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ABSTRACT. A class of linear elliptic Wick-stochastic boundary value problems is considered. The problems are formulated in a variational (or weak) form and existence and uniqueness of a solution to this variational formulation is proved under general assumptions on the data. Furthermore, a Galerkin type of finite element method is formulated and presented in an algorithmic form. As an illustration the algorithm is then applied to the Wick-stochastic pressure equation in one and two dimensions.

1. INTRODUCTION

In this paper we study of Wick-stochastic boundary value problems of the type

$$\begin{aligned} L^{\diamond}u &= f \text{ on } D, \\ u &= 0 \text{ on } \partial D, \end{aligned} \tag{1.1}$$

where $D \subset \mathbb{R}^d$ is some bounded domain with Lipschitz boundary ∂D and the linear Wick-stochastic differential operator given as

$$L^{\diamond}u = -\sum_{i,j=1}^{d} D_i(a^{ij}\diamond D_j u) + \sum_{i=1}^{d} b^i\diamond D_i u + c\diamond u, \tag{1.2}$$

with all coefficients allowed to be generalized stochastic variables. We assume that the generalized expectation of the differential operator give a continuous, coercive and bilinear form. Note, as argued in [14], that even if we restrict the coefficients to be ordinary stochastic variables, a solution to an equation like (1.1) exists in many cases only as a generalized stochastic variable. The Wick product $\diamond$ is introduced as way to handle multiplication of such generalized variables. This product can be understood as a renormalisation procedure of the product of generalized variables A detailed discussion on the modeling properties of the Wick product is beyond the scope of this paper and we refer the interested reader to e.g. [3, 14, 15]. It should however be noted that the Wick product makes (1.1) well-defined for very general data. We define and give some properties of the Wick product in Section 3.

One example of an equation like (1.1) is the Wick-stochastic pressure equation

$$\begin{aligned} -\nabla(K\diamond\nabla p) &= f \text{ on } D, \\ p &= 0 \text{ on } \partial D, \end{aligned} \tag{1.3}$$

where p denotes the pressure, K the permeability and the right hand side f denotes the source rate of the fluid. The stochasticity of the equation is usually a result of letting the permeability be some given (positive) random field. The equation has been studied extensively, see e.g. [3, 8, 17, 15, 24].

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It was introduced by Holden et al. [17] as a stochastic model describing the pressure in a fluid that flow through a porous medium. The authors proved existence and uniqueness of a solution in the Kondratiev space $(\mathcal{S})^{-1}$ (see [14, 18] or [19] for a definition) and gave an explicit formula for the solution. Other examples for Wick-type stochastic partial differential equations include [5, 8, 9, 16, 18], see also [14] where the theory of Wick-stochastic (both ordinary and partial) differential equations is treated.

One important background for the work in this paper is [24] (see also [25]). Here the author formulates a variational interpretation of (1.3) on a family of Hilbert spaces (defined below), and he gives conditions securing the existence of a unique solution on a subset of these spaces. The motivation for the present paper was to extend the results from [24] to equations on the form in (1.1). Furthermore, inspired by [1, 2, 13], we wanted to provide a numerical approximation method based on the finite element method for this class of equations.

A numerical solution is interesting because it provides an approximation which can be used to get information about the true solution, even in the cases where the true solution is not known. We can e.g. use the approximation to do stochastic simulation and investigate the statistical properties of the solution. It seems to be better to apply a finite element method approach (see [4, 23] for the deterministic case) to this kind of equations, rather than a finite difference approach, because we avoid dealing with discretisation of the Wick product, see [13] for an example where the finite difference approach is used. It is also an advantage that our approach enables us to obtain a priori error bounds from general theory and these results are independent of the actual shape of the domain (as long as the domain is regular enough). Furthermore, there is already a wide range of software available for solving deterministic equations with the finite element method (e.g. [22]), and as we will see, the method we present in this paper makes it possible to take advantage of such software. This makes it straightforward to start experimenting with stochastic simulations and investigations of a problem once a deterministic finite element method has been developed, something we believe can be useful when a practical problem is to be investigated.

We extend the results from [24] to boundary value problems on the general form given in (1.1). We present a variational formulation of (1.1) on the same family of spaces as in [24]. In order to give insight in the numerical method used later, we reformulate this (stochastic) variational problem to an infinite set of deterministic variational problems, using the properties of the Wick product. Each of these variational problems will give one of the coefficients in the Wiener-Itô chaos expansion of the solution of (1.1). We then give conditions on the data to secure existence of a unique solution to this (infinite) set of problems. This will again imply uniqueness for the original stochastic variational problem, but *not* existence. For this existence we also need to control the growth in the chaos coefficients. We proceed to prove this existence of a solution to the original (stochastic) variational problem, thus localising the formal Wiener-Itô chaos expansion of the solution to some specific family of spaces. This proof applies the Lax-Milgram Theorem and is based on the same ideas as in [24]. We include the proof since we have formulated it differently and more general than what was given in [24]. Next, with background in the ideas from Benth and Gjerde [2, 1] we show how one can apply a finite element method to solve the variational problem numerically. This reduces to solving a finite series of deterministic linear systems where each equation gives the classical finite element approximation [4, 23] of one coefficient in the Wiener-Itô chaos expansion of the solution. Finally, we present this numerical solution method in the form of an algorithm and we apply this to the model problem given in (1.3).

We give an outline of the paper: In Section 2 we introduce notation and a few necessary preliminary results. Then in Section 3 we present the family of Hilbert spaces we use to form the variational equation. We also define the Wick product and give some basic properties. In Section 4 we give a variational formulation of (1.1) on this family of Hilbert spaces. We reformulate this (stochastic) variational problem to an infinite set of deterministic variational problems, and we consider the uniqueness of a solution to this set of problems. Next, in Section 5 we present a proof of the existence (and uniqueness) of a solution of the original variational formulation using the Lax-Milgram Theorem. In Section 6 we show how to apply a Galerkin finite element method to solve the variational problem numerically, and we present convergence results securing that the method converges in the norm of the underlying Hilbert space. In Section 7 we formulate the resulting fully discrete problem as a finite set of deterministic variational problems and we give an algorithm that describe how to solve this discrete problem. Finally, in Section 8 we apply the given algorithm to the model problem (1.3) for a special choice of data and present results from simulations in one and two dimensions.

2. Preliminaries

We present the definitions and results needed for this paper. For the interested reader we recommend the introduction to white noise analysis given in Hida et al. [12], other references include [11, 14, 19, 20, 21]. The application of white noise analysis to Wick-stochastic partial differential equations is treated in Holden et al. [14]. For an introduction to the finite element method see e.g. Brenner and Scott [4] or Quarteroni and Valli [23].

Let $\mathcal{S}$ denote the Schwarz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing C^∞ functions on $\mathbb{R}^d$. The dual space $\mathcal{S}'$ equipped with the weak-star topology is the space of tempered distributions. By Bochner-Minlos theorem there exists a unique probability measure μ on the members of the family $\mathcal{B}(\mathcal{S}')$ of Borel subsets of $\mathcal{S}'$ such that

$$E[e^{i\langle\cdot,\phi\rangle}] := \int_{\mathcal{S}'} e^{i\langle\omega,\phi\rangle}d\mu(\omega) = e^{-\|\phi\|_0^2/2} \tag{2.1}$$

where $\|\phi\|_0 = (\phi,\phi)_0 = \int_{\mathbb{R}^d}\phi(x)^2dx$. The triplet $(\mathcal{S}',\mathcal{B},\mu)$ forms our basic probability space.

Lemma 2.1. *Let $\phi_1,\cdots,\phi_n$ be n functions in $\mathcal{S}(\mathbb{R}^d)$ that are orthonormal in $L^2(\mathbb{R}^d)$. Then the stochastic variable*

$$\omega \mapsto (\langle\omega,\phi_1\rangle,\cdots,\langle\omega,\phi_n\rangle)$$

is standard Gaussian (or Normal) distributed on $\mathbb{R}^n$.

The Lemma follows from (2.1) and a proof is presented [14].

We will use the following multi-index notation. Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of all multi-indices $\alpha = (\alpha_1,\alpha_2,\dots)$ where all $\alpha_i \in \mathbb{N}_0$ and only finitely many $\alpha_i \neq 0$. For each $\alpha,\beta\in\mathcal{I}$ we define the usual operations $\alpha+\beta = (\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots)$, $\alpha! = \alpha_1!\alpha_2!\cdots$, $|\alpha| := \sum_j \alpha_j$. Furthermore, we use ϵ_i to denote the multi-index with one in the ith positions and zero in all the others and we write 0 for the multi-index containing only zeros. For each $\alpha,\gamma\in\mathcal{I}$ we say $\alpha\preceq\gamma$ if and only if $\alpha_i\le\gamma_i$ for all $i\in\mathbb{N}$. Clearly $\preceq$ is a partial ordering of $\mathcal{I}$. Let the relation $\prec$ on $\mathcal{I}$ be defined in the analogue way using strict inequality term-wise. It follows from the definition that $\alpha\preceq\gamma$ ($\alpha\prec\gamma$) if and only if there exists $\beta\in\mathcal{I}$ such that $\alpha+\beta=\gamma$ ($\alpha+\beta=\gamma$ and $\beta\neq 0$).

For each $\alpha\in\mathcal{I}$ define the stochastic variable

$$H_\alpha(\omega) := \prod_{j=1}^{\infty} h_{\alpha_j}(\langle\omega,\eta_j\rangle), \tag{2.2}$$

where h_n denote the Hermite polynomial

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}) \qquad (n\in\mathbb{N}_0), \tag{2.3}$$

and the family $\{\eta\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. This orthonormal family is constructed from the Hermite functions

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2}e^{-x^2/2}h_{n-1}(\sqrt{2}x) \qquad (x\in\mathbb{R},\ n\in\mathbb{N}) \tag{2.4}$$

in the following way: Let $\delta = (\delta_1,\cdots,\delta_d)\in\mathbb{N}_0^d$ be the d-dimensional multi-indices and let $\{\delta^{(i)}\}$ ($i\in\mathbb{N}$) be some fixed ordering of these multi-indices such that $i<j \Rightarrow |\delta^{(i)}|\le|\delta^{(j)}|$. Then we defien η_j as the tensor product

$$\eta_j := \xi_{\delta^{(j)}} := \xi_{\delta_1^{(j)}}\otimes\cdots\otimes\xi_{\delta_d^{(j)}} \qquad (j\in\mathbb{N}). \tag{2.5}$$

see also [14, p. 19]. Note that the family $\{\xi_n\}_{n=1}^{\infty}$ is a subset of $\mathcal{S}(\mathbb{R})$, and it and forms an orthonormal basis for $L^2(\mathbb{R})$. Thus, $\{\eta\}_{j=1}^{\infty}$ is a subset of $\mathcal{S}(\mathbb{R}^d)$ and forms an orthonormal basis for $L^2(\mathbb{R}^d)$.

The following Theorem is one of the corner-stones for our finite element approximation. A proof is given in e.g. [14].

Theorem 2.2 (Wiener-Itô chaos expansion theorem). *Every $f \in L^2(\mu)$ has a unique Wiener-Itô chaos expansion*

$$f(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha H_\alpha(\omega) \quad \textit{where } c_\alpha \in \mathbb{R}. \tag{2.6}$$

In addition, the family $\{H_\alpha/\sqrt{\alpha!}\}_{\alpha\in\mathcal{I}}$ constitutes an orthonormal basis for $L^2(\mu)$ and we have

$$||f||^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \alpha! \tag{2.7}$$

for every $f \in L^2(\mu)$.

For completeness we also include a definition of coercivity of a bilinear form and the Lax-Milgram theorem.

Definition 2.3. Let V be a real Hilbert space and let $B : V \times V \mapsto \mathbb{R}$ be some given bilinear form. Then B is said to be coercive on V if there exits a constant $\theta > 0$ such that

$$B(v,v) \geq \theta ||v||_V^2$$

for all v in V. Here $||\cdot||_V$ denotes the norm on V.

Theorem 2.4 (Lax-Milgram). *Let V be a real Hilbert space, let F be in V', and let $B : V\times V \mapsto \mathbb{R}$ be some given continuous, coercive and bilinear form. Then there exists a unique $u \in V$ such that*

$$B(u,v) = F(v)$$

for all v in V.

A proof based on the contraction mapping principle can be found in e.g. [4]. The theorem can be generalized to the complex case [26], but this will not be needed in this paper.

3. A family of stochastic Hilbert spaces

To be able to apply the Lax-Milgram Theorem [4] and use the finite element method, we want to study stochastic equations in a Hilbert space structure which captures properties in the physical space. This is the reason for introducing the following family of Hilbert spaces $(\mathcal{S})^{\rho,k,V}$.

Definition 3.1. Let $(V, (\cdot,\cdot)_V)$ be any real separable Hilbert space and choose $\rho \in [-1,1], k \in \mathbb{R}$. Then the stochastic Hilbert space $(\mathcal{S})^{\rho,k,V}$ is defined as the set of all (formal) sums

$$f = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha, \quad \text{where } f_\alpha \in V \text{ for all } \alpha \in \mathcal{I},$$

such that the norm

$$||f||_{\rho,k,V} := (\sum_{\alpha\in\mathcal{I}} ||f_\alpha||_V^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha})^{1/2} \tag{3.1}$$

is finite. The weights are defined as $(2\mathbb{N})^{k\alpha} := \prod_{j=1}^{\infty}(2j)^{k\alpha_j}$.

Note that the norm $||\cdot||_{\rho,k,V}$ is induced by the inner product $(\cdot,\cdot)_{\rho,k,V}$ defined as

$$(f,g)_{\rho,k,V} := \sum_{\alpha\in\mathcal{I}} (f_\alpha, g_\alpha)_V (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \tag{3.2}$$

for $f = \sum_\alpha f_\alpha H_\alpha$ and $g = \sum_\alpha g_\alpha H_\alpha$ given in $(\mathcal{S})^{\rho,k,V}$. This inner product is well defined because $f_\alpha, g_\alpha \in V$ for all $\alpha \in \mathcal{I}$ and V is a Hilbert space.

It is clear from the definition of the norm in (3.1) that given $k_1, k_2 \in \mathbb{R}$ such that if $k_1 \le k_2$, then $\|f\|_{\rho,k_1,V} \le \|f\|_{\rho,k_2,V}$. It follows that

$$(\mathcal{S})^{\rho,k_2,V} \subset (\mathcal{S})^{\rho,k_1,V}.$$

Furthermore, for any $k \in \mathbb{R}$ and $\rho \in [0,1]$ we clearly have

$$(\mathcal{S})^{1,k,V} \subset (\mathcal{S})^{\rho,k,V} \subset (\mathcal{S})^{0,k,V} \subset (\mathcal{S})^{-\rho,k,V} \subset (\mathcal{S})^{-1,k,V}.$$

For given $k \ge 0$ and $\rho \in [0,1]$ the space $(\mathcal{S})^{-\rho,-k,V}$ is dual to $(\mathcal{S})^{\rho,k,V}$ under the pairing defined by

$$\langle\langle F, f\rangle\rangle := \sum_{\alpha\in\mathcal{I}} (F_\alpha, f_\alpha)_V \alpha! \tag{3.3}$$

for any $F \in (\mathcal{S})^{-\rho,-k,V}$ and $f \in (\mathcal{S})^{\rho,k,V}$.

If V is the Sobolev space $H^m(D)$ we denote $(\mathcal{S})^{\rho,k,V}$ by $(\mathcal{S})^{\rho,k,m,D}$, suppressing D when it is clear from the context. Likewise, for $V = H_0^m(D)$ or $V = \mathbb{R}$ we use the notation $(\mathcal{S})_0^{\rho,k,m,D}$ and $(\mathcal{S})^{\rho,k}$, respectively. Furthermore, for the Hilbert spaces $L^2(D)$ and $H^m(D)$ we denote the corresponding inner products by $(\cdot,\cdot)_{0,D}$ and $(\cdot,\cdot)_{m,D}$, respectively. We write $\|\cdot\|_{m,D}$ for the norm induced by the inner product on $H^m(D)$ (or $H_0^m(D)$), suppressing D when the set we integrate over is clear from the context.

From (2.7) it is clear that $(\mathcal{S})^{0,0} = L^2(\mu)$. To give some relation to the Kondratiev spaces used by Holden et al. [17], note that for given $\rho \in [0,1]$ the Kondratiev test function spaces are defined as $(\mathcal{S})^\rho := \bigcap_{k\ge 0} (\mathcal{S})^{\rho,k}$ equipped with the projective limit topology [7], and the Kondratiev distribution spaces are defined as $(\mathcal{S})^{-\rho} := \bigcup_{k\le 0} (\mathcal{S})^{-\rho,k}$ equipped with the inductive limit topology.

We have the following result due to Våge [24].

Proposition 3.2. *The space $(\mathcal{S})^{\rho,k,V}$ forms a separable Hilbert space, and it is isomorphic to $V \otimes (\mathcal{S})^{\rho,k}$.*

The proof by Våge is presented for the spaces $V = H^m$ and $V = H_0^m$, but it is easily adapted to fit any separable Hilbert space.

Remark 3.3. The weights $(2\mathbb{N})$ used in our definition of $(\mathcal{S})^{\rho,k,V}$ are different from those used by Våge in [24]. He use the weights

$$\Delta^{k\alpha} := \prod_{j=1}^{\infty} (2^d \delta_1^{(j)} \cdots \delta_d^{(j)})^{k\alpha_j}$$

in the corresponding definition. The resulting spaces are only slightly different in the sense that

$$(2\mathbb{N})^{\alpha/d} \le \Delta^\alpha \le (2\mathbb{N})^{d\alpha}$$

for all $\alpha \in \mathcal{I}$ and with our ordering of the multi-indices δ used in (2.5). See also page 34 in [14]. The results from [24] that are used in this paper are not affected by this difference in the weights.

Definition 3.4. Let $f(x,\omega) = \sum_\alpha f_\alpha(x) H_\alpha(\omega) \in (\mathcal{S})^{\rho,k,m,D}$, where D is a open subset of $\mathbb{R}^d$ and m is a non-negative integer, and define the derivative $D^\beta f$ to be

$$(D^\beta f)(x) = \sum_\alpha (D^\beta f_\alpha)(x) H_\alpha$$

for any multi-index $\beta \in \mathbb{N}_0^d$ and with $|\beta| \le m$. Here $D^\beta f_\alpha$ denotes the derivative in the usual weak (or distributive) sense.

The definition makes sense because for a given f in $(\mathcal{S})^{\rho,k,m,D}$, all coefficients f_α are in $H^m(D)$ by assumption, and thus all weak derivatives $D^\beta f_\alpha$ exist for $|\beta| \leq m$ and are in $H^{m-|\beta|}(D) \subset L^2(\mathbb{R}^d)$. It follows that the operator

$$D^\beta : (\mathcal{S})^{\rho,k,m,D} \longrightarrow (\mathcal{S})^{\rho,k,m-|\beta|,D}$$

is linear and continuous.

Example 3.5. Let $\phi \in \mathcal{S}$, D bounded, and define $\phi_x(y) = \phi(y-x)$. The smoothed white noise process is defined by

$$W_\phi(x,\omega) := \langle \omega, \phi_x \rangle = \sum_{i=1}^{\infty} (\phi_x, \eta_i) H_{\epsilon_i}(\omega) \qquad (x \in D)$$

and can be shown to belong to $(\mathcal{S})^{1,k,m,D}$ for any $k \leq 0$ and any $m \in \mathbb{N}_0$. The singular white noise process is defined by the formal expansion

$$W(x) := \sum_{i=1}^{\infty} \eta_i(x) H_{\epsilon_i} \qquad (x \in D)$$

and can be shown to belong to $(\mathcal{S})^{1,k,1,D}$ for $k < -2/3$.

Definition 3.6. The Wick product $f \diamond g$ of two elements

$$f = \sum_\alpha f_\alpha H_\alpha \in (\mathcal{S})^{-1,k,0} \text{ and } g = \sum_\alpha g_\alpha H_\alpha \in (\mathcal{S})^{-1,k,0}$$

is defined as

$$f \diamond g := \sum_{\alpha,\beta \in \mathcal{I}} f_\alpha g_\beta H_{\alpha+\beta}. \tag{3.4}$$

This product is associative, commutative and distributive. Note that if one of the terms are deterministic, then the Wick product coincide with the ordinary point-wise product. Also note that we multiply functions in $L^2(D)$ here. Therefore, each term $f_\alpha g_\beta$ is integrable, but in general the product is not square integrable. Thus, to make sure that the operator $g \mapsto f \diamond g$ is bounded and continuous on $(\mathcal{S})^{-1,k,0}$ we introduce the Banach spaces $\mathcal{F}_l(D)$:

$$\begin{aligned}\mathcal{F}_l(D) := \{f(x) = \sum_\alpha f_\alpha(x) H_\alpha : f_\alpha(x) \text{ measurable}, \\ \|f\|_{l,*} := \sup_{x \in D} (\sum_\alpha |f_\alpha(x)| (2\mathbb{N})^{l\alpha}) < \infty\}.\end{aligned} \tag{3.5}$$

These spaces where first given in [24]. The following proposition secures the continuity of the Wick product for $k \leq 2l$.

Proposition 3.7. *Let $D \subset \mathbb{R}^d$ be open and choose $l \in \mathbb{R}$. Then $f \in \mathcal{F}_l$ defines a continuous linear operator on $(\mathcal{S})^{-1,k,0}$ by $g \mapsto f \diamond g$ as long as $k \leq 2l$. Furthermore, we have*

$$\|f \diamond g\|_{-1,k,0} \leq \|f\|_{k/2,*} \|g\|_{-1,k,0} \leq \|f\|_{l,*} \|g\|_{-1,k,0}, \tag{3.6}$$

for all $g \in (\mathcal{S})^{-1,k,0}$.

A proof can be found in [24] and is based on a simple application of Young's Inequality for convolutions [6].

Definition 3.8. For any element $f = \sum_\alpha f_\alpha H_\alpha \in (\mathcal{S})^{\rho,k,m}$ we define the generalized expectation as

$$E[f] := f_0 \tag{3.7}$$

That is, $E[f]$ equals the zeroth Wiener-Itô chaos coefficient of f.

Note that this definition coincides with ordinary expectation if $f \in L^2(\mu)$. Also note that from the orthogonality of H_α we have $E[f \diamond g] = E[f]E[g]$ for any given pair of (generalized) stochastic variables f and g in $(\mathcal{S})^{\rho,k,m}$. This property is a consequence of the definition of the Wick product.

4. A variational formulation of the problem

We give a variational (or weak) formulation of the Wick-stochastic boundary value problem (1.1) and show that a solution is unique. In the following Section 5 we will provide sufficient conditions on the data to secure existence of a solution.

Formally, assuming enough regularity on the data and the solution, we may take the inner product of (1.1) with any v in $(\mathcal{S})_0^{\rho,k,1}$. This gives

$$(L^\diamond u, v)_{\rho,k,0} = (f, v)_{\rho,k,0}, \qquad v \in (\mathcal{S})_0^{\rho,k,1}. \tag{4.1}$$

Now, consider the left-hand side in (4.1) term vise; using the definition of the inner product the first term results in

$$\sum_\gamma \sum_{i,j=1}^d (-D_i(a^{ij}\diamond D_j u)_\gamma, v_\gamma)_{0,D}(\gamma)^{1+\rho}(2\mathbb{N})^{k\gamma}$$

for each $v \in (\mathcal{S})_0^{\rho,k,1}$ and where $(\cdot,\cdot)_{0,D}$ denotes the usual inner product on $L^2(D)$. By convention the expression $(a^{ij}\diamond D_j u)_\gamma$ denote the γth chaos coefficient of $a^{ij}\diamond D_j u$. Integrating each coefficient by parts gives

$$\sum_\gamma \sum_{i,j=1}^d \Big[((a^{ij}\diamond D_j u)_\gamma, D_i v_\gamma)_{0,D} - ((a^{ij}\diamond D_j u)_\gamma n_i, v_\gamma)_{0,\partial D}\Big](\gamma)^{1+\rho}(2\mathbb{N})^{k\gamma}$$

where $n = (n_i)$ is the unit normal vector pointing out of D. Note that the integral over the boundary vanishes because by assumption each $v_\gamma \in H_0^1(D)$. Summing the contributions from the second and third term in (1.2) gives the bilinear form

$$\mathcal{A}_{\rho,k}(u,v) := \sum_{i,j=1}^d (a^{ij}\diamond D_j u, D_i v)_{\rho,k,0} + \sum_{i=1}^d (b^i\diamond D_i u, v)_{\rho,k,0} + (c\diamond u, v)_{\rho,k,0} \tag{4.2}$$

defined for any $u, v \in (\mathcal{S})_0^{\rho,k,1}$. Here we only assume enough regularity on a^{ij}, b^i, c $(i,j = 1,\dots,n)$ to secure that (4.2) defines a continuous bilinear form. Lemma 5.1 below give sufficient conditions for this continuity. We define the variational formulation of (1.1).

Definition 4.1. For a given pair (ρ,k) with $\rho \in [-1,1]$ and $k \in \mathbb{R}$ the variational (or weak) formulation of (1.1) is

$$\text{Find } u \in (\mathcal{S})_0^{\rho,k,1} \text{ such that } \mathcal{A}_{\rho,k}(u,v) = (f,v)_{\rho,k,0} \quad \text{for all } v \in (\mathcal{S})_0^{\rho,k,1}. \tag{4.3}$$

and a solution of (4.3) is called a variational (or weak) solution of (1.1).

Before we proceed to consider the existence of a unique solution to (4.3) we take a closer look at the problem we are trying to solve. Let u be a solution to (4.3) for some pair (ρ,k). By the definition of the Wick product we have (formally)

$$\begin{aligned}
\mathcal{A}_{\rho,k}(u,v) &= \sum_\gamma \Big[\sum_{i,j=1}^d ((a^{ij}\diamond D_j u)_\gamma, D_i v_\gamma)_0 + \sum_{i=1}^d ((b^i\diamond D_i u)_\gamma, v_\gamma)_0 + \\
&\qquad ((c\diamond u)_\gamma, v_\gamma)_0\Big](\gamma!)^{1+\rho}(2\mathbb{N})^{k\gamma} \\
&= \sum_\gamma \sum_{\alpha+\beta=\gamma} \Big[\sum_{i,j=1}^d (a_\beta^{ij} D_j u_\alpha, D_i v_\gamma)_0 + \sum_{i=1}^d (b_\beta^i D_i u_\alpha, v_\gamma)_0 + \\
&\qquad (c_\beta u_\alpha, v_\gamma)_0\Big](\gamma!)^{1+\rho}(2\mathbb{N})^{k\gamma} \\
&= \sum_\gamma \sum_{\alpha \preceq \gamma} B_{\gamma-\alpha}(u_\alpha, v_\gamma)(\gamma!)^{1+\rho}(2\mathbb{N})^{k\gamma}
\end{aligned}$$

for all v in $(\mathcal{S})_0^{\rho,k,1}$. Here

$$\mathcal{B}_\beta(g,h) := \sum_{i,j=1}^d (a_\beta^{ij} D_j g, D_i h)_0 + \sum_{i=1}^d (b_\beta^i D_i g, h)_0 + (c_\beta g, h)_0 \tag{4.4}$$

defines a bilinear form on $H_0^1(D)$ for each $\beta \in \mathcal{I}$ and we use the partial ordering $\preceq$ given in Section 2. If we choose $v = wH_\gamma$ with w in $H_0^1(D)$ it follows that the sequence $\{u_\gamma\}_{\gamma\in\mathcal{I}}$ of chaos coefficients of the solution satisfy the following (infinite) set of variational problems

$$\begin{aligned} &\text{For each } \gamma \in \mathcal{I} \text{ find } u_\gamma \in H_0^1(D) \text{ such that} \\ &\mathcal{B}_0(u_\gamma, w) = (f_\gamma, w)_0 - \sum_{\alpha\prec\gamma} \mathcal{B}_{\gamma-\alpha}(u_\alpha, w) \text{ for all } w \text{ in } H_0^1(D). \end{aligned} \tag{4.5}$$

Theorem 4.2. *Let $a_\alpha^{ij}, b_\alpha^i, c_\alpha$ be in $L^\infty(D)$ for $i,j = 1,\dots,d$ and all $\alpha \in \mathcal{I}$. Let f_γ be in $L^2(D)$ for all $\gamma \in \mathcal{I}$ and let $\mathcal{B}_0$ (defined in (4.4)) be a coercive, bilinear form on $H_0^1(D)$. Then there exist a unique set of functions $\{u_\gamma \in H_0^1(D) : \gamma \in \mathcal{I}\}$ solving the variational problems* (4.5).

Proof. First note that by assumption $\mathcal{B}_0$ is a continuous, coercive and bilinear form on $H_0^1(D)$ and for each fixed $\gamma \in \mathcal{I}$ the right hand side in (4.5) is a continuous linear operator on $H_0^1(D)$, provided the set $\{u_\alpha : \alpha \prec \gamma\}$ is known and a subset of $H_0^1(D)$. The main idea in the proof is that we may order the set of multi-indices in such a way that when we reach the γth variational problem in (4.5) we have already solved the variational problems corresponding to all multi-indices α such that $\alpha \prec \gamma$. The proof follows by induction on γ and repeated use of the Lax-Milgram Theorem. □

Remark 4.3. It should be noted that Theorem 4.2 does not imply existence of a solution to the variational problem given in (4.3). For this we also need that the growth condition

$$\sum_{\gamma\in\mathcal{I}} \|u_\gamma\|_1^2 \gamma!^{1+\rho}(2\mathbb{N})^{\gamma k} < \infty$$

is satisfied for the pair (ρ, k) given for (4.3). The theorem does however imply uniqueness of a solution to (4.3), as the following Corollary 4.4 shows. In Section 5 below we will prove the existence of a solution to (4.3). This will then localise of our formal expansion of the solution to some space $(\mathcal{S})_0^{\rho,k,1}$.

Corollary 4.4. *Given the assumptions in Theorem 4.2. Then any solution to* (4.3) *has the (formal) chaos expansion*

$$u = \sum_{\alpha\in\mathcal{I}} u_\alpha H_\alpha,$$

where $\{u_\gamma : \gamma \in \mathcal{I}\}$ is the set of functions solving (4.5). *This expansion is independent of (ρ, k).*

Proof. The corollary follows directly from the calculations that lead us to (4.5) and the uniqueness of $\{u_\gamma : \gamma \in \mathcal{I}\}$ from Theorem 4.2. □

We want to remark that the bilinear form $\mathcal{B}_0$ given in (4.4) is the variational form corresponding to the (generalized) expectation of (1.1). This follows since

$$\begin{aligned} E[L^\diamond u] &= -\sum_{i,j=1}^d E[D_i(a_{ij}\diamond D_j u)] + \sum_{i=1}^d E[b^i\diamond D_i u] + E[c\diamond u] \\ &= -\sum_{i,j=1}^d D_i(a_0^{ij} D_j u_0) + \sum_{i=1}^d b_0^i D_i u_0 + c_0 u_0. \end{aligned}$$

There is a range of well-known conditions on the coefficients a_0^{ij}, b_0^i and c_0 securing that $\mathcal{B}_0$ is coercive on $H_0^1(D)$. See for example [4, 23] and the references therein. Furthermore, note that from a modeling point of view, where we think of the stochasticity as a small perturbation of some original deterministic model, it is natural to assume that $\mathcal{B}_0$ is coercive. Thus securing existence of a unique (variational) solution to the averaged equation.

5. Existence of a solution

We show that we may choose ρ and k so that the bilinear form $\mathcal{A}_{\rho,k}(\cdot,\cdot)$ is both continuous and coercive on the space $(\mathcal{S})_0^{\rho,k,1}$. From this the existence (and uniqueness) of a solution of follows directly using the Lax-Milgram Theorem. As was noted in Remark 4.3 we already know that a solution is unique. The proof presented here is a generalisation of the proof given in [24].

First, we consider the question of continuity.

Lemma 5.1. *If $a^{ij}, b^i, c \in \mathcal{F}_l(D)$ for $i,j = 1,\dots,d$ and some l such that $k \leq 2l$. Then the bilinear form $\mathcal{A}_{-1,k}$ given in (4.2) is continuous and there exists a constant $C_L < \infty$ such that*

$$\mathcal{A}_{-1,k}(u,v) \leq C_L \|u\|_{-1,k,1}\|v\|_{-1,k,1}$$

for all $u, v \in (\mathcal{S})^{-1,k,1}$.

Proof. It suffices to show that $\mathcal{A}_{-1,k}$ is bounded. Let $u, v \in (\mathcal{S})^{-1,k,1}$ and $k \leq 2l$, then

$$\begin{aligned}
|\mathcal{A}_{-1,k}(u,v)| &\leq \sum_{i,j=1}^{d} \|a^{ij}\diamond D_j u\|_{-1,k,0}\|D_i v\|_{-1,k,0} + \sum_{i=1}^{d} \|b^i\diamond D_i u\|_{-1,k,0}\|v\|_{-1,k,0} \\
&\quad + \|c\diamond u\|_{-1,k,0}\|v\|_{-1,k,0} \\
&\leq \sum_{i,j=1}^{d} \|a^{ij}\|_{l,*}\|D_j u\|_{-1,k,0}\|D_i v\|_{-1,k,0} + \sum_{i=1}^{d} \|b^i\|_{l,*}\|D_i u\|_{-1,k,0}\|v\|_{-1,k,0} \\
&\quad + \|c\|_{l,*}\|u\|_{-1,k,0}\|v\|_{-1,k,0} \\
&\leq C_L\|u\|_{-1,k,1}\|v\|_{-1,k,1},
\end{aligned}$$

using Cauchy-Schwarz, Proposition 3.7 and the definition of $\|\cdot\|_{-1,k,1}$. By our assumption on the data $C_L := \sum_{i,j=1}^{d} \|a^{ij}\|_{l,*} + \sum_i \|b^i\|_{l,*} + \|c\|_{l,*} < \infty$. □

Next, we give an intermediate lemma before we state the result about the coercivity of $\mathcal{A}_{-1,k}$. Recall that we use f_0 to denote the zeroth order term in the Wiener-Itô chaos expansion of a stochastic variable f and that this equals the (generalized) expected value of f.

Lemma 5.2. *Let $f \in \mathcal{F}_l(D)$ for some real l and let $v, w \in (\mathcal{S})^{-1,k,0}$. Then*

$$\begin{aligned}
&(f\diamond v, w)_{-1,k,0} \geq \\
&\quad \sum_{\gamma}(f_0 v_\gamma, w_\gamma)_0 (2\mathbb{N})^{k\gamma} - \frac{1}{2}2^{k/2-l}\|f\|_{l,*}(\|v\|^2_{-1,k,0} + \|w\|^2_{-1,k,0})
\end{aligned} \tag{5.1}$$

Proof. The proof follows the ideas in the proof of Proposition 6 in [24]. First note that if $f \in \mathcal{F}_l(D)$ for some real l then

$$\sup_{x\in D}\sum_{|\alpha|>0}|f_\alpha(x)|(2\mathbb{N})^{k\alpha/2} = \sup_{x\in D}\sum_{|\alpha|>0}|f_\alpha(x)|(2\mathbb{N})^{l\alpha}(2\mathbb{N})^{(k/2-l)\alpha} \leq 2^{k/2-l}\|f\|_{l,*} \tag{5.2}$$

Next, choose $v, w \in (\mathcal{S})^{-1,k,0}$ and consider the left-hand side of (5.1). Using the definition in (3.2) we get

$$\begin{aligned}
(f\diamond v, w)_{-1,k,0} &= \sum_\gamma \int_D (f\diamond v)_\gamma w_\gamma dx\,(2\mathbb{N})^{k\gamma} \\
&= \sum_\gamma \int_D \Big(\sum_{\alpha+\beta=\gamma} f_\alpha v_\beta\Big) w_\gamma dx\,(2\mathbb{N})^{k\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{\gamma} \int_D \big(f_0 v_\gamma w_\gamma - \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|>0}} |f_\alpha||v_\beta||w_\gamma|\big)\, dx\, (2\mathbb{N})^{k\gamma} \\
&= \sum_{\gamma} \int_D f_0 v_\gamma w_\gamma dx\, (2\mathbb{N})^{k\gamma} - \sum_{\gamma} \int_D \big(\sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|>0}} a_{\alpha\gamma} b_{\alpha\beta}\big)\, dx
\end{aligned} \tag{5.3}
$$

where $a_{\alpha\gamma} = (|f_\alpha|(2\mathbb{N})^{k\alpha/2})^{1/2}|w_\gamma|(2\mathbb{N})^{k\gamma/2}$ and $b_{\alpha\beta} = (|f_\alpha|(2\mathbb{N})^{k\alpha/2})^{1/2}|v_\beta|(2\mathbb{N})^{k\beta/2}$. Applying the inequality $ab \leq a^2/2 + b^2/2$ in the last term of (5.3) give

$$
\sum_{\gamma} \int_D \big(\sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|>0}} a_{\alpha\gamma} b_{\alpha\beta}\big)\, dx \leq \frac{1}{2} \sum_{\gamma} \int_D \big(\sum_{|\alpha|>0} a_{\alpha\gamma}^2\big)\, dx + \frac{1}{2} \sum_{\beta} \int_D \big(\sum_{\alpha} b_{\alpha\beta}^2\big)\, dx,
$$

after some reordering and adding of positive terms on the right-hand side. Consider this bound term-wise. From (5.2) and the definition of $a_{\alpha\gamma}$ we have

$$
\begin{aligned}
\frac{1}{2} \sum_{\gamma} \big(\sum_{|\alpha|>0} a_{\alpha\gamma}^2\big)\, dx &\leq \frac{1}{2} \sum_{\gamma} \int_D \sup_{x\in D} \big(\sum_{|\alpha|>0} |f_\alpha(x)|(2\mathbb{N})^{k\alpha/2}\big)|w_\gamma|^2 dx\, (2\mathbb{N})^{k\gamma} \\
&\leq \frac{1}{2} 2^{k/2-l} \|f\|_{l,*} \sum_{\gamma} \int_D |w_\gamma|^2 dx\, (2\mathbb{N})^{k\gamma} \\
&\leq \frac{1}{2} 2^{k/2-l} \|f\|_{l,*} \|w\|_{-1,k,0}^2.
\end{aligned} \tag{5.4}
$$

A similar argument gives

$$
\frac{1}{2} \sum_{\beta} \int_D \big(\sum_{\alpha} b_{\alpha\beta}^2\big)\, dx \leq \frac{1}{2} 2^{k/2-l} \|f\|_{l,*} \|v\|_{-1,k,0}^2. \tag{5.5}
$$

Summing (5.4) and (5.5) and using this estimate in (5.3) proves the lemma. □

Now we state the result about the coercivity of $\mathcal{A}_{-1,k}$.

Lemma 5.3. *If $\mathcal{B}_0$ is coercive on $H_0^1(D)$, then there are constants $k_0 \leq 2l$ and $\theta > 0$ such that the bilinear form $\mathcal{A}_{-1,k}$ given in (4.2) satisfies*

$$
\mathcal{A}_{-1,k}(v,v) \geq \theta \|v\|_{-1,k,1}^2 \tag{5.6}
$$

for all v in $(\mathcal{S})_0^{-1,k,1}$ and all $k < k_0$.

Proof. The proof is easy. First, use (5.1) repeatedly and obtain

$$
\mathcal{A}_{-1,k}(g,g) \geq \sum_{\gamma} \mathcal{B}_0(g_\gamma, g_\gamma)(2\mathbb{N})^{k\gamma} - 2^{k/2-l} C_L \|g\|_{-1,k,1}^2 \tag{5.7}
$$

where $C_L = \sum_{i,j} \|a^{ij}\|_{l,*} + \sum_i \|b^i\|_{l,*} + \|c\|_{l,*} < \infty$. Next, since we assumed $\mathcal{B}_0$ to be coercive, there is a constant $\theta_0 > 0$ such that $\mathcal{B}_0(h,h) \geq \theta_0 \|h\|_1^2$ for all h in $H_0^1(D)$. Thus

$$
\mathcal{A}_{-1,k}(g,g) \geq \sum_{\gamma} \theta_0 \|g_\gamma\|_1^2 (2\mathbb{N})^{k\gamma} - 2^{k/2-l} C_L \|g\|_{-1,k,1}^2 = \theta \|g\|_{-1,k,1}^2
$$

for all g in $(\mathcal{S})_0^{-1,k,1}$. The constant $\theta := \theta_0 - 2^{k/2-l} C_L$ can be made positive if we choose k small enough. It suffices to have

$$
k < k_0 := 2l + \frac{2}{\ln 2} \ln\Big(\frac{\theta_0}{C_L}\Big) \leq 2l.
$$

□

Theorem 5.4. *Under the assumptions in Lemma 5.1 and 5.3, and with $\rho = -1$ and k_0 chosen small enough, the variational problem (4.3) has a unique solution for each $k < k_0$.*

Proof. The continuity of $\mathcal{A}_{-1,k}$ follows from Lemma 5.1, and the coercivity from Lemma 5.3. Thus, by the Lax-Milgram Theorem we have existence and uniqueness of the solution to (4.3). □

Remark 5.5. From Corollary 4.4 we know that the solution (when it exist) is independent of our choice of (ρ, k).

Example 5.6. Consider the pressure equation (1.3) with the singular permeability given by the Wick exponential

$$K(x,\omega) := \exp^{\diamond}(W(x)) := \sum_{n=0}^{\infty} \frac{W(x)^{\diamond n}}{n!}$$

where $W(x)$ is the singular white noise defined in Example 3.5. A straightforward calculation shows that

$$K(x,\omega) = \sum_{n=0}^{\infty} \frac{\eta^{\alpha}}{\alpha!} H_{\alpha}(\omega) \text{ where } \eta^{\alpha}(x) := \prod_{i=1}^{\infty} \eta_i^{\alpha_i}(x).$$

It follows that $\mathcal{B}_{\beta}$ from (4.4) can be written as

$$\mathcal{B}_{\beta}(g,h) = \int_D \frac{\eta^{\beta}(x)}{\beta!} \nabla g(x) \nabla h(x) dx.$$

The domain D is a bounded domain with Lipschitz boundary ∂D by assumption. Thus, by the Poincaré inequality it follows that $\mathcal{B}_0$ is coercive on $H_0^1(D)$. By Theorem 5.4 there exists a unique solution to the variational form of the pressure equation.

Remark 5.7. The approach to finding the variational formulation and showing continuity and coercivity of the bilinear form described in this paper can be adapted to other type of boundary conditions such as Neumann or mixed conditions. The variational formulation and the space $(\mathcal{S})_0^{-1,k,1}$ changes to account for the difference in boundary conditions. Apart from this the approach is similar to what we described above.

6. A FINITE ELEMENT APPROXIMATION

From the previous section we know that the variational problem (4.3) has a unique solution in $(\mathcal{S})_0^{-1,k,1}$ as long as we choose k small enough. We want to investigate this solution, and do so by solving the variational problem numerically using a finite element method.

The finite element method is based on a finite dimensional approximation of the space $(\mathcal{S})_0^{-1,k,1}$. By constructing a basis for this finite space, and using the linearity of the equations, we can express the approximated problem as a set of linear equations that give the coefficients of the finite basis expansion of the solution. We show how a finite dimensional space can be constructed, give an appropriate basis for this space, present the resulting finite dimensional problem, and give an error estimate securing convergence in $(\mathcal{S})_0^{-1,k,1}$.

A natural approach to construct a finite dimensional subspace of $(\mathcal{S})_0^{-1,k,1}$ is indicated by Proposition 3.2. We have the isometry

$$(\mathcal{S})_0^{-1,k,1} \cong H_0^1(D) \otimes (\mathcal{S})^{-1,k}.$$

Thus, we may construct our subspace by using a finite dimensional subspace of the space $(\mathcal{S})^{-1,k}$ and some classical finite element approximation of $H_0^1(D)$ [4, 23].

First, any $f \in (\mathcal{S})^{-1,k}$ may be written as a formal sum $f = \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ where $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ and all the coefficients $f_{\alpha} \in \mathbb{R}$. Following the ideas in [1] a finite dimensional subspace is then constructed by restricting the allowed multi-indices to a finite set. Define the cutting $\mathcal{I}_{N,K} \subset \mathcal{I}$ as

$$\mathcal{I}_{N,K} := \{0\} \cup \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \{\alpha \in \mathbb{N}_0^k \mid |\alpha| = n \text{ and } \alpha_k \neq 0\}. \tag{6.1}$$

The resulting space

$$(\mathcal{S}_{N,K})^{\rho,k} := \{\, f = \sum_{\alpha \in \mathcal{I}_{N,K}} f_\alpha H_\alpha : f_\alpha \in \mathbb{R}, \|f\|_{\rho,k} < \infty\} \tag{6.2}$$

is clearly a subset of $(\mathcal{S})^{\rho,k}$ for any choice of N, K and can be shown to have dimension $\binom{N+K}{K}$.

Now for the construction of a subspace of $H_0^1(D)$. This problem has been studied extensively, and here we follow the approach described in [4]. First we give some basic notation and definitions. For given $n \in \mathbb{N}_0$ and some domain $T \subset \mathbb{R}^d$ define the function spaces

$$\mathbb{P}_n(T) := \{ \sum_{\beta \in \mathbb{N}^d, |\beta| \leq n} c_\beta x^\beta : c_\beta \in \mathbb{R}, x \in T\},$$

that is, $\mathbb{P}_n(T)$ is the family of polynomials of degree less or equal to n defined on the domain T. A triangulation of a polygonal domain $\mathcal{D}$ is a finite collection of open triangles $\{T_i\}_{i=0}^N$ such that

(a) $T_i \cap T_j =$ if $i \neq j$ and $\cup \bar{T}_i = \bar{\mathcal{D}}$.
(b) No vertex of any triangle lies in the interior of an edge of any other triangle.

The family of triangulations $\mathcal{T}_h$ for $h \in (0,1]$ of the domain $\mathcal{D}$ is said to be non-degenerate if

(a) $\max\{\operatorname{diam} T : T \in \mathcal{T}_h\} \leq h \operatorname{diam} \mathcal{D}$
(b) There exists a $\rho > 0$ such that for all $T \in \mathcal{T}_h$ and all $h \in (0,1]$

$$\operatorname{diam} B_T \geq \rho \operatorname{diam} T$$

where B_T is the greatest ball contained in the triangle T.

We call h the grid-size of a given triangulation T_h.

Now we proceed with the approximation of $H_0^1(D)$. We assume spatial dimension $d = 1, 2$ or 3 and for convenience we assume that D is a polyhedral domain in $\mathbb{R}^d$. Let $\mathcal{T}_h$ be some non-degenerate family of triangulations of D and for each $T_h \in \mathcal{T}_h$ define

$$V_M := \{v \in C^0(\bar{D}) : v|_{\partial D} = 0, v \in \mathbb{P}_1(K) \text{ for each } K \in T_h\}. \tag{6.3}$$

We let M denote the number of internal nodes in the triangulation. Note that M is the dimension of the finite element space V_M and it inversely proportional to the grid-size h. Furthermore, note that the continuity of the functions in V_M secures that V_M is a subspace of $H_0^1(D)$, as an easy computation using the Green formula shows.

The approximations (6.2)–(6.3) may now be put together to define the finite dimensional space

$$(\mathcal{S}_{N,K,M})_0^{-1,k,1} := V_M \otimes (\mathcal{S}_{N,K})^{-1,k}. \tag{6.4}$$

Because $V_M \subset H_0^1(D)$ and $(\mathcal{S}_{N,K})^{-1,k} \subset (\mathcal{S})^{-1,k}$ we clearly have that

$$(\mathcal{S}_{N,K,M})_0^{-1,k,1} \subset (\mathcal{S})_0^{-1,k,1}. \tag{6.5}$$

for any choice of N, M, K. Let f be in $(\mathcal{S})_0^{-1,k,1}$. Then the projection of f into $(\mathcal{S}_{N,K,M})_0^{-1,k,1}$ is given by

$$f^{N,K,M}(x) := \sum_{\alpha \in \mathcal{I}_{N,K}} f_\alpha^M(x) H_\alpha \tag{6.6}$$

where $f_\alpha^M(x) \in V_M$ denotes the projection of $f_\alpha \in H_0^1(D)$ into V_M. The following result is due to Benth and Gjerde [2].

Theorem 6.1. *Let $k, q \geq 0$ be given and assume $r := k - q > r^*$, where r^* solves*

$$\frac{r^*}{2^{r^*}(r^* - 1)} = 1 \qquad (r^* \approx 1.54).$$

Then there exists a constant $C > 0$ depending on $\mathcal{T}_h$ such that

$$\|f - f^{N,K,M}\|_{-1,-k,1} \leq B_{N,K,r}\|f\|_{-1,-q,1} + Ch\|f\|_{-1,-k,2},$$

and for any $g \in (\mathcal{S})^{1,k,1}$

$$|\langle\langle f - f^{N,K,M}, g\rangle\rangle| \le (B_{K,N,r}\|f\|_{-1,-q,1} + Ch\|f\|_{-1,-k,2})\|g\|_{1,k,1}$$

where

$$\begin{aligned} B_{K,N,r} =& \sqrt{C_1(r)K^{1-r} + C_2(r)(\frac{r}{2^r(r-1)})^{N+1}} \\ C_1(r) =& \frac{1}{2^r(r-1)-r} \\ C_2(r) =& 2^r(r-1)C_1(r). \end{aligned} \tag{6.7}$$

Note that we have changed notation slightly here, assuming both $k, q \ge 0$ and explicitly writing the minus signs in the norms.

Using the space (6.4) the Galerkin approximation problem [4], derived from the variational formulation in (4.3), becomes

$$\begin{aligned} &\text{Find } u^{N,K,M} \in (\mathcal{S}_{N,K,M})_0^{-1,-k,1} \text{ such that} \\ &\mathcal{A}_{-1,-k}(u^{N,K,M}, v) = (f, v)_{-1,-k,0} \text{ for all } v \in (\mathcal{S}_{N,K,M})_0^{-1,-k,1} \end{aligned} \tag{6.8}$$

Existence of a unique solution to (6.8) is a result of the Lax-Milgram Theorem together with (6.5) and the results about coercivity and continuity of the bilinear form. Furthermore, by Céas Theorem [4] we have the following estimate on the error

$$\|u - u^{N,K,M}\|_{-1,-k,1} \le \frac{C_L}{\theta}\min\{\|u - v\|_{-1,-k,1} : v \in (\mathcal{S}_{N,K,M})_0^{-1,k,1}\}, \tag{6.9}$$

where $C_L < \infty$ is the continuity constant from Lemma 5.1 and $\theta > 0$ the coercivity constant from Lemma 5.3.

Assuming $u \in (\mathcal{S})_0^{-1,-q,1} \cap (\mathcal{S})^{-1,-q,2}$ then by Theorem 6.1 together with (6.9) we get the error bound

$$\|u - u^{N,K,M}\|_{-1,-(q+r),1} \le \frac{C_L}{\theta}(B_{N,K,r}\|u\|_{-1,-q,1} + Ch\|u\|_{-1,-(q+r),2}), \tag{6.10}$$

for any M, N, K and q big enough.

This bound on the error implies convergence of the numerical method in $(\mathcal{S})_0^{-1,-k,1}$, because by (6.7) it is clear that $B_{N,K,r} \to 0$ as $N, K \to \infty$ as $M \to \infty$ (or equivalently as $h \to 0$).

7. An algorithm for the numerical solution

We now turn our attention to the finite dimensional Galerkin variational problem (6.8). The solution of (6.8) has the chaos expansion

$$u^{N,K,M} = \sum_{\alpha \in \mathcal{I}_{N,K}} u_\alpha^M H_\alpha, \tag{7.1}$$

where u_α^M are in $V_M \subset H_0^1(D)$ and the cutting $\mathcal{I}_{N,K}$ is defined in (6.1). The formulation corresponding to (4.5) becomes

$$\begin{aligned} &\text{For each } \gamma \in \mathcal{I}_{N,K} \text{ find } u_\alpha^M \in V_M \text{ such that} \\ &\mathcal{B}_0(u_\alpha^M, w) = (f_\gamma, w)_0 - \sum_{\alpha \prec \gamma} \mathcal{B}_{\gamma-\alpha}(u_\alpha^M, w) \text{ for all } w \in V_M. \end{aligned} \tag{7.2}$$

Note that now we also need the property

$$\gamma \in \mathcal{I}_{N,K} \Rightarrow \{\alpha \in \mathcal{I} : \alpha \prec \gamma\} \subset \mathcal{I}_{N,K}. \tag{7.3}$$

in order to have a well-posed problem. The cutting defined in (6.1) satisfy this property. Since this is a finite dimensional problem both existence and uniqueness of a solution follows by the induction argument in Theorem 4.2.

We rewrite (7.2) to a series of linear systems of equations. Let $\{\phi_n\}_{n=1}^M$ be some basis for V_M. Then each coefficient u_α^M can be written as

$$u_\alpha^M(x) = \sum_{n=1}^{M} c_\alpha^n \phi_n(x) \qquad (c_\alpha^n \in \mathbb{R}). \tag{7.4}$$

Substituting this sum into (7.2) and choosing $w = \phi_m$ gives us the following series of M-dimensional linear systems:

$$A_0 \mathbf{c}_\gamma = \mathbf{f}_\gamma - \sum_{\alpha \prec \gamma} A_{\gamma-\alpha} \mathbf{c}_\beta, \tag{7.5}$$

where the matrices $A_{\gamma-\alpha} := [\mathcal{B}_{\gamma-\alpha}(\phi_n, \phi_m)]$ are in $\mathbb{R}^{M\times M}$ and the vectors $\mathbf{f}_\gamma := [(f_\gamma, \phi_n)_0]$ and $\mathbf{c}_\gamma := [c_\gamma^n]$ are in $\mathbb{R}^M$.

Remark 7.1. Since the equation for $\mathbf{c}_\gamma$ depends on $\mathbf{c}_\alpha^M$ for $\alpha \prec \gamma$, we need to traverse the multi-indices in such a way that the necessary information is available when we reach the γth equation.

Summarising these ideas give the following algorithm for the numerical solution of (1.1). Here we assume that an appropriate triangulation T_h has been chosen, with the corresponding function space V_M and basis $\{\phi_n\}$ $(n = 1, \ldots, N)$. Furthermore, we assume that there has been chosen $N, K \in \mathbb{N}$ to get a cutting as described in (6.1), and that the set of variational equations (7.2) has been formed.

Algorithm 7.2.

1. *Form the ordered set* $\mathcal{I}_{N,K}$
2. *Start with* $\gamma = (0, \cdots, 0) \in \mathcal{I}_{N,K}$
3. *If* $\gamma \in \mathcal{I}_{N,K}$ *do*
    - *3.1 Calculate* $\mathbf{f}_\gamma = [(f_\gamma, \phi_n)_{0,D}]$.
    - *3.2 Find the set* $\mathcal{L}_\gamma = \{\alpha \in \mathcal{I}_{N,K} : \alpha \prec \gamma\}$
    - *3.3 For each* $\alpha \in \mathcal{L}_\gamma$
        - *3.3.1 Calculate the matrices* $A_{\gamma-\alpha} = [\mathcal{B}_{\gamma-\alpha}(\phi_n, \phi_m)]$.
        - *3.3.2 Update the right hand side* $\mathbf{f}_\gamma := \mathbf{f}_\gamma - A_{\gamma-\alpha}\mathbf{c}_\alpha$
    - *3.5. Solve the problem* $A_0\mathbf{c}_\gamma = \mathbf{f}_\gamma$
4. *Find the next* γ *and go to 3.*

We now comment on the different parts of this algorithm:

First, note that there is a one-to-one correspondence between each α in $\mathcal{I}_{N,K}$ and the binary number given by

$$\operatorname{bin}(\alpha) := \underbrace{11\cdots1}_{\alpha_K}0\underbrace{11\cdots1}_{\alpha_{K-1}}0\cdots0\underbrace{11\cdots1}_{\alpha_1}. \tag{7.6}$$

In step 1. we order the set $\mathcal{I}_{N,K}$ in the following way: Given $\alpha, \beta \in \mathcal{I}_{N,K}$ then

$$\alpha < \beta \text{ if } \begin{cases} |\alpha| < |\beta|, \text{ or} \\ |\alpha| = |\beta|, \operatorname{bin}(\alpha) < \operatorname{bin}(\beta) \end{cases} \tag{7.7}$$

This relation makes $\mathcal{I}_{N,K}$ into a totally ordered set. Clearly, $\alpha \prec \gamma$ imply $\alpha < \gamma$ so (7.7) provides a good way of choosing the next γ in step 4.

Next, considering the number of required computations. A straight forward calculation shows that the algorithm requires $\binom{N+K}{N}$ matrix- and vector-definitions (cf. step 3.1 and 3.3.1), $\binom{N+2K}{2K}$ matrix-vector products and vector additions (cf. step 3.3.2), and $\binom{N+K}{K}$ solutions of a linear system (cf. step 3.5). The exact amount of numerical work required depends on choice of basis $\{\phi_n\}$, numerical integration method in step 3.1 and 3.3.1, and how one solves the linear system in step 3.5. When deciding these parameters for a given case, the above results are helpful when trying to minimise the numerical work. Note that by storing the matrices after they have been calculated in step 3.3.1, instead of recalculation them when we use them again later, we reduce the number of matrix-definitions from $\binom{N+2K}{2K}$ to $\binom{N+K}{K}$. This saves us a considerable amount of computational time, but makes the algorithm need more memory.

Usually one will calculate $\mathbf{f}_\alpha$ and A_β using numerical integration over all the elements in T_h. This kind of integration is a standard tool in (deterministic) finite element methods and there are many available software-packages for this (we used Diffpack [22]). Our opinion is that the possibility to use ready software made for deterministic problems is an advantage for the method. Furthermore, we did a LU factorisation of A_0 to solve the equation in step 3.5. If M is large it is preferable to apply an iterative method like the Conjugated Gradient (CG) method here. See Golub and Van Loan [10] for both LU-factorisation and CG.

Once all the coefficients u_α^M $(\alpha \in \mathcal{I}_{N,K})$ have been computed, one can do stochastic simulations of $u^{N,K,M}(x,\omega)$ by simulating $H_\alpha(\omega)$ for each $\alpha \in \mathcal{I}_{N,K}$ and summing like described in (7.1). The simulation of

$$H_\alpha(\omega) = \prod_j h_{\alpha_j}(\langle \omega, \eta_j \rangle) \tag{7.8}$$

can be done using Lemma 2.1. Draw K independent standard Gaussian distributed variables and form the product (7.8) for each $\alpha \in \mathcal{I}_{N,K}$.

Finally, we would like to point out the parallelism inherent in the problem. Since $\mathbf{c}_\gamma$ only depends on $\{\mathbf{c}_\alpha : \alpha \prec \gamma\}$ it is clear that we can solve for $\{\mathbf{c}_\gamma : |\gamma| = k\}$ in parallel.

8. A NUMERICAL EXAMPLE

The main motivation for this Section was to provide an application of Algorithm 7.2. We also wanted to investigate the properties of the solution to the pressure equation (1.3).

Let the permeability and source term given by (slightly misusing notation)

$$K(\omega, x) = \exp^\diamond W(x,\omega) = \sum_{\alpha \in \mathcal{I}} \frac{\eta^\alpha(x)}{\alpha!} H_\alpha(\omega) \quad \text{and} \quad f(x,\omega) = 1,$$

respectively $(x \in D, \omega \in \mathcal{S}')$. The above permeability corresponds to the singular permeability case studied in [17], where the authors prove existence of a strong solution in $(\mathcal{S})^{-1}$. It should be noted that this solution has later been shown to be in a much smaller space $\mathcal{G}^{-1}$, see [3] for details.

We applied Algorithm 7.2 on the variational formulation arising from the pressure equation (1.3). The properties of the solution was investigated through direct simulations of the pressure, and by plotting the norm $\|u_\gamma^M\|_\infty$ $(\gamma \in \mathcal{I}_{N,K})$ against increasing multi-index. This plot is interesting because of the bound

$$\|u\|^2_{-1,-k,0} \leq |D|^2 \sum_\alpha \|u_\alpha\|^2_\infty (2\mathbb{N})^{-k\alpha},$$

thus if the numerical results indicate that $\|u_\alpha^M\|_\infty$ is uniformly bounded in α, then this would indicate that the solution is in $(\mathcal{S})^{-1,-k,0}$ for some (small) $k > 0$.

We solved three specific cases: Case A, B and C, corresponding to the one-dimensional (A,B) and two-dimensional (C) case with given choices of cutting (N, K) and grid size M. The specific data are given in Table 1. We did more simulations than those we report here, but we picked these examples to illustrate the typical behaviour of the solution.

Now we have the following comments on the simulations.

First, the one-dimensjonal cases (A and B): We start with the stochastic simulation of the pressure. In Figure 1 and 2 we show typical behaviour for the simulated pressure. These simulations are done as described in the comments to Algorithm 7.2. Note here how the approximation in case

Case	A	B	C
Domain (D)	$[-5,5]$	$[-5,5]$	$[-1,1]\times[-1,1]$
Nodes (M)	100	100	256
Cutting (N,K)	$(3,15)$	$(15,3)$	$(3,15)$
Element	Lagrange interval	Lagrange interval	Lagrange triangle

TABLE 1. Data used in the numerical simulations.

B ($N=15, K=3$) in Figure 2 is much smoother than for case A ($N=3, K=15$) in Figure 1. This has the explanation that in case B we have $K=3$, so there are fewer random variables used when simulating each $H_\gamma(\omega)$. Another contribution to this behaviour of the simulated pressure is that coefficients degrease much faster for increasing N compared to increasing K, see discussion below.

Now we consider the $\|\cdot\|_\infty$ norm of chaos coefficients. In Figure 4 and Figure 5 we give plots of $\|u_\alpha^M\|_\infty$ against the ordering of the multi-indices $\alpha\in\mathcal{I}_{N,K}$ described in (7.7). Note how the plot in Figure 4 shows three distinct levels for the values of $\|u_\alpha^M\|_\infty$. These levels corresponds to different order ($|\alpha|$) of the multi-indices α. By comparing Figure 4 with the rapid decrease in the values of $\|u_\alpha^M\|_\infty$ that can be seen in Figure 5, the conclusion we get from our numerical data is: When we increase $|\alpha|$, thus moving to higher order chaos of the solution, the values of $\|u_\alpha^M\|_\infty$ are bounded (and even seem to decrease).

Thus, our numerical results indicate that the solution is in fact quite regular. Since the $u_\alpha\|_\infty$ are bounded (and even decreasing) we expect the real solution to be in some space $(\mathcal{S})^{-1,-k,0}$ for quite small $k>0$.

This argument on the behaviour of the coefficients is also supported by looking at the error term in (6.10). As M is chosen big compared to N and K, the right hand side in the error (6.10) is dominated by the constant

$$\sqrt{\frac{C_1}{K^{r-1}}+C_2\Big(\frac{r}{2^r(r-1)}\Big)^{N+1}}, \qquad (r>r^*\approx 1.54).$$

Thus, it is no surprise that the coefficients decrease faster when we increase N (case B) compared to the increase in K (case A).

To give an idea of how the shape of coefficients u_α, we included in Figure 4 some typical chaos coefficients u_γ^M plotted against the space variable, for the one-dimensional case A1.

Next, for the two-dimensional case: In Figure 6 we present a plot that corresponds to Figure 4 and 5 for the two one-dimensional cases. We can still see the same bounded (and decreasing) behaviour of $\|u_\alpha^M\|_\infty$ as for the one-dimensional case, and the conclusion from this numerical investigation is the same as for the one-dimensional case.

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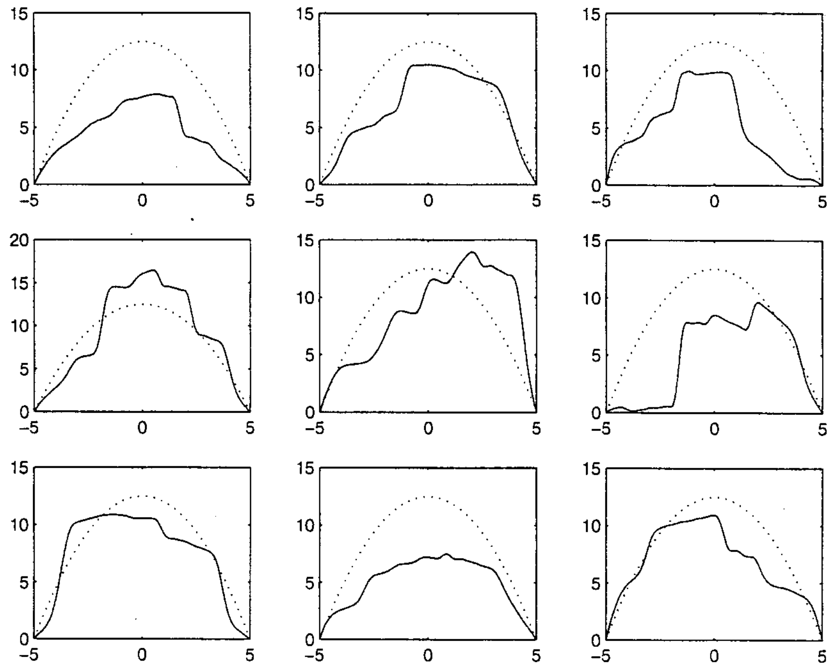

FIGURE 1. *This plot shows nine different simulations of the pressure in the one-dimensional case A* ($N = 3, K = 15$). *As a comparison we included a plot of the expected solution* u_0.

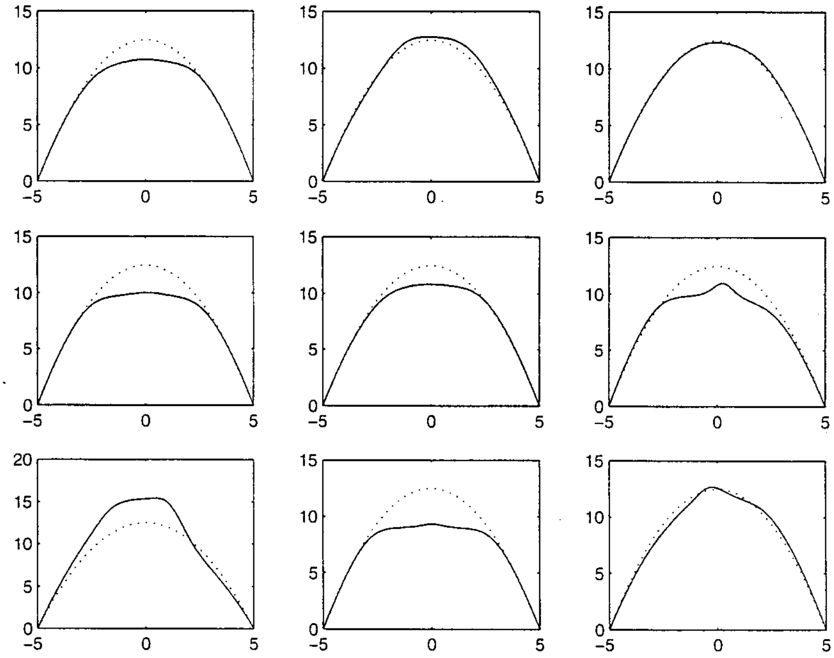

FIGURE 2. *Same kind of plot as Figure 1 above for the one-dimensional case B* ($N = 15, K = 3$).

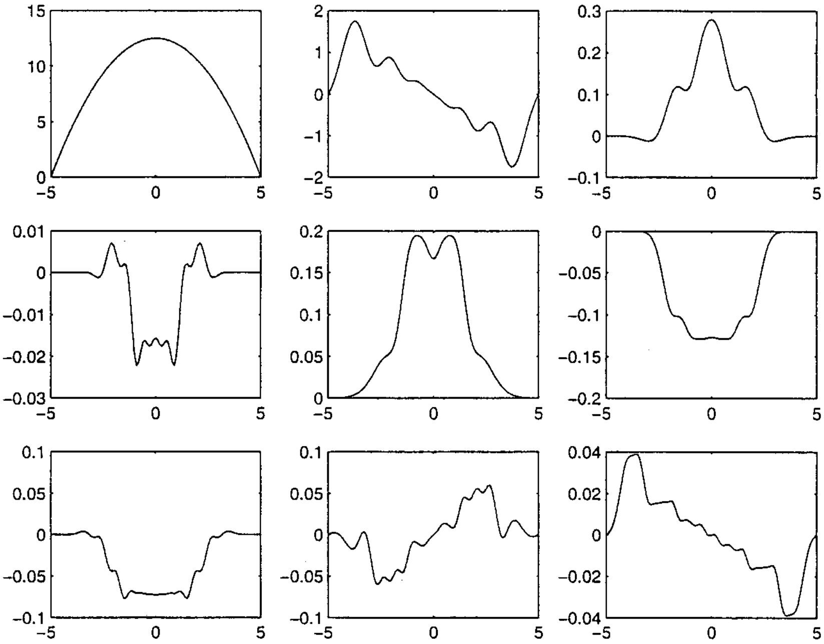

FIGURE 3. *The figure shows some typical coefficients u_α^M, plotted against space in the one-dimensional case A ($N = 3, K = 15$). Counted from right to left we have coefficients numbered 1, 13, 59, 201, 274, 387, 431, 611 and 797. The simulations of the pressure is then a sum of these (and many more) coefficients, with a stochastic weight (cf.* (7.1) *).*

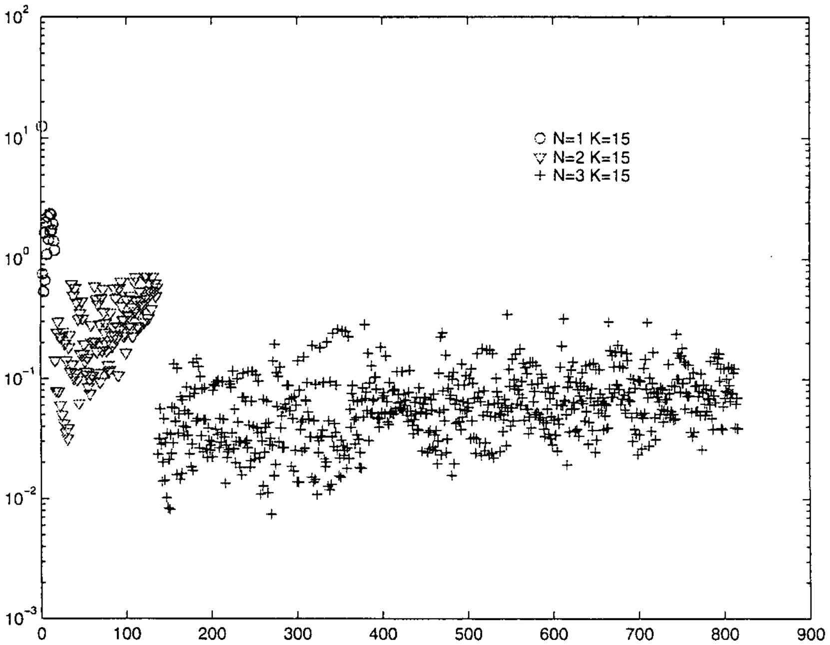

FIGURE 4. *The figure shows $\|u_\gamma^M\|_\infty$ plotted against the ordering of the multi-indices in $\mathcal{I}_{N,K}$ for the one-dimensional case A ($N = 3, K = 15$). Note the three distinct levels in the values, as the order $|\alpha|$ increase.*

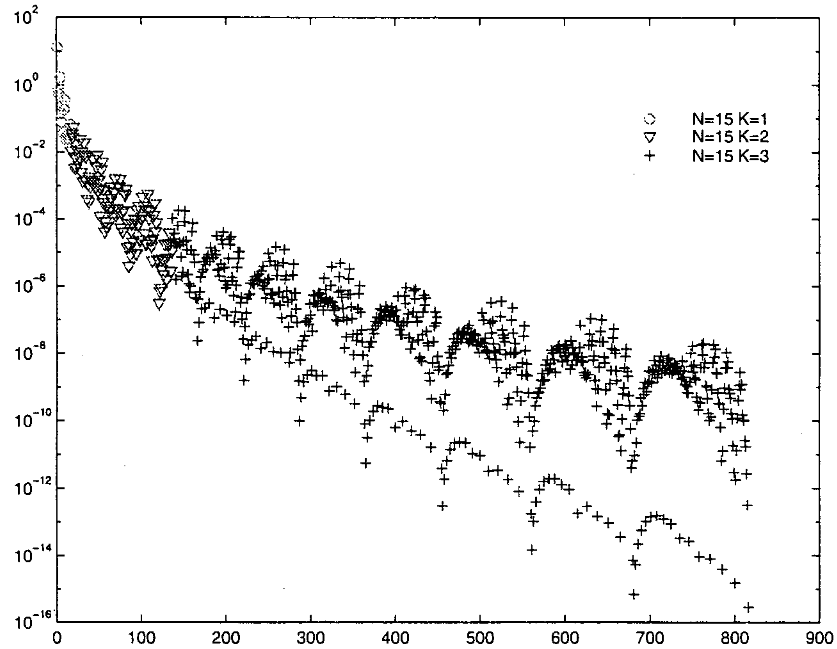

FIGURE 5. *Same plot as in Figure 4 above, but now for the one-dimensional case B* $(N = 15, K = 3)$. *Notice that there is a clear trend towards smaller* $\|u_\gamma^M\|_\infty$ *as we move towards bigger* α.

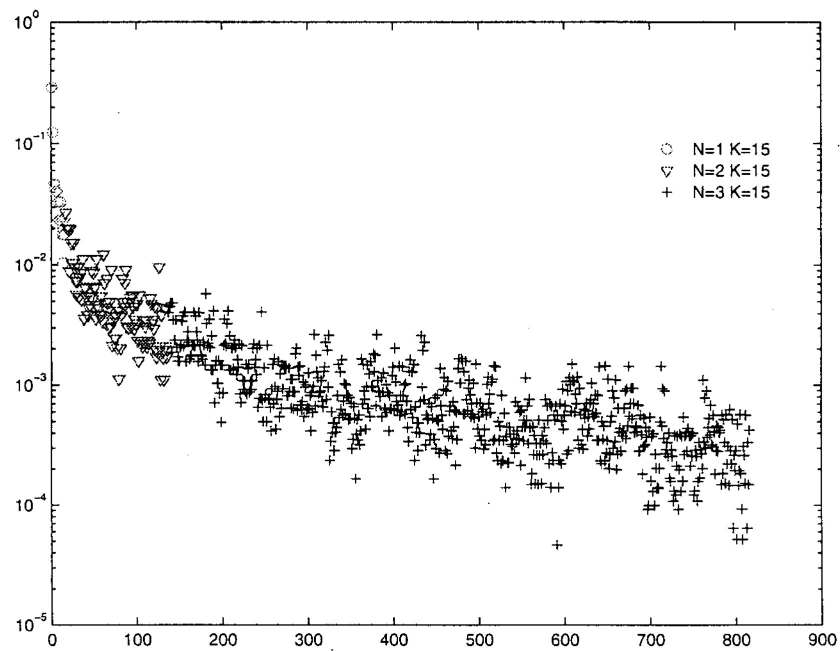

FIGURE 6. *The figure shows* $\|u_\gamma^M\|_\infty$ *plotted against the ordering of the multi-indices in* $\mathcal{I}_{N,K}$ *in the two-dimensional case C. We see the same behaviour as for the corresponding one-dimensional case A.*