EQUATIONALLY COMPACT ARTINIAN RINGS

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Nr. 13 (1971)

Diese Arbeit bildet Teil der Dissertation des Verfassers.

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By a Noetherian (Artinian) ring $R = \langle R;+,-,0,\cdot\rangle$ we mean an associative ring satisfying the ascending (descending) chain condition on left ideals. An arbitrary ring R is said to be *equationally compact* if every system of ring polynomial equations with constants in R is simultaneously solvable in R provided every finite subset is. (The reader is referred to [2], [8], [13] and [14] for terminology and relevant results on equational compactness, and to [4] for unreferenced ring-theoretical results.) In this report a characterization of equationally compact Artinian rings is given - roughly speaking, these are the finite direct sums of finite rings and Prüfer groups; as consequences it is shown that an equationally compact ring satisfying both chain conditions is always finite, as is any Artinian ring which is a compact topological ring; further, using a result of S. Warner [11], we give a necessary and sufficient condition for an equationally compact Noetherian ring with identity to be a compact topological ring; a few remarks on the embedding of certain rings into equationally compact rings are made, and we obtain also here generalizations of known results on compact topological rings.

The material forms a part of the author's Ph.D. thesis.

Preliminary results. We begin by deriving a few useful tools. Let R be a ring and A an ideal of R ("ideal" always means two-sided ideal), and let Σ be a system of equations with constants in A. If $(x_0,x_1,\ldots,x_\gamma,\ldots)_{\gamma<\alpha}$ are the variables occurring in Σ, then the solution set of Σ in R is a certain subset S of R^α. If such a system Σ exists such that the projection of S onto the first component is the ideal A, then we shall say that A *is expressible by equations*. For example, if R has an identity and A is finitely generated as a left ideal, then A is expressible by the equation $x_0 = x_1a_1+\cdots+x_na_n$, where $a_1,\ldots,a_n$ generate A.

If x is a variable and A is an ideal of R, then "$x \in A$" will denote, quite naturally, the relational predicate $A(x)$ for the unary relation A on R.

We will make recurrent use of the following observation:

Remark. Let R be an equationally compact ring. Suppose $(A_i \mid i \in I)$ is a family of ideals of R, each of which is expressible by equations, and suppose $(x_i \mid i \in I)$ is a family of variables. Let Σ be a set of equations with constants in R. Then the system of formulas

$$\Omega := \Sigma \cup \{x_i \in A_i;\ i \in I\}$$

is solvable in R provided it is finitely solvable in R.

proof: Let A_i be expressible by the system Σ_i, $i \in I$; let $(x_{oi}, x_{1i}, \ldots, x_{\gamma i}, \ldots)_{\gamma < \alpha_i}$ denote the variables appearing in Σ_i, whereby it is assumed that the variables $x_{\gamma i}$ and $x_{\delta j}$ are distinct if $i \neq j$ or $\gamma \neq \delta$, and that no $x_{\gamma i}$ occurs in Σ. Now the finite solvability of Ω implies the finite solvability of the system of equations

$$\bigcup(\Sigma_i \mid i \in I) \cup \{x_i = x_{oi};\ i \in I\} \cup \Sigma,$$

which is then solvable by the equational compactness of R, and a solution obviously yields a solution of Ω in R. q.e.d.

Proposition 1. Let R be a ring and A an ideal of R such that A is expressible by equations and R is equationally compact. Then R/A and A are equationally compact rings.

proof: Suppose $\Sigma = \{\Phi_i = 0;\ i \in I\}$ is a system of equations with constants in R/A and finitely solvable in R/A. Now each Φ_i induces a polynomial in R, say Φ_i', by replacing the constants by arbitrary representatives in R. If z_i, $i \in I$, are variables not occurring in Σ, then the system

$$\{\Phi_i' = z_i;\ i \in I\} \cup \{z_i \in A;\ i \in I\}$$

is clearly finitely solvable in R, hence (by the last Remark) solvable in R, and any solution taken modulo A yields a solution for Σ in R/A. Thus R/A is equationally compact, and a similar argument shows that A is equationally compact.

Next we derive a useful remark on matrix rings.

Proposition 2. Let R be a ring with identity, let $\mathfrak{m}$ be a nonzero cardinal and let $S = M_{\mathfrak{m}\times\mathfrak{m}}(R)$ (i.e., S is the ring of linear transformations on the free R-module F on $\mathfrak{m}$ generators). Then S is equationally compact if and only if R is equationally compact and $\mathfrak{m}$ is finite.

proof: Sufficiency. If Σ is a finitely solvable system of equations with constants in S then by replacing each variable x by the variable matrix $(x_{ij} \mid 1 \leq i,j \leq \mathfrak{m})$, every equation in Σ reduces in the obvious fashion to a system over R, finitely solvable in R, hence solvable in R; such a solution yields a solution for Σ in S.

Necessity. Let I be a set with cardinality $\mathfrak{m}$ and let $\{e_i; i \in I\}$ be a basis for F. Fix $i_0 \in I$. For each $i \in I$ define $\pi_i \in S$ as follows: $\pi_i(e_j) = \delta_{ji_0} e_i$, for all $j \in I$. Let p_i be the retraction of F onto Re_i. Then the system

$$\Sigma = \{p_i x = \pi_i;\ i \in I\}$$

is finitely solvable (for a finite subset $J \subseteq I$ of indices appearing, take x as follows: $x(e_{i_0}) = \sum_{i \in J} e_i$, and $x(e_j) = 0$ for $j \neq i_0$). However Σ forces x to be such that $x(e_{i_0}) = \sum_{i \in I} e_i$ which is impossible unless $\mathfrak{m}$ is finite. To see that R is equationally compact, consider a system Σ of equations with constants in R and finitely solvable in R. For $r \in R$ let $e(r)$ denote the matrix (a_{ij}) where $a_{11} = r$ and $a_{ij} = 0$ otherwise. Replace every constant $r \in R$ appearing in Σ by $e(r)$ and every variable x by $e(1)\cdot x \cdot e(1)$. Then Σ is finitely solvable in S, hence solvable in S. Taking the upper left hand entries from a solution in S yields obviously a solution of Σ in R. q.e.d.

If $R = \langle R; +, -, 0, \cdot \rangle$ is a ring, we denote by R^+ the underlying additive abelian group $\langle R; +, -, 0 \rangle$.

Proposition 3. Let R be an equationally compact ring and let $\mathcal{D} = \langle D; +, -, 0 \rangle$ be the largest divisible subgroup of R^+. Then $R \cdot D = D \cdot R = \{0\}$. In particular, $\mathcal{D}$ is an ideal of R. Moreover, the ring $R/\mathcal{D}$ is equationally compact.

proof: Let $d \in D$ and $r \in R$. Consider the system of equations

$$\Sigma = \{(x_i - x_j)x_{ij} = r\cdot d;\ i,j \in I,\ i \neq j\}$$

where I is a set with cardinality larger than $|R|$. Σ is finitely solvable in R, since for any finite subset of indices $J \subseteq I$, choose n_i, $i \in J$, to be distinct natural numbers, set $c_i = n_i r$, and pick d_{ij} such that $(n_i - n_j)d_{ij} = d$ for $i \neq j$. Then clearly $(c_i - c_j)d_{ij} = r\cdot d$ for all $i,j \in J$, $i \neq j$. Thus Σ must be solvable in R. However Σ implies $x_i = x_j$ for some $i \neq j$, because of the cardinality of I, hence $r\cdot d = 0$. An almost identical argument shows that $d\cdot r = 0$.

We recall that an abelian group G is *algebraically compact* (in the sense of Kaplansky [6]) if

$$G \simeq C \oplus (\Pi(G_p |\ p = \text{prime}))$$

where C is divisible and each G_p is a p-primary group complete in its p-adic topology and containing no nonzero element which is divisible by all powers of p. The group R^+ is equationally compact and therefore algebraically compact as was shown by S. Balcerzyk in [1]; thus in view of the latter condition on the G_p's the subgroup D under discussion equals C and is expressible by the equations

$$\{x_0 = n\cdot x_n;\ n \in \mathbb{N}\}.$$

Thus, R/D is equationally compact by Proposition 1.

Proposition 4. Let R be an equationally compact ring such that R^+ is a bounded torsion group. Then there exists an equationally compact ring S with identity such that R is an ideal in S of finite index.

proof: Let n be a natural number such that $n\cdot R = (0)$, and let Z_n denote the integers modulo n. Define $S = \langle R\times Z_n; +, -, 0, \cdot\rangle$ as follows: + is the usual direct sum addition, and

$$(r,l)\cdot(s,k) := (r\cdot s + l\cdot s + k\cdot r,\ l\cdot k).$$

The map $r \mapsto (r,0)$ is a ring embedding of R into S, making R clearly an ideal of S of finite index.

Now let Σ be a system of equations with constants in S, finitely solvable in S. Let $(x_0, x_1, \ldots x_\gamma, \ldots)_{\gamma<\alpha}$ be the variables appearing in Σ. Replace each variable x_γ by (y_γ, z_γ), inducing the system Σ_0 with the obvious interpretation of solvability (i.e., y_γ must be replaced by an element of R and z_γ by an element of Z_n). We construct by transfinite induction a sequence $(n_0, n_1, \ldots n_\gamma, \ldots)_{\gamma<\alpha} \in Z_n^\alpha$, such that $\Sigma_0((z_\gamma \to n_\gamma)_{\gamma<\alpha})$ is finitely solvable ("$z_\gamma \to n_\gamma$" means that the variable z_γ is replaced by n_γ). Let β be an ordinal and let n_γ, $\gamma<\beta$, be already constructed such that

$$\Sigma_\beta := \Sigma_0((z_\gamma \to n_\gamma)_{\gamma<\beta})$$

is finitely solvable. (For $\beta = 0$ the construction is trivial.) Suppose for each $m \in Z_n$ the system $\Sigma_\beta(z_\beta \to m)$ is not finitely solvable; i.e., for each $m \in Z_n$ there exists a finite subset $\Sigma_{\beta,m}$ of Σ_β such that $\Sigma_{\beta,m}(z_\beta \to m)$ is not solvable. But then the finite system

$$\bigcup_{m \in Z_n} \Sigma_{\beta,m} \subseteq \Sigma_\beta$$

is clearly not solvable. This is a contradiction, so there exists $n_\beta \in Z_n$ such that $\Sigma_\beta(z_\beta \to n_\beta)$ is finitely solvable, and the induction step is complete. Thus $\Sigma_1 := \Sigma_0((z_\gamma \to n_\gamma)_{\gamma<\alpha})$ is a finitely solvable system involving only the variables $(y_\gamma)_{\gamma<\alpha}$. Now any $\Phi \in \Sigma_1$ is equivalent to a pair of equations (Φ_1, Φ_2), where Φ_1 is an equation with constants in R and involving the variables $(y_\gamma)_{\gamma<\alpha}$, and Φ_2 involves only constants (from Z_n). Therefore Σ_1 is solvable because R is equationally compact. q.e.d.

Semisimplicity. A ring R is *semisimple* if its Jacobson radical $J(R)$ is zero. We consider now the impact of this condition on equationally compact Artinian and Noetherian rings.

Recall that an element r of a ring R is *left quasi-regular* if there exists an element $y \in R$ with $r + y + y \cdot r = 0$. It is well-known that $J(R)$ is the largest left quasi-regular left ideal in R; that is, $r \in J(R)$ if and only if the left ideal generated by r is left quasi-regular. Hence $J(R)$ is expressible by the set of equations

$$\{s \cdot x_0 + z \cdot x_0 + y_{s,z} + y_{s,z} \cdot (s \cdot x_0 + z \cdot x_0) = 0;\ s \in R,\ z \in \mathbb{Z}\},$$

and in view of Proposition 1 we have

Proposition 5. If the ring R is equationally compact, then so are the rings $R/J(R)$ and $J(R)$.

Lemma 1. A semisimple Artinian ring R is equationally compact if and only if it is finite.

proof: Sufficiency. It is perhaps appropriate at this point to remark that an arbitrary universal algebra $A = \langle A;F\rangle$ which is also a compact topological algebra (i.e., A can be endowed with a compact Hausdorff topology compatible with the algebraic structure) is equationally compact (see [8]). Indeed, the solution set of any equation is a closed subset of an appropriate power of A endowed with the Tychonov product topology.

As a special case, any finite algebra, hence any finite ring, is equationally compact.

Necessity. It is easily seen that a finite direct sum of rings is equationally compact if and only if every summand is. By Wedderburn's theorem R is a finite direct sum of matrix rings over division rings, each of which, therefore, is equationally compact. By Proposition 2 the respective divisions rings are equationally compact. However, equationally compact division rings are known to be finite (consider, for example, the system $\Sigma = \{(x_i - x_j)y_{ij} = 1;\ i,j\in I,\ i\neq j\}$ for suitably large I). Thus R is finite.

Proposition 6. Let R be an equationally compact semisimple Noetherian ring with identity. Then R is finite.

In view of the fact that equationally compact Noetherian rings with identity are necessarily linearly compact for the discrete topology, Proposition 6 follows from D. Zelinsky's decomposition of linearly compact semisimple rings [15,Prop.11] and Lemma 1. For completeness' sake we give a proof, which is in the spirit of an argument of S. Warner [12, p.55].

Lemma 2. Let R be as above but, in addition, a primitive ring. Then R is finite (and hence simple Artinian).

proof: By the Jacobson-Chevalley Density Theorem R is a dense ring of linear transformations on a vector space V with basis, say, $\{e_i;\ i\in I\}$. For each $i\in I$, let

$$A_i = \{\phi \in R;\ \phi(e_i) = 0\}.$$

A_i is a left ideal, hence finitely generated, and therefore expressible by equations. Let $(v_i)_{i\varepsilon I} \varepsilon V^I$ be chosen arbitrarily. By denseness there exists for each $i\varepsilon I$ $\phi_i \varepsilon R$ such that $\phi_i(e_i) = v_i$. Thus the system of equations

$$\Sigma = \{x = \phi_i + z_i;\ i\varepsilon I\} \cup \{z_i \ \varepsilon\ A_i;\ i\varepsilon I\}$$

is finitely solvable (again by denseness) and hence solvable. However Σ implies that x must map each e_i to v_i. Thus R is the complete transformation ring, and therefore by Prop. 2 and Lemma 1 a finite matrix ring over a division ring. q.e.d.

<u>proof of Proposition 6</u>: As is well-known R is a subdirect product of a family of primitive rings $\{R/A_i;\ i\varepsilon I\}$ where the A_i's are ideals of R. Since R is Noetherian with identity, each A_i is expressible by equations, so R/A_i is equationally compact by Proposition 1 and Noetherian. Hence by Lemma 2 R/A_i is finite, simple and Artinian. Hence the A_i's are maximal ideals. Let $r = (r_i + A_i)_{i\varepsilon I} \ \varepsilon\ \Pi(R/A_i \mid i\varepsilon I)$. The system

$$\Sigma = \{x = r_i + z_i;\ i\varepsilon I\} \cup \{z_i \ \varepsilon\ A_i;\ i\varepsilon I\}$$

is finitely solvable by the Chinese Remainder Theorem, hence solvable in R. But Σ imples $x = r$, so $r \ \varepsilon\ R$. Hence R is the full direct product and so I must be finite because R is Noetherian. q.e.d.

We summarize these results in the following

<u>Theorem 1</u>. For an equationally compact semisimple ring R the following are equivalent:

- (i) R is finite.
- (ii) R is Artinian.
- (iii) R is Noetherian with identity.

<u>Noetherian rings</u>. Although we are not able to characterize structurally those Noetherian rings with identity which are equationally compact, Theorem 1 and a crucial result of Warner yield a pleasant criterium relating equational compactness and topological compactness in this class of rings. We paraphrase the relevant result:

Proposition 7 [11, Theorem 2]. Let R be a topological Noetherian ring with identity. Then R is topologically compact if and only if the topology of R is the radical topology T, R is complete for that topology and $R/J(R)$ is a finite ring.

Now let R be an equationally compact Noetherian ring with identity. By Theorem 1, $R/J(R)$ is finite. Now the topology T defined by taking the powers of $J(R)$ as a neighbourhood base of 0 is not necessarily Hausdorff. However, we shall show that the space (R,T) is complete. To see this, consider a Cauchy sequence $(r_i)_{i=1,2,...}$ in R. For each natural number n choose i_n such that the subsequence $(r_i | i \geq i_n)$ is $J(R)^n$-close. Since R is Noetherian with identity, the ideal $J(R)^n$ is expressible by equations, so we have the system of equations

$$\Sigma = \{x = r_{i_n} + z_n;\ n \in \mathbb{N}\} \cup \{z_n \in J(R)^n;\ n \in \mathbb{N}\}$$

which is finitely solvable (if m is the largest index appearing in a finite subset, set $x = r_{i_m}$ and $z_n = r_{i_m} - r_{i_n}$ for all $n \leq m$). Hence Σ is solvable and obviously any solution is a limit of $(r_i)_{i=1,2,...}$. As a matter of fact, T is compact. To see this we quote the following

Lemma 3. Let R be a ring with identity, A and B two ideals such that B is finitely generated as a left ideal and both R/A and R/B are finite. Then $R/A \cdot B$ is finite.

The proof is a straightforward counting of cosets as given in the proof of [10, Lemma 4], where the hypothesized commutativity is not used.

Now by Lemma 3 and induction, we see that $J(R)^n$ has finite index in R for each n. This means that the family of cosets $F = \{r + J(R)^n;\ r \in R,\ n \in \mathbb{N}\}$ is a subbase of closed sets for the topology T, and by the Alexander Subbase Theorem T is compact if every subfamily of F with the finite intersection property has a nonempty intersection. The latter is however clear by equational compactness of R and the fact that each $J(R)^n$ is expressible by equations. In view of Proposition 7 we have proved

Theorem 2. Let R be an equationally compact Noetherian ring with identity. Then the radical topology is a complete and compact topology on R, and $R/J(R)$ is finite. Moreover, R is a compact topological ring if and only if $\bigcap(J(R)^n \mid n \in \mathbb{N}) = \{0\}$.

Remark. By [3], equational and topological compactness coincide when R is a commutative Noetherian ring with identity. In general, I do not know of an equationally compact Noetherian ring with identity which is not topologically compact.

Artinian rings. As an immediate consequence of Theorem 2 we have the following

Corollary 1. An equationally compact Artinian ring R with identity is finite.

proof: Two well-known results assert that R is Noetherian and $J(R)$ is nilpotent. Hence $J(R)^n = (0)$ for some n, thus the radical topology is discrete and, by theorem 2, compact, which forces R to be finite.

Corollary 2 [11, Theorem 2, Corollary]. A compact topological Artinian ring with identity is finite.

The case of arbitrary Artinian rings requires a closer look.

Lemma 4. If R is an equationally compact Artinian ring such that R^+ is a bounded torsion group, then R is finite.

proof: By Proposition 4 there is an equationally compact ring with identity S, such that R is an ideal of S and S/R is finite. Thus R is an Artinian S-module, as is the finite S-module S/R, and so S is an Artinian S-module, i.e., S is an Artinian ring. But then S is finite by Corollary 1. q.e.d.

Lemma 5. Let R be an equationally compact torsion-free Artinian ring. Then $R = (0)$.

proof: A torsion-free Artinian ring has, as well-known, a left identity e and is an algebra over the rationals. But then the system of equations

$$\{(x_i - x_j)y_{ij} = e;\ i,j \in I,\ i \neq j\}$$

is finitely solvable in R, hence solvable in R; taking $|I| > |R|$ forces $e = 0$, i.e., $R = (0)$.

Recall that the Prüfer group $Z(p^\infty)$ is the subgroup of the unit circle in the complex plane consisting of all p^n-th roots of unity for all natural numbers n and fixed prime p.

Theorem 3. For an Artinian ring R the following are equivalent:

(i) R is equationally compact.

(ii) $R^+ \simeq B \oplus P$ where $B = \langle B;+,.,0\rangle$ is a finite group, $P = \langle P;+,-,0\rangle$ is a finite direct sum of Prüfer groups, and $R \cdot P = P \cdot R = \{0\}$.

(iii) R is (algebraic) retract of a compact topological ring.

proof: (iii) $\Rightarrow$ (i) holds for arbitrary universal algebras (see [8]).

(i) $\Rightarrow$ (ii): By a result of F. Szász [9, Satz 4] every Artinian ring is the ring direct sum of its torsion ideal T and some torsion-free ideal D. But D is then an equationally compact torsion-free Artinian ring, so must be (0) by Lemma 5. Hence $R = T$. Let $R^+ = B \oplus P$ be the (group) decomposition of R^+ into its divisible part P and reduced part B. As a torsion divisible abelian group P is, as well-known, a direct sum of Prüfer groups. Now by Proposition 3 $R \cdot P = P \cdot R = \{0\}$. Thus every subgroup of P is an ideal of R and therefore P is a finite direct sum, because R is Artinian.

Now the family $F = \{n \cdot B \oplus P;\ n \in \mathbf{N}\}$ is easily seen to be a downward directed set of ideals of R, hence has a smallest element $n_0 \cdot B \oplus P$ since R is Artinian. However $n_0 \cdot B \oplus P$ is clearly divisible, being the meet of F, and so $n_0 \cdot B = (0)$ as B is reduced. Thus B is a bounded torsion group. The quotient R/P is Artinian and, again by Proposition 3, equationally compact; moreover, $(R/P)^+ \simeq B$. Hence B is finite by Lemma 4, and we are done.

(ii) $\Rightarrow$ (iii): Let $R^+ \simeq B \oplus P_1 \oplus \cdots \oplus P_n$ where B is finite and $P_i = \mathbf{Z}(p_i^\infty)$, $i = 1,\dots,n$. Each P_i is divisible, hence injective and therefore retract of every extending abelian group - e.g., the compact topological circle group C. Let $f_i : C \to P_i$

be a retraction. Endowing B with the discrete topology, we have then a (group) retraction

$$f: H \to R^+$$

where H is the compact topological group $B \oplus (\oplus(C|\ i=1,..,n))$ and $f = id_B \oplus f_1 \oplus \cdots \oplus f_n$.

If multiplication is defined on H by letting every element of $\oplus(C|\ i=1,\ldots,n)$ annihilate H and then extending by distributivity, H clearly becomes a ring. Moreover H is a topological ring under the given topology, because the inverse image under the multiplication map of any subset of H is the finite union of sets of the form $A_1 \times A_2$ where each A_j is a coset of $\oplus(C|\ i=1,\ldots,n)$ in H, all of which, however, are closed; thus multiplication is continuous. By a straightforward calculation one sees that f is a ring homomorphism, and the proof is complete.

Remark. It is not possible, in general, to obtain a ring-direct sum in the decomposition given in condition (ii). Consider, for example, the ring R, where $R^+ = Z_2 \oplus Z(2^\infty)$, $R \cdot Z(2^\infty) = Z(2^\infty) \cdot R = \{0\}$, and $(1,0)\cdot(1,0)$ is defined to be the primitive square root of unity in $Z(2^\infty)$. Here we have a nonzero divisible element appearing as a product of two nondivisible elements.

The following improves Corollary 2:

Corollary 3. A compact topological Artinian ring R is finite.

proof: By Theorem 3 we have $R^+ \simeq B \oplus P_1 \oplus \cdots \oplus P_n$, where B is finite and $P_i = Z(p_i^\infty)$. Let P_i^k be the subgroup of P_i consisting of all p_i^k-th roots of unity, and let

$$R^k = B \oplus P_1^k \oplus \cdots \oplus P_n^k.$$

Now $R = \bigcup(R^k|\ k=1,2,3,\ldots)$, that is, the intersection of the complements $R \setminus R^k$ is empty. By the Baire Category Theorem [7, p.200] at least one of the sets $R \setminus R^k$ is not dense in R, i.e., for some k_o the finite subgroup R^{k_o} contains a nonempty open set; this forces the topology to be discrete and therefore by compactness R must be finite.

Corollary 4. An equationally compact ring satisfying both chain conditions is finite.

proof: clear.

Compactifications. We conclude with a few remarks on the question of embedding rings into equationally compact ones. Following the terminology of [14] we define, for a fixed universal algebra A, a *compactification* of A to be an algebra B such that B is equationally compact and A is a subalgebra of B. B is a *quasi-compactification* of A if A is a subalgebra of B and every system of equations with constants in A and finitely solvable in A is solvable in B. The classes of compactifications resp. quasi-compactifications of A are denoted by Comp(A) resp. c(A). Clearly Comp(A) $\subseteq$ c(A). A *positive formula* is a formula of the first order predicate calculus which is built up from polynomial equations (of a fixed algebraic type) by application of the logical connectives $\forall$, $\exists$, $\wedge$, $\vee$ in a finite number of steps. We quote the following result of G.H. Wenzel:

Proposition 8 [14, Theorems 8.10,12]. Let A be an algebra and let K be one of Comp(A) or c(A). If K is not empty then there is an algebra B in K such that B satisfies every positive formula with constants in A which is satisfiable in A.

Proposition 9. Let R be a ring and Δ an infinite division ring. If R contains Δ as a subring, then c(R) = Φ. In particular, an infinite semisimple Artinian ring cannot be quasi-compactified, and hence not (algebraically) embedded into a compact topological ring. If R is an algebra over Δ and $R^2 \neq \{0\}$, then c(R) = Φ. If D denotes any divisible subgroup of R^+ and $R \cdot D \neq \{0\}$, then c(R) = Φ. In particular, if R is a subring of a compact topological ring, then $R \cdot D = D \cdot R = \{0\}$.

proof: If c(R) $\neq \Phi$, then c(R) contains a ring by Proposition 8; the proofs are then implicit in Proposition 3.

Proposition 10. Let R be an infinite Artinian ring with identity. Then Comp(R) = Φ. In particular, R cannot be (algebraically) embedded in a compact topological ring.

proof: R is Noetherian by a well-known result; hence R has finite length. If n is the (unique!) length of a maximal chain of left ideals then as is easily checked, the property of "maximal length of at most n" is characterized by the positive formula

$$\Psi = (\forall x_1)\cdots(\forall x_{n+2})(\exists y_1)\cdots(\exists y_{n+2})\left(\bigvee_{1<k\leq n+2} x_k = y_1x_1+\cdots+y_{k-1}x_{k-1}\right)$$

Thus if Comp(R) $\neq \Phi$, there is by Proposition 8 an $S \in$ Comp(R) satisfying Ψ, i.e., of finite length. But this cannot be, since by Corollary 4 S would be finite.

I am indebted to my supervisor, G.H. Wenzel, for his invaluable advice and assistance, and in particular for numerous corrections and improvements in the foregoing.

REFERENCES

[1] S. Balcerzyk, On the Algebraically Compact Groups of I. Kaplansky, Fund. Math. 44 (1957) pp.91-93

[2] G. Grätzer, Universal Algebra, The University Series in Higher Mathematics, Van Nostrand, 1968

[3] D.K. Haley, On Compact Commutative Noetherian Rings, Math. Ann. 189 (1970) pp. 272-274

[4] I.N. Herstein, Noncommutative Rings, The Carus Mathematical Monographs, John Wiley & Sons, 1968

[5] I. Kaplansky, Topological Rings, Am. J. Math. 69 (1947) pp. 153-183

[6] _________, Infinite Abelian Groups, Univ. of Michigan Press, Ann Arbor, 1954

[7] J.L. Kelley, General Topology, The University Series in Higher Mathematics, Van Nostrand, 1955

[8] J. Mycielski, Some Compactifications of General Algebras, Coll. Math. 13 (1964) pp. 1-9

[9] F. Szász, Über Artinsche Ringe, Bull. Acad. Polon. Sci. Ser. math. astr. phys. 11 (1963) pp.351-354

[10] S. Warner, Compact Noetherian Rings, Math. Ann. 141 (1960) pp. 161-170

[11] _________, Compact Rings, Math. Ann. 145 (1962) pp. 52-63

[12] _________, Linearly Compact Noetherian Rings, Math.Ann. 178 (1968) pp. 53-61

[13] B. Weglorz, Equationally Compact Algebras I, Fund. Math. 59 (1966) pp. 289-298

[14] G.H. Wenzel, Equational Compactness in Universal Algebras, Manuskripte der Fakultät f. Math. und Informatik d. Univ. Mannheim, Nr. 8, 1971

[15] D. Zelinsky, Linearly Compact Modules and Rings, Am. J. Math. 75 (1953) pp. 79-90