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The Contraction Property is sufficient to guarantee the Uniqueness of Fixed Points of Endofunctors in a Category of Complete Metric Spaces

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Summary

In [3] de Bakker and Zucker proposed to use complete metric spaces for the semantic definition of programming languages that allow for concurrency and synchronisation. The use of the tools of metric topology has been advocated by Nivat and his colleagues already in the seventies and metric topology was successfully applied to various problems [12,13]. Recently, the question under which circumstances fixed point equations involving complete metric spaces can be (uniquely) solved has attracted attention, e.g. [1,11]. The solution of such equation provides the basis for the semantics of a given language and is hence of practical relevance. In [1], a criterion for the existence of a solution, namely that the respective functor is contracting, is provided. This property together with an additional criterion, namely that the respective functor is hom-contracting, was shown in [1] to guarantee uniqueness. In this paper we show that the contraction property is already sufficient to guarantee the uniqueness.

1. Introduction

In [3] de Bakker et al. proposed a promising approach to the definition of semantics for programming languages that allow for concurrency and communication. It is based on the concept of complete metric spaces. The technique has been successfully applied to various languages involving concurrency, e.g. to different variants of Hoare's communicating sequential processes [8]. In [14] the method is used to specify a language that includes dynamic process creation. The idea of the approach of [3] is to establish a meaning function Me that maps a program P (in a language $\mathcal{L}$) to its meaning $Me[P]$, which is an element of a certain complete metric space $M_{\mathcal{L}}$, Me : Programs $\rightarrow M_{\mathcal{L}}$. $M_{\mathcal{L}}$ is constructed as solution of a certain fixed point equation

$$\mathcal{F}_{\mathcal{L}}X = X$$

called process domain equation.

Here $\mathcal{F}_{\mathcal{L}}$ is a functor in a suitable category of metric spaces. Ever since the question of existence and uniqueness of solutions of such fixed point equations as well as the connection to other models for semantics, e.g. the denotational approach based on CPO's (complete partial orders) has attracted interest [5,6,9,10,11,15,16,17,18,19]. In [1] America and Rutten developed general criteria to ensure existence and uniqueness of fixed points. They show that contracting functors have fixed points and that contracting functors that are hom–contracting have unique fixed points (up to isometry). In a previous paper [9] we showed that contracting functors have "minimal" fixed points and contracting functors that map the empty space to a one–element space already possess unique fixed points. In this paper we prove the stronger result that the contraction property is already sufficient to guarantee uniqueness. By this we solve an open problem of [1], namely the problem to establish a contracting functor that has nonisometric fixed points: such functors do not exist. It should be noted that – for practical applications –, one does not really have to construct the solution $M_{\mathcal{L}}$ of the process domain equation in order to assign a meaning to programs in a language $\mathcal{L}$. One only has to ensure the contraction property of the respective functor $\mathcal{F}_{\mathcal{L}}$ and can then make use of the fact that $M_{\mathcal{L}}$ is the unique space that satisfies the equation $\mathcal{F}_{\mathcal{L}}X = X$ for further purposes.

2. Mathematical Preliminaries

A metric space is a pair (M, d) with M a set and d a mapping, $d : M \times M \to [0,1]$ which satisfies [1]

(a) $\forall x, y \in M \quad (d(x,y) = 0 \Leftrightarrow x = y)$,

(b) $\forall x, y \in M \quad d(x,y) = d(y,x)$,

(c) $\forall x, y, z \in M \quad d(x,y) \leq d(x,z) + d(z,y)$.

A sequence (x_i) in a metric space (M, d) is a Cauchy sequence, whenever $\forall \epsilon > 0$ $\exists N \in \mathbb{N} \quad \forall n, m > N \quad d(x_n, x_m) < \epsilon$. The metric space (M, d) is called complete if every Cauchy sequence converges to an element of M .

Let (M_1, d_1) , (M_2, d_2) be metric spaces. A function $f : M_1 \to M_2$ is called non distance increasing, if $\forall x, y \in M_1$

$$d_2(f(x), f(y)) \leq d_1(x,y) .$$

f is called (isometric) embedding, if $\forall x, y \in M_1$

$$d_2(f(x), f(y)) = d_1(x,y) .$$

f is called an isometry, if f is onto and an (isometric) embedding.

It is well known, [4], that every metric space (M, d) can be embedded into a "unique" "minimal" complete metric space, called the completion of (M, d) .

Let M denote the category that has metric spaces as objects and non distance increasing functions as arrows.

A sequence $((M_i, d_i))_{i \geq 0}$ of metric spaces together with a sequence of embeddings $(e_i)_{i \geq 0}$, $e_i : M_i \to M_{i+1}$, is called an embedding sequence. Let $((M_i, d_i))_{i \geq 0}$ with $(e_i)_{i \geq 0}$ be an embedding sequence. The direct limes of (M_i) in $\mathbf{M}$ with respect to (e_i) exists and is denoted by $(\cup M_i, \cup d_i)$.

Let C denote the category that has complete metric spaces as objects. The arrows in C are the non distance increasing functions. The empty space $\emptyset$ is an initial object in C.

Let M_1 , M_2 be complete metric spaces, and let

$e : M_1 \to M_2$ be an embedding

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(1) $0 \leq d(x,y) \leq 1$ can be always obtained for an arbitrary metric $\widehat{d} : M \times M \to \mathbb{R}$ by substituting $\widehat{d}(x,y)$ by $\frac{\hat{d}(x,y)}{\hat{d}(x,y)+1}$.

and $c : M_2 \to M_1$ be a non distance increasing function.
c is called a <u>cut</u> for e if $e \circ c = id_{M_1}$ [(2)]. For embedding $e : M_1 \to M_2$ with cut c we put $\iota = (e, c)$ and write $M_1 \longrightarrow^{\iota} M_2$ and define

$$\underline{\delta(\iota)} = \sup_{x \in M_2} \{ d_{M_2}(x, e(c(x))) \} \quad .$$

We say that a functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ <u>preserves embeddings</u> iff $\mathcal{F}e$ is an embedding for every embedding e. If a functor $\mathcal{F}$ preserves embeddings then $\mathcal{F}c$ is a cut for $\mathcal{F}e$, whenever c is a cut for e. A functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ that preserves embeddings is called <u>contracting</u> if there exists an ϵ, $0 \leq \epsilon < 1$, such that for all $D \longrightarrow^{\iota} E \in \mathbf{C}$, $\iota = (e, c)$,

$$\delta(\mathcal{F}\iota) \leq \epsilon \cdot \delta(\iota)$$

where $\mathcal{F}\iota = (\mathcal{F}e, \mathcal{F}c)$.
Please note that we have modified the definition of [1] slightly, just in order to be able to include the empty space as an object.

(2) Throughout this text the composition $f \circ g$ of functions stands for $\lambda x . g(f(x))$.

3. Uniqueness of Fixpoints of Contracting Functors

In this section we show that the contraction property of a functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ is already sufficient to guarantee uniqueness of fixed points. By this we solve an open problem of [1] to establish a contracting functor that has nonisometric fixed points.

Remark 1

Let (M, d) be a complete metric space. Let

$$\wp_c(M) = \{U \subset M \ : U \text{ is non-empty and closed}\}$$

and let $\tilde{d}_M$ denote the Hausdorff metric on $\wp_c(M)$, i.e. for $x, y \in M$, $X, Y \in \wp_c(M)$ let

$$\Delta(x, Y) = \inf_{y \in Y} d(x, y)$$
$$\tilde{d}_M(X, Y) = \max\left\{ \sup_{x \in X}\{\Delta(x, Y)\}, \sup_{y \in Y}\{\Delta(y, X)\} \right\}.$$

It is well-known that $(\wp_c(M), \tilde{d}_M)$ is a complete metric space, see e.g. [4].

Let now $D \longrightarrow^{\iota} E \in \mathbf{C}$, $\iota = (e, c)$. In particular D is isometrically embedded into E via e and we can view D as a closed subset of E, i.e. an element of $\wp_c(E)$. Let $\tilde{d}_E$ be the Hausdorff metric on $\wp_c(E)$ then we can talk about

$$\tilde{d}_E(D, E)$$

or

$$\tilde{d}(D, E)$$

for ease of notation, considering D and E as elements of $\wp_c(E)$. (3)

Remark 2

Let $D \longrightarrow^{\iota} E \in \mathbf{C}$, $\iota = (e, c)$. If $\forall x \in E$

$$d_E\big(x, e(c(x))\big) \le \mu$$

(3) Strictly speaking the "distance of D and E as elements of $\wp_c(E)$" also depends on the choice of the embedding e. But as this choice will always be clear from the context, we omit its indication.

then

$$\tilde{d}(D,E) \leq \mu .$$

This can be easily seen by the definition of the Hausdorff metric.

Remark 3

Let $D \longrightarrow^{\iota} E \in \mathbf{C}$ then

$$\tilde{d}(D,E) \leq \delta(\iota)$$

by the above remark and the definition of $\delta(\iota)$.

Lemma

Let $\left((M_i, d_i)\right)_{i\geq 0}$ with $(e_i)_{i\geq 0}$ be an embedding sequence, where every (M_i, d_i) is a complete metric space. The completion M of $(\cup M_i, \cup d_i)$ is the direct limes of the $\left((M_i, d_i)\right)_{i\geq 0}$ in the category $\mathbf{C}$ (with respect to the $(e_i)_{i\geq 0}$).

Proof

By the universal properties of the completion and the continuity of the metric. See also [19].

Theorem

Let $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}$ be a contracting functor with $\mathcal{F}\emptyset \neq \emptyset$. Then $\mathcal{F}$ has a fixed point that is unique (up to isomorphism).

Proof

We choose a one-element space $S_0 = \{x_0\}$ and let $S_i = \mathcal{F}S_{i-1}$, $i \geq 1$. Clearly S_i is a complete metric space. In addition let $M_0 = \emptyset$, $M_i = \mathcal{F}M_{i-1}$, $i \geq 1$, and let

$$e_0 : M_0 \rightarrow M_1$$

be the unique embedding and

$$e_i = \mathcal{F}e_{i-1}, \quad i \geq 1 .$$

There is a unique embedding

$$i_0 : M_0 \rightarrow S_0 .$$

We choose in addition an embedding

$$\lambda_0 : S_0 \rightarrow M_1$$

which is possible as $S_0 = \{x_0\}$ and $M_1 = \mathcal{F}\emptyset \neq \emptyset$ by assumption. From the initiality of M_0 we obtain

$$e_0 = i_0 \circ \lambda_0 \, . \qquad (I)$$

We now put $\sigma_0 = \lambda_0 \circ \mathcal{F} i_0$ and $c_0 = \lambda x . x_0$

$$\sigma_0 : S_0 \to S_1 \qquad c_0 : S_1 \to S_0$$

and $\sigma_i = \mathcal{F}\sigma_{i-1}$, $c_i = \mathcal{F} c_{i-1}$, $i \geq 1$, having thus turned the sequence S_i into an embedding sequence (with embeddings σ_i).

Let S denote the completion of $\cup S_i$ and let $k_i : S_i \to S$ be the canonical embeddings, $i \geq 0$. It has been shown in [1,11] that S is a fixed point of $\mathcal{F}$. Let now N be another fixed point of $\mathcal{F}$. Hence there is an isometry

$$h : \mathcal{F}N \to N \, .$$

Let $j_0 : \emptyset \to N$ be the unique morphism then by the initiality of M_0 we have

$$j_0 = e_0 \circ \mathcal{F} j_0 \circ h \, . \qquad (II)$$

We define now $\tau_0 : S_0 \to N$

$$\tau_0 = \lambda_0 \circ \mathcal{F} j_0 \circ h$$

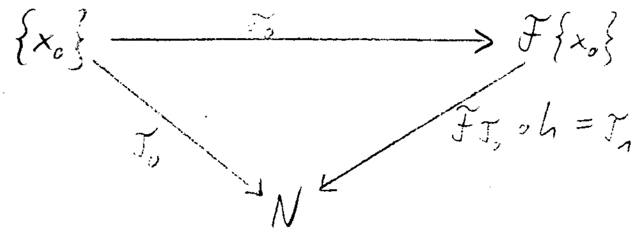
and set $\tau_i : S_i \to N$

$$\tau_i = \mathcal{F}\tau_{i-1} \circ h$$

and get

$$\begin{aligned}
\sigma_0 \circ \tau_1 &= \sigma_0 \circ (\mathcal{F}\tau_0 \circ h) \quad \text{by Def. of } \tau_1 \\
&= (\lambda_0 \circ \mathcal{F} i_0) \circ (\mathcal{F}\tau_0 \circ h) \quad \text{by Def. of } \sigma_0 \\
&= \lambda_0 \circ \mathcal{F}(i_0 \circ \tau_0) \circ h \\
&= \lambda_0 \circ \mathcal{F}(i_0 \circ \lambda_0 \circ \mathcal{F} j_0 \circ h) \circ h \quad \text{by Def. of } \tau_0 \\
&= \lambda_0 \circ \mathcal{F}(\epsilon_0 \circ \mathcal{F} j_0 \circ h) \circ h \quad \text{by } (I) \\
&= \lambda_0 \circ \mathcal{F}(j_0) \circ h \quad \text{by } (II) \\
&= \tau_0 \quad \text{by Def. of } \tau_0
\end{aligned}$$

hence τ_0 is an embedding such that the following diagram



commutes. Consequently

$$\tau_i = \sigma_i \circ \tau_{i+1}$$

by induction. Hence, as S is the direct limit of the S_i with respect to the σ_i we conclude that there is a unique embedding

$$f : S \to N$$

with $\tau_i = k_i \circ f$, $i \geq 0$.

It remains to show that the embedding sequence S_i (with respect to σ_i) converges to N . For this we define

$$g_0 : N \to S_0$$

$$g_0 = \lambda x . x_0$$

and

$$g_i = h^{-1} \circ \mathcal{F} g_{i-1} \; , \quad i \geq 1$$

$$g_i : N \to S_i$$

Clearly $\tau_0 \circ g_0 = 1$ and $\delta((\tau_0, g_0)) \leq 1$. By induction

$$\tau_i \circ g_i = 1 \; , \quad i \geq 0 \; .$$

As $\mathcal{F}$ is contracting there is ϵ, $0 \leq \epsilon < 1$, with $\delta(\mathcal{F}\iota) \leq \epsilon \cdot \delta(\iota)$. By induction $\delta((\tau_i, g_i)) \leq \epsilon^i$, as for all $x \in N$

$$\begin{aligned} d_N\Big(x, \tau_{i+1}\big(g_{i+1}(x)\big)\Big) &= d\Big(x, h\big(\mathcal{F}\tau_i(\mathcal{F}g_i(h^{-1}(x)))\big)\Big) \\ &= d_N\Big(h(y), h\big(\mathcal{F}\tau_i(\mathcal{F}g_i(y))\big)\Big) \qquad \text{where } h(y) = x \\ &= d_{\mathcal{F}(N)}\Big(y, \mathcal{F}\tau_i\big(\mathcal{F}g_i(y)\big)\Big) \end{aligned}$$

as h is an isometry.

Hence $d_{\tau_i}(S_i, N) \leq \epsilon^i$ by Remark 3, hence N and S coincide up to isometry.

Example

The functor

$$\mathcal{F}(X) = \{p_0\} \cup \Big(A \to \wp_c\big(\{f, \delta, \bot\} \cup (A \cup C \cup \wp(I)) \times (X \cup (V \times X) \cup (V \to X))\big)\Big)$$

that is the basis for a semantic definition of CSP in [19] satisfies the conditions of the theorem.

Remark 4

It should be noted that such a uniqueness result cannot be obtained by a general category–theoretic investigation as e.g. in [18].

4. References

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