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**Foundations for semantics in complete metric spaces**

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0. Introduction

In the last years a series of papers appeared that deal with the semantics of those languages or systems that allow for some notion of concurrency [1-8, 13-18, 22]. The approach of Francez et al. [7] e.g. is based on complete partial orders, the work of de Bakker and Zucker [6] is based on complete metric spaces, Plotkin presents an operational approach [17], axiomatic methods can be found in [1, 3, 10, 13, 15, 18]. The connection between some of the approaches has been investigated in [8, 12].

We are here presenting an investigation and foundation of the metric space approach of [6].

In order to do so we briefly sketch how semantics is defined in [6]. The basic concepts of [6] are the notion of a "process domain" and a "domain equation". Given a language  $L$  for which semantics is to be defined, the authors suggest to construct a suitable equation  $P = \mathcal{F}(P)$ , called domain equation, such that the solution of this equation (a complete metric space) provides a domain for the interpretation of programs, i.e. the meaning function maps programs to elements of this solution.

The authors demonstrate their ideas concerning the solution of such equations by considering the following four prototypes

$$P = \{p_0\} \cup (A \times P) \quad (1)$$

$$P = \{p_0\} \cup \wp_c(A \times P) \quad (2)$$

$$P = \{p_0\} \cup (A \rightarrow \wp_c(B \times P)) \quad (3)$$

$$P = \{p_0\} \cup (A \rightarrow \wp_c((B \times P) \cup (C \rightarrow P))) \quad (4)$$

where e.g. the Cartesian product is used to model the sequencing of actions, the powerset construction  $\wp_c$  (see section I.) and the function space construction is used to model nondeterminism, concurrency and communication.

For each equation  $P = \mathcal{F}_i(P)$ ,  $i = 1, 2, 3$ , a solution is constructed as follows (the last equation is left to the reader): A sequence  $((P_n, d_n))$  of metric spaces is constructed,  $P_\omega$  is defined as  $(\cup P_n, \cup d_n)$ . It is then shown that the completion  $(P, d)$  of  $P_\omega$  is a solution of the given equation. The thus constructed solutions serve as semantic domains for various sample languages.

When looking closer at the proposed handling of process domain equations, a number of questions arise immediately:

*Is the thus constructed solution the only solution? If not, what features characterize the constructed solution? And most important, under what conditions is*

it possible to give a solution of an equation  $P = \mathcal{F}(P)$  in such a way? What properties must the operator  $\mathcal{F}$  have in order to guarantee the existence of a solution altogether?

In this paper we are dealing with these questions. In particular, we establish a framework for discussing the existence of solutions of equations as discussed above. This is an important task, because, when we are trying to apply the techniques of [6] to some nontrivial language like CSP [8, 11, 12] we have to have some criterion to decide if the respective equation does have a solution at all.

This problem already occurs with such simple-looking equations as equation (4), the solution of which is left to the reader in [6].

We will prove that this equation cannot be solved in the way claimed in [6].

This is interesting, as the associated operator  $\mathcal{F}$  does not satisfy our conditions for existence of fixed points given in theorem 12 and theorem 14.

We finally make two observations. First, there is a strong analogy between the construction of a fixed point theory for the category *CPO* of complete partial orders from the theory of fixed points in complete partial orders on one side and our ideas on the other. Second, everyone who wants to use complete metric spaces to define the semantics of some language does not have to go into details about existence proofs of fixed points. One only has to ensure some contraction property of the operator involved according to theorem 12 or theorem 14. This can be easily done by Lemma 11.

The paper is divided into five sections. Section I contains the definitions and elementary statements. In section II we establish conditions for existence and uniqueness of fixed points. Section III is devoted to equation (4) from above. Section IV deals with the special role of the  $\varphi_c$ -operator and section V creates the connection to related work.

## I. Definitions and Elementary Properties

### Definition 1

Let  $(N, d_N)$  and  $(M, d_M)$  be metric spaces. A function  $f : N \rightarrow M$  is called a weak contraction, if  $\forall x \in N \forall y \in N$

$$d_M(f(x), f(y)) \leq d_N(x, y).$$

### Remark 1

In the following we will consider only metric spaces  $(N, d_N)$  for which

$$\forall x \forall y \quad d_N(x, y) \leq 1$$

holds. This is no restriction for our purposes, as  $d_N$  and  $d_N/(1 + d_N)$  yield the same topology on  $N$ .

### Definition 2

Let  $(M, d_M)$  be a metric space ( $d_M \leq 1$ ), let  $\wp(M)$  denote the collection of all subsets of  $M$  and let  $\wp_c(M)$  denote the collection of all closed subsets of  $M$ . The Hausdorff metric on  $\wp(M)$  is given by

$$d(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\right\}$$

for  $X, Y \in \wp(M)$ .

It is well known that  $(\wp_c(M), d)$  yields a metric space. Moreover it has been shown by Hahn [9]:

### Remark 2

If  $(M, d_M)$  is complete, so is  $(\wp_c(M), d)$ .

### Definition 3

Let  $(N, d_N)$ ,  $(M, d_M)$  be metric spaces. A weak contraction  $f : N \rightarrow M$  is called an embedding, if it preserves distances, i.e. if  $d_M(f(x), f(y)) = d_N(x, y) \quad \forall x, y \in N$ . If the embedding  $f$  is onto,  $f$  is called an isometry.

### Remark 3

Let  $(N, d_N)$ ,  $(M, d_M)$  be complete metric spaces. If  $e : N \rightarrow M$  is an embedding then  $N$  can be identified with the closed subset  $e(N)$  of  $M$ . Hence, we can talk about

the distance of  $N$  and  $M$  (as elements of  $\wp_c(M)$ ) with respect to the embedding  $e$ , denoted by  $d_e(N, M) = d(e(N), M)$ . The subscript  $e$  will often be omitted, if no ambiguity arises.

Lemma 1

Let  $N, M, Z$  be complete metric spaces. Let  $e : N \rightarrow M, f : M \rightarrow Z$  be embeddings then

$$d_e(N, M) \leq d_{e \circ f}(N, Z) \quad ^1).$$

Proof:

$$\begin{aligned} d_e(N, M) &= d(e(N), M) \\ &= \sup_{x \in M} \inf_{y \in e(N)} d(x, y) \\ &= \sup_{x \in M} \inf_{y \in e(N)} d(f(x), f(y)) \\ &= \sup_{x \in f(M)} \inf_{y \in f(e(N))} d(x, y) \\ &\leq \sup_{x \in Z} \inf_{y \in f(e(N))} d(x, y) \\ &= d(f(e(N)), Z) \\ &= d_{e \circ f}(N, Z). \end{aligned}$$

Hence, if  $N$  can be embedded into  $M$  and  $M$  into  $Z$  we will write

$$d(N, M) \leq d(N, Z)$$

bearing in mind that the assumed embedding of  $N$  into  $Z$  is the functional composition of the two given embeddings.

Definition 4

A sequence  $((M_i, d_i))_{i \geq 0}$  of metric spaces together with a sequence of embeddings  $(e_i)_{i \geq 0}, e_i : M_i \rightarrow M_{i+1}$ , is called an embedding sequence.

Definition 5

Let  $(N, d_N), (M, d_M)$  be metric spaces,  $e : N \rightarrow M$  an embedding. A weak contraction  $c : M \rightarrow N$  is called a  $\mu$ -cut for  $e$  if,

- i)  $\forall x \in N \quad c(e(x)) = x$
- ii)  $\forall x \in M \quad d_M(x, e(c(x))) \leq \mu.$

<sup>1)</sup>  $e \circ f$  denotes the composition of  $e$  and  $f$  such that first  $e$  is applied and then  $f$ .

Let  $((M_i, d_i))_{i \geq 0}$  with  $(e_i)_{i \geq 0}$  be an embedding sequence with associated  $\mu_i$ -cuts  $c_i$  then

$$c_{mn} : M_m \rightarrow M_n$$

is defined by

$$c_{mn} = \begin{cases} id, & \text{if } m = n \\ c_{m-1} \circ \dots \circ c_n, & \text{if } m > n \\ e_m \circ \dots \circ e_{n-1}, & \text{if } m < n. \end{cases}$$

Remark 4

Let  $(N, d_N)$ ,  $(M, d_M)$  be metric spaces,  $e : N \rightarrow M$  an embedding,  $c : M \rightarrow N$  a weak contraction such that (i) holds. One may interpret  $c(x)$  as an approximation of  $x$  in  $N$ . Then (ii) implies that the approximation is at least as good as  $\mu$ .

Lemma 2

Let  $(N, d_N)$ ,  $(M, d_M)$  be complete metric spaces,  $e : N \rightarrow M$  an embedding with  $\mu$ -cut  $c$  then

$$d_e(N, M) \leq \mu.$$

Proof:

By remark 3 and definition 2.

In order to be able to formulate the fixed point problem we have to define a suitable category in which the equations have to be solved.

Definition 6

The category MS is defined as follows: the objects of MS are the metric spaces ( $d \leq 1$ ), the morphisms are the weak contractions. The category CMS has as objects complete metric spaces, the morphisms are the weak contractions. The subcategory of CMS that has the same objects and embeddings as morphisms is called CMS<sub>E</sub>.

Remark 5

In MS and CMS the one-element spaces are terminal, the empty set is initial.

Remark 6

In MS products exist.

Remark 7

Let  $((M_i, d_i))$  with  $(e_i)$  be an embedding sequence in  $MS$ . Then the direct limes of  $(M_i)$  in  $MS$  with respect to  $(e_i)$  exists and is denoted by  $(\cup M_i, \cup d_i)$ .

Lemma 3

Let  $((M_i, d_i))_{i \geq 0}$  be an embedding sequence in  $CMS$ . Let  $M$  denote the completion of the direct limes  $(\cup M_i, \cup d_i)$  of  $((M_i, d_i))$  in  $MS$ . If there exists a  $0 \leq k < 1$  such that  $d(M_i, M_{i+1}) \leq k \cdot d(M_{i-1}, M_i)$  for all  $i$  then  $M = \lim M_i$  in  $\wp_c(M)$ .

Proof:

Obviously each  $M_i$  can be embedded into  $M$ , hence  $(M_i)_{i \geq 0}$  is a Cauchy sequence in  $(\wp_c(M), d)$ . By Hahn's Theorem [9] one concludes that its limes  $N$  equals

$\{x : x = \lim x_n, (x_n) \text{ Cauchy sequence, } x_n \in M_n\}$  and hence  $N \subseteq M$ . To show that  $M \subseteq N$  consider  $x \in M$ ,  $x = \lim y_n$ ,  $y_n \in \cup M_i$ ,  $(y_n)$  Cauchy-sequence and construct a suitable subsequence  $(x_n)$  with  $x_n \in M_n$ .

Lemma 4

Let  $((M_i, d_i))_{i \geq 0}$  with  $(e_i)_{i \geq 0}$  be an embedding sequence in  $CMS$ . The completion  $M$  of  $(\cup M_i, \cup d_i)$  is the direct limes of  $(M_i)$  in  $CMS$ .

Proof:

By the universal properties of the completion and the continuity of the metric.

In the following we will be interested in such solutions of equations that are complete metric spaces as in [6]. The reason why fixed points that are not complete metric spaces are not interesting for the semantic specification of programming languages is easily understood by the following example.

Example 1 (see [6]):

Let  $(X, d)$  be a metric space,  $d \leq 1$ ,  $p_0$  a distinguished element,  $A$  an arbitrary set. Consider  $Y = \{p_0\} \cup A \times X$  together with the metric

$$d(p_0, p_0) = 0,$$

$$d(p_0, y) = d(y, p_0) = 1 \text{ for } y \neq p_0,$$

$$d((a, x), (a', x')) = \begin{cases} 1, & \text{if } a \neq a' \\ \frac{1}{2}d(x, x'), & \text{if } a = a'. \end{cases}$$

Let  $\mathcal{F}$  be the functor in  $MS$  that maps  $X$  to  $\{p_0\} \cup A \times X$ . Define

$$P_0 = \{p_0\}, P_{i+1} = \mathcal{F}(P_i), i \geq 0$$

and  $P_\omega = \bigcup_{i \geq 0} P_i$  with the inherited metric then clearly there is an isometry between  $P_\omega$  and  $\mathcal{F}(P_\omega)$ , hence  $P_\omega$  is a fixed point of  $\mathcal{F}$ . If, however,  $P_\omega$  is to be used as a semantic domain for the interpretation of programs, the problem arises that nonterminating program executions cannot be handled. This can be achieved by taking the completion of  $P_\omega$  as a semantic domain. A nonterminating computation can then be modelled by the limes of the Cauchy sequence of its finite approximations.

Definition 7

Let  $\mathcal{F} : MS \rightarrow MS$  be a functor. A metric space  $X$  in  $MS$  is called a prefixed point of  $\mathcal{F}$ , if there is an embedding  $e : X \rightarrow \mathcal{F}(X)$ . A prefixed point  $X$  is called a fixed point if  $e$  is an isometry.

Definition 8

Let  $n \geq 1$  and let

$$\mathcal{F} : \underbrace{MS \times \dots \times MS}_{n \text{ times}} \rightarrow MS$$

be a functor.  $\mathcal{F}$  preserves completeness, if for  $M_1, \dots, M_n$  in  $CMS$ ,  $\mathcal{F}(M_1, \dots, M_n)$  is an object in  $CMS$ .  $\mathcal{F}$  preserves embeddings if, given embeddings  $e_i : N_i \rightarrow M_i$ ,  $i = 1, \dots, n$ ,  $\mathcal{F}(e_1, \dots, e_n)$  is an embedding from  $\mathcal{F}(N_1, \dots, N_n)$  to  $\mathcal{F}(M_1, \dots, M_n)$ . If  $\mathcal{F}$  preserves embeddings we say that  $\mathcal{F}$  preserves  $\mu$ -cuts if, given embeddings  $e_i$  with  $\mu$ -cuts  $c_i$ ,  $i = 1, \dots, n$ , then  $\mathcal{F}(c_1, \dots, c_n)$  is a  $\mu$ -cut for  $\mathcal{F}(e_1, \dots, e_n)$ .

Lemma 5

Let  $\mathcal{F} : MS \rightarrow MS$  be a functor that preserves completeness and embeddings. Then there is a prefixed point of  $\mathcal{F}$  in  $CMS$ .

Proof:

If  $\mathcal{F}\emptyset = \emptyset$  nothing has to be shown. Let now  $\mathcal{F}\emptyset \neq \emptyset$ , hence  $\mathcal{F}(X) \neq \emptyset$  for all  $X$  in  $MS$ . Let  $M_0$  be any metric space consisting of one element and define  $M_i = \mathcal{F}(M_{i-1})$ ,  $i \geq 1$ . As  $M_0$  is complete, so are the  $M_i$ . Moreover,  $M_0$  can be embedded into  $\mathcal{F}(M_0)$ ,  $e_0 : M_0 \rightarrow \mathcal{F}(M_0) = M_1$ , hence there is an embedding  $e_i : M_i \rightarrow M_{i+1}$ . Let  $M$  denote the completion of the direct limes of the  $(M_i)$ . There is a canonical embedding  $h_i : M_i \rightarrow M$ , hence there is an embedding  $\mathcal{F}(h_i) : \mathcal{F}(M_i) = M_{i+1} \rightarrow \mathcal{F}(M)$ ,  $i \geq 0$ . We conclude that  $\bigcup M_i$  can be embedded into  $\mathcal{F}(M)$ ,  $e : \bigcup M_i \rightarrow \mathcal{F}(M)$ . By the universal properties of the completion, the fact that  $\mathcal{F}(M)$  is complete and the continuity of the metric we conclude the existence of an embedding from  $M$  to  $\mathcal{F}(M)$ .



Corollary 6

Every functor  $\mathcal{F} : CMS \rightarrow CMS$  that preserves embeddings has a prefixed point.

Corollary 7

Every functor  $\mathcal{F} : CMS_E \rightarrow CMS_E$  has a prefixed point.

Remark 8

The existence of a weak contraction from  $M$  to  $\mathcal{F}(M)$  (Notation of Lemma 5) can be concluded from the fact that  $M$  is the direct limes of the  $M_i$ 's. This can be easily seen by observing that

$$h_{i-1} = e_{i-1} \circ h_i$$

implies

$$\begin{aligned} \mathcal{F}(h_{i-1}) &= \mathcal{F}(e_{i-1}) \circ \mathcal{F}(h_i) \\ &= e_i \circ \mathcal{F}(h_i). \end{aligned}$$

Definition 9

i) Let  $A$  be a set,  $(X, d)$  a metric space. Define a metric on  $A \times X$  by

$$d((a, x), (a', x')) = \begin{cases} 1 & \text{if } a \neq a' \\ \frac{1}{2}d(x, x') & \text{else.} \end{cases}$$

ii) Let  $A$  be a set,  $(X, d)$  a metric space.  $A \rightarrow X$  is the set of functions from  $A$  to  $X$ . Define a metric on  $A \rightarrow X$  by

$$d(f, g) = \sup_{a \in A} d(f(a), g(a)).$$

iii) Let  $(X_1, d_1)$ ,  $(X_2, d_2)$  be metric spaces. Define a metric on  $M_1 \times M_2$  by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Lemma 8

Let the endofunctor  $\mathcal{F}$  in  $MS$  be defined by

$$\begin{aligned} \mathcal{F}(X) &= A \times X \\ \mathcal{F}(f) &= \lambda(a, x)(a, f(x)) \end{aligned}$$

then  $\mathcal{F}$  preserves completeness, embeddings and  $\mu$ -cuts.

Proof:

Let  $\{y_n\}$  be a Cauchy sequence in  $A \times X$ ; from the definition of the metric it follows that there is  $n_0 \in \mathbb{N}$  and  $a \in A$  such that  $y_n = (a, x_n)$  for  $n > n_0$  and  $\{x_n\}$  is a Cauchy sequence in  $X$ . Hence  $\{y_n\}$  converges to  $(a, \lim x_n)$ . Let  $e : X \rightarrow Y$  be an embedding then

$$\begin{aligned} d(\mathcal{F}(e)((a, x)), \mathcal{F}(e)((a', x'))) &= d((a, e(x)), (a', e(x'))) \\ &= \begin{cases} 1 & \text{if } a \neq a' \\ \frac{1}{2}d((a, x), (a', x')) & \text{else} \end{cases} \\ &= d((a, x), (a', x')). \end{aligned}$$

Let  $c$  be a  $\mu$ -cut for  $e$ , i.e.

$$\begin{aligned} c(e(x)) &= x \quad \forall x \in X \\ d(y, e(c(y))) &\leq \mu \quad \forall y \in Y. \end{aligned}$$

Let  $z \in \mathcal{F}(X) = A \times X$ ,  $z = (a, x)$ , then

$$\begin{aligned} \mathcal{F}(c)(\mathcal{F}(e)(z)) &= (a, c(e(x))) \\ &= (a, x) \\ &= z. \end{aligned}$$

and for  $z \in \mathcal{F}(Y) = A \times Y$ ,  $z = (a, y)$

$$\begin{aligned} d(z, \mathcal{F}(e)\mathcal{F}(c)(z)) &= d((a, y), (a, e(c(y)))) \\ &= \frac{1}{2}d(y, e(c(y))) \\ &\leq \mu. \end{aligned}$$

Lemma 9

The endofunctor  $\mathcal{F} : MS \rightarrow MS$

$$\begin{aligned} \mathcal{F}(X) &= A \rightarrow X \\ \mathcal{F}(f) &= \lambda g \lambda a f(g(a)) \end{aligned}$$

preserves completeness, embeddings and  $\mu$ -cuts.

Proof:

in analogy to Lemma 5.

Lemma 10

The functor  $\mathcal{F} : MS \times MS \rightarrow MS$

$$\begin{aligned} \mathcal{F}(X_1, X_2) &= X_1 \times X_2 \\ \mathcal{F}(f_1, f_2) &= \lambda(x, y) (f_1(x), f_2(y)) \end{aligned}$$

preserves completeness, embeddings and  $\mu$ -cuts.

Proof:

in analogy to Lemma 5.

By now, we have treated some examples of functors that are relevant for the definition of the semantics of programming languages. One functor of interest in this context, the functor  $\wp_c$ , is given special treatment in section IV.

## II. The Existence and Uniqueness of Fixed Points

In this section we are going to derive conditions for the existence of fixed points.

In analogy to the classical case of fixed points in complete metric spaces we establish conditions that guarantee that

- i) a sequence  $\{M_i\}$  of metric spaces generated by iteration as in Lemma 5 is a "Cauchy sequence"

and

- ii) its "limes" is a fixed point.

The first criterion is derived from the Banach principle.

### Definition 10

Let  $\mathcal{F} : MS \rightarrow MS$  be a functor that preserves completeness and embeddings.  $\mathcal{F}$  is called a contraction functor, if there exists a  $k$ ,  $0 \leq k < 1$ , such that for all  $N, M$  in  $CMS$  and all embeddings  $e : N \rightarrow M$

$$d_{\mathcal{F}(e)}(\mathcal{F}(N), \mathcal{F}(M)) \leq k \cdot d_e(N, M).$$

### Definition 11

Let  $\mathcal{F} : MS \rightarrow MS$  be a functor that preserves completeness, embeddings and  $\mu$ -cuts.  $\mathcal{F}$  is called cut-contractive, if there is a  $k$ ,  $0 \leq k < 1$ , such that for every embedding  $e$  with  $\mu$ -cut  $c$ ,  $\mathcal{F}(e)$  is a  $(k \cdot \mu)$ -cut for  $\mathcal{F}(e)$ .

For practical purposes there is an easy way to determine contractiveness of a given functor:

### Lemma 11

Let  $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$  or  $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$  where  $\mathcal{F}_i$  is an endofunctor in  $MS$ ,  $i = 1, 2$ , that preserves embeddings.

- a) If  $\mathcal{F}_1$  is a contraction functor,  $\mathcal{F}_2$  preserves completeness and satisfies a weak contraction property; i.e.  $d_{\mathcal{F}_2(e)}(\mathcal{F}_2(N), \mathcal{F}_2(M)) \leq d_e(N, M)$  for every embedding  $e$ , then  $\mathcal{F}$  is a contraction functor.
- b) If  $\mathcal{F}_1$  is cut-contractive and  $\mathcal{F}_2$  preserves  $\mu$ -cuts then  $\mathcal{F}$  is cut-contractive.

Proof:

a) Let  $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ ,  $e: N \rightarrow M$  an embedding.

$$\begin{aligned} d_{\mathcal{F}(e)}(\mathcal{F}(N), \mathcal{F}(M)) &= d_{\mathcal{F}_2(\mathcal{F}_1(e))}(\mathcal{F}_2(\mathcal{F}_1(N)), \mathcal{F}_2(\mathcal{F}_1(M))) \\ &\leq d_{\mathcal{F}_1(e)}(\mathcal{F}_1(N), \mathcal{F}_1(M)) \\ &\leq k \cdot d_e(N, M). \end{aligned}$$

b) Let  $\mathcal{F} = \mathcal{F}_1 \circ \mathcal{F}_2$ ,  $e: N \rightarrow M$  an embedding with  $\mu$ -cut  $c: M \rightarrow N$ . We have to show that there is a  $k$  such that  $\mathcal{F}(c)$  is a  $(k \cdot \mu)$ -cut. Clearly  $\mathcal{F}(c)(\mathcal{F}(e)(x)) = x$ .

Consider

$$\begin{aligned} d(\mathcal{F}(c)(\mathcal{F}(e)(x)), x) &= d(\mathcal{F}_2(\mathcal{F}_1(c))(\mathcal{F}_2(\mathcal{F}_1(e)(x))), x) \\ &\leq k \cdot \mu \end{aligned}$$

as  $\mathcal{F}_1(c)$  is a  $(k \cdot \mu)$ -cut for  $\mathcal{F}_1(e)$  and  $\mathcal{F}_2$  preserves this property.

Theorem 12

Let  $\mathcal{F}: MS \rightarrow MS$  be a contraction functor then  $\mathcal{F}$  has a fixed point in  $CMS$ . Moreover, this fixed point is unique up to isometry among the objects of  $CMS$ . In other words,  $\mathcal{F}$  considered as functor from  $CMS$  to  $CMS$  has a unique fixed point.

Proof:

Clearly  $\mathcal{F}\emptyset \neq \emptyset$ . As a first step we construct an embedding sequence  $(M_i)$  as in Lemma 5 by choosing  $M_0$  as a one-element space and  $M_i = \mathcal{F}(M_{i-1})$ ,  $i \geq 1$ ; each  $M_i$  is complete and can be identified with an element of  $\wp_c(M)$ , where  $M$  is the completion of  $\cup M_i$ . We already know by Lemma 5 that  $M$  is a prefixed point.

$$\begin{aligned} d_{h_{n+1}}(M_{n+1}, M) &= d_{h_{n+1}}(\mathcal{F}(M_n), M) \\ &\leq d_{h_{n+1} \circ e}(\mathcal{F}(M_n), \mathcal{F}(M)) \text{ by Lemma 1} \\ &= d_{\mathcal{F}(h_n)}(\mathcal{F}(M_n), \mathcal{F}(M)) \text{ by the universal properties} \\ &\quad \text{of the direct limit} \\ &\leq k \cdot d_{h_n}(M_n, M), \end{aligned}$$

where the  $h_i$  are the canonical embeddings. Continuing this argument we get

$$d(M_{n+1}, M) \leq k^{n+1} d(M_0, M),$$

hence  $M$  is the limes of the  $M_i$  in  $\wp_c(M)$ .

On the other hand

$$\begin{aligned} d(M_{n+1}, \mathcal{F}(M)) &= d_{\mathcal{F}(h_n)}(\mathcal{F}(M_n), \mathcal{F}(M)) \\ &\leq k \cdot d_{h_n}(M_n, M), \end{aligned}$$

hence  $\mathcal{F}(M)$  is the limes of the  $M_i$  in  $\wp_c(\mathcal{F}(M))$  from which we conclude that  $M$  is a fixed point.

Let  $N$  be another fixed point of  $\mathcal{F}$  in  $CMS$ . Chose  $y_0 \in N$  and construct  $Y_0 = \{y_0\}$ ,  $Y_i = \mathcal{F}(Y_{i-1})$ ,  $i \geq 1$ . Then the completion  $Y$  of  $\cup Y_i$  is a fixed point and can be embedded into  $N$ , hence

$$\begin{aligned} d(Y, N) &= d(\mathcal{F}(Y), \mathcal{F}(N)) \\ &\leq k \cdot d(Y, N) \end{aligned}$$

from where  $N = Y$  follows. As  $M_i$  is isometrical to  $Y_i$  we conclude that  $M$  is isometrical to  $N$ .

Remark

Obviously definitions 10 and 11, as well as lemma 11 can be adapted to  $n$ -ary functors.

Example 2

The functor  $\mathcal{F}$  given in Example 1 satisfies the conditions of Theorem 12 with  $k = \frac{1}{2}$  as contraction constant.

Example 3

The functor  $\mathcal{G} : MS \rightarrow MS$  given by

$$\mathcal{G}(X) = \{p_0\} \cup A \times (X \cup (B \times X))$$

and suitably defined for morphisms satisfies the conditions of the theorem.

By applying lemma 10 various functors can be shown to satisfy the conditions of Theorem 12. There are, however, interesting cases for which the conditions of Theorem 12 are too strong, e.g. functors that are built with the  $\wp_c$ -functor. For these cases we use cut-contractiveness.

Lemma 13

Let  $\mathcal{F} : MS \rightarrow MS$  be a cut-contractive functor. Let  $M_0$  be a complete metric space such that there is an embedding  $e_0 : M_0 \rightarrow \mathcal{F}(M_0)$  with  $\mu$ -cut  $c_0 : \mathcal{F}(M_0) \rightarrow M_0$ . Let  $M_i = \mathcal{F}(M_{i-1})$ ,  $i \geq 1$ ,  $e_i = \mathcal{F}(e_{i-1})$ ,  $c_i = \mathcal{F}(c_{i-1})$ ,  $i \geq 1$ . Let  $M$  be the completion of  $\cup M_i$  and let  $h_i : M_i \rightarrow M$  be the canonical embedding. Then there is a  $\mu_i$ -cut  $l_i : M \rightarrow M_i$  for  $h_i$  with  $\lim_{i \rightarrow \infty} \mu_i = 0$ .

Proof:

From the properties of  $\mathcal{F}$  it is clear that  $c_i$  is a  $(\mu \cdot k^i)$ -cut for  $e_i$ . For fixed  $n$  we consider the family of morphisms  $(c_{mn})_{m \geq 0}$  as given in Definition 5,  $c_{mn} : M_m \rightarrow M_n$ . As  $M$  is the direct limes of the  $M_i$  according to Lemma 4 there is a uniquely determined contraction  $l_n : M \rightarrow M_n$  such that

$$c_{mn} = h_m \circ l_n \quad m \geq 0.$$

From this we immediately get that

$$l_n h_n(x) = x \quad \forall x \in M_n.$$

It remains to evaluate  $d(x, h_n l_n(x))$  for  $x \in M$ . For this let  $n \geq 0$  and  $x \in M_{n+2}$ . As  $c_n$  is a  $(\mu \cdot k^n)$ -cut

$$d\left(c_{n+1}(x), e_n\left(c_n(c_{n+1}(x))\right)\right) \leq \mu \cdot k^n$$

hence

$$d\left(e_{n+1}(c_{n+1}(x)), e_{n+1}\left(e_n\left(c_n(c_{n+1}(x))\right)\right)\right) \leq \mu \cdot k^n$$

implying

$$\begin{aligned} d\left(x, e_{n+1}\left(e_n\left(c_n(c_{n+1}(x))\right)\right)\right) &\leq d\left(x, e_{n+1}(c_{n+1}(x))\right) \\ &\quad + d\left(e_{n+1}(c_{n+1}(x)), e_{n+1}\left(e_n\left(c_n(c_{n+1}(x))\right)\right)\right) \\ &\leq \mu \cdot k^{n+1} + \mu \cdot k^n \\ &= \mu \cdot (k^n + k^{n+1}) \end{aligned}$$

thus

$$d(x, c_{n+2,n}(x)) \leq \mu \cdot (k^n + k^{n+1})$$

and in general

$$d(x, c_{mn}(x)) \leq \mu \cdot (k^n + k^{n+1} + \dots + k^{m-1})$$

for all  $m \geq n$ . Put

$$\begin{aligned} \mu_n &= \mu \cdot \sum_{m \geq n} k^m \\ &= \mu \cdot \left(\frac{k^n}{1-k}\right) \end{aligned}$$

Let now  $x \in M$ ,  $x = \lim x_m$ ,  $x_m \in M_m$ .

$$\begin{aligned} d(x, l_n(x)) &= d\left(\lim_m x_m, \lim_m c_{mn}(x_m)\right) \\ &= \lim_m d(x_m, c_{mn}(x_m)) \\ &\leq \mu_n, \end{aligned}$$

omitting the explicit notation of the canonical embeddings.

Theorem 14

Let  $\mathcal{F} : MS \rightarrow MS$  be a cut-contractive functor and  $\mathcal{F}\emptyset \neq \emptyset$ . Then  $\mathcal{F}$  has a fixed point in  $CMS$  that is unique up to isometry among the objects in  $CMS$ .

Proof:

Let  $M_0 = \{x_0\}$  be a one-element space,  $e_0 : M_0 \rightarrow \mathcal{F}(M_0)$  an embedding and define  $c_0 : \mathcal{F}(M_0) \rightarrow M_0$  by

$$c_0 = \lambda x . x_0$$

$c_0$  is a 1-cut for  $e_0$ . Let  $M_i = \mathcal{F}(M_{i-1})$ ,  $i \geq 1$ , and pull up the  $e_i$  and  $c_i$  analogously. Let  $M$  be the completion of  $\cup M_i$ . From Lemma 13 we conclude the existence of  $\mu_n$ -cuts  $l_n$ ,  $l_n : M \rightarrow M_n$ , for the canonical embeddings  $h_n$ ,  $h_n : M_n \rightarrow M$ , where  $\mu_n = \mu \cdot \frac{k^n}{1-k}$ . By Lemma 2 we conclude

$$d(M_n, M) \leq \mu_n,$$

hence  $M = \lim M_n$  in  $\wp_c(M)$ . On the other hand  $d_{\mathcal{F}(h_n)}(\mathcal{F}(M_n), \mathcal{F}(M)) \leq k \cdot \mu_n$  as  $\mathcal{F}$  is cut-contractive, hence  $\mathcal{F}(M) = \lim M_n = M$  up to isometry. Uniqueness is shown as in Theorem 12.

Remark 9

Under the conditions of Theorem 14 one can see that  $M$  is the inverse limes of the  $M_i$  in  $CMS$  with respect to the  $\mu_n$ -cuts  $c_n$ . As the  $l_n$  induce contractions  $\mathcal{F}(l_n) : \mathcal{F}(M) \rightarrow \mathcal{F}(M_n)$  we conclude the existence of a unique contraction  $f$  from  $\mathcal{F}(M)$  to  $M$ . The direct limes property of  $M$  guarantees the existence of a unique contraction  $g$  from  $M$  to  $\mathcal{F}(M)$ . Showing that  $f \circ g = \text{id}$  and  $g \circ f = \text{id}$  and observing that the isomorphisms of  $MS$  are exactly the isometries yields a category-theoretic proof of the above theorem.

Remark 10

As  $\wp_c$  preserves  $\mu$ -cuts (see section IV) Theorem 14 together with Lemma 11 allow us to handle a variety of interesting functors.

Example 4

The functor  $\mathcal{F}(X) = \{p_0\} \cup \left( A \rightarrow \wp_c(B \times (P \cup (C \rightarrow P))) \right)$  satisfies the conditions of Theorem 14.



Example 5

The functor

$$\mathcal{F}(X) = \{p_0\} \cup \left( A \rightarrow \wp_c(\{f, \delta, \perp\} \cup (A \cup C \cup \wp(I)) \times (X \cup (V \times X) \cup (V \rightarrow X))) \right)$$

that is the basis for a semantic definition of *CSP* in [12] satisfies the conditions of Theorem 14.

III. The Equation  $P = \{p_0\} \cup (A \rightarrow \wp_c((B \times P) \cup (C \rightarrow P)))$

We claimed in the introduction that the above equation in [6], the solution of which is left to the reader, cannot be solved as proposed by [6], namely by putting

$$Y_0 = \{p_0\}$$

$$Y_i = \{p_0\} \cup \left( A \rightarrow \wp_c((B \times Y_i) \cup (C \rightarrow Y_i)) \right)$$

and showing that the completion  $Y$  of  $\cup Y_i$  is a solution of the above equation by establishing an isometry between  $Y$  and  $\mathcal{F}(Y)$ .

We do not claim that the equation does not have a solution at all. We do claim that  $Y$  cannot be one.

Let us consider the functor  $\mathcal{F}(X) = \{p_0\} \cup \left( A \rightarrow \wp_c((B \times X) \cup (C \rightarrow X)) \right)$  in more detail.  $\mathcal{F}$  clearly preserves completeness and embeddings and according to Lemma 5,  $Y$  is a prefixed point of  $\mathcal{F}$ , i.e. there is an embedding  $\Phi : Y \rightarrow \mathcal{F}(Y)$ . In order to establish that  $\Phi$  is an isometry, we have to show that  $\Phi$  is onto.

We claim that this cannot be the case. Let us for simplicity only consider the case where  $A$ ,  $B$  and  $C$  are finite sets.

We define the infinite set

$$S_\infty = \{\lambda c p_0, \lambda c \lambda a \lambda c p_0, \lambda c \lambda a \lambda c \lambda a \lambda c p_0, \dots\}$$

and observe

- i)  $S_\infty \subset (C \rightarrow Y)$ ,
- ii)  $S_\infty$  is closed, as there do not exist any nontrivial convergent sequences in  $S_\infty$ , i.e.  $S_\infty \in \wp_c((B \times Y) \cup (C \rightarrow Y))$ ,
- iii)  $S_\infty$  has non countably many infinite subsets  $T_\infty$ , each of which is closed, as there are no nontrivial convergent sequences.

To see this, remember that the metric on  $Y_{n+1}$  is given by

$$d_{n+1}(p, p_0) = d(p, p_0) = 1 \quad p \neq p_0$$

$$d_{n+1}(p, p') = \sup_{a \in A} \{d(p'(a), p(a))\}$$

and for  $x, y \in (B \times Y_n) \cup (C \rightarrow Y_n)$

$$d(x, y) = \begin{cases} 1, & \text{if } x \in B \times Y_n, y \in C \rightarrow Y_n \text{ or viceversa} \\ 1, & \text{if } x, y \in B \times Y_n, x = (a, x'), y = (b, y'), a \neq b \\ \frac{1}{2}d(x', y'), & \text{if } x = (a, x'), y = (a, y') \\ \sup_{c \in C} (x(c), y(c)), & \text{if } x, y \in C \rightarrow Y_n. \end{cases}$$

Let us assume that there is an isometry  $\Phi : Y \rightarrow \mathcal{F}(Y)$ . We consider the family of functions

$$g = \lambda a S_\infty$$

$$g_{T_\infty} = \lambda a T_\infty$$

where  $T_\infty$  is an infinite subset of  $S_\infty$ . Clearly  $g$  and all  $g_T$  are elements of  $\mathcal{F}(Y)$ . If  $\Phi : Y \rightarrow \mathcal{F}(Y)$  is onto there must be an  $f \in Y$  such that  $\Phi(f) = g$ .  $f \in Y$  implies that either

$$f \in \bigcup_{i \geq 0} Y_i$$

or

$$f = \lim f_n \quad f_n \in Y_n.$$

Assume that  $f = \lim f_n$  and  $f \notin \cup Y_i$ ; then we get

$$\begin{aligned} 0 &= \lim d(\Phi(f), \Phi(f_n)) \\ &= \lim d(g, \Phi(f_n)) \end{aligned}$$

yielding  $\Phi(f_n) \rightarrow g$  and hence a contradiction, because only a trivial (finally constant) sequence can converge towards  $g$ . On the other hand  $\Phi$  is one-to-one and  $\{f_n\}$  cannot be trivial because  $f \notin \cup Y_i$  was assumed.

So we conclude that no element in  $Y \setminus \bigcup_{i \geq 0} Y_i$  can be mapped to  $g$  or analogously to any  $g_{T_\infty}$ , thus only remain the elements of  $\cup Y_i$  as candidates. But from the definition of the functor it is clear that  $\cup Y_i$  has only countably many elements. Hence there cannot exist an isometry.

IV. The functor  $\wp_c$

In this section we deal with the operator  $\wp_c$  that deserves some special consideration because it cannot be simply considered as an endofunctor in  $MS$ , as, in general, an arbitrary morphism in  $MS$

$$f : N \rightarrow M$$

will not yield a morphism from  $\wp_c(N)$  to  $\wp_c(M)$  via  $\lambda U f(U)$ . So  $\wp_c$  has to be restricted to those morphisms  $f : N \rightarrow M$  that are closed, i.e. that they map closed subsets of  $N$  to closed subsets of  $M$ . If we denote by  $MS_c$  the subcategory of  $MS$  that has the same objects as  $MS$  and closed morphisms as morphisms then  $\wp_c$  is a functor from  $MS_c$  to  $MS$ .

Clearly all the definitions of prefixed point, completeness preserving etc. can be easily adapted to the case of such a "partial" functor.

Lemma 15:

The functor  $\wp_c : MS_c \rightarrow MS$  preserves completeness, embeddings and  $\mu$ -cuts.

Proof:

$\wp_c$  preserves completeness according to Remark 2. Preservation of embeddings is trivial, preservation of  $\mu$ -cuts follows from the definition of the Hausdorff metric.

For functors  $\mathcal{F}$  that arise from combination of  $\wp_c$  with other functors it has to be ensured that the construction of fixed points by iteratively defining an embedding sequence  $(M_i)$  with respective  $\mu$ -cuts is not affected.

We have to establish that starting with

$$\begin{aligned} e_0 : M_0 &\rightarrow \mathcal{F}(M_0) & M_0 &= \{x_0\} \\ c_0 : \mathcal{F}(M_0) &\rightarrow M_0 & c_0 &= \lambda x x_0 \end{aligned}$$

we can always apply  $\mathcal{F}$  iteratively to get

$$\begin{aligned} e_i &= \mathcal{F}^i(e_0) \\ c_i &= \mathcal{F}^i(c_0). \end{aligned}$$

Definition 12:

A metric space  $(X, d)$  has the minimum distance property, if there exists  $\delta \in \mathbb{R}$ ,  $\delta \geq 0$ , such that for all  $x, y \in X$ ,  $x \neq y$ ,  $d(x, y) \geq \delta$ .

Remark 14:

The topology of a metric space with the minimum distance property is the discrete topology, as every one-element set is open.

Lemma 16:

The functors  $\mathcal{F}_1(X) = A \times X$ ,  $\mathcal{F}_2(X) = A \rightarrow X$ ,  $\mathcal{F}_3(X) = \wp_c(X)$ ,  $\mathcal{F}_4(X_1, X_2) = X_1 \cup X_2$ ,  $\mathcal{F}_5(X_1, X_2) = X_1 \times X_2$  preserve the minimum distance property, i.e. if the arguments of  $\mathcal{F}_i$  inherit the minimum distance property, so does the resulting metric space.

Proof:

As an example we treat the case of  $\mathcal{F}_2$ . Let  $(X, d)$  be a metric space and  $\delta \in \mathbb{R}$ ,  $\delta > 0$  such that  $d(x, y) \geq \delta \ \forall x, y \in X$ . Let  $f, g \in \mathcal{F}_2(X)$

$$\begin{aligned} d(f, g) &= \sup_{a \in A} d(f(a), g(a)) \\ &\geq \delta. \end{aligned}$$

Lemma 17:

Let  $\mathcal{F}$  be a functor that is composed of functors in  $\{\mathcal{F}_1, \dots, \mathcal{F}_5\}$  (see Lemma 16). Let  $N$  be a metric space that has the minimum distance property and  $g : M \rightarrow N$  a contraction, then  $\mathcal{F}$  is defined for  $g$ .

Proof:

For ease of notation we only treat unary functors in  $\{\mathcal{F}_1, \dots, \mathcal{F}_5\}$ . Let hence

$\mathcal{F} = \mathcal{G}_1 \circ \mathcal{G}_2 \circ \dots \circ \mathcal{G}_k$  with  $\mathcal{G}_i$  (unary) in  $\{\mathcal{F}_1, \dots, \mathcal{F}_5\}$ ,  $1 \leq i \leq k$ . As  $N$  has the minimum distance property so does  $\mathcal{G}_1(N)$ ,  $\mathcal{G}_2(\mathcal{G}_1(N))$ , etc. and finally  $\mathcal{F}(N)$  by Lemma 16. Hence the topology of  $\mathcal{G}_1(N)$ ,  $\mathcal{G}_2(\mathcal{G}_1(N))$  etc. is the discrete topology by Remark 11. As  $N$  has the discrete topology we conclude that  $g$  is a closed morphism, hence  $\mathcal{G}_1$  is defined for  $g$ ,  $\mathcal{G}_1(g) : \mathcal{G}_1(M) \rightarrow \mathcal{G}_1(N)$ . Similarly  $\mathcal{G}_1(N)$  has the discrete topology, hence  $\mathcal{G}_2$  is defined for  $\mathcal{G}_1(g)$  and so on.

Corollary 18:

Let  $\mathcal{F}$  be as in Lemma 17,  $N$  a metric space that has the minimum distance property,  $g : M \rightarrow N$  a morphism. Then  $\mathcal{F}^n$  is defined for  $g$  for all  $n \geq 1$ .

The above observations guarantee that our results also hold for functors that are composed from  $\wp_c$  and others. Obviously the above results can be extended to any other functors that preserve the minimum distance property.

V. Related Work

Recursive specification of “domains” plays a crucial role in the denotational semantics based on metric spaces [6] as well as in the denotational semantics as developed by Scott and Strachey. First approaches of Scott to solve recursive equations were his inverse limit construction [20], which were later substituted by using a universal domain and a fixed point construction [21].

The categorical aspects of these approaches were studied e.g. by Reynolds [19] and Wand [24]. These investigations typically stuck to one fixed category, e.g. the category *CPO* of complete partial orders with strict continuous functions or the category of countably based continuous lattices and continuous functions, and are at the same level of abstraction as our work presented here.

In [23] and [24] a further abstraction step is initiated to develop a theory of solving recursive equations for general categories. For this [23] elaborate a basic lemma:

Basic lemma [23]

“Let  $k$  be a category with initial object  $\perp_k$  and let  $\mathcal{F} : k \rightarrow k$  be a functor. Define the  $\omega$ -chain  $\Delta$  to be  $(\mathcal{F}^n(\perp_k), \mathcal{F}^n(\perp_{F\perp}))$ . Suppose that both  $\mu : \Delta \rightarrow A$  and  $\mathcal{F}\mu : \mathcal{F}\Delta \rightarrow \mathcal{F}A$  are colimiting cones then the initial fixed point exist.”

In the sequel [23] discuss how the conditions of the lemma can be satisfied for the class of *O*-categories, i.e. categories that exhibit certain order structures in their hom-sets. If we compare our procedure with that implied by the basic lemma, then obviously choosing  $k = CMS$  our  $M$  (the completion of  $\cup M_i$  in theorems 12 and 14) plays the role of  $A$  and we know that  $M$  is the direct limes of  $(M_i)$  in *CMS*. In order to prove the fixed point property, however, we do not show that  $\mathcal{F}(M)$  is direct limes of  $\mathcal{F}(M_i)$ , but rather show that the distance between  $\mathcal{F}(M_i) = M_{i+1}$  and  $\mathcal{F}(M)$  (understood as elements in  $\wp_c(\mathcal{F}(M))$ ) tends to zero as  $i \rightarrow \infty$ . Having then established the fixed point property of  $M$  we get as a trivial conclusion that  $\mathcal{F}(M)$  is the direct limes of  $\mathcal{F}(M_i)$ . So,  $M$  is a fixed point if and only if  $\mathcal{F}(M)$  is direct limes of  $(M_i)$ . In addition, in *CMS* besides existence the uniqueness of fixed points is guaranteed for functors with a contraction property.

VI. Conclusion

We have proposed a rigorous framework within which the problem of solving recursive equations such that the solution constitutes a complete metric space can be formulated and discussed. We established conditions, that are very easy to verify for a given functor, see Lemma 11, under which the (unique) existence of a solution is guaranteed. For example, all equations in [6] - except for equation (4) from our introduction - satisfy either the conditions of Theorem 12 or Theorem 14. Equation (4) has been investigated and it has been shown that the methods of [6] do not apply to it. The question if this equation does have a solution at all is open. We have also given special attention to the functor  $\rho_c$  because of its partiality and we pointed out some connection to related work.

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