Long-Range and Periodic Solutions of Parabolic Problems

by

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1. Introduction

The existence of periodic solutions of parabolic problems has been proved by Šmulev [26], Kružkov [19], and Kolesov [17]. Friedman [8 - 12] and others [6, 13, 18, 21, 24, 28] studied the asymptotic behaviour of solutions of parabolic initial boundary value problems and showed that these functions $u: (x,t) \longmapsto u(x,t)$ frequently converge in time t to a steady state $u^*: x \longmapsto u^*(x)$ which may be gained as solution of a boundary value problem. The present paper concerns the numerical solution of such problems.

For the practical computation the knowledge of the steady state u^* has been used first by Greenspan [15] and later by Carasso-Parter [4] and Carasso [5] in two important papers. They put in this 'boundary value technique' $u(\cdot,T) = u^*$ for sufficiently large T and approximate by this way the original initial boundary value problem by a boundary value problem. In solving the latter the characteristic difficulties of stability of initial boundary problems appear in some milder form while the computational effort depends on the speed with which $u(\cdot,t)$ converges to u^*. The approximating boundary problem is solved by finite difference methods as in elliptic problems. We drop in this paper some restricting assumptions [5,p. 307] by choosing a norm more adequate to the given problem. Further, the proof of convergence in [5] is simplified by means of the monotonicity principle of Minty [22, 23]. Moreover, parabolic problems with degenerating differential equation are handled also.

First suggestions for the practical treatment of parabolic problems with periodic solutions have been made by Tee [27] and Osborne [25]. But they confined themself on the equation $u_t = u_{xx}$ and substituted u_t by a finite difference approximation of order one (compare also [14]). We start out from more general equations and study three approximations of order two. It results that in the most usual case of these finite difference approximations the periodic problems are numerically rather related to the approximating boundary value problem in the mentioned boundary value method.

Proving the stability of the approximations we can no longer work in this paper with the notions 'positive definite' and 'M-matrix'; we must use instead the monotonicity in the meaning of Minty and a result of [2, 4] which we call the principle of Carasso-Parter.

2. Mildly Nonlinear Problems

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N} = \{1,2,...\}$, $G = \{x \in \mathbb{R},\ 0 < x < 1\}$, $Z^+ = G\times(0,\infty)$, and let $\overline{Z^+}$ be the closure of Z^+. The derivative of a function $f\colon \mathbb{R} \ni x \longmapsto f(x) \in \mathbb{R}$ is denoted by f_x. We consider first the initial boundary value problem

$$(1)\quad \begin{aligned} &a(x)u_t - [b(x,t)u_x]_x + c(x,t)u_x + f(x,t,u) = 0,\ (x,t) \in Z^+,\\ &u(x,0) = r(x),\ x \in G,\ u(0,t) = s^-(t),\ u(1,t) = s^+(t),\ 0 \le t,\end{aligned}$$

satisfying the following conditions:

Assumption I. Suppose there exists a unique classical solution $u\colon (x,t) \longmapsto u(x,t)$ of (1) which converges to a known steady state u^*. Let be known an increasing positive function $t\colon \mathbb{R}^+ \ni \varepsilon \longmapsto t(\varepsilon) \in \mathbb{R}^+$ with

$$|u(\cdot,t(\varepsilon)) - u^*|_\infty := \sup_{x\in G} |u(x,t(\varepsilon)) - u^*(x)| \le \varepsilon.$$

Assumption II. (i) Let a, b, c, r, s^-, s^+ be continuous and bounded in $\overline{Z^+}$; let f be continuous in $Z^+\times\mathbb{R}$ and bounded in $\overline{Z^+}$ for bounded $u \in \mathbb{R}$.

(ii) Let c be continuously differentiable in Z^+, and let

$$0 < \beta \le b(x,t),\ |c_x(x,t)| \le \gamma, \qquad (x,t) \in Z^+,$$

$$(f(x,t,v) - f(x,t,w))(v - w) \ge 0, \qquad (x,t) \in Z^+,\quad v,w \in \mathbb{R}.$$

(iii) Let u_{ttt}, u_{xxxx}, b_{xxx} as well as all lower derivatives of u and b be continuous and bounded in $\overline{Z^+}$.

Further, let Δt and $\Delta x = 1/(M + 1)$, $M \in \mathbb{N}$, be small increments of the variables x, t, and let $N\colon \Delta t \longmapsto N(\Delta t) \in \mathbb{N}$ be a function which we shall define more exactly later. Put

(2) $$v_k^n = v(k\Delta x, n\Delta t),\ V^n = (v_1^n, \dots, v_M^n)^T,\ V = (V^1, \dots, V^{N(\Delta t)})^T,$$
$$(V,W) = \frac{\Delta x}{N(\Delta t)} \sum_{n=1}^{N(\Delta t)} (V^n)^T W^n, \quad |V|_2^2 = (V,V).$$

Following Greenspan [15], Carasso-Parter [4], and Carasso [5] we approximate the problem (1) by

(3) $$a_k \frac{v_k^{n+1} - v_k^{n-1}}{2\Delta t} - \frac{b_{k+1/2}^n (v_{k+1}^n - v_k^n) - b_{k-1/2}^n (v_k^n - v_{k-1}^n)}{\Delta x^2}$$
$$+ c_k^n \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} + f_k^n(v_k^n) = 0, \qquad k = 1,\dots,M,\quad n = 1,\dots,N(\ t),$$

with the boundary conditions

(4) $$v_k^o = r(k\Delta x),\ v_k^{N(\Delta t)+1} = u^*(k\Delta x),\ k = 0,\dots,M+1,$$
$$v_o^n = s^-(n\Delta t),\ V_{M+1}^n = s^+(n\Delta t),\ n = 1,\dots,N(\Delta t).$$

We collect the $M \times N(\Delta t)$ equations (3) in the usual way, i.e. keeping first $n = 1$ fixed and going from $k = 1$ to $k = M$ then keeping $n = 2$ fixed and going from $k = 1$ to $k = M$ and so on. So we obtain the following system of equations

(5) $$P_1(V) := (AS_1 + B + C)V + F(V) = H_1.$$

Here, H_1 and $F(V)$ are vectors of the form of V in (2), H_1 contains only known quantities and $F(V)$ contains the elements $f_k^n(v_k^n) = f(k\Delta x, n\Delta t, v_k^n)$. A, B, C are block diagonal matrices
$$A = (A^1, \dots, A^{N(\Delta t)})$$
with the submatrices $A^n = (a_{ik}^n)_{i,k=1,\dots,M}$ etc., and

$$a_{ik}^n = a_k \delta_{ik}/2\Delta t = a(k\Delta x)\delta_{ik}/2\Delta t \qquad (\delta_{ik} \text{ Kroneckers symbol}),$$

$$b^n_{ik} = \begin{cases} + (b^n_{k+1/2} + b^n_{k-1/2})/\Delta x^2 & i = k \\ - b^n_{k+1/2}/\Delta x^2 & i = k+1 \\ - b^n_{i+1/2}/\Delta x^2 & k = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$c^n_{ik} = \begin{cases} c^n_i/2\Delta x & k = i+1 \\ - c^n_i/2\Delta x & i = k+1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_1 = \begin{pmatrix} 0 & I & & \bigcirc \\ -I & 0 & \ddots & \\ & \ddots & \ddots & I \\ \bigcirc & & -I & 0 \end{pmatrix}$$

(I identity matrix of dimension M).

In the following theorems U denotes the blockvector (2) of the solution of the analytic problem at the mesh points $(k\Delta x, n\Delta t)$ and const is a generic positive constant independent of Δx and Δt.

<u>Theorem 1</u>. Suppose the initial boundary value problem (1) satisfies Assumptions I and II and assume there exists a $\varepsilon > 0$ with $\beta\pi^2 - (\gamma/2) - \varepsilon > 0$. Choose N such that

(6) $\quad |u(\cdot,(N(\Delta t)+1)\Delta t) - u^*|_\infty \leq \text{const } \Delta t^{5/2}$,

$N(\Delta t) \geq \Delta t^{-1}$. If $0 < \Delta x \leq (12\varepsilon/\beta\pi^4)^{1/2}$ and $\Delta t/\Delta x = \text{const}$ then there exists a unique solution V_Δ of (5) and

(7) $\quad |V_\Delta - U|_2 \leq \text{const } \Delta t^2$.

<u>Proof</u>. The matrix AS_1 is skew-symmetric and by Carasso [3,5] we

$$(P_1(Y) - P_1(Z), Y - Z) \geq \alpha_\varepsilon |Y - Z|_2^2 \qquad Y,Z \in \mathbb{R}^{M\times N(\Delta t)}$$

where $\alpha_\varepsilon = \beta\pi^2 - (\gamma/2) - \varepsilon > 0$ independent of Δx and Δt. Therefore, by Browder [1] and Minty [22, 23] $P_1^{-1}: \mathbb{R}^{M\times N(\Delta t)} \to \mathbb{R}^{M\times N(\Delta t)}$ exists and is continuous; furthermore $|V_\Delta - U|_2 = |P_1^{-1}(P_1(V_\Delta)) - P_1^{-1}(P_1(U))|_2 \leq \alpha_\varepsilon^{-1}|P_1(U) - H_1|_2$.

The block vector $\Theta = P_1(U) - H_1$ consists of the local truncation error $\theta_{t,k}^n$, $k = 1,\ldots,M$, $n = 1,\ldots,N(\Delta t)$, and the boundary error

$$(8) \qquad \theta_{b,k}^{N(\Delta t)} = \frac{u(k\Delta x,(N(\Delta t)+1)\Delta t) - u^*(k\Delta x)}{2\Delta t}, \; k = 1,\ldots,M.$$

$\theta_{t,k}^n$ is a linear combination of $\Delta t^2 u_{ttt}$, $\Delta x^2 u_{xxxx}$, and $\Delta x^2 b_{xxx}$, the derivatives being evaluated at points of Z^+ intermediate to successivemesh points, therefore $|\theta_{t,k}^n| \leq$ const Δt^2. Putting $|\Theta^n|_2^2 = \Delta x(\Theta^n)^T\Theta^n$ and appraising (8) by means of (6) we obtain

$$|\Theta|_2^2 = \frac{1}{N(\Delta t)}\sum_{n=1}^{N(\Delta t)} |\Theta^n|_2^2 = \frac{1}{N(\Delta t)}\sum_{n=1}^{N(\Delta t)-1} |\Theta_t^n|_2^2 + \frac{1}{N(\Delta t)}|\Theta_t^{N(\Delta t)} + \Theta_b^{N(\Delta t)}|_2^2$$

$$\leq \frac{1}{N(\Delta t)}\sum_{n=1}^{N(\Delta t)} |\Theta_t^n|_2^2 + 2\Delta t|\Theta_t^{N(\Delta t)}|_2|\Theta_b^{N(\Delta t)}|_2 + \Delta t|\Theta_b^{N(\Delta t)}|_2^2 \leq \text{const } \Delta t^4$$

which proves the theorem.

<u>Remark 1</u>. If the solution u of (1) satisfies $u(x,t) = u^*(x) + t^{-r}k(x,t)$ with $|k(\cdot,t)|_\infty \leq$ const as $t \to \infty$ then (6) means

$$N(\Delta t) \geq (\Delta t)^{-(1 + 5/2r)} - 1.$$

If $|u(\cdot,t) - u^*|_\infty \leq$ const $e^{-\alpha t}$ $(\alpha > 0)$ then we must have

$$N(\Delta t) \geq -\frac{5}{2\alpha\Delta t}(\log\Delta t) - 1.$$

Let now $Z = G\times(-\infty,\infty)$ and consider the parabolic problem

$$(9)\quad \begin{aligned} &a(x)u_t - [b(x,t)u_x]_x + c(x,t)u_x + f(x,t,u) = 0, \; (x,t) \in Z, \\ &u(0,t) = s^-(t), \; u(1,t) = s^+(t), \quad -\infty < t < \infty, \end{aligned}$$

satisfying Assumption II and

Assumption III. Suppose the functions a, b, c, f, s^-, s^+ are periodic in t with period τ and assume there exists a unique classical solution u of the problem (9) periodic in t with period τ.

Because of the periodicity it suffices to approximate the solution u at the mesh points of $G\times(0,\tau]$. We choose Δx as above and $\Delta t = \tau/N$, $N \in \mathbb{N}$. Collecting the $M\times N$ finite difference approximations (3) of the differential equation in (9) in the same way as before we obtain the system

$$(10)\qquad P_2(V) := (AS_2 + B + C)V + F(V) = H_2.$$

Observe that now no boundary values are given on the lines $n = 0$ and $n = N$, instead we must put $V^0 = V^N$. Thus, in comparison with (5) nothing changes but S_2 and H_2. H_2 contains again only known values and S_2 is the skew-symmetric block matrix

$$S_2 = \begin{pmatrix} 0 & I & & O & -I \\ -I & 0 & \ddots & & O \\ & O & \ddots & \ddots & I \\ I & & O & -I & 0 \end{pmatrix}.$$

Therefore, if we take into account the periodicity of U in the appraisal of $|P_2(U) - H_2|_2$ then it results just as before

Theorem 2. Suppose the boundary value problem (9) satisfies Assumptions II and III and assume there exists a $\varepsilon > 0$ with $\beta\pi^2 - (\gamma/2) - \varepsilon > 0$. If $0 < \Delta x \leq (12\varepsilon/\beta\pi^4)^{1/2}$ then a unique solution V_Δ of (10) exists and

$$|V_\Delta - U|_2 \leq \text{const } (\Delta x^2 + \Delta t^2).$$

Remark 2. Let

$$|V_x^n|_2^2 = \Delta x \sum_{k=1}^{M} \frac{|v_{k+1}^n - v_k^n|^2}{\Delta x^2}, \quad |V_x|_2^2 = \frac{1}{N(\Delta t)} \sum_{n=1}^{N(\Delta t)} |V_x^n|_2^2 ,$$

where $v_0^n = v_{M+1}^n = 0$. Carasso [5, p.311, 312] deduces from (7) that $|V_{\Delta,x} - U_x|_2 \leq \text{const } \Delta t^2$, $\sup_n |V_{\Delta,x}^n - U^n|_2 \leq \text{const } \Delta t^{3/2}$, and

(11) $$\max_{k,n} \{|v_k^n - u_k^n|\} \leq \text{const } \Delta t^{3/2}$$

in case $\Delta t/\Delta x = \text{const}$. The same results are naturally valid for Theorem 2.

3. The Principle of Carasso-Parter

Let be given the problem

$$(12)\quad \begin{aligned} &u_t - [b(x)u_x]_x + c(x)u_x + g(x)u + e(x,t) = 0,\ (x,t) \in Z^+, \\ &u(x,0) = r(x),\ x \in G,\ u(0,t) = s^-(t),\ u(1,t) = s^+(t),\ 0 \leq t, \end{aligned}$$

satisfying Assumptions I and II ($f(x,t,u) = g(x)u + e(x,t)$); but instead of $|c_x|_\infty \leq \gamma$ we assume here c continuous and $|c|_\infty \leq \mu$. In place of (5) we have now

$$(13)\qquad P_1 V := \left(\frac{1}{2\Delta t} S_1 + B + C + G\right)V = H_3 .$$

The block diagonal matrices B and C as well as G, which is now independent of V, consist of identical submatrices $\tilde{B}$, $\tilde{C}$, and $\tilde{G}$ respectively because the functions b, c, and g in (12) are independent of t. We use the norms

$$|V|_\infty = \max_{1 \leq k \leq s} \{|v_k|\},\ V \in \mathbb{R}^s,\ |Q|_\infty = \max_{1 \leq i \leq s} \{\sum_{k=1}^{s} |q_{ik}|\},\ Q\ s\times s\text{-matrix}.$$

Carasso [2 , Lemma 1] has shown that for $0 < \Delta x \leq \beta/\mu$ there exists a diagonal matrix $\tilde{D}$ with the properties:

$$\tilde{D}^{-1}(\tilde{B} + \tilde{C})\tilde{D} \text{ symmetric},\ |\tilde{D}|_\infty \leq \text{const},\ |\tilde{D}^{-1}|_\infty \leq \text{const} .$$

Thus the $M\times M$-matrix $\tilde{D}^{-1}(\tilde{B} + \tilde{C})\tilde{D}$ has a complete set of orthonormal eigenvalues W_k corresponding to the eigenvalues λ_k. Let $\tilde{\Lambda}$ be the diagonal matrix of the λ_k,

$$\tilde{W} = (W_1, \ldots, W_M),$$

and let Λ, D, and W be the block diagonal matrices with the submatrices $\tilde{\Lambda}$, $\tilde{D}$, and $\tilde{W}$ respectively. Then we may expand

$$D^{-1}(P_1 U - H_3) = W\Psi, \quad D^{-1}P_1 D = WT_1W^{-1}$$

where Ψ is a block vector because the eigenvectors W_k are orthonormal and T_1 is a certain block matrix. Consequently, we have in case T_1^{-1} exists and all λ_k are positive

$$|V_\Delta - U|_\infty = |DD^{-1}P_1^{-1}(P_1U - H_3|_\infty \leq |D|_\infty |W\, \Lambda^{-1}\Lambda\, T_1^{-1}\Psi|_\infty$$

(14)
$$\leq |D|_\infty \sum_{k=1}^{M} \frac{|W_k|_\infty}{\lambda_k} |\hat{T}_{1,N}^{-1}(\lambda_k)\Psi_k|_\infty$$

where $\Psi_k = (\psi_k^1, \ldots, \psi_k^N)^T$. $\hat{T}_{1,N}(\lambda)$ is a certain N×N-matrix which results from T_1 and may be gained easy. On the other side T_1^{-1} and P_1^{-1} exist if all λ_k are positive and $\hat{T}_{1,N}^{-1}(\lambda_k)$ exists. Now, Carasso [2] has proved the important result that under the above conditions $\sum_{k=1}^{M}(|W_k|_\infty/\lambda_k)$ remains bounded for $\Delta x \longrightarrow 0$ if the W_k are normalized by $|W_k|_2 = 1$. But then

$$|\Psi_k^n| = |(D^{-1}(P_1U - H_3)^n)^T W_k / W_k^T W_k|$$

(15)
$$\leq |D^{-1}|_\infty |(P_1U - H_3)^n|_2 \;.$$

We formulate the result in the already established notations as

Principle of Carasso-Parter. Let $0 < \Delta x = 1/(M+1) \leq \beta/\mu$. If $|\hat{T}_{1,N}^{-1}(\lambda)|_\infty \leq$ const for $\lambda \geq \lambda_0 > 0$, $\Delta t > 0$, $N \in \mathbb{N}$, and if $|(P_1U - H_3)^n|_2 \leq$ const $(\Delta x^2 + \Delta t^2)$, $n = 1,\ldots,N$, then $|V_\Delta - U|_\infty \leq$ const $(\Delta x^2 + \Delta t^2)$.

Theorem 3 (Carasso-Parter [4]). Suppose the initial boundary value problem (12) satisfies Assumptions I and II but let $|c|_\infty \leq \mu$ instead of $|c_x|_\infty \leq \gamma$. Choose N such that

(16) $\quad |u(\cdot,(N(\Delta t)+1)\Delta t) - u^*|_\infty \leq$ const Δt^3,

$N(\Delta t) \geq \Delta t^{-1}$. If $0 < \Delta x \leq \beta/\mu$ and $\Delta t/\Delta x$ = const then there exists a unique solution V_Δ of (13) and

$$|V_\Delta - U|_\infty \leq \text{const } \Delta t^2.$$

<u>Proof</u>. Appraising (14) by (15) and Assumption II we obtain $|\psi_k^n| \leq \text{const } \Delta t^2$. $\hat{T}_{1,N}(\lambda)$ has the form

$$\hat{T}_{1,N}(\lambda) = (I + \frac{1}{2\lambda\Delta t} S_{1,N}), \quad S_{1,N} = \begin{pmatrix} 0 & 1 & & 0 \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{pmatrix}$$

and it is to be shown that $|\hat{T}_{1,N}^{-1}(\lambda)|_\infty$ remains bounded. This is a result of Carasso-Parter [4, Lemma 2.1].

Just so we obtain

<u>Theorem 4</u>. Suppose the boundary value problem (9) with the differential equation of (12) satisfies Assumptions II and III but let $|c|_\infty \leq \mu$ instead of $|c_x|_\infty \leq \gamma$. If $0 < \Delta x \leq \beta/\mu$ then a unique solution V_Δ of the corresponding system (10) exists and

$$|V_\Delta - U|_\infty \leq \text{const}(\Delta x^2 + \Delta t^2).$$

<u>Proof</u>. The corresponding matrix $\hat{T}_{2,N}(\lambda)$ has the form

$$\hat{T}_{2,N}(\lambda) = (I + \frac{1}{2\lambda\Delta t} S_{2,N}), \quad S_{2,N} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

$|\hat{T}_{2,N}^{-1}(\lambda)|_\infty \leq \text{const}$ is proved in a similar way as [4, Lemma 2.1]. computing $\hat{T}_{2,N}^{-1}(\lambda)$ explicitly.

Remark 3. By the transformation

$$u(x,t) = \left[\exp\left(\frac{1}{2}\int_0^x \frac{c(s)}{b(s)}\,ds\right)\right]v(x,t)$$

we obtain from the boundary value problem (9) with the differential equation (12) the equally periodic boundary value problem

$$v_t - [b(x)v_x]_x + \left[g(x) - \frac{1}{2}c_x(x) + \frac{c^2(x)}{4b(x)}\right]v$$

$$+ \left[\exp\left(-\frac{1}{2}\int_0^x \frac{c(s)}{b(s)}\,ds\right)\right]e(x,t) = 0, \quad (x,t) \in Z,$$

$$v(0,t) = s^-(t), v(1,t) = s^+(t)\left[\exp\left(-\frac{1}{2}\int_0^1 \frac{c(s)}{b(s)}\,ds\right)\right], \quad -\infty < t < \infty.$$

We may apply Theorem 2 to this problem if

$$g(x) - \frac{1}{2}c_x(x) + \frac{c^2(x)}{4b(x)} \geq 0, \qquad x \in G.$$

But observe that (11) is slightly weaker than Theorem 4.

4. Further Applications of the Principle of Carasso-Parter

Suppose once more that the periodic boundary value problem

$$(17)\quad \begin{aligned} &u_t - [b(x)u_x]_x + c(x)u_x + g(x)u + e(x,t) = 0, \quad (x,t) \in Z, \\ &u(0,t) = s^-(t), \ u(1,t) = s^+(t), \ -\infty < t < \infty, \end{aligned}$$

satisfies Assumptions II and III. As in the method of Crank-Nicolson (compare Lees [20]) we approximate (17) in $G\ (0,\tau]$ by

$$(18)\quad \begin{aligned} &\frac{v_k^n - v_k^{n-1}}{\Delta t} \\ &- \frac{1}{2}\left[\frac{b_{k+1/2}(v_{k+1}^n - v_k^n) - b_{k-1/2}(v_k^n - v_{k-1}^n)}{\Delta x^2}\right. \\ &\qquad\qquad \left. + \frac{b_{k+1/2}(v_{k+1}^{n-1} - v_k^{n-1}) - b_{k-1/2}(v_k^{n-1} - v_{k-1}^{n-1})}{\Delta x^2}\right] \\ &- \frac{1}{2} c_k\left[\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} + \frac{v_{k+1}^{n-1} - v_{k-1}^{n-1}}{2\Delta x}\right] + \frac{1}{2} g_k\left[v_k^n + v_k^{n-1}\right] + e_k^{n-1/2} = 0, \\ &\qquad\qquad k = 1,\ldots,M, \ n = 1,\ldots,N, \end{aligned}$$

with the boundary values $v_o^n = s^-(n\Delta t)$, $v^n_{M+1}(n\Delta t)$, $n = 1,\ldots,N$, and $V^o = V^N$. Again we may write (18) in matrix-vector notation as

$$(19)\quad P_3 V := \frac{1}{\Delta t}(I - S_3)V + \frac{1}{2}(B + C + G)(I + S_3)V = H_4.$$

Here, the block diagonal matrices B, C, and G are already known, and S_3 is the N-cyclic matrix

$$S_3 = \begin{pmatrix} O & & & I \\ I & O & & \\ & \ddots & \ddots & \\ & & I & O \end{pmatrix}$$

(compare Varga [29, p. 35], I identity matrix of dimension M) with the eigenvalues

$$\lambda_n(S_3) = \exp(2\pi i n/N), \qquad n = 0,\dots,N-1.$$

Theorem 5. Suppose the boundary value problem (17) satisfies Assumptions II and III but let $|c|_\infty \leq \mu$ instead of $|c_x|_\infty \leq \gamma$. If $0 < \Delta x \leq \beta/\mu$ then a unique solution V_Δ of (19) exists and

$$(20) \qquad |V_\Delta - U|_\infty \leq \mathrm{const}\left(1 + \frac{\Delta t}{\Delta x^2}\right)(\Delta x^2 + \Delta t^2).$$

Proof. The block diagonal matrix $Q = B + C + G$ consists of identical submatrices $\tilde{Q}$ and, as already shown, the eigenvalues $\lambda_k(\tilde{Q})$, $k = 1,\dots,M$, are positive in case $0 < \Delta x \leq \beta/\mu$. Therefore, the eigenvalues of $(I + \Delta t Q/2)^{-1}(I - \Delta t Q/2)S_3$ are less than one in absolute value and

$$P_3^{-1} = [I - \{(\tfrac{1}{\Delta t}I + \tfrac{1}{2}Q)^{-1}(\tfrac{1}{\Delta t}I - \tfrac{1}{2}Q)S_3\}]^{-1}\,[\tfrac{1}{\Delta t}I + \tfrac{1}{2}Q]^{-1}$$

exists. Let

$$S_{3,N} = \begin{pmatrix} 0 & & \bigcirc & 1 \\ 1 & 0 & & \bigcirc \\ & \ddots & \ddots & \\ \bigcirc & & 1 & 0 \end{pmatrix}$$

and

$$\hat{T}_{3,N}(\lambda) = \frac{1}{\lambda\Delta t}(I - S_{3,N}) + \frac{1}{2}(I + S_{3,N})$$

be matrices of dimension N. According to the Principle of Carasso-Parter we have to show that $|\hat{T}_{3,N}^{-1}(\lambda)|_\infty$ remains bounded for the eigenvalues $\lambda = \lambda(\tilde{Q})$ and $\tau/\Delta t = N \in \mathbb{N}$. Put $\sigma = 2/\lambda\Delta t$ then

$$\hat{T}_{3,N}^{-1}(\lambda) = \frac{2}{1+\sigma}\left(I + \frac{1-\sigma}{1+\sigma}S_{3,N}\right)^{-1}.$$

Using $|S^m_{3,N}|_\infty = 1$, $m \in \mathbb{N}$, we obtain

$$(21) \qquad |\hat{T}^{-1}_{3,N}(\lambda)|_\infty \leq \frac{2}{1+\sigma} \frac{1}{1 - \left|\frac{1-\sigma}{1+\sigma}\right|} \leq 1 + \frac{\lambda \Delta t}{2} .$$

Now, a result of Carasso [2, Lemma 3] says that $0 < \lambda_k(\tilde{Q}) \leq \text{const}\, M^2$ is true for the eigenvalues of $\tilde{Q}$ if Δx is sufficiently small. Inserting $\lambda = \lambda_k(\tilde{Q})$ in (21) we obtain

$$|\hat{T}^{-1}_{3,N}(\lambda_k)|_\infty \leq 1 + \kappa_1 \frac{\Delta t}{\Delta x^2} \leq \kappa_2 (1 + \frac{\Delta t}{\Delta x^2})$$

($\kappa_2 = \max\{1, \kappa_1\}$). Finally, we obtain $|P_3 U - H_4|_\infty \leq \text{const}(\Delta x^2 + \Delta t^2)$ as Lees [20] taking into account $U^0 = U^N$ again; this completes the proof.

Remark 4. The normalized eigenfunctions of the Sturm-Liouville eigenvalue problem

$$(22) \qquad \begin{aligned} &-[b(x)u_x]_x + c(x)u_x + g(x)u = \lambda u, \quad 0 < x < 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

where $b(x) \geq \beta > 0$, $g(x) \geq 0$ and b, c, g are all smooth functions, are uniformly bounded in the supremum norm (see Courant-Hilbert [7]). The same is true for the discrete eigenvalue problem

$$-\frac{v_{k+1} - 2v_k + v_{k-1}}{\Delta x^2} + g_k v_k = \lambda v_k, \qquad k = 1,\ldots,M,$$

$$v_0 = v_{M+1} = 0,$$

corresponding to (22) with the equation $u_{xx} + g(x)u = \lambda u$ (see for instance Isaacson-Keller [16]). Let us suppose for the moment that the normalized eigenvectors $\tilde{D}\,W_k$ of the matrix $(\tilde{B} + \tilde{C})$ as defined in §3 remain equally bounded in the supremum norm independent of M: $|W_k|_\infty \leq \kappa$, $k = 1,\dots,M$. Then we obtain by (14) and (21)

$$|V_\Delta - U|_\infty \leq |D|_\infty \kappa \sum_{k=1}^{M} \left[\frac{1}{\lambda_k}\left(1 + \kappa_1 \frac{\Delta t\,\lambda_k}{2}\right)|\Psi_k|_\infty\right]$$

$$\leq \mathrm{const}\left(1 + \frac{\Delta t}{\Delta x}\right)(\Delta x^2 + \Delta t^2),$$

thus

$$|V_\Delta - U|_\infty \leq \mathrm{const}(\Delta x^2 + \Delta t^2)$$

instead of (20) provided that $\Delta t/\Delta x = \mathrm{const}$.

Hitherto we have $u_t(x,t)$ replaced by the finite difference approximation $(u(x, t + \delta) - u(x, t - \delta))/2\delta$ $(\delta \to 0)$. We consider as before the periodic problem (17) and approximate the differential equation as in (3) but for u_t we choose finally the well known approximation

$$u_t(x,t) = \frac{3u(x,t) - 4u(x,t-\Delta t) + u(x,t-\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2).$$

Collecting the M×N differene equations in the described way we obtain instead of (10) the following system

(23) $\quad P_4V := \left(\frac{1}{2\Delta t}S_4 + B + C + G\right)V = H_5$

where S_4 is the N/2-cyclic matrix

$$S_4 = \begin{pmatrix} 3I & & & & I & -4I \\ -4I & \ddots & & 0 & & I \\ I & \ddots & \ddots & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & I & -4I & 3I \end{pmatrix}.$$

(I identity matrix of dimension M).

<u>Theorem 6</u>. Suppose the boundary value problem (17) satisfies Assumptions II and III but let $|c|_\infty \leq \mu$ instead of $|c_x|_\infty \leq \gamma$. If $0 < \Delta x \leq \beta/\mu$ and $(\tau/\Delta t =)\ N \in \mathbb{N}$ even then a unique solution V_Δ of (23) exists and

$$|V_\Delta - U|_\infty \leq \text{const}(\Delta x^2 + \Delta t^2).$$

<u>Proof</u>. Let $S_{4,N}$ be the matrix of dimension N corresponding to S_4 but with units instead of the matrices I. By the Principle of Carasso-Parter we have to show that the inverse of

$$\hat{T}_{4,N}(\lambda) = I + \frac{1}{2\lambda\Delta t} S_{4,N}$$

is bounded in the supremum norm independent of $\lambda > 0$ and $\Delta t = \tau/N > 0$.

First, we show that $\hat{T}^{-1}_{4,N}(\lambda)$ exists. Let $\sigma = 1/2\lambda\Delta t$ and

$$\tilde{L} = \begin{pmatrix} 1+3\sigma & 0 \\ -4\sigma & 1+3\sigma \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} \sigma & -4\sigma \\ 0 & \sigma \end{pmatrix}, \quad \tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let

$$Z = \begin{pmatrix} 0 & & & \tilde{I} \\ \tilde{I} & \ddots & & \\ & \ddots & \ddots & \\ & & \tilde{I} & 0 \end{pmatrix}$$

be of dimension N. Then $\hat{T}_{4,N}(\lambda) = L + R\,Z$ where L and R are block diagonal matrices of dimension N with the identical submatrices $\tilde{L}$ and $\tilde{R}$ respectively. For the eigenvalues of $L^{-1}RZ$ we obtain

$$\lambda(L^{-1}RZ) = \lambda(\tilde{L}^{-1}\tilde{R})\exp(4\pi i n/N), \qquad n = 0,\dots,N-1;$$

and the eigenvalues

$$\lambda_{1,2}(\tilde{L}^{-1}\tilde{R}) = (1 + 3\sigma)^{-2}(-5\sigma^2 + \sigma \pm 4\sigma(\sigma^2 - \sigma)^{1/2})$$

of $\tilde{L}^{-1}\tilde{R}$ are less than one in absolute value for $\sigma \geq 0$. Thus $\hat{T}^{-1}_{4,N}(\lambda) = (I + L^{-1}RZ)^{-1}L^{-1}$ exists for all $\sigma \geq 0$.

Secondly, we show that $|\hat{T}^{-1}_{4,N}(\lambda)|_\infty \leq \text{const}_\varepsilon$ if $0 < \sigma < 1 - \varepsilon$ or $1 + \varepsilon < \sigma$ $(\varepsilon > 0)$. Let $\alpha = \lambda_1(L^{-1}R)$, $\beta = \lambda_2(L^{-1}R)$,

$$\tilde{\Lambda} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

and

$$\tilde{X} = \begin{pmatrix} \frac{2\sigma+(\sigma^2-\sigma)^{\frac{1}{2}}}{\sigma} & \frac{2\sigma-(\sigma^2-\sigma)^{\frac{1}{2}}}{\sigma} \\ 1 & 1 \end{pmatrix} = 2(\sigma^2-\sigma)^{\frac{1}{2}} \begin{pmatrix} \sigma & -2\sigma+(\sigma^2-\sigma)^{\frac{1}{2}} \\ -\sigma & 2\sigma+(\sigma^2-\sigma)^{\frac{1}{2}} \end{pmatrix}^{-1}$$

the matrix of the eigenvectors of $\tilde{L}^{-1}\tilde{R}$, and let X, Λ be the corresponding block diagonal matrices of dimension N. Then

$$\hat{T}_{4,N}(\lambda) = LX(I + \Lambda Z)X^{-1}.$$

Now recall that the element in the ith row and kth column of a matrix A^{-1} in case of existence consists of the cofactor of the element in the kth row and ith column of A divided by the determinant of A. Using this fact we can construct $(I + \Lambda Z)^{-1}$ explicitly for general dimension N; by further computation it results then that $|(I + \Lambda Z)^{-1}|_\infty$ remains bounded independent of $\sigma \geq 0$ and $N \in \mathbb{N}$. This is however a very tedious work and it shall be suppressed here. L^{-1} and X are bounded in the supremum norm independent of $\sigma \geq 0$. So the proof would be complete if X^{-1} would have the same property; but X^{-1} is bounded in the supremum norm only if σ remains bounded away from one.

Thirdly, we show that $|\hat{T}^{-1}_{4,N}(\lambda)|_{\infty} \leq$ const for $\sigma = 1$ independent of N. In this case

$$\tilde{L}^{-1}\tilde{R} = \tilde{Y}\ \tilde{\Omega}\ \tilde{Y}^{-1} = \begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4} & 0 \\ 1 & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} .$$

and

$$\hat{T}_{4,N}(\lambda) = L\ Y(I + \Omega Z)Y^{-1}$$

where Y and Ω are the corresponding block matrices again. In an equally tedious computation as above we can obtain $(I + \Omega Z)^{-1}$ explicitly and it can then be shown that the appraisal $|(I + \Omega Z)^{-1}|_{\infty} \leq$ const is true independent of the dimension N. The exact proof is omitted here also.

Combining the three parts of the proof we obtain the boundedness of $\hat{T}^{-1}_{4,N}(\lambda)$ in the supremum norm since $\hat{T}^{-1}_{4,N}(\lambda)$ exists and depends continuously on $\sigma \geq 0$. The rest of the proof follows in the known way.

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