On the Convergence of the von Neumann Difference Approximation to Hyperbolic Initial Boundary Value Problems

by

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1. Introduction

The present contribution is concerned with the hyperbolic initial boundary problem

$$
\begin{aligned}
& u_{tt} - [a(x)u_x]_x + b(x)u_x + c(x)u = d(x,t), \quad (x,t) \in (0,1)\times(0,\infty), \\
(1) \quad & u(x,0) = f(x), \; u_t(x,0) = g(x), \qquad 0 < x < 1, \\
& u(0,t) = h^-(t), \; u(1,t) = h^+(t), \qquad 0 \leq t,
\end{aligned}
$$

where $0 < \alpha_o \leq a(x)$ and $c(x) \geq 0$. For an arbitrary but fixed T let $\Delta t = T/N$ and $\Delta x = 1/(M+1)$, $M,N \in \mathbb{N}$, be small increments of the variables t and x, and let

$$v_k^n = v(k\Delta x, n\Delta t), \qquad V^n = (v_1^n,\ldots,v_M^n)^T,$$

$$V = (V^o, V^1, \ldots, V^N)^T.$$

We consider thewell-known implicit finite-difference approximation of (1) devised by von Neumann (cf. O'Brien et al. [9])

$$
(2) \quad
\begin{aligned}
& \frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\Delta t^2} + \\
& \omega[l_\Delta v^{n+1}]_k + (1-2\omega)[l_\Delta v^n]_k + \omega[l_\Delta v^{n-1}]_k = d_k^n, \\
& \qquad k = 1,\ldots,M, \quad n = 2,3,\ldots \quad ,
\end{aligned}
$$

with the initial and boundary values

(3) $v_k^o = f_k$, $k = 1,\dots,M$, $v_o^n = (h^-)^n$, $v_{M+1}^n = (h^+)^n$, $n = 0,1,\dots$,

(4) $v_k^1 = f_k + \Delta t g_k + \frac{\Delta t^2}{2}[a_k(f_{xx})_k - b_k(f_x)_k - c_k f_k + d_k^o]$,

$k = 1,\dots,M$,

where l_Δ denotes the finite-difference operator corresponding to the elliptical part of the differential equation in (1), i.e.

$$[l_\Delta v]_k = - \frac{a_{k+1/2}(v_{k+1}-v_k) - a_{k-1/2}(v_k-v_{k-1})}{\Delta x^2} + b_k\frac{v_{k+1}-v_{k-1}}{2\Delta x} + c_k v_k,$$

and ω is a relaxation factor. Further, we denote by $U^n = (u_1^n,\dots,u_M^n)^T$ the vector obtained from the solution u of the non-discrete problem (1) by mesh-point evaluation on the line $t = n\Delta t$, and we use the following norms

$$|V^n|_\infty = \max_{1\le k\le M}\{|v_k^n|\}, \qquad |V|_\infty = \max_{o\le n\le N}\{|V^n|_\infty\},$$

$$|a|_\infty = \sup_{o<x<1}|a(x)|, \qquad |V^n|_2^2 = \Delta x \sum_{k=1}^{M}|v_k^n|^2.$$

Lees [7] applies slight modifications of von Neumann's method to the problem (1) with the differential equation

$$u_{tt} - [a(x,t)u_x]_x + b(x,t)u_x + c(x,t)u + e(x,t)u_t = d(x,t).$$

By means of discrete energy inequalities he obtains the estimation

(5) $$|U^N - V^N|_2 \le \kappa_1 e^{\kappa_2 T}(\Delta t^2 + \Delta x^2), \quad \kappa_1,\kappa_2 \text{ constants},$$

(V^N denoting the solution of the discrete problem on the line t = T) in case $\omega \geq 1/4$ and $\Delta t/\Delta x$ arbitrary but fixed as $\Delta x \longrightarrow 0$. Friberg [4] applies von Neumann's method to the problem (1) with the differential equation

$$u_{tt} - a^2 u_{xx} = d(x,t)$$

and derives by similar methods as in [7] the estimation (5) in case $0 \leq \omega < 1/4$ and $\Delta x^2/\Delta t^2 \geq (1 - 4\omega)a^2$. Since $\Delta x|U^N - V^N|_\infty^2 \leq |U^N - V^N|_2^2$ we deduce from (5) immediately that

$$|U^N - V^N|_\infty \leq \frac{\kappa_1 e^{\kappa_2 T}}{\Delta x^{1/2}}(\Delta t^2 + \Delta x^2).$$

Thus the estimation $|U^N - V^N|_\infty = \mathcal{O}(\Delta t)$ of Friberg [4] is to coarse. On the other hand , as consequence of his estimation (5) Lees [7, Theorem 3] states without proof that $|U^N - V^N|_\infty = \mathcal{O}(\Delta t^2 + \Delta x^2)$. However, the author of the present paper was not able to verify this assertion via discrete energy inequalities.

Here we do not use discrete energy inequalities as Lees [7,8] and Friberg [4] nor we study the behaviour of the discrete solution in transition from the line $t = n\Delta t$ to the line $t = (n+1)\Delta t$. Instead we consider the approximative solution on the lines $t = n\Delta t$, $n = 0,1,\ldots,N$, together and expand the error in the eigenvalues of the elliptical part of the hyperbolic differential equation as Carasso-Parter [2] did in proving the convergence of the "boundary value technique" for parabolic initial boundary problems (see also [5]). By this way we establish

$$|U^N - V^N|_\infty \leq \kappa_3 \frac{T}{\Delta x^\varepsilon} (\Delta t^2 + \Delta x^2) \sum_{p=1}^{M} p^{-(1+\varepsilon)}$$

($\varepsilon \searrow 0$). So the convergence on the line $t = T$ reveals to be proportional to T itself.

The method of estimation applied in this paper is by no means limited to hyperbolic marching procedures (see Carasso-Parter [2], Carasso [3], and [6]). In a subsequent paper A-stability of finite-difference approximations to parabolic initial boundary problems is studied by this way.

2. Stability and Convergence

We suppose that the non-discrete problem (1) satisfies the following assumption

Assumption. (i) Let $0 < \alpha_0 \leq a(x)$, $0 \leq c(x)$.

(ii) Let $a, \ldots, a_{xxx}$, b, b_x, c, f, f_x, f_{xx}, g be continuous and bounded in $G = \{x, 0 < x < 1\}$, say $a(x) \leq \alpha_1$, $|b(x)| \leq \beta$, $|b_x(x)| \leq \mu$, $c(x) \leq \gamma$. Let d, h^-, h^+ be continuous in $G \times (0,\infty)$ and d bounded in $G \times (0,T)$ for every $T > 0$.

(iii) Let $f(0) = h^-(0)$, $f(1) = h^+(0)$, $g(0) = h_t^-(0)$, $g(1) = h_t^+(0)$. Let all coefficients of the problem (1) be sufficiently smooth that a solution $u: (x,t) \longmapsto u(x,t)$ exists in the classical sense having in $G \times (0,\infty)$ four continuous time derivatives and four continuous space derivatives.

We insert the known quantities (3) as far as possible in the equations (2) and collect the equations (2), (3), and (4). The result is

$$(6) \qquad V^0 = H^0, \qquad H^0 = (f_1, \ldots, f_M)^T, \qquad V^1 = H^1$$

where the components of the vector H^1 are the values on the right side of equation (4), and

$$(7) \quad \frac{1}{\Delta t^2}(V^n - 2V^{n-1} + V^{n-2}) + \omega \tilde{T} V^n + (1-2\omega)\tilde{T} V^{n-1} + \omega \tilde{T} V^{n-2} = H^n, \quad n = 2, \ldots, N,$$

where $\tilde{T} = A + B + C$, $A = (a_{ik})_{i,k=1,\dots,M}$ etc.,

$$a_{ik} = \begin{cases} (a_{k+1/2}+a_{k-1/2})/\Delta x^2 \\ -\,a_{k+1/2}/\Delta x^2 \\ -\,a_{i+1/2}/\Delta x^2 \\ 0 \end{cases}, \quad b_{ik} = \begin{cases} 0 & i = k \\ -\,b_i/2\Delta x & i = k+1 \\ -\,b_i/2\Delta x & k = i+1 \\ 0 & \text{otherwise} \end{cases},$$

$$c_{ik} = c_k \delta_{ik}$$

(δ_{ik} denotes the Kronecker symbol), and $H^n = D^{n-1} + \dot{\tilde{T}}\,\dot{V}^{n-1}$,

$$\dot{\tilde{T}}\,\dot{V}^n = (\underbrace{(2a_{1/2}+\Delta x b_1)(h^-)^n, 0, \dots, 0, (2a_{M+1/2}-\Delta x b_M)(h^+)^n}_{M\ \text{components}})^T/2\Delta x^2.$$

Collecting the N+1 (partly trivial) systems of equations (6) and (7) we obtain for the block vector V the following system

$$(8) \qquad P\,V := (\Gamma\,S + T\,\Omega)V = H$$

where $\Gamma = (I, I, \Delta t^{-2}I, \dots, \Delta t^{-2}I)$ and $T = (\tilde{T}, \dots, \tilde{T})$ are block diagonal matrices,

$$S = \begin{pmatrix} I & & & & \\ 0 & I & & & \\ I & -2I & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & -2I & I \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ \omega I & (1-2\omega)I & \omega I & & \\ & \ddots & \ddots & \ddots & \\ & & \omega I & (1-2\omega)I & \omega I \end{pmatrix}$$

(I identity matrix of dimension M). The finite-dimensional eigenvalue problem $\tilde{T}\,Y = \lambda\,Y$ is a discrete analog to a nonselfadjoint Strrm-Liouville eigenvalue problem. Subsequently we need some facts on the eigenvalues and eigenvectors of the matrix $\tilde{T}$ which shall now be stated. Assuming $0 < \Delta x < 2\alpha_o/\beta$ Carasso [1, Lemma 1] constructed a real diagonal matrix $\tilde{D}$ with the properties

$$\tilde{D}^{-1}\tilde{T}\,\tilde{D} \text{ symmetric}, \quad |\tilde{D}|_\infty \leq \kappa_4, \quad |\tilde{D}^{-1}|_\infty \leq \kappa_5$$

where κ_4, κ_5 are constants not depending on M. Following this result $\tilde{D}^{-1}\tilde{T}\,\tilde{D}$ has a complete system of orthonormal eigenvectors $W_p = (w_1^{(p)},\dots,w_M^{(p)})^T$, $p = 1,\dots,M$, and the corresponding eigenvalues λ_p are real. Furthermore, we obtain $\lambda_p > 0$, $p = 1,\dots,M$, by means of Varga [10, Theorem 1.8]. Thus, as the eigenvalues λ_p converge to the (positive) eigenvalues of the corresponding analytical problem (cf. Carasso [1, Theorem 1]), the eigenvalues λ_p are bounded below by a positive constant if $0 < \Delta x \leq \alpha_o/\beta$.

Lemma 1 (Carasso [1, Lemma 3], Gekeler [6]). Suppose that the problem (1) satisfies the Assumption and let $0 < \Delta x \leq \alpha_o/\beta$. Let λ_p be the eigenvalues of $\tilde{T}$ and let W_p be the eigenvectors of $\tilde{D}^{-1}\tilde{T}\,\tilde{D}$ normalized so that $|W_p|_2 = 1$, $p = 1,\dots,M$. Then there exists a positive integer p_o independent of M such that

$$\kappa_6\, p^2\, \pi^2 \leq \lambda_p \leq \kappa_7\, p^2\, \pi^2, \qquad p_o \leq p \leq M;$$

furthermore

$$|W_p|_\infty \leq \kappa_8,$$

where κ_6, κ_7, κ_8 are positive constants independent of M.

As a consequence of Lemma 1 we have for $\varepsilon > 0$

(9) $$\sum_{p=1}^{M} \frac{|W_p|_\infty}{\lambda_p^{(1+\varepsilon)/2}} \leq \kappa_9 \sum_{p=1}^{M} p^{-(1+\varepsilon)} .$$

Let now $\tilde{\Lambda}$ be the diagonal matrix of the eigenvalues λ_p, let $\tilde{W} = (W_1,\dots,W_M)$ be the matrix of eigenvectors of $\tilde{D}^{-1}\tilde{T}\,\tilde{D}$, let D, W be the block diagonal matrices with the submatrices $\tilde{D}$, $\tilde{W}$ respectively, and denote by Λ the block diagonal matrix $\Lambda = (I, I, \tilde{\Lambda},\dots, \tilde{\Lambda})$. Then we may write

$$D^{-1}(PU - H) = W\Phi, \quad D^{-1}P\,D = W\,Q\,W^{-1}$$

which implies

(10) $$|U - V|_\infty = |D\,D^{-1}P^{-1}(PU - H)|_\infty \leq |D|_\infty |W\Lambda^{-1}\Lambda\; Q^{-1}\Phi|_\infty .$$

Here Φ is a block vector (and not a block matrix because the eigenvectors W_p are orthonormal; this is the reason why the diagonal matrix $\tilde{D}$ must be introduced). More exactly,

$$\Phi = (\Phi^o,\dots,\Phi^N)^T, \quad \Phi^n = (\phi_1^n,\dots,\phi_M^n)^T,$$

$$|\phi_p^n| = |(\tilde{D}^{-1}(PU - H)^n)^T W_p / W_p^T W_p| \leq |\tilde{D}^{-1}|_\infty |(PU - H)^n|_2 .$$

Thus

$$|\phi_p^o| = 0,$$

(11) $$|\phi_p^1| \leq \kappa_5 K_1(\Delta t)\Delta t^3,$$

(12) $$|\phi_p^n| \leq \kappa_{10}(K_2(T,\Delta t)\Delta t^2 + K_{3,\omega}(T,\Delta x)\Delta x^2), \quad p = 1,\dots,M, \quad n = 2,3,\dots .$$

Since we have not supposed that h^-, h^+, and the solution u remain bounded in the cylinder $G\times(0,\infty)$ the bounds K_1, K_2, K_3 are the following quantities

$$K_1(\Delta t) = \max_{0\le\tau\le\Delta t} \{\tfrac{1}{6}|u_{ttt}(\cdot,\tau)|_\infty\} ,$$

$$K_2(T,\Delta t) = \max_{0\le t\le T} \{\tfrac{1}{12}|u_{tttt}(\cdot,t)|_\infty\},$$

$$K_{3,\omega}(T,\Delta x) = \max_{0\le t\le T} \Bigg\{ \frac{\beta|u_{xxx}(\cdot,t)|_\infty}{6} + \frac{|a_x(\cdot)u_{xxx}(\cdot,t)|_\infty}{6} + \frac{|a_{xx}(\cdot)u_{xx}(\cdot,t)|_\infty}{8} + \frac{|a|_\infty|u_{xxxx}(\cdot,t)|_\infty}{12} + \frac{\Delta x|a_x|_\infty|u_{xxxx}(\cdot,t)|_\infty}{24} + \frac{\Delta x^2|a_{xx}|_\infty|u_{xxxx}(\cdot,t)|_\infty}{96} + \frac{|u_x(\cdot,t)|_\infty|a_{xxx}|_\infty}{24} + \frac{\Delta x|a_{xxx}|_\infty|u_{xx}(\cdot,t)|_\infty}{48}\Bigg\} (2\omega + |1-2\omega|).$$

The block matrix Q consists of diagonal submatrices only, i.e.

$$Q = \Gamma\ S + \Lambda \begin{pmatrix} 0 & & & \\ 0 & 0 & & \bigcirc \\ \omega I & (1-2\omega) & \omega I & \\ & \ddots & \ddots & \ddots \\ \bigcirc & \omega I & (1-2\omega)I & \omega I \end{pmatrix} .$$

Our aim is now to derive a bound for the right side of inequality (10) using the estimation (9). Hereto we put for the moment $\Psi = \Lambda\ Q^{-1}\Phi$, i.e. $\Psi^0 = \Phi^0 = 0$, $\Psi^1 = \Phi^1 = \mathcal{O}(\Delta t^3)$, and $\Psi_p = (\psi_p^o,\ldots,\psi_p^N)^T$. Then, of course,

$$|\Psi_p|_\infty = \max_{o\le n\le N}\{|\psi_p^n|_\infty\}, \qquad p = 1,\ldots,M.$$

Consequently

$$(13)\qquad \begin{aligned} |W\Lambda^{-1}\Psi|_\infty &= \max\{|\tilde{W}\Phi^1|_\infty, \max_{2\le n\le N}\{|\tilde{W}\tilde{\Lambda}^{-1}\Psi^n|_\infty\}\} \\ &\le \max\{|\tilde{W}\Phi^1|_\infty, \max_{o\le n\le N}\{|\tilde{W}\tilde{\Lambda}^{-1}\Psi^n|_\infty\}\}, \end{aligned}$$

and

$$\max_{o\le n\le N}\{|\tilde{W}\tilde{\Lambda}^{-1}\Psi^n|_\infty\} \le \sum_{p=1}^{M}\frac{|W_p|_\infty}{\lambda_p}\max_{o\le n\le N}\{|\psi_p^n|\} = \sum_{p=1}^{M}\frac{|W_p|_\infty}{\lambda_p}|\Psi_p|_\infty .$$

But $\Psi_p = \hat{P}(\tau_p)^{-1}\Phi_p$ where

$$(14)\qquad \tau_p = -2 + \frac{\lambda_p\Delta t^2}{1+\omega\lambda_p\Delta t^2}$$

and $\hat{P}(\tau)$ is the $(N+1)\times(N+1)$-matrix

$$(15)\qquad \hat{P}(\tau) = \Theta\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & \tau & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \tau & 1 \end{pmatrix}$$

Θ denoting the diagonal matrix $\Theta = (1, 1, (2+\tau)^{-1},\ldots,(2+\tau)^{-1})$.

Summarizing the above results we find that

$$(16)\qquad |\tilde{W}\tilde{\Lambda}^{-1}\Psi^n|_\infty \le \sum_{p=1}^{M}\frac{|W_p|_\infty}{\lambda_p^{(1+\varepsilon)/2}}\left|\frac{1}{\lambda_p^{(1-\varepsilon)/2}}\hat{P}(\tau_p)^{-1}\right|_\infty |\Phi_p|_\infty .$$

Therefore, since by (11)

$$(17)\qquad |\tilde{W}\Phi^1|_\infty = \mathcal{O}(\Delta t^3)/\Delta x,$$

we have shown that under the above assumptions an estimation of $|U - V|_\infty$ depends essentially on a suitable bound for $|\hat{P}(\tau)^{-1}|_\infty$.

Lemma 2. Let $-2 < \tau < 2$ and let $\hat{P}(\tau)$ be the $(N+1)\times(N+1)$-matrix (15) with N odd. Then

(18) $$|\hat{P}(\tau)^{-1}|_\infty \leq \kappa_{11}\left[\frac{2+\tau}{2-\tau}\right]^{1/2} \frac{T}{\Delta t}$$

where κ_{11} is a constant independent of M.

Proof. Set

$$\tilde{L} = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let

$$Z = \begin{pmatrix} 0 & & & \\ \tilde{I} & 0 & & \text{O} \\ & \ddots & \ddots & \\ \text{O} & & \tilde{I} & 0 \end{pmatrix}$$

be a matrix of dimension N+1. Then $\hat{P}(\tau) = \Theta L(L^{-1}RZ + I)$ where $L = (\tilde{I}, \tilde{L},\ldots, \tilde{L})$, $R = (\tilde{R},\ldots,\tilde{R})$ are block diagonal matrices of dimension $(N+1)/2$. The eigenvalues of $\tilde{L}^{-1}\tilde{R}$ are

$$\eta_{1,2} = (2 - \tau^2 \pm \tau(\tau^2 - 4)^{1/2})/2,$$

and the matrix of its eigenvectors reads

$$\tilde{X} = \begin{pmatrix} \frac{-\sqrt{\tau^2-4} - \tau}{2} & \frac{\sqrt{\tau^2-4} - \tau}{2} \\ 1 & 1 \end{pmatrix} = \sqrt{\tau^2-4} \begin{pmatrix} -1 & \frac{\sqrt{\tau^2-4} - \tau}{2} \\ 1 & \frac{\sqrt{\tau^2-4} + \tau}{2} \end{pmatrix}^{-1} .$$

Define $\tilde{U}$ as the diagonal matrix $\tilde{U} = (\eta_1, \eta_2)$ and denote by X and U the block diagonal matrices with the identical submatrices $\tilde{X}$ and $\tilde{U}$ respectively. Then, we have $\hat{P}(\tau)^{-1} = X(UZ + I)^{-1}X^{-1}L^{-1}\Theta^{-1}$. But for $-2 < \tau < 2$

$$(19) \quad |X|_\infty \leq \kappa_{12}, \ |L^{-1}|_\infty \leq \kappa_{13}, \ (2+\tau)|X^{-1}|_\infty \leq \kappa_{14}\left[\frac{2 + \tau}{2 - \tau}\right]^{1/2} .$$

Thus, since $|\tilde{U}|_\infty = 1$ for $-2 < \tau < 2$, an explicit computation of $(UZ + I)^{-1}$ combined with the bounds (19) provides the estimation (18).

Theorem. Suppose that the hyperbolic initial boundary problem (1) satisfies the Assumption. Let $0 < \Delta x = 1/(M+1) \leq \alpha_o/\beta$, $\Delta t = T/N$, N odd. Let U^N be the vector of dimension M obtained from the solution of the problem (1) by mesh-mesh-point evaluation on the line $t = N\Delta t = T$. Let V^n, $n = 1,\dots,N$, be the solution of the system (8). Then

$$|U^N - V^N|_\infty \leq \kappa_{15}\frac{T}{\Delta x^\varepsilon}(\max\{K_1(\Delta t), K_2(T,\Delta t)\}\Delta t^2 + K_{3,\omega}(T,\Delta x)\Delta x^2) \times \sum_{p=1}^{M} p^{-(1+\varepsilon)}$$

($\varepsilon > 0$, κ_{15} constant independent of T, Δx, and Δt) for

$\omega \geq 1/4$ if $\Delta t/\Delta x$ is kept arbitrary but fixed as $\Delta x \longrightarrow 0$ (unconditional stability), and for $0 \leq \omega < 1/4$ if

$$\frac{\Delta t^2}{4} \left(\frac{4\alpha_1}{\Delta x^2} + \frac{\mu}{2} + \gamma \right) \leq 1 - \delta$$

($\delta > 0$) as $\Delta x \longrightarrow 0$ (conditional stability).

Proof. By Varga [10, p. 219] and Carasso [3, p. 310] we find that $|\tilde{T}|_2 \leq 4\alpha_1/\Delta x^2 + \mu/2 + \gamma$. Hence, as $|\lambda| \leq |\tilde{T}|_2$ is true for every eigenvalue λ of $\tilde{T}$, we have $-2 < \tau < 2 - 4\delta$ in case $0 \leq \omega < 1/4$ if $\max\limits_{1 \leq p \leq M} \{|\lambda_p \Delta t^2|\} < 4 - 4\delta$. Moreover, we have $-2 < \tau < 2 - \delta_o$ for a certain $\delta_o > 0$ independent of $\lambda\Delta t^2$ in case $\omega \geq 1/4$ and $\Delta t/\Delta x$ fixed as $\Delta x \longrightarrow 0$ since $\lambda\Delta x^2$ bounded independently of M by Lemma 1. Consequently the assumption of Lemma 2 is satisfied in both cases. However by Lemma 2 and (14)

$$\frac{1}{\lambda^{(1-\varepsilon)/2}} |\hat{P}(\tau)^{-1}|_\infty \leq \frac{\kappa_{11} T}{\lambda^{(1-\varepsilon)/2} \Delta t} \left[\frac{\lambda \Delta t^2}{4(1 + \omega\lambda\Delta t^2) - \lambda\Delta t^2} \right]^{1/2}$$

$$\leq \kappa_{16}\, T\, \Delta x^{-\varepsilon} \frac{1}{[4(1 + \omega\lambda\Delta t^2) - \lambda\Delta t^2]^{1/2}} .$$

This result together with the estimations (9), (10), (12), (13), (16), and (17) proves the Theorem.

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