On a Question of G.H. Wenzel

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Nr. 21 (1971)

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§1. Introduction

A relational system $\mathfrak{A} = \langle A;F;R \rangle$ is called atomic compact iff the following condition is satisfied: if $\mathfrak{A} \in K(\tau)$ and if Σ is a set of atomic formulas of type τ (possibly modified by the substitution of elements of A as "constants") such that each finite subset of Σ has a simultaneous solution in $\mathfrak{A}$, then Σ itself has a simultaneous solution in $\mathfrak{A}$. If $R = \phi$ then the (universal) algebra $\mathfrak{A}$, should it be atomic compact, is called equationally compact. One should consult [2], [5], [7] and [8] for further basic information.

Let $\mathfrak{A}$ be a relational system and let λ be any ordinal. In [4] and [9] a mapping $L_\lambda : A^{\omega_\lambda} \longrightarrow A$ is called a λ-limit on $\mathfrak{A}$ provided it has the following three properties:

(1) L_λ is a weak homomorphism from $\mathfrak{A}^{\omega_\lambda}$ into $\mathfrak{A}$; i.e. L_λ is a homomorphism of the associated algebras, and if r is an n-ary relational symbol in R, and if $f_0,\ldots,f_{n-1}$ satisfy $r(f_0(\gamma),\ldots,f_{n-1}(\gamma))$ for all $\gamma < \omega_\lambda$, then $r(L_\lambda(f_0),\ldots,L_\lambda(f_{n-1}))$ holds.

(2) If $f(\gamma) = a \in A$ for all $\gamma < \omega_\lambda$, then $L_\lambda(f) = a$.

(3) If $f,g \in A^{\omega_\lambda}$ and if there exists $\alpha < \omega_\lambda$ such that $f(\gamma) = g(\gamma)$ for all $\gamma \geq \alpha$, then $L_\lambda(f) = L_\lambda(g)$.

G.H. Wenzel has recently proved, as part of the main theorem of [9], that an algebra $\mathfrak{A}$ is equationally compact if and only if there exists a λ-limit on $\mathfrak{A}$ for all $\lambda \leq \alpha$, where $\mathfrak{A}$ is of the (infinite) cardinality $\aleph_\alpha$. He then asks whether the existence of an α-limit on $\mathfrak{A}$ is sufficient for the equational compactness of an algebra $\mathfrak{A}$ of power $\aleph_\alpha$. (To quote Wenzel: "I conjecture a negative answer, although I have no counterexample.") This note provides various examples that settle positively the above conjecture.

§2. An Example in the Class of Algebras of Type $\langle 1 \rangle$

Let C_n $(n<\omega_0)$ be disjoint sets $C_n = \{a_{n,0},\ldots,a_{n,n}\}$ of $n+1$ elements such that each C_n is disjoint from the set $\omega_1 + 1 = \{\gamma \mid \gamma<\omega_1\} \cup \{\omega_1\}$. Let $C = \bigcup (C_n \mid n<\omega_0)$ and let $A = (\omega_1 + 1) \cup C$. Let $\mathfrak{A}$ be the mono-unary algebra $\langle A;\langle f \rangle\rangle$ where the mapping $f : A \longrightarrow A$ is defined by:

$$f(x) = \begin{cases} \gamma + 1 & \text{if } x = \gamma < \omega_1 \\ \omega_1 & \text{if } x = \omega_1 \\ 0 & \text{if } x = a_{n,0} \text{ for some } n<\omega_0 \\ a_{n,i-1} & \text{if } x = a_{n,i} \text{ for some } n<\omega_0 \\ & \text{and for some } i \geq 1. \end{cases}$$

Thus $\mathfrak{A}$ may be represented by the following diagram:

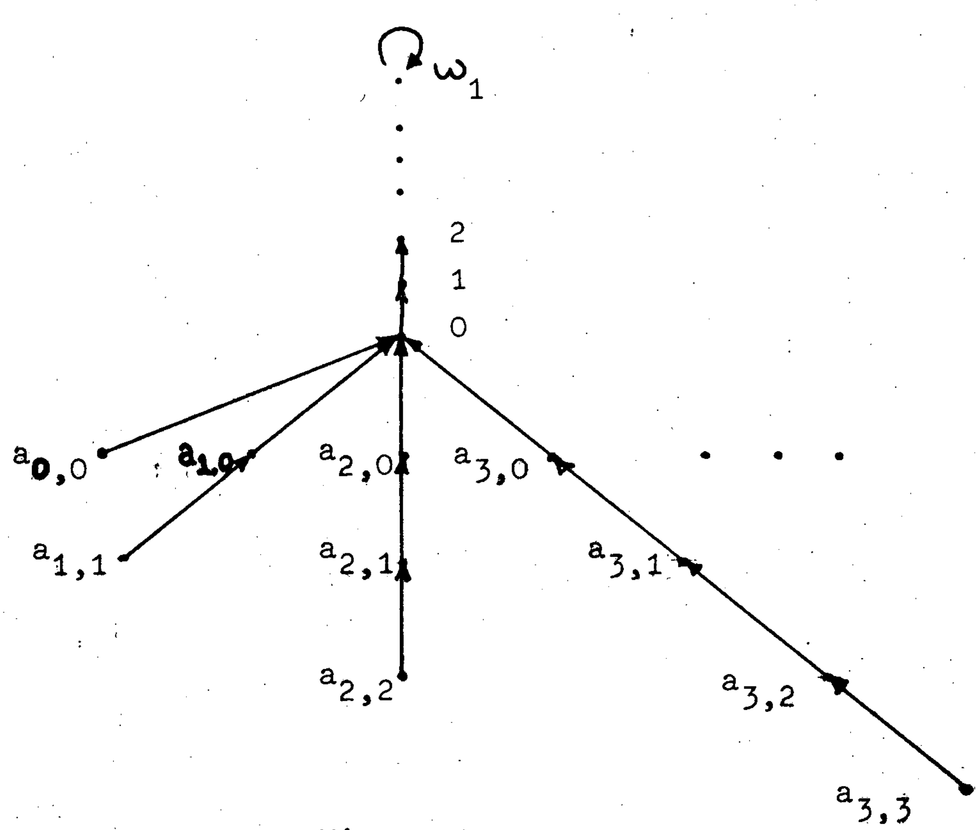

Figure 1

The algebra $\mathfrak{A}$ just defined will be shown to admit a 1-limit, although it has cardinality $\aleph_1$ and is not equationally compact. That $\mathfrak{A}$ is not equationally compact may be shown either by displaying a suitable set of equations or by applying the characterization theorem of G.H. Wenzel [10] of equationally compact algebras of type $\langle 1 \rangle$. A proof more in keeping with the nature of this report is the following: if $\mathfrak{A}$ were equationally compact, then a 0-limit L_0 would exist on $\mathfrak{A}$, by the result of Wenzel mentioned previously. Necessarily we would have $L_0((a_{n,0})_{n<\omega_0}) = a_{m,0}$ for some $m<\omega_0$. Defining $b_n = a_{n,m+1}$ for all $n \geq m + 1$, and letting $b_0, b_1, \ldots, b_m$ be arbitrary elements of A one finds that

$$f^{m+1}(L_0((b_n)_{n<\omega_0})) = a_{m,0}$$

which is impossible, since $a_{m,0} \neq f^{m+1}(x)$ for all $x \in A$.

A 1-limit will now be constructed on $\mathfrak{A}$. The following notational conventions will be adopted. An ω_1-sequence $(x_\gamma)_{\gamma<\omega_1}$ will be denoted simply by (x_γ), and if F is a mapping from A^{ω_1} to A then $F((x_\gamma))$ will be shortened to $F(x_\gamma)$. Frequently fx will be written instead of $f(x)$. The set A will be endowed with the following linear order:

(1) The elements of $\omega_1 + 1$ retain their usual ordering;

(2) Every element of C is less than 0;

<u>and</u> (3) $a_{n,i} < a_{m,j}$ iff $i > j$ or $[i = j$ and $m > n]$.

A sequence (x_γ) will be called <u>often</u> <u>x</u> for $x \in A$ iff $|\{\gamma<\omega_1 \mid x_\gamma = x\}| = \aleph_1$; (x_γ) will be called <u>minimally</u> <u>often</u> <u>x</u> for $x \in A$ iff (x_γ) is <u>often</u> <u>x</u>, and (x_γ) is not <u>often</u> <u>y</u>, for all $y < x$. The 1-limit L_1 is defined now as follows:

$$L_1(x_\gamma) = \begin{cases} x & \text{if } (x_\gamma) \text{ is } \underline{\text{minimally often}}\ \underline{x}. \\ \omega_1 & \text{if there exists no } x \in A \text{ such that } (x_\gamma) \text{ is } \underline{\text{minimally often}}\ \underline{x}. \end{cases}$$

To verify that L_1 is indeed a 1-limit on $\mathfrak{A}$, one may concentrate on showing that it is a homomorphism, the other parts of the definition of 1-limit being trivial to check. It must be shown that $L_1(fx_\gamma) = fL_1(x_\gamma)$ for all $(x_\gamma) \in A^{\omega_1}$. The verification is divided into three cases:

Case I: $L_1(x_\gamma) = \omega_1$.

If (x_γ) is minimally often $\underline{\omega_1}$, then so also is (fx_γ) and the conclusion follows. The remaining possibility is that there exists no $x \in A$ such that (x_γ) is minimally often $\underline{x}$. In this eventuality, the same is true for the sequence (fx_γ) and the desired result appears.

Case II: $L_1(x_\gamma) = \delta$ for some $0 \leq \delta < \omega_1$.

This case is trivially disposed of.

Case III: $L_1(x_\gamma) = a_{n,i}$ for some $n, i < \omega_0$.

If $i = 0$ then (x_γ) is minimally often $\underline{a}_{n,0}$, and hence (fx_γ) is minimally often $\underline{0}$. Consequently $L_1(fx_\gamma) = 0 = fa_{n,0} = fL_1(x_\gamma)$.

If $i > 0$ then (x_γ) is minimally often $\underline{a}_{n,i}$, which clearly implies that (fx_γ) is minimally often $\underline{a}_{n,i-1}$. Again the required equality can be calculated.

The verification of this example is now complete.

§3. An Example in the Class of Algebras of Type $\langle 1,1 \rangle$

To begin with, a strictly relational system $\mathfrak{A} = \langle A; \langle R \rangle\rangle$

where R is a binary relation on A will be constructed such that $|A| = \aleph_1$, such that the system $\mathfrak{A}$ is not atomic compact, but such that a 1-limit exists on $\mathfrak{A}$. Then, from $\mathfrak{A}$ an algebra $\mathfrak{B} = \langle B; \langle f,g \rangle\rangle$ of type $\langle 1,1\rangle$ with the same properties will be determined. (For an earlier example of such a conversion, see [6]).

Let $A = \omega_1 + 1$, and define a binary relation R on A by: $\langle x,y\rangle \in R$ iff $x < y$ or $x = y = \omega_1$. That $\mathfrak{A} = \langle A; \langle R\rangle\rangle$ is not atomic compact can be deduced by considering the following system Σ of atomic formulas with constants in A:

$$R(x_1, \omega_0)$$

$$R(x_2, x_1)$$

$$\vdots$$

$$R(x_{n+1}, x_n)$$

$$\vdots$$

However, retaining the notions of <u>often x</u> and <u>minimally often x</u> introduced in §2, it is not difficult to verify that $L : A^{\omega_1} \longrightarrow A$ defined by

$$L(x_\gamma) = \begin{cases} x & \text{if } (x_\gamma) \text{ is minimally often } x \\ \omega_1 & \text{if there exists no } x \in A \text{ such that } (x_\gamma) \text{ is minimally often } x \end{cases}$$

is a 1-limit on $\mathfrak{A}$.

The algebra $\mathfrak{B} = \langle B; \langle f,g\rangle\rangle$ of type $\langle 1,1\rangle$ mentioned above is defined in the following fashion. Let π_1 and π_2 denote the first and second coordinate projections respectively of the graph of R (here denoted also by R) to the set A, where A and R are as in the preceding paragraph. Let B be the set $A \cup R \cup \{z\}$, where z belongs neither to A nor to R. The mappings $f, g : B \longrightarrow B$ are defined by

$$f(x) = \begin{cases} \pi_1(x) & \text{if } x \in R \\ z & \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} \pi_2(x) & \text{if } x \in R \\ z & \text{otherwise.} \end{cases}$$

That $\mathcal{L}$ is not equationally compact can be seen from the following system Σ' of equations with constants in B (which corresponds to the system Σ already considered for the relational system $\mathfrak{A}$):

$$g(z_1) = \omega_0$$

$$f(z_1) = g(z_2)$$

$$f(z_2) = g(z_3)$$

$$\vdots$$

$$f(z_n) = g(z_{n+1})$$

$$\vdots$$

To complete this example, a 1-limit $\bar{L}$ on the algebra $\mathcal{L}$ is now given. Let $(x_\gamma) \in B^{\omega_1}$. The standard terminology that (x_γ) is <u>eventually in the set X</u> iff there exists $\alpha < \omega_1$ such that $x_\gamma \in X$ for all $\gamma > \alpha$ will be used. $\bar{L}(x_\gamma)$ is defined according to the nature of (x_γ) as follows:

(1) <u>(x_γ) is eventually in A</u>; <u>i.e.</u> there exists $\gamma_0 < \omega_1$ such that $x_\gamma \in A$ for all $\gamma \geq \gamma_0$. Let (a_γ) be an arbitrary element of A^{ω_1}, and define $(y_\gamma) \in A^{\omega_1}$ by $y_\gamma = x_\gamma$ if $\gamma \geq \gamma_0$, and $y_\gamma = a_\gamma$ if $\gamma < \gamma_0$. Define $\overline{L}(x_\gamma) = L(y_\gamma)$.

(2) <u>(x_γ) is eventually in R</u>; <u>i.e.</u> there exists $\gamma_0 < \omega_1$ such that $x_\gamma = \langle u_\gamma, v_\gamma \rangle \in R$ for all $\gamma \geq \gamma_0$. Define (s_γ), $(t_\gamma) \in A^{\omega_1}$ as follows:

$$s_\gamma = \begin{cases} 0 & \text{if } \gamma < \gamma_0 \\ u_\gamma & \text{if } \gamma \geq \gamma_0 \end{cases}$$

$$t_\gamma = \begin{cases} 1 & \text{if } \gamma < \gamma_0 \\ v_\gamma & \text{if } \gamma \geq \gamma_0. \end{cases}$$

Then $\langle (s_\gamma),(t_\gamma) \rangle \in A^{\omega_1} \times A^{\omega_1}$ is in the relation R (interpreted as a binary relation on A^{ω_1} in the natural way) and since L is a homomorphism of $\mathfrak{A}^{\omega_1}$ into $\mathfrak{A}$, the element $r = \langle L(s_\gamma), L(t_\gamma) \rangle$ belongs to R (this time reading $R \subseteq A \times A$). Define $\overline{L}(x_\gamma) = r$.

(3) <u>(x_γ) is not eventually in A and not eventually in R</u>. In this case define $\overline{L}(x_\gamma) = z$.

The verification that $\overline{L}$ is a 1-limit can be carried out in similar fashion to the verification of the corresponding result of §2.

§4. <u>Concluding Remarks</u>

Assuming the Continuum Hypothesis we have an example of a <u>semilattice</u> $\mathcal{T} = \langle S; \{\wedge\} \rangle$ such that $\mathcal{T}$ has a 1-limit, $|S| = \aleph_1$, and $\mathcal{T}$ is not equationally compact. To produce this example, let S be the set of non-negative real numbers, let $\wedge$ denote the usual binary "minimum" operation on S, and let $L_1(x_\gamma) = \lim\inf (x_\gamma)$ for any $(x_\gamma) \in S^{\omega_1}$. It can be shown that this limit always exists and is in fact a 1-limit for $\mathcal{T}$. However $\mathcal{T}$ is not equationally compact, since by [2] equationally compact $\wedge$-semilattices must contain $\bigvee C$ for every chain $C \subseteq S$.

Can the above example be converted to an example of power $\aleph_1$? The fact that lim inf exists for ω_1-sequences relies on the axiom of Archimedes; and it is not altogether clear that there exists an uncountable Archimedean order type which is conditionally complete for all ω_1-sequences unless it has already the power of the continuum. For that matter, we may ask whether any of the examples given in this note can be turned into semilattice examples.

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