Then* $(D, \tilde{\pi})$ *is a complete rank ordered set, i.e.* $(D, d[\tilde{\pi}])$ *is a complete metric space. If* $(x_n)_{n\geq 0}$ *is monotone Cauchy sequence in* D *then*

$$\lim_{n\to\infty} x_n \;=\; \bigsqcup_{n\geq 0} x_n.$$

Proof: Let $d \;=\; d[\tilde{\pi}]$. First we show the completeness of D as a metric space: Let (x_m) be a Cauchy sequence in D. W.l.o.g. $\;d(x_m, x_{m+1}) \;\leq\; 1/2^m\;$ for all $m \geq 0$. Then $\pi_n(x_m) \;=\; \pi_n(x_n)$ and

$$\pi_m(x_m) \;=\; \pi_m(x_{m+1}) \;=\; \pi_m(\;\pi_{m+1}(x_{m+1})\;) \;\sqsubseteq\; \pi_{m+1}(x_{m+1})$$

for all $m \geq n \geq 0$. Since D is a cpo and $(\pi_m(x_m))_{m\geq 0}$ is monotone $x \;=\; \bigsqcup \pi_m(x_m)$ exists. Since π_n is continuous we get:

$$\pi_n(x) \;=\; \bigsqcup_{m\geq 0} \pi_n(\;\pi_m(x_m)\;) \;=\; \bigsqcup_{m\geq n} \pi_n(\;\pi_m(x_m)\;) \;=\; \pi_n(x_n).$$

We conclude: $d(x, x_n) \le 1/2^n$ for all $n \ge 0$, i.e. $\lim x_n = x$.

Now we assume that (x_m) is a monotone Cauchy sequence in D, $x = \lim x_m$ and $y = \bigsqcup x_m$. We have to show that $x = y$.

First we show that $x \sqsubseteq y$. There exists a sequence $(m_k)_{k \ge 0}$ of natural numbers $m_0 < m_1 < m_2 < \ldots$ with $d(x, x_m) \le 1/2^k$ for all $m \ge m_k$. Hence

$$\pi_k(x) = \pi_k(x_{m_k}) \sqsubseteq x_{m_k}$$

for all $k \ge 0$. Therefore

$$x = \bigsqcup_{k \ge 0} \pi_k(x) \sqsubseteq \bigsqcup_{k \ge 0} x_{m_k} = \bigsqcup_{m \ge 0} x_m = y.$$

Now we show that $y \sqsubseteq x$. Let $(m_n)_{n \ge 0}$ be a sequence in $I\!N_0$ as above. If $n \ge m$ then $m_n \ge n \ge m$ and hence $x_m \sqsubseteq x_{m_n}$. Then $\pi_n(x_m) \sqsubseteq \pi_n(x_{m_n}) = \pi_n(x)$ for all $n \ge m$. Therefore for all $m \ge 0$:

$$x_m = \bigsqcup_{n \ge m} \pi_n(x_m) \sqsubseteq \bigsqcup_{n \ge m} \pi_n(x) = x.$$

I.e. x is an upper bound of $(x_m)_{m \ge 0}$. Hence $y \sqsubseteq x$. □

Definition 2.24 *Let $\tilde{\pi}$ be a pseudo rank ordering on a pointed poset $(D, \sqsubseteq)$. An element $x \in D$ is called* `approximable` *(w.r.t. $\tilde{\pi}$) iff*

$$x = \bigsqcup_{n \ge 0} \pi_n(x).$$

$\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ denotes the set of approximable elements in D.

The following theorem shows that adding a continuous weight on a pointed poset is a special case of adding a pseudo rank ordering.

Theorem 2.25 *Let ρ be a continuous weight on a pointed poset $(D, \sqsubseteq)$. Then*

$$\tilde{\mu} = (\mu_n^\rho)_{n \ge 0}$$

is a pseudo rank ordering on $(D, \sqsubseteq)$. In addition we have:

$$\mathcal{M}(D) = \mathcal{M}(D, \sqsubseteq, \tilde{\mu})$$

and

$$d[\rho](x, y) = d[\tilde{\mu}](x, y)$$

for all x, $y \in \mathcal{M}(D)$.

Proof: easy verification. Uses the fact that if x, y are approximable and $x[n] = y[n]$ for all $n \ge 0$ then

$$x = \bigsqcup_{n \ge 0} x[n] = \bigsqcup_{n \ge 0} y[n] = y.$$

□

Remark 2.26 Let $\tilde{\pi}$ be a pseudo rank ordering on a pointed poset $(D, \sqsubseteq)$ with $\tilde{\pi} = (\mu_n^\rho)$ for some weight ρ on D then

$$\rho(x) = \inf \{ n : \pi_n(x) = x \}$$

for all $x \in D$ (where $\inf \emptyset = \infty$). On the other hand, if $\tilde{\pi}$ is a pseudo rank ordering on D then in general the function

$$\rho[\tilde{\pi}] : D \to \mathbb{N}_0 \cup \{\infty\}, \quad \rho[\tilde{\pi}](x) = \inf \{ n : \pi_n(x) = x \}$$

is not a length on D since we cannot guarantee the monotonicity of $\rho[\tilde{\pi}]$. If we require the monotonicity of $\rho[\tilde{\pi}]$ then $\rho[\tilde{\pi}]$ is a continuous weight on D with $\tilde{\pi} = (\mu_n^{\rho[\tilde{\pi}]})_{n \geq 0}$.

Next we show that for pseudo rank ordered cpo's we get a similar result as in Theorem 2.13:

Lemma 2.27 *Let $\tilde{\pi}$ be a pseudo rank ordering on a pointed poset $(D, \sqsubseteq)$. Then:*

(a) If (x_n) is a monotone sequence in $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ and $x = \bigsqcup x_n$ exists in D then $x \in \mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ and x is the least upper bound of (x_n) in $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$. If in addition (x_n) is a Cauchy sequence in $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ then $\lim x_n$ exists and $x = \lim x_n$.

(b) If $(D, \sqsubseteq)$ is a cpo then also $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ is a cpo and the inclusion $\mathcal{M}(D, \sqsubseteq, \tilde{\pi}) \to D$ is continuous.

(c) If (x_n) is a monotone Cauchy sequence in $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ such that $\lim x_n$ exists then $\bigsqcup x_n$ exists and $\bigsqcup x_n = \lim x_n$.

(d) $\tilde{\pi}$ is a rank ordering on $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$.

Here we assume that the partial order on $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ is the restriction of $\sqsubseteq$ on $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ and the functions π_n are considered as functions $\mathcal{M}(D, \sqsubseteq, \tilde{\pi}) \to \mathcal{M}(D, \sqsubseteq, \tilde{\pi})$.

Proof: Let $M = \mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ and $d = d[\tilde{\pi}]$.

(a) Let $x = \bigsqcup x_n$, $x_n \in M$. We have to show that $x \in M$.
Since $\pi_m \sqsubseteq id_D$ we have: $\pi_m(x) \sqsubseteq x$. I.e. x is an upper bound of $(\pi_m(x))_{m \geq 0}$. Since π_m is continuous

$$\pi_m(x) = \bigsqcup_{n \geq 0} \pi_m(x_n).$$

If $y \in D$, $\pi_m(x) \sqsubseteq y$ for all $m \geq 0$ then $\pi_m(x_n) \sqsubseteq y$ for all $m \geq 0$ and $n \geq 0$. Hence for all $n \geq 0$:

$$x_n = \bigsqcup_{m \geq 0} \pi_m(x_n) \sqsubseteq y.$$

Therefore $x = \bigsqcup x_n \sqsubseteq y$. We conclude: $x = \bigsqcup \pi_m(x) \in M$.

(b) follows by (a).

(c) Let (x_n) be a monotone Cauchy sequence in M and $x = \lim x_n$. W.l.o.g.

$$d(x_n, x_m) \le \frac{1}{2^m}$$

for all $n \ge m \ge 0$. Then $d(x_m, x) \le 1/2^m$ and therefore

$$\pi_m(x_n) = \pi_m(x_m) = \pi_m(x) \sqsubseteq x$$

for all $n \ge m \ge 0$. If $0 \le n > m$ then (since $x_n \sqsubseteq x_m$):

$$\pi_m(x_n) \sqsubseteq \pi_m(x_m) = \pi_m(x) \sqsubseteq x$$

Hence $\pi_m(x_n) \sqsubseteq x$ for all $m \ge 0$ and $n \ge 0$. Since x_n is approximable we get:

$$x_n = \bigsqcup_{m \ge 0} \pi_m(x_n) \sqsubseteq x$$

for all $n \ge 0$. I.e. x is an upper bound of the sequence (x_n).
If $y \in D$, $x_n \sqsubseteq y$ for all $n \ge 0$ then for all $m \ge 0$:

$$\pi_m(x) = \pi_m(x_m) \sqsubseteq \pi_m(y) \sqsubseteq y$$

Since x is approximable: $x = \bigsqcup \pi_m(x) \sqsubseteq y$. I.e. $x = \bigsqcup x_n$.

(d) It is clear that $\tilde{\pi}$ is a pseudo rank ordering on M. By Lemma 2.22(c): $\tilde{\pi}$ is a rank ordering on M. □

Theorem 2.28 *If $\tilde{\pi}$ be a pseudo rank ordering on a cpo $(D, \sqsubseteq)$ then $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ endowed with the distance $d[\tilde{\pi}]$ is a complete metric space and*

$$\lim_{n \to \infty} x_n = \bigsqcup_{n \ge 0} x_n$$

for each Cauchy sequence in $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ which is monotone in D.

Proof: follows immediately by Lemma 2.27 and 2.23. □

Lemma 2.29 *Let $\tilde{\pi} = (\pi_n)_{n \ge 0}$ be a pseudo rank ordering on a cpo $(D, \sqsubseteq)$ such that for all $n \ge 0$ the set $\pi_n(D)$ is finite. Then $\mathcal{M}(D, \sqsubseteq, \tilde{\pi})$ is a compact metric space.*

Proof: similar to Lemma 2.18. □

3 Metric and partial order semantics

Semantic operators on a given domain model the syntactic operators of a given language. In the case of the domain being a cpo continuity of the operators guarantees the existence of the semantic model. In this chapter we investigate the conditions which a semantic operator has to satisfy in the case of a weighted poset resp. a pseudo rank ordered cpo. Here we would like to characterize the properties that ensure that the semantic operator can be viewed as one in the poset framework as well as one in the induced metric framework. Moreover the two semantic models obtained by the two views should coincide. We show that continuous and contracting operators do achieve the desired behaviour.

In the following *lfp*(f) resp. *fix*(f) denotes the least resp. unique fixed point (if it exists) of a function $f : D \to D$ resp. $f : M \to M$ where D is a pointed poset and M a metric space.

Let Σ be a nonempty set of operator symbols. $|\omega|$ denotes the arity of $\omega \in \Sigma$. *Idf* is a nonempty set of variables. Then the language $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ is given by the production system

$$s \ ::= \ \omega(s_1, \dots, s_n) \mid \xi \mid \mathit{fix}(\xi = s)$$

where $\omega \in \Sigma$, $|\omega| = n$, $\xi \in Idf$. A semantics for $\mathcal{L}$ in some semantic domain A can be defined as a function $\Phi : \mathcal{L} \to (Env[A] \to A)$ where $Env[A]$ is the set of environments, i.e. the set of functions $\sigma : Idf \to A$. If $\sigma : Idf \to A$ is an environment and $\xi \in Idf$, $x \in A$ then the environment $\sigma[x/\xi] : Idf \to A$ is defined by

$$\sigma[x/\xi](\eta) \ = \ \begin{cases} \sigma(\eta) & : \ \text{if } \eta \neq \xi \\ x & : \ \text{if } \eta = \xi. \end{cases}$$

If $(D, \sqsubseteq)$ is a cpo together with a continuous operator $\omega_D : D^n \to D$ for each $\omega \in \Sigma$, $|\omega| = n$, then a denotational semantics for $\mathcal{L}$ on D can be defined by structural induction and Tarski's fixed point theorem. The meaning function $\Phi^D : \mathcal{L} \to (Env[D] \to D)$ is given by [3]:

- $\Phi^D(\xi)(\sigma) \ = \ \sigma(\xi)$
- $\Phi^D(\ \omega(s_1, \dots, s_n)\ )(\sigma) \ = \ \omega_D\left(\Phi^D(s_1)(\sigma), \dots, \Phi^D(s_n)(\sigma)\right)$
- $\Phi^D(\ \mathit{fix}(\xi = s)\ )(\sigma) \ = \ \mathit{lfp}(f_\sigma^D[s, \xi])$

where $f_\sigma^D[s, \xi] : D \to D$ is given by

$$f_\sigma^D[s, \xi](x) \ = \ \Phi^D(s)(\sigma[x/\xi]).$$

More generally: Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$ which is closed under the operator symbols $\omega \in \Sigma$ (i.e. $\omega(s_1, \dots, s_n) \in \mathcal{L}'$ if $s_i \in \mathcal{L}'$, $i = 1, \dots, n$). For each $s \in \mathcal{L}'$ let $I(s)$ be the set of identifiers $\xi \in Idf$ with $\mathit{fix}(\xi = s) \in \mathcal{L}'$. Then:

If $(D, \sqsubseteq)$ is a poset (which might be not complete) and $\omega_D : D^n \to D$, $\omega \in \Sigma$, $|\omega| = n$, are semantic operators on D such that for all $s \in \mathcal{L}'$ and $\xi \in I(s)$ the functions $f_\sigma^D[s, \xi]$ (defined as above) have a least fixed point in D then we also get a partial order semantics

$$\Phi^D : \mathcal{L}' \to (Env[D] \to D)$$

which is defined as above.

The metric approach works analogously [3]: here we consider non-distance-increasing operators instead of continuous operators, restrict recursion to guarded recursion which ensures the existence of fixed points and may substitute least fixed points by unique fixed points which is guaranteed by Banach's fixed point theorem.

For each $\omega \in \Sigma$ let $deg(\omega)$ the degree of guardedness of ω, i.e. $deg(\omega)$ is a natural number between 0 and $|\omega|$. If $deg(\omega) = k$, $|\omega| = n$, then we say that ω ensures guardedness in its last k arguments. We define guardedness of a variable ξ in a term $s \in \mathcal{L}$ by structural induction:

1. ξ is guarded in each constant symbol $\omega \in \Sigma$ (i.e. in each operator symbol of the arity 0).
2. If $\omega \in \Sigma$, $|\omega| = n \geq 1$, $deg(\omega) = k$, then ξ is guarded in $\omega(s_1, \ldots, s_n)$ iff whenever ξ occurs in a subterm s_i then either $n - k + 1 \leq i \leq n$ or ξ is guarded in s_i.
3. ξ is guarded in a term $\mathit{fix}(\eta = s)$ iff either ξ is guarded in s or $\xi = \eta$.

$\mathcal{L}^g$ denotes the set of guarded terms, i.e. $\mathcal{L}^g$ is the set of terms $s \in \mathcal{L}$ such that for each subterm $\mathit{fix}(\xi = t)$ of s the variable ξ is guarded in t.

Example 3.1 The prefixing operator of CCS [21] has the degree 1 of guardedness. All other CCS operators have degree of guardedness 0. This leads to the usual definition of guardedness of a variable ξ in a CCS term s: ξ is guarded in s if and only if each free occurrence of ξ in s is in the scope of a prefixing operator.

The sequential operator ; of CSP [13] has the degree of guardedness 1. Then a variable ξ is guarded in a term $s;t$ if and only if either s is closed (i.e. each occurence of a variable η is within a subterm $\mathit{fix}(\eta = t)$) or ξ is guarded in s.

Let (M, d) be a complete metric space together with non-distance-increasing operators $\omega_M : M^n \to M$ which are contracting in those arguments in which ω ensures guardedness. More precisely, for each $\omega \in \Sigma$, $|\omega| = n$, $deg(\omega) = k$, there exists a constant C with $0 \leq C < 1$ such that

$$d(\omega_M(\tilde{x}, \tilde{x}'), \omega_M(\tilde{y}, \tilde{y}')) \ \leq \max\ \{d(\tilde{x}, \tilde{y}),\ C \cdot d(\tilde{x}', \tilde{y}')\ \}$$

for all $\tilde{x}$, $\tilde{y} \in M^{k-l}$, $\tilde{x}'$, $\tilde{y}' \in M^l$. A denotational semantics for $\mathcal{L}^g$ on M can be defined by structural induction and Banach's fixed point theorem:

$$\Psi^M \ : \ \mathcal{L}^g \to (\mathit{Env}[M] \to M)$$

is given by:

- $\Psi^M(\xi)(\sigma) \ = \ \sigma(\xi)$
- $\Psi^M(\ \omega(s_1, \ldots, s_n)\ )(\sigma) \ = \ \omega_M\left(\Psi^M(s_1)(\sigma), \ldots, \Psi^M(s_n)(\sigma)\right)$
- $\Psi^M(\ \mathit{fix}(\xi = s)\ )(\sigma) \ = \ \mathit{fix}(f_\sigma^M[s, \xi])$

where $f_\sigma^M[s,\xi] : M \to M$ is given by

$$f_\sigma^M[s,\xi](x) \;=\; \Psi^M(s)(\sigma[x/\xi]).$$

It is easy to see that for all $s \in \mathcal{L}^g$, $\xi \in \mathit{Idf}$ and $\sigma \in \mathit{Env}[M]$ the function $f_\sigma^M[s,\xi]$ is non-distance-increasing. If ξ is guarded in s then $f_\sigma^M[s,\xi]$ is contracting.

Now we assume that $(D,\sqsubseteq)$ is a cpo, $\Phi^D : \mathcal{L} \to (\mathit{Env}[D] \to D)$ a denotational semantics and d a metric on a subset M of D (induced by a length or a pseudo rank ordering) such that M is closed w.r.t. the semantic operators ω_D, $\omega \in \Sigma$, and the restriction of ω_D on M, i.e. the function

$$\omega_M : M^n \to M, \;\; \omega_M(x_1,\ldots,x_n) \;=\; \omega_D(x_1,\ldots,x_n) \quad \text{where } n = |\omega|,$$

is non-distance-increasing and contracting in its last k arguments where $k = \mathit{deg}(\omega)$. If M is complete then the metric denotational semantics $\Psi^M : \mathcal{L}^g \to (\mathit{Env}[M] \to M)$ can be defined as described above. If M is incomplete we get a metric semantics $\Psi^{\overline{M}}$ on the metric completion of M where we use the canonical extensions of the semantic operators ω_M. The question arises in which way the cpo semantics Φ^D and the metric semantics $\Psi = \Psi^M$ resp. $\Psi = \Psi^{\overline{M}}$ are related. Our aim is to find conditions which ensure that

$$\Phi^D(s)(\sigma) \;=\; \Psi(s)(\sigma)$$

for all guarded statements s and environments $\sigma : \mathit{Idf} \to M$. We observe that this consistency result is equivalent to the following: For each term $\mathit{fix}(\xi = s)$ in $\mathcal{L}^g$ the function $f_\sigma^M[s,\xi]$ has a unique fixed point in M and

$$\mathit{lfp}(f_\sigma^D[s,\xi]) \;=\; \mathit{fix}(f_\sigma^M[s,\xi]).$$

Lemma 3.2 *Let $(D,\sqsubseteq)$ be a cpo, (M,d) a complete metric space and $\varphi : M \to D$ a function such that $\bot_D \in \varphi(M)$ and $\varphi(\lim x_n) \;=\; \bigsqcup \varphi(x_n)$ for each Cauchy sequence (x_n) in M with $\varphi(x_0) \sqsubseteq \varphi(x_1) \sqsubseteq \ldots$. Then:*

If $f : D \to D$ is a continuous function, $F : M \to M$ a contracting function with $\varphi \circ F \;=\; f \circ \varphi$ then

$$\mathit{lfp}(f) \;=\; \varphi(\mathit{fix}(F)).$$

Proof: Let $x_0 \in \varphi^{-1}(\bot)$, $x_{n+1} = F(x_n)$ and $y_0 = \bot_D$, $y_{n+1} = f(y_n)$. Then (by induction on n):

$$y_n \;=\; \varphi(x_n)$$

Hence

$$\varphi(\mathit{fix}(F)) \;=\; \varphi(\lim x_n) \;=\; \bigsqcup_{n\ge 0} \varphi(f(x_n)) \;=\; \bigsqcup_{n \ge 0} y_n \;=\; \mathit{lfp}(f).$$

□

Theorem 3.3 *Let $(D,\sqsubseteq)$ be a cpo, (M,d) a complete metric space and $\varphi : M \to D$ a function such that $\bot_D \in \varphi(M)$ and $\varphi(\lim x_n) \;=\; \bigsqcup \varphi(x_n)$ for each Cauchy sequence (x_n) in M with $\varphi(x_0) \sqsubseteq \varphi(x_1) \sqsubseteq \ldots$. Let $\mathcal{L} = \mathcal{L}(\Sigma, \mathit{Idf})$ be a language as above and for each $\omega \in \Sigma$, $|\omega| = n$, $\mathit{deg}(\omega) = k$, let $\omega_D : D^n \to D$ be a continuous operator on D*

and $\omega_M : M^n \to M$ *a non-distance-increasing operator on* M *which is contracting in its last* k *arguments such that*

$$\varphi\left(\omega_M(x_1, \dots, x_n)\right) = \omega_D\left(\varphi(x_1), \dots, \varphi(x_n)\right)$$

for all $x_1, \dots, x_n \in M$. *Then*

$$\Phi^D(s)(\varphi \circ \sigma) = \varphi\left(\Psi^M(s)(\sigma)\right)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

Proof: follows by structural induction on $s \in \mathcal{L}^g$ and Lemma 3.2. □

As a special case of Theorem 3.3 we get with $M \subseteq D$ and the inclusion $\varphi : M \to D$ the following consistency result for denotational semantics:

Theorem 3.4 *Let* D *be a cpo and* M *a complete metric space such that* $\perp_D \in M \subseteq D$ *and*

$$\lim_{n\to\infty} x_n = \bigsqcup_{n\geq 0} x_n$$

for each Cauchy sequence (x_n) *in* M *which is monotone in* D.

Let $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ *be a language as above and let* ω_D, $\omega \in \Sigma$, *be continuous semantic operators on* D *such that for all* $\omega \in \Sigma$: $\omega_D(M^n) \subseteq M$ *and*

$$\omega_M : M^n \to M, \quad \omega_M(x_1, \dots, x_n) = \omega_D(x_1, \dots, x_n)$$

is non-distance-increasing and contracting in its last k *arguments where* $k = \deg(\omega)$ *and* $n = |\omega|$. *Then*

$$\Phi^D(s)(\sigma) = \Psi^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

Now we omit the assumption that M is complete and we consider the metric semantics $\Psi^{\overline{M}}$ on the metric completion of M.

Theorem 3.5 *Let* $(D, \sqsubseteq)$ *be a cpo and* (M, d) *a metric space such that* $\perp_D \in M \subseteq D$. *Let* $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ *be a language as above and let* ω_D, $\omega \in \Sigma$, *be continuous semantic operators on* D *such that for all* $\omega \in \Sigma$ *and* $\sigma : Idf \to M$:

(i) $\omega_D(M^n) \subseteq M$ *and*

$$\omega_M : M^n \to M, \quad \omega_M(x_1, \dots, x_n) = \omega_D(x_1, \dots, x_n)$$

is non-distance-increasing and contracting in its last k *arguments where* $k = \deg(\omega)$ *and* $n = |\omega|$.

(ii) *For each term* $\mathit{fix}(\xi = s)$ *in* $\mathcal{L}^g$ *the function* $f^M_\sigma[s, \xi] : M \to M$ *has a unique fixed point in* M *and*

$$\mathit{lfp}(f^D_\sigma[s, \xi]) = \mathit{fix}(f^M_\sigma[s, \xi]).$$

Then the cpo semantics Φ^D *on* D *and the metric semantics* $\Psi^{\overline{M}}$ *on the metric completion of* M *(which is defined by using the canonical extensions of the semantic operators* ω_M*) coincide:*

$$\Phi^D(s)(\sigma) \;=\; \Psi^{\overline{M}}(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

Proof: by structural induction on $s \in \mathcal{L}^g$. Uses the fact that $f_\sigma^M[s,\xi]$ is the restriction of $f_\sigma^{\overline{M}}[s,\xi]$ on M and that

$$lfp(f_\sigma^D[s,\xi]) \;=\; fix(f_\sigma^M[s,\xi]) \;=\; fix(f_\sigma^{\overline{M}}[s,\xi]).$$

□

3.1 Metric and partial order semantics on weighted posets

In Theorem 3.6 we present conditions which ensure that for weighted posets the partial order on D and the metric semantics on $\mathcal{M}(D)$ coincide. Remark 3.12 shows that this result carries over to metric semantics on $\mathcal{M}_{\text{fin}}(D)$.

Theorem 3.6 *Let* $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ *be a language as before,* ρ *a weight on a pointed poset* $(D, \sqsubseteq)$ *and* $M = \mathcal{M}(D)$. *For each* $\omega \in \Sigma$, $|\omega| = n$, $deg(\omega) = k$, *let* $\omega_D : D^n \to D$ *be an operator such that* $\omega_D(M^n) \subseteq M$ *and*

$$\omega_M : M^n \to M, \;\; \omega_M(x_1, \ldots, x_n) \;=\; \omega_D(x_1, \ldots, x_n)$$

is non-distance-increasing and contracting in its last k *arguments. Then:*

(a) If D *and* M *are complete and* ω_D *is continuous for all* $\omega \in \Sigma$ *then the cpo semantics on* D *and the metric semantics on* M *are the same. More precisely:*

$$\Phi^D(s)(\sigma) \;=\; \Psi^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

(b) If D *is a cpo and* ω_D *is continuous for all* $\omega \in \Sigma$ *then the cpo semantics* Φ^D *on* D *and the metric semantics on the completion* $\overline{M}$ *of* M *are the same. More precisely:*

$$\Phi^D(s)(\sigma) \;=\; \Psi^{\overline{M}}(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

(c) If M *is complete and the operators* ω_D *are monotone then the partial order semantics* Φ^D *can be defined for the sublanguage* $\mathcal{L}^g$. *The metric semantics* Ψ^M *and the partial order semantics* Φ^D *are the same. More precisely:*

$$\Phi^D(s)(\sigma) \;=\; \Psi^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ *and* $\sigma : Idf \to M$.

Theorem 3.6 follows by Theorem 3.4, Theorem 3.5, Lemma 2.4 and the results which are presented in Lemma 3.10 and Lemma 3.11.

Definition 3.7 *Let ρ_D resp. ρ_C be weights on pointed posets $(D, \sqsubseteq_D)$ resp. $(C, \sqsubseteq_C)$ and let $M_D = \mathcal{M}(D)$, $M_C = \mathcal{M}(C)$. Let $k \geq 1$ and $k \geq l \geq 0$. A function $f : D^k \to C$ is called* `cut-preserving of the degree` *l iff*

$$f(M_D^k) \subseteq M_C$$

$$f(x_1, \ldots, x_k)[n] = f(x_1[n], \ldots, x_{k-l}[n], x_{k-l+1}[n-1], \ldots, x_k[n-1])[n]$$

for all $n \geq 1$ and $x_1, \ldots, x_k \in M_D$. If $l = k$ then f is called `strong cut-preserving`. *We say f is* `cut-preserving` *iff f is cut-preserving of degree 0.*

Notation 3.8 *Let ρ be a weight on a pointed poset D and $k \geq 1$, $\tilde{z} = (z_1, \ldots, z_n) \in D^k$ and $\tilde{x} = (x_1, \ldots, x_k)$, $\tilde{y} = (y_1, \ldots, y_k) \in M^k$ where $M = \mathcal{M}(D)$. Then we put:*

$$\tilde{z}[n] = (z_1[n], \ldots, z_k[n])$$

$$d[\rho](\tilde{x}, \tilde{y}) = \max \{ d[\rho](x_i, y_i) : 1 \leq i \leq k \}$$

Lemma 3.9 *Let ρ_D resp. ρ_C be weights on pointed posets $(D, \sqsubseteq_D)$ resp. $(C, \sqsubseteq_C)$. Let $k \geq 1$, $k \geq l \geq 0$ and $f : D^k \to C$ a function with $f(M_D^k) \subseteq M_C$ where $M_D = \mathcal{M}(D)$, $M_C = \mathcal{M}(C)$. Then:*

f is strong cut-preserving of the degree l if and only if $f|M_D^k \to M_C$ is non-distance-increasing and contracting in the last l arguments. In particular:

(a) f is cut-preserving if and only if $f|M_D^k \to M_C$ is non-distance-increasing.

(b) f is strong cut-preserving if and only if $f|M_D^k \to M_C$ is contracting with contracting constant $1/2$.

Proof: Let $d_D = d[\rho_D]$, $d_C = d[\rho_C]$. If f is cut-preserving of the degree l then we have to show that for each natural number $n \geq 1$:

$$d_C\left(f(\tilde{x}, \tilde{x}'),\ f(\tilde{y}, \tilde{y}')\right) \leq \frac{1}{2^n} \iff d_D(\tilde{x}, \tilde{y}) \leq \frac{1}{2^n} \wedge d_D(\tilde{x}', \tilde{y}') \leq \frac{1}{2^{n-1}}$$

Since f is cut-preserving of degree l we have

$$f(\tilde{z}, \tilde{z}')[n] = f(\tilde{z}[n], \tilde{z}'[n-1])[n]$$

where $\tilde{z} \in M_D^{k-l}$, $\tilde{z}' \in M_D^l$. We get:

If $d_D(x_i, y_i) \leq 1/2^n$, $i = 1, \ldots, k-l$, and $d_D(x_i, y_i) \leq 1/2^{n-1}$, $i = k-l+1, \ldots, k$, then

$$\tilde{x}[n] = \tilde{y}[n], \quad \tilde{x}'[n-1] = \tilde{y}'[n-1].$$

Hence

$$f(\tilde{x}, \tilde{x}')[n] = f(\tilde{x}[n], \tilde{x}'[n-1])[n] = f(\tilde{y}[n], \tilde{y}'[n-1])[n] = f(\tilde{y}, \tilde{y}')[n]$$

and therefore $d_C\left(f(\tilde{x},\tilde{x}'),\ f(\tilde{y},\tilde{y'})\right) \le 1/2^n$.

If f is non-distance-increasing and contracting in the last l arguments then for all $\tilde{x} \in D'^{k-l}$, $\tilde{x}' \in M_D^l$:

$$d_D(\tilde{x},\tilde{x}[n]) \le \frac{1}{2^n},\quad d_D(\tilde{x}',\tilde{x}'[n-1]) \le \frac{1}{2^n}$$

Here we use the fact that contracting w.r.t. d_D and d_C implies contracting with contracting constant $1/2$. Then:

$$d_C\left(f(\tilde{x},\tilde{x}'),\ f(\tilde{x}[n],\tilde{x}'[n-1])\right) \le \frac{1}{2^n}$$

and therefore $f(\tilde{x},\tilde{x'})[n] = f(\tilde{x}[n],\tilde{x}'[n-1])[n]$. We get that f is cut-preserving of degree l. □

In the following two lemmas we present conditions which ensure the existence of a unique resp. least fixed point of a contracting resp. monotone operator $\mathcal{M}(D) \to \mathcal{M}(D)$ resp. $D \to D$ in absence of the assumption that D resp. $\mathcal{M}(D)$ is complete.

Lemma 3.10 *Let ρ be a weight on a cpo D, $M = \mathcal{M}(D)$ and $f : D \to D$ a continuous and strong cut-preserving function. Then:*

(a) $lfp(f) \in M$

(b) If $x_0 \in M$, $x_{n+1} = f(x_n)$ then the sequence $(x_n)_{n\ge 0}$ is a Cauchy sequence in the complete metric space M and

$$x = \lim_{n\to\infty} x_n$$

is the unique fixed point of the contracting function $f|M \to M$. (Note that we do not require the completeness of M.)

Proof: Let $x = lfp(f)$. Then (by Tarski's fixed point theorem):

$$x = \bigsqcup_{n\ge 0} x_n$$

where $x_0 = \perp_D$ and $x_{n+1} = f(x_n)$. Since $f(M) \subseteq M$ and $\perp_D \in M$ we get by induction on n that $x_n \in M$. By Lemma 2.10: $x \in M$. I.e. x is a fixed point of $f|M \to M$. By Lemma 3.9: $f|M \to M$ is contracting. It can be shown by induction on n that

$$d[\rho](x,x_n) \le \frac{1}{2^n}.$$

Hence $x = \lim x_n$. If $x' \in M$ is also a fixed point of f then

$$d[\rho](x,x') = d[\rho](f(x),f(x')) \le \frac{1}{2}\cdot d[\rho](x,x').$$

Hence $d[\rho](x,x') = 0$, i.e. $x = x'$. □

Lemma 3.11 *Let ρ be a weight on a pointed poset $(D,\sqsubseteq)$ such that $M = \mathcal{M}(D)$ is a complete metric space. Let $f : D \to D$ be a monotone and strong cut-preserving function.*

(a) *Then lfp(f) exists and*

$$\mathit{lfp}(f) \;=\; \bigsqcup_{n\geq 0} x_n$$

where $x_0 \;=\; \bot_D$ *and* $x_{n+1} \;=\; f(x_n)$.

(b) *lfp(f) is the unique fixed point of f in* $\mathcal{M}(D)$.

Proof: Since M is complete and $f|M \to M$ contracting (Lemma 3.9) f has a unique fixed point x in M and

$$x \;=\; \lim_{n\to\infty} x_n$$

where $x_0 \in M$ and $x_{n+1} \;=\; f(x_n)$. Now we assume that $x_0 = \bot_D$. Since f is monotone we get:

$$\bot \;=\; x_0 \;\sqsubseteq\; x_1 \;\sqsubseteq\; x_2 \;\sqsubseteq\; \ldots$$

By Lemma 2.4 we get: $x \;=\; \bigsqcup x_n$. If $y \in D$ is also a fixed point of f then we can show by induction on n that $x_n \;\sqsubseteq\; y$: The basis of induction $n = 0$ is clear since $x_0 = \bot$. In the induction step $n \Longrightarrow n+1$ we use the monotony of f:

$$x_{n+1} \;=\; f(x_n) \;\sqsubseteq\; f(y) \;=\; y.$$

Hence $x \;=\; \bigsqcup x_n \;\sqsubseteq\; y$. We conclude: $x \;=\; \mathit{lfp}(f)$. □

Remark 3.12 Let $\mathcal{L}$, ρ, D and M be as in Theorem 3.6 and let $M_{\text{fin}} \;=\; \mathcal{M}_{\text{fin}}(D)$. Then: If M is a complete metric space then also M_{fin} (as a closed subspace of M) is a complete metric space. If in addition the semantic operators ω_D preserve finitely approximability (i.e. $\omega_D({M_{\text{fin}}}^n) \subseteq M_{\text{fin}}$) then we get a metric semantics

$$\Psi^M_{\text{fin}} \;:\; \mathcal{L}^g \;\to\; (\mathit{Env}[M_{\text{fin}}] \to M_{\text{fin}})$$

using the semantic operators $\omega_D|{M_{\text{fin}}}^n \to M_{\text{fin}}$. In this case we have:

$$\Phi^D(s)(\sigma) \;=\; \Psi^M(s)(\sigma) \;=\; \Psi^M_{\text{fin}}(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ and $\sigma : \mathit{Idf} \to M_{\text{fin}}$. Here Φ^D is as in Theorem 3.6(a) or (c).

3.2 A consistency result for partial order semantics on weighted posets

In Theorem 3.14 we present a condition which guarantees the consistency of two partial order semantics on weighted posets. The following lemma relates the least fixed points of monotone and strong cut-preserving functions on weighted posets (which exist by Lemma 3.11).

Lemma 3.13 *Let* ρ *resp.* ρ' *be weights on pointed posets* D *resp.* D', $M = \mathcal{M}(D)$, $M' \;=\; \mathcal{M}(D')$, *and let* $\varphi : D \to D'$ *a continuous function with*

$$\varphi(x[n]) \;=\; \varphi(x)[n]$$

for all $x \in D$, $n \geq 0$. *Then:*

(a) $\varphi(M) \subseteq M'$

(b) *If M and M' are complete metric spaces and $f : D \to D$, $f' : D' \to D'$ are monotone and strong cut-preserving with $\varphi \circ f = f' \circ \varphi$ then*

$$lfp(f') = \varphi(\ lfp(f)\ ).$$

(c) *If D and D' are cpo's and $f : D \to D$, $f' : D' \to D'$ are continuous with $\varphi \circ f = f' \circ \varphi$ then*

$$lfp(f') = \varphi(\ lfp(f)\ ).$$

Proof: If $x \in M$ then $x = \bigsqcup x[n]$. Since φ is continuous we get

$$\varphi(x) = \bigsqcup_{n \geq 0} \varphi(x[n]) = \bigsqcup_{n \geq 0} \varphi(x)[n] \in M'.$$

Now we assume that M and M' are complete and f, f' monotone and strong cut-preserving resp. that D and D' are cpo's and f, f' continuous with $\varphi \circ f = f' \circ \varphi$. Then $lfp(f)$ and $lfp(f')$ exist by Lemma 3.11 resp. Tarski's fixed point theorem and

$$lfp(f) = \bigsqcup_{n \geq 0} x_n, \quad lfp(f') = \bigsqcup_{n \geq 0} x'_n$$

where $x_0 = \perp_D$, $x'_0 = \perp_{D'}$ and $x_{n+1} = f(x_n)$, $x'_{n+1} = f'(x'_n)$. Since $\varphi \circ f = f' \circ \varphi$ it can be shown by induction on n that $x'_n = \varphi(x_n)$. Hence

$$lfp(f') = \bigsqcup_{n \geq 0} x'_n = \bigsqcup_{n \geq 0} \varphi(x_n) = \varphi\left(\bigsqcup_{n \geq 0} x_n\right) = \varphi(lfp(f)).$$

□

Theorem 3.14 *Let ρ resp. ρ' be weights on pointed posets $(D, \sqsubseteq)$ resp. $(D', \sqsubseteq')$, such that $M = \mathcal{M}(D)$ and $M' = \mathcal{M}(D')$ are complete metric spaces. Let $\varphi : D \to D'$ a surjective and continuous function with*

$$\varphi(x[n]) = \varphi(x)[n]$$

for all $x \in D$, $n \geq 0$. Then:

(a) *If $\omega : D^n \to D$ is a monotone operator on D such that for all $x_1, \dots x_n$, $y_1, \dots y_n \in D$:*

$$\varphi(x_i) = \varphi(y_i),\ i = 1, \dots, n \quad \Longrightarrow \quad \varphi(\ \omega(x_1, \dots, x_n)\ ) = \varphi(\ \omega(y_1, \dots, y_n)\ )$$

then $\varphi[\omega] : D'^n \to D'$, $\varphi[\omega](\ \varphi(x_1), \dots, \varphi(x_n)\ ) = \varphi(\ \omega(x_1, \dots, x_n)\ )$, is welldefined and a monotone operator on D'. If ω is cut-preserving of degree l then also $\varphi[\omega]$ is cut-preserving of degree l.

(b) *Let $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ be a language as before and for each operator symbol ω of $\mathcal{L}$, $|\omega| = n$, $deg(\omega) = l$, let $\omega_D : D^n \to D$ be a monotone operator on D which is cut-preserving of degree l and which satisfies the condition of (a). Then the partial order semantics Φ^D and $\Phi^{D'}$ can be defined for the language $\mathcal{L}^g$ (w.r.t. the semantic operator ω_D on D and the operators $\varphi[\omega_D]$ on D') and Φ^D resp. $\Phi^{D'}$ are consistent w.r.t. φ. I.e.*

$$\varphi\left(\Phi^D(s)(\sigma) \right) = \Phi^{D'}(s)(\varphi \circ \sigma)$$

for all $s \in \mathcal{L}^g$, $\sigma : Idf \to D$.

(c) *Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be cpo's and let $\mathcal{L}$ be a language as above. For each operator symbol ω of $\mathcal{L}$, $|\omega| = n$, let $\omega_D : D^n \to D$ be a continuous operator on D which satisfies the condition of (a) and such that the operator $\varphi[\omega_D]$ is continuous on D'. Then the cpo semantics Φ^D and $\Phi^{D'}$ (w.r.t. the semantic operator ω_D on D and $\varphi[\omega_D]$ on D') are consistent w.r.t. φ, i.e.*

$$\varphi(\Phi^D(s)(\sigma)) = \Phi^{D'}(s)(\varphi \circ \sigma)$$

for all $s \in \mathcal{L}$, $\sigma : Idf \to D$.

Proof: (a) is an easy verification. (b) and (c) follow by (a) and Lemma 3.13(b) where we use the following facts:

- The functions $f^D_\sigma[s, \xi] : D \to D$ resp. $f^{D'}_{\sigma'}[s, \xi] : D' \to D'$ are monotone and cut-preserving resp. continuous for all $s \in \mathcal{L}^g$, $\sigma : Idf \to D$, $\sigma' : Idf \to D'$ and all identifiers ξ. If ξ is guarded in s then $f^D_\sigma[s, \xi]$ and $f^{D'}_{\sigma'}[s, \xi]$ is strong cut-preserving.
- $\varphi \circ f^D_\sigma[s, \xi] = f^{D'}_{\varphi\circ\sigma}[s, \xi] \circ \varphi$ for all $s \in \mathcal{L}^g$ resp. $s \in \mathcal{L}$ and $\sigma : Idf \to D$.

Then we use Lemma 3.13(b) resp. (c). □

Remark 3.15 Let ρ be a continuous weight on a cpo D, $M = \mathcal{M}(D)$, $M_{\text{fin}} = \mathcal{M}_{\text{fin}}(D)$. Let $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ be a language as before. For each operator symbol $\omega \in \Sigma$, $|\omega| = n$, $deg(\omega) = k$, let $\omega_D : D^n \to D$ be a continuous operator which is cut-preserving of degree k. By Remark 2.17 ρ is a continuous weight on the cpo M and on the pointed poset M_{fin}. Since D is a cpo D and then also M_{fin} are ρ-complete (Remark 2.17).

1. Let Φ^D denote the cpo semantics on D for the language $\mathcal{L}$ (using the semantic operators ω_D).
2. Let Φ^M resp. Ψ^M denote the cpo semantics resp. metric semantics on M for $\mathcal{L}$ resp. $\mathcal{L}^g$. In both cases we use the semantic operators $\omega_D|M^n \to M$ where $\omega \in \Sigma$, $|\omega| = n$.

Applying Theorem 3.6(b) to the cpo's D and M we get

$$\Phi^D(s)(\sigma) = \Psi^M(s)(\sigma) = \Phi^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ and $\sigma : Idf \to \mathcal{M}$. Applying Theorem 3.14(c) to the cpo's D and M and the function

$$\varphi = \mu^\rho : D \to M$$

yields

$$\mu^\rho\left(\Phi^D(s)(\sigma)\right) = \Phi^M(s)(\mu^\rho \circ \sigma)$$

for all $s \in \mathcal{L}$ and $\sigma : Idf \to D$. Here we use the fact that $\mu^\rho[\omega_D]$ is the restriction of ω_D on approximable elements. We conclude that the cpo semantics on D and M are the same, i.e. if $\sigma : Idf \to M$ then

$$\Phi^D(s)(\sigma) = \Phi^M(s)(\sigma)$$

for all $s \in \mathcal{L}$. Here we use the fact that $\mu^\rho \circ \sigma = \sigma$ and Lemma 2.10:

$$lfp(f) = \bigsqcup_{n \geq 0} f^n(\bot) = \bigsqcup_{n \geq 0} f'^n(\bot) = lfp(f')$$

where $f = f_\sigma^D[s,\xi]$ and $f' = f_\sigma^M[s,\xi]$.

If each of the operators ω_D preserves finitely approximability (i.e. $\omega_D({M_{\mathsf{fin}}}^n) \subseteq M_{\mathsf{fin}}$ where $|\omega| = n$) then

$$\omega_D | {M_{\mathsf{fin}}}^n \to M_{\mathsf{fin}}$$

is monotone and cut-preserving of degree $deg(\omega)$ and non-distance-increasing and contracting in its last $deg(\omega)$ arguments. Hence the partial order semantics Φ_{fin}^M and the metric semantics Ψ_{fin}^M on M_{fin} can be defined for the sublanguage $\mathcal{L}^g$. By Theorem 3.6(c):

$$\Phi_{\mathsf{fin}}^M(s)(\sigma) = \Psi_{\mathsf{fin}}^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ and $\sigma : Idf \to M_{\mathsf{fin}}$. By Remark 3.12:

$$\Phi_{\mathsf{fin}}^M(s)(\sigma) = \Phi^M(s)(\sigma) = \Phi^D(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ and $\sigma : Idf \to M_{\mathsf{fin}}$.

3.3 Metric and partial order semantics on pseudo rank ordered cpo's

Similary to the result of Theorem 3.6(a) we get for pseudo rank ordered cpo's the consistency of the cpo semantics and the metric semantics:

Lemma 3.16 *Let $(M, \tilde{\pi})$ and $(N, \tilde{\mu})$ be pseudo rank ordered sets and $f : M^k \to N$ a function where $k \geq 1$. Then: f is non-distance-increasing and contracting in its last l arguments (w.r.t. $d[\tilde{\pi}]$ resp. $d[\tilde{\mu}]$) if and only if*

$$\mu_n \circ f \circ (\pi_n^{k-l}, \pi_{n-1}^l) = \mu_n \circ f$$

for all $n \geq 0$.

Here $\pi_m^l : D^l \to D^l$ is given by $\pi_m^l(x_1, \ldots, x_l) = (\pi_m(x_1), \ldots, \pi_m(x_l))$.

Proof: similar to Lemma 3.9. □

Theorem 3.17 *Let $\mathcal{L} = \mathcal{L}(\Sigma, Idf)$ be a language as before and $(D, \sqsubseteq, \tilde{\pi})$ a pseudo rank ordered cpo and $M = \mathcal{M}(D, \sqsubseteq, \tilde{\pi})$. For each operator symbol $\omega \in \Sigma$ let $\omega_D : D^n \to D$ be a continuous semantic operator on D with*

$$\omega_D(M^n) \subseteq M$$

$$\pi_m \circ \omega_D \circ (\pi_m^{n-k}, \pi_{m-1}^k) = \pi_m \circ \omega_D$$

for all $m \geq 1$ where $|\omega| = n$, $k = deg(\omega)$. Then the denotational cpo semantics on $(D, \sqsubseteq)$ and the denotational metric semantics on $(M, d[\tilde{\pi}])$ are the same:

$$\Phi^D(s)(\sigma) = \Psi^M(s)(\sigma)$$

for all $s \in \mathcal{L}^g$ and $\sigma : Idf \to M$.

Proof: follows by Theorem 3.4, Lemma 3.16 and Theorem 2.28. □

4 Examples: Strings, traces, trees, event structures and pomsets

In this section we show how the semantic domains of strings (i.e. sequences of actions) and Mazurkiewicz traces [20] (i.e. equivalence classes of strings w.r.t. the equivalence relation induced by an independency) together with the prefixing order, the semantic domains of labelled trees, prime event structures and pomsets together with Winskels partial orders [30, 31] fit in our framework. In the following *Act* is a nonempty set of actions.

4.1 Strings

Let Act^∞ denote the set of (finite or infinite) sequences over *Act*. $\sqsubseteq$ denotes the prefixing order on Act^∞, i.e. $\omega \sqsubseteq \omega'$ iff ω is a prefix of ω'. Then $(Act^\infty, \sqsubseteq)$ is a cpo with bottom element $\emptyset$ (the empty sequence). The function

$$|\cdot| : Act^\infty \to \mathbb{N}_0 \cup \{\infty\}$$

(where $|x|$ denotes the usual length of the string x) is a continuous weight on $(Act^\infty, \sqsubseteq)$. The n-cut $x[n]$ of $x \in Act^\infty$ is given by

$$x[n] = \begin{cases} x & : \text{ if } |x| \leq n \\ \alpha_1\alpha_2\ldots\alpha_n & : \text{ if } \alpha_1\ldots\alpha_n \text{ is a prefix of } x \text{ (where } \alpha_i \in Act) \end{cases}$$

$\downarrow^n (x)$ is the set of prefixes of x of the length $\leq n$. It is easy to see that $x = \bigsqcup x[n]$ for all $x \in Act^\infty$. $\downarrow^n (x)$ is a finite set and all sequences $x \in Act^\infty$ are finitely approximable.

$$\mathcal{M}(Act^\infty) = \mathcal{M}_{\text{fin}}(Act^\infty) = Act^\infty$$

The induced ultrametric d on Act^∞ is the usual metric on sequences:

$$d(x,y) = \inf\left\{\frac{1}{2^n} : x[n] = y[n]\right\}$$

By Theorem 2.13 we get the wellknown result that (Act^∞, d) is a complete ultrametric space.

4.2 Mazurkiewicz traces

Let (Act, ι) be a `concurrent alphabet`, i.e. ι is an irreflexive and symmetric relation on Act (called `independency`). A `trace` is an equivalence class $[x]$ of a finite string x over Act where the underlying equivalence relation $\equiv$ is the reflexive, transitive closure of $\equiv'$ which is given by:

$$x \equiv' y \quad :\Longleftrightarrow \quad \exists \alpha, \beta \in Act,\ z, w \in Act^* \ : \ \alpha \ \iota \ \beta \ \wedge \ x = z\alpha\beta w \ \wedge \ y = z\beta\alpha w$$

If $\sigma = [x]$ is a trace then $|\sigma| = |x|$ where $|x|$ means the usual length of x. In the following D denotes the set of traces w.r.t. a fixed concurrent alphabet (Act, ι) and $\sqsubseteq$ means the lifting of the prefixing ordering on Act^* to D. I.e.

$$[x] \sqsubseteq [y] \quad :\Longleftrightarrow \quad \exists x', y', z \in Act^* \ : \ x' \equiv x \ \wedge \ y' \equiv y \ \wedge \ y' = x'z$$

If $n \in I\!N$ then

$$\sigma^{(n)} = \{ \ \sigma' \in D \ : \ \sigma' \sqsubseteq \sigma \ \ \wedge \ \ |\sigma'| \leq n \ \}.$$

[16] considers the metric

$$d(\sigma, \tau) = \inf \left\{ \frac{1}{2^n} \ : \ \sigma^{(n)} = \tau^{(n)} \right\}$$

This is the metric $d[\rho]$ where the length ρ is given by $\rho(\sigma) = |\sigma|$. If $\iota \neq \emptyset$ then ρ is not a weight, e.g. if α, $\beta \in Act$, $\alpha \ \iota \ \beta$ then

$$\downarrow^1 (\ [\alpha\beta]\ ) = \{\perp, [\alpha], [\beta]\}$$

does not contain a greatest element since $[\alpha]$, $[\beta]$ are incomparable. Since we only deal with finite traces D is not ρ-complete. In [17] it is shown how the concept of infinite traces as proposed in [16] fits in our framework.

4.3 Trees

Let *Tree* denote the set of countably branching trees with labelled edges. Formally, a tree is a 4-tupel (N, K, k, v_0) where N is a set of nodes, $K \subseteq N \times N$ is a set of edges such that (N, K) is a tree in the graph-theoretical sense, $k : K \to Act$ is a labelling function and $v_0 \in N$ is the root. *height*(T) denotes the usual height of T, i.e. the length of a longest path in T.

To ensure that *Tree* is a set we assume that $N \subseteq Nodes$ where *Nodes* is a fixed uncountable set of nodes which contains a fixed element v_0. In addition we require that always v_0 is the root of a tree.

The partial order $\sqsubseteq$ on *Tree* is defined as in [31]:

$$T_1 \sqsubseteq T_2 \quad :\Longleftrightarrow \quad T_i = (N_i, K_i, k_i, v_0),\ i = 1,2 \text{ and } T_1 \ = \ T_2\lceil N_1.$$

Here $T_1 = T_2\lceil N_1$ means that $N_1 \subseteq N_2$, $K_1 = K_2 \cap N_1 \times N_1$ and $k_1 = k_2|K_1$. $(\mathit{Tree}, \sqsubseteq)$ is a cpo where the bottom element is the tree $T_\perp = (\{v_0\}, \emptyset, \emptyset, v_0)$. If (T_i) is a monotone sequence in *Tree* (where $T_i = (N_i, K_i, k_i, v_0)$) then the supremum of (T_i) in *Tree* is

$$\left(\bigcup N_i, \bigcup K_i, \bigcup k_i, v_0\right).$$

height is a continuous weight on the cpo $(\mathit{Tree}, \sqsubseteq)$. The n-cut $T[n]$ of a tree is the tree which arises from T by removing all nodes of the depth $\geq n+1$. I.e. $T[n] = T\lceil N[n]$ where $N[n]$ is the set of nodes $v \in N$ such that the depth of v in T is at most n. Here the depth of a node v in T is defined as the length of the path from the root to v. If T is a tree then $\bigsqcup T[n] = T$. I.e. all trees are approximable:

$$\mathcal{M}(\mathit{Tree}) \ = \ \mathit{Tree}$$

The induced metric d on *Tree* coincides with the usual metric on trees:

$$d(T_1, T_2) \ = \ \inf \left\{ \frac{1}{2^n} \ : \ T_1[n] \ = \ T_2[n] \right\}$$

A tree T is finitely approximable if and only if for each $n \geq 0$ the set

$$\downarrow^n (T) \ = \ \{ S \in \mathit{Tree} : S \sqsubseteq T[n]\}$$

is finite. This is the case if and only if $N[n]$ is finite for all $n \geq 0$ if and only if T is finitely branching. In the following $\mathit{Tree}_{\mathsf{fin}}$ denotes the subspace of finitely branching trees. Then:

$$\mathcal{M}_{\mathsf{fin}}(\mathit{Tree}) \ = \ \mathit{Tree}_{\mathsf{fin}}$$

By Remark 2.17 *height* is a continuous weight on the (incomplete) pointed poset $(\mathit{Tree}_{\mathsf{fin}}, \sqsubseteq)$ which is *height*-complete. Hence $\mathit{Tree}_{\mathsf{fin}}$ is a complete metric space.

Using trees as semantic domain we are not interested in the names of the nodes. Hence we abstract from the names which means that we deal with isomorphism classes. I.e. we consider the semantic domain

$$\mathit{TREE} \ = \ \mathit{Tree}/\simeq$$

instead of *Tree* where $\simeq$ means isomorphism of trees. It can be shown that the 'lifting' of $\sqsubseteq$ on *TREE* is a preorder but not a partial order on *TREE*. Here by the 'lifting' we mean the following relation (which we also denote by $\sqsubseteq$) on *TREE*:

$$\mathcal{T}_1 \sqsubseteq \mathcal{T}_2 \quad :\Longleftrightarrow \quad \text{There exists representants } T_i \text{ of } \mathcal{T}_i \text{ such that } T_1 \sqsubseteq T_2.$$

It can be shown that $\sqsubseteq$ as an ordering on $\mathit{TREE}_{\mathsf{fin}} = \mathit{Tree}_{\mathsf{fin}}/\simeq$ is an incomplete partial order. The lifting of the weight *height* on $\mathit{TREE}_{\mathsf{fin}}$ yields a continuous weight on $\mathit{TREE}_{\mathsf{fin}}$ and $\mathit{TREE}_{\mathsf{fin}}$ is *height*-complete. All elements of $\mathit{TREE}_{\mathsf{fin}}$ are finitely approximable. Hence $\mathit{TREE}_{\mathsf{fin}}$ is a complete metric space. The n-cut of $[T]_\simeq$ is the isomorphism class of $T[n]$. The canonical function

$$\varphi \ : \ \mathit{Tree}_{\mathsf{fin}} \ \to \ \mathit{TREE}_{\mathsf{fin}}, \ \ \varphi(T) \ = \ [T]_\simeq$$

is continuous and $\varphi(T[n]) = \varphi(T)[n]$. Let

$$tree_{cpo} : CCS \to Tree, \quad tree_{cms} : GCCS \to TREE_{fin}$$

denote Winskels cpo semantics for CCS resp. the metric semantics for guarded CCS on $TREE_{fin}$ where Winskels semantic operators lifted to isomorphism classes of finitely branching trees are used. Then by Theorem 3.14, Remark 3.12 and Remark 3.15:

$$[\ tree_{cpo}(s)(\sigma)\ ]_{\simeq} = tree_{cms}(s)(\varphi \circ \sigma)$$

for all $s \in GCCS$ and $\sigma : Idf \to Tree_{fin}$.

4.4 Event structures

Let $(Ev, \sqsubseteq)$ denote the cpo of prime event structures as defined in [30]. A prime event structure (or shortly event structure) is a 4-tupel $E = (N, \leq, \#, l)$ where N is a set of events, $\leq$ a partial order on N, $\#$ is a binary symmetric, irreflexive relation on N and $l : N \to Act$ is a labelling function such that for each $e \in N$ the set $\{e' \in N : e' \leq e\}$ is finite and for all e, e', $e'' \in N$:

$$e \leq e' \ \wedge\ e\#e'' \implies e'\#e''$$

The partial order $\sqsubseteq$ on Ev of [30] is given by

$$E' \sqsubseteq E \quad :\Longleftrightarrow \quad E' = E\lceil N'$$

where $E' = E\lceil N$ iff $E = (N, \leq, \#, l)$ and N' is a leftclosed subset of N such that

$$E' = (\ N',\ \leq\ \cap\ N' \times N',\ \#\ \cap\ N' \times N',\ l|N'\ ).$$

The depth of event structures is a continuous weight on Ev. Here the depth of an event structure E is given by

$$depth(E) = \sup \{depth_E(e) : e \in N\}$$

where $E = (N, \leq, \#, l)$ and

$$depth_E(e) = \max\ \{n \in I\!N_0 : \exists e_1, \ldots e_n \in N \text{ with } e_1 < \ldots < e_n = e\ \}$$

for all $e \in N$. $e < e'$ means $(e \leq e') \wedge (e \neq e')$. The n-cut of E is $E\lceil N[n]$ where $N[n]$ denotes the set of all events $e \in N$ with $depth_E(e) \leq n$. All event structures are approximable, i.e.

$$\mathcal{M}(Ev) = Ev.$$

The set of finitely approximable elements of Ev is the set of event structures E where $E[n]$ is finite for all $n \geq 0$. Here $E = (N, \leq, \#, l)$ is called finite iff N is a finite set. Let Ev_{fin} denote the set of finitely approximable event structures. By Remark 2.17 *depth* is a continuous weight on the incomplete pointed poset $(Ev_{fin}, \sqsubseteq)$ which is *depth*-complete.

Let $EV = Ev/\simeq$ where $\simeq$ means that isomorphism, i.e. we abstract from the names of the events. Similary to the situation above where we consider isomorphism classes of trees we get the following results: The 'lifting' of $\sqsubseteq$ to EV (which we also denote by $\sqsubseteq$) yields a preorder on isomorphism classes of event structures. The restriction of $\sqsubseteq$ on $EV_{\mathsf{fin}} = Ev_{\mathsf{fin}}/\simeq$ is an incomplete partial order and

$$depth([E]_{\simeq}) \;=\; depth(E)$$

is a continuous weight on EV_{fin}. EV_{fin} is *depth*-complete. The induced complete metric space coincides with the metric space considered in [11]. If $ev_{\mathsf{cpo}} : CCS \to Ev$ denotes Winskels cpo semantics for CCS and $ev_{\mathsf{cms}} : GCCS \to EV_{\mathsf{fin}}$ is the metric semantics for guarded CCS where Winskels semantic operators lifted to EV_{fin} are used then we get the following consistency result (by Theorem 3.14, Remark 3.12 and Remark 3.15):

$$[ev_{\mathsf{cpo}}(s)(\sigma)]_{\simeq} \;=\; ev_{\mathsf{cms}}(s)([\sigma]_{\simeq})$$

for all $s \in GCCS$ and $\sigma : Idf \to Ev_{\mathsf{fin}}$.

4.5 Pomsets

Following the idea of [23] in [7, 8] sets of pomsets are used to describe the linear time and true parallelism behaviour of CCS-/CSP-like processes. Pomsets can be defined as event structures without conflicts. Here we only deal with finitely approximable pomsets: In our setting a pomset is a tripel $p = (N, \leq, l)$ such that $(N, \leq, \emptyset, l) \in Ev_{\mathsf{fin}}$. Let *Pom* denote the set of pomsets. It is easy to see that *Pom* endowed with the restriction of the partial order $\sqsubseteq$ on *Ev* to *Pom* is a pointed poset. The least upper bound of a monotone sequence in *Pom* (if it exists) equals the least upper bound in *Ev*. *depth* is a continuous weight on *Pom*. The n-cut of a pomset p in *Pom* coincides with its n-cut in *Ev*. *Pom* is *depth*-complete and all pomsets are finitely approximable.

$$\mathcal{M}_{\mathsf{fin}}(Pom) \;=\; \mathcal{M}(Pom) \;=\; Pom.$$

Dealing with isomorphism classes of pomsets we get a subspace $POM = Pom/\simeq$ of EV_{fin}. Then *POM* is a pointed poset (but not a cpo) and the least upper bound of a monotone sequence in *POM* (if it exists) coincides with its least upper bound in EV_{fin}. *depth* is a continuous weight on *POM*. *POM* is *depth*-complete and $\mathcal{M}_{\mathsf{fin}}(POM) = POM$. The associated metric space coincides with the metric space of (isomorphism classes of) pomsets as defined in [7].

5 Characterization of *SFP* domains as rank ordered cpo's

We show that the *SFP* domains of Plotkin [22] can be characterized as special kinds of rank ordered cpo's.

A *SFP* domain is a cpo D which is the inverse limit of some embedding sequence of finite cpo's (in the category CPO^E of cpo's and embedding projection pairs). An embedding projection pair $D \to D'$ is a pair $< e, p >$ of continuous functions $e : D \to D'$, $p : D' \to D$

such that $p \circ e = id_D$ and $e \circ p \sqsubseteq id_{D'}$. An embedding sequence means a sequence $(D_n, \iota_n)_{n\geq 0}$ of cpo's D_n and embedding projection pairs $\iota_n : D_n \to D_{n+1}$. For further details see [22, 26, 28].

If $\tilde{\pi} = (\pi_n)_{n\geq 0}$ is a rank ordering on a cpo D then $(\pi_n(D))_{n\geq 0}$ can be considered as an embedding sequence where the embedding projection pair $\iota_n : \pi_n(D) \to \pi_{n+1}(D)$ is given by: $\iota_n = < i_n, j_n >$ where $i_n : \pi_n(D) \to \pi_{n+1}(D)$ denotes the inclusion and $j_n : \pi_{n+1}(D) \to \pi_n(D)$, $j_n(x) = \pi_n(x)$.

Lemma 5.1 *If $\tilde{\pi} = (\pi_n)_{n\geq 0}$ is a rank ordering on a cpo D then D is the inverse limit of $(\pi_n(D))_{n\geq 0}$.*

Proof: It is clear that $\gamma_n = < e_n, p_n >: D_n \to D$ is an embedding projection pair where $e_n : \pi_n(D) \to D$ is the inclusion and $p_n = \pi_n | D \to \pi_n(D)$. In addition we have:

$$\gamma_{n+1} \circ \iota_n = \gamma_n$$

If D' is a cpo and $\gamma'_n = < i'_n, j'_n >: D_n \to D'$ are embedding projection pairs with $\gamma'_{n+1} \circ \iota_n = \gamma'_n$ then it can be shown that $< e, p >: D \to D'$ which is given by

$$e(x) = \bigsqcup_{n\geq 0} i'_n(\pi_n(x)), \quad p(y) = \bigsqcup_{n\geq 0} j'_n(y)$$

is the unique embedding projection pair with $< e, p > \circ \gamma_n = \gamma'_n$. Hence (D, γ_n) is the inverse limit. □

Definition 5.2 *A rank ordering $\tilde{\pi} = (\pi_n)$ on a pointed poset $(D, \sqsubseteq)$ is called* `finitary` *iff for each $n \geq 0$ the set $\pi_n(D)$ is finite.*

Lemma 5.3 *Let (D_n, ι_n) be an embedding sequence of finite cpo's. Then there exists a finitary rank ordering on the inverse limit D of (D_n, ι_n).*

Proof: Let (D, γ_n) be the inverse limit where $\gamma_n = < e_n, p_n >$. Then it is easy to see that $(e_n \circ p_n)_{n\geq 0}$ is a finitary rank ordering on D. □

Theorem 5.4 *Let D be a cpo. Then D is a SFP domain if and only if there exists a finitary rank ordering on D.*

Proof: follows by Lemma 5.1 and Lemma 5.3. □

A similar result is presented in [1] where bifinite domains are described in terms of directed sets of so-called idempotent deflations (which can be considered as a generalization of finitary rank orderings).

6 The Lawson and the metric topology on weighted cpo's

In this section we discuss the connection between the Lawson topology on an algebraic dcpo and the topology induced by the metric $d[\rho]$ resp. $d[\tilde{\pi}]$ where ρ is a length and $\tilde{\pi}$ a rank ordering.

The topology on a metric space (M, d) is the topology induced by the basis of open balls, i.e. the sets

$$B(x, r) = \{ y \in M : d(x, y) < r \}$$

where $x \in D$ and $r > 0$.

Let D be an algebraic dcpo. I.e. for each element $x \in D$ the set

$$\mathcal{K}(x) = \{ \xi \in \mathcal{K}(D) : \xi \sqsubseteq x \}$$

is directed and $x = \mathrm{lub}(\mathcal{K}(x))$. Here $\mathcal{K}(D)$ denotes the set of compact (or finite or isolated) elements of D. An element ξ of D is called compact (or finite or isolated) iff whenever X is a directed subset of D with $\xi \sqsubseteq \mathrm{lub}(X)$ then $\xi \sqsubseteq x$ for some $x \in X$. D is called ω-algebraic iff D is algebraic and $\mathcal{K}(D)$ countable. The Lawson topology on an algebraic dcpo D is defined to be the topology induced by the subbasis $\xi \uparrow$, $D \setminus \xi \uparrow$ where $\xi \in \mathcal{K}(D)$.

Lemma 6.1 *Let $(D, \sqsubseteq)$ be an algebraic dcpo, ρ a weight on D then $D = \mathcal{M}(D)$ if and only if $\mathcal{K}(D) \subseteq Fin(D)$.*

If ρ is a length on D and $\mathcal{K}(D) \subseteq Fin(D)$ then $D = \mathcal{M}(D)$.

Proof: First we assume that ρ is a length on D and $\mathcal{K}(D) \subseteq Fin(D)$. For each element $x \in D$ we have:

$$\mathcal{K}(x) \subseteq \downarrow^{\mathrm{fin}}(x)$$

Since x is an upper bound of $\downarrow^{\mathrm{fin}}(x)$ and since $x = \mathrm{lub}(\mathcal{K}(x))$ we get:

$$x = \mathrm{lub}(\downarrow^{\mathrm{fin}}(x))$$

Therefore $D = \mathcal{M}(D)$.

Now we assume that ρ is a weight on D and $D = \mathcal{M}(D)$. Let $\xi \in \mathcal{K}(D)$. Then $\downarrow^{\mathrm{fin}}(\xi)$ is directed (since ρ is a weight) and

$$\xi = \mathrm{lub}\left(\downarrow^{\mathrm{fin}}(\xi)\right).$$

Since ξ is compact there exists $x \in \downarrow^{\mathrm{fin}}(\xi)$ with $\xi \sqsubseteq x$. Therefore $\xi = x \in Fin(D)$. $\square$

Lemma 6.2 *Let $(D, \sqsubseteq)$ be an algebraic dcpo and ρ a weight on D with $\mathcal{K}(D) \subseteq Fin(D)$. Then the $d[\rho]$-topology is finer than the Lawson topology.*

Proof: We have to show that the sets $\xi \uparrow$ and $D \setminus \xi \uparrow$ (where $\xi \in \mathcal{K}(D)$) are open w.r.t. $d[\rho]$. Let $\xi \in \mathcal{K}(D)$ and $n = \rho(\xi)$. By assumption $n < \infty$.

If $y \in \xi \uparrow$ then for all $x \in B(y, 1/2^n)$:

$$\xi \sqsubseteq y[n+1] = x[n+1] \sqsubseteq x$$

Hence $x \in B(y, 1/2^n)$. Therefore $B(y, 1/2^n) \subseteq \xi \uparrow$.

If $y \in D \setminus \xi \uparrow$ then for all $x \in B(y, 1/2^n)$:

$$\xi \not\sqsubseteq y[n+1] = x[n+1]$$

Therefore $\xi \not\sqsubseteq x$. Hence $B(y, 1/2^n) \subseteq D \setminus \xi \uparrow$. $\square$

Lemma 6.3 *Let $(D, \sqsubseteq)$ be an algebraic dcpo and ρ a length on D such that*

$$D[n] \;=\; \{\; x \in D \;:\; \rho(x) \leq n \;\}$$

is finite for all $n \geq 0$ and such that $Fin(D) \;=\; \mathcal{K}(D)$. Then the Lawson topology on D is finer than the $d[\rho]$-topology on D.

Proof: We have to show that the open balls $B(y, r)$ are open w.r.t. the Lawson topology. Let $y \in D$, $r > 0$ and $x \in B(y, r)$. Then $1/2^n < r$ for some natural number $n \geq 0$. Let

$$U \;=\; \bigcap \{\; \xi \uparrow :\; \xi \in \downarrow^n (x) \;\},$$

$$V \;=\; \bigcap \{\; \xi \uparrow :\; \xi \in D[n],\; \xi \not\sqsubseteq x \;\}$$

Then U and V are Lawson open (since $D[n]$ is finite and $D[n] \subseteq Fin(D) = \mathcal{K}(D)$). It is clear that

$$y \;\in\; U \cap V \;=\; B\left(x, \frac{1}{2^n}\right) \;=\; B\left(y, \frac{1}{2^n}\right) \;\subseteq\; B(y, r).$$

Hence $B(x, r)$ is Lawson open. □

Theorem 6.4 *Let ρ be a weight on an algebraic dcpo $(D, \sqsubseteq)$ such that $\mathcal{K}(D) \;=\; Fin(D)$ and such that for all $n \geq 0$ the set*

$$D[n] \;=\; \{\; x \in D \;:\; \rho(x) \leq n \;\}$$

is finite. Then $D \;=\; \mathcal{M}(D)$ and the Lawson topology on D agrees with the topology induced by the metric $d[\rho]$.

Proof: follows by Lemma 6.1, Lemma 6.2 and Lemma 6.3. □

Dealing with rank orderings instead of weights we get similar results. In [26] it is shown that whenever (D, γ_n) is the inverse limit of an embedding sequence (D_n, ι_n) where D_n are ω-algebraic dcpo's then D is ω-algebraic and

$$\mathcal{K}(D) \;=\; \bigcup_{n \geq 0} e_n(\mathcal{K}(D_n))$$

where $\gamma_n \;=\; < e_n, p_n >$. Since finite posets are always ω-algebraic dcpo's where all elements are compact we obtain by Lemma 5.1:

Lemma 6.5 *If $\tilde{\pi} \;=\; (\pi_n)_{n \geq 0}$ is a finitary rank ordering on a cpo D then D is an ω-algebraic dcpo and*

$$\mathcal{K}(D) \;=\; \bigcup_{n \geq 0} \pi_n(D).$$

Lemma 6.6 *Let $\tilde{\pi}$ be a rank ordering on an algebraic dcpo such that $\pi_n(D) \subseteq \mathcal{K}(D)$ for all $n \geq 0$. Then the $d[\tilde{\pi}]$-topology on D is finer than the Lawson-topology on D.*

Proof: analogous to Lemma 6.2 where we have to deal with $\pi_n(x)$ instead of $x[n]$. □

Lemma 6.7 *Let $\tilde{\pi}$ be a finitary rank ordering on a cpo D. Then the Lawson topology on D is finer than the $d[\tilde{\pi}]$-topology on D.*

Proof: analogous to Lemma 6.3 where we have to deal with $\pi_n(x)$ instead of $x[n]$. □

Theorem 6.8 *Let $\tilde{\pi}$ be a finitary rank ordering on a cpo D. Then the Lawson topology on D agrees with the topology induced by the metric $d[\tilde{\pi}]$.*

Proof: follows by Lemma 6.6 and Lemma 6.7. □

By Lemma 2.29, Theorem 5.4 and Theorem 6.8 we conclude that each *SFP*-domain D (endowed with the Lawson topology) is a compact, metrizable topological space.

7 Related work and future research

Various other authors have attempted to build a bridge between cpo and metrics. E.g. Matthews [19] introduces the notion of partial metrics and quasi metrics in order to obtain a topology that is not Hausdorff. Smyth [26] introduces quasi uniformities for the same propose. In [17] we show that a finite length on a pointed poset induces a continuous weight (and hence a metric) on the ideal completion and we discuss the relationship between the metric and the ideal completion as a metric space. There we also show the connection to the approach of [10] where a metric on the ideal completion of a countable poset is defined: If ρ is a finite length on D such that $D[n]$ is finite for all $n \geq 0$ then the metric of [10] on the ideal completion is equivalent our metric. In [17] we also discuss the relation to the approach of Weihrauch and Schreiber [29]. We recall the results: [29] start with a partial order $(D, \sqsubseteq)$ with a function $|\cdot| : D \to [0, \infty]$ that obeys:

$$(*) \quad x \sqsubseteq y \implies |x| \geq |y|$$

From this they construct a distance d:

$$d(x,y) \ = \ \inf \left\{ \sum_{i=1}^{k-1} |z_i| \ : \ z_0, z_1, \ldots, z_k \text{ is a path from } x \text{ to } y \right\}$$

A path from x to y is a (finite) sequence $z_0, z_1, \ldots, z_k$ in D such that $z_0 = x$, $z_k = y$ and such that for all i there exists an upper bound of z_i and z_{i+1} in D. Those elements x in D with $|x| = 0$ form a pseudometric space. [29] describes a method to select from every distance-0-equivalence class a member and obtain a subset of D that is a metric space. Condition $(*)$ tells us that $\perp$ is the 'heaviest' element and that the 'largest' elements are the lightest. E.g. given an alphabet A and choosing $D = A^\infty$ and $|x| = 1/l(x)$ where $l(x)$ means the length of a string x we see that the constructed metric space will consist of infinite strings only. Choosing $D = A^*$ with $|x| = 1/l(x)$ there are no elements with $|x| = 0$ and the constructed metric space of [29] is empty whereas we obtain the cpo and complete metric space A^∞ of all sequences.

In [3, 4] we discuss the relation between denotational semantics in the cpo and metric approach. At present we are studying the connection between initial solutions of domain equations for cpo's and unique solutions of domain equations for complete metric spaces [5]. In contrast to [24] where fixed point theorems for locally continuous and locally contractive endofunctors of the category of quasi ultrametric spaces are established (and hence combine the results of [28] and [2]) we show how the solutions of 'corresponding' domain equations $D \cong \mathcal{G}(D)$ for cpo's (which are solved by the method of [28]) and $M \cong \mathcal{H}(M)$ complete metric spaces (which are solved by the methods of [2, 18, 25]) are related.

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