On a Marinescu structure on C(X)

by

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The purpose of this note is to introduce a natural Marinescu structure [7] (an inductive limit of locally convex topological vector spaces in the category of convergence spaces) on $C(X)$, where $C(X)$ denotes the $\mathbb{R}$-algebra of all continuous real-valued function on a completely regular topological space X. The structure in question is closely related to $C_c(X)$, the algebra $C(X)$ endowed with the continuous convergence structure [1].

1.1. Definition of the convergence structure

Let X be a completely regular topological space. We denote the Stone-Cech compactification of X by βX. It is well-known that every continuous map from X into a compact space C can be extended to a continuous map from βX into C. Since X is a dense subspace of βX, this extension is unique.

By $C(X)$, we mean the $\mathbb{R}$-algebra of all continuous real-valued functions on X (under the pointwise defined operations). Every function f in $C(X)$ can be regarded as a map from X into $\dot{R}$, the one point compactification of the reals. Hence we can extend f to a function from βX into $\dot{R}$. Clearly if f is bounded, then the extension is still real-valued. For any $f \in C(X)$, the extension of f to βX, as a function with values in $\dot{R}$, is again denoted by f. Let $K_f \subset \beta X$ be the pre-image under f of the point $\infty \in \dot{R}$. Since $f\colon \beta X \longrightarrow \dot{R}$ is continuous, K_f is a compact subset of βX. The function f restricted to X is of course real-valued, and thus K_f must be a subset of $\beta X \setminus X$, the

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complement of X in βX. For any space Y such that

$$X \subset Y \subset \beta X ,$$

we identify each continuous real-valued function on Y with its restriction to X. Therefore given any compact set $K \subset \beta X \setminus X$, the algebra $C(\beta X \setminus K)$ is contained in $C(X)$. In particular, the subalgebra $C(\beta X \setminus K_f)$ contains f. We now conclude that

$$C(X) = \bigcup_{K \subset \beta X \setminus X} C(\beta X \setminus K) ,$$

where K ranges through all compact subsets of $\beta X \setminus X$.

By $C_{co}(\beta X \setminus K)$, we mean the algebra $C(\beta X \setminus K)$ endowed with the topology of compact convergence. The convergence structure, being the subject of our investigation, is the finest of all convergence structures on $C(X)$ making the inclusion maps from $C_{co}(\beta X \setminus K)$ into $C(X)$ continuous for every compact subset $K \subset \beta X \setminus X$. We denote the algebra $C(X)$ together with this convergence structure by $C_I(X)$, and notice that this is simply the inductive limit, in the category of convergence spaces, (see [7]) of the family

(*) $\{C_{co}(\beta X \setminus K) : K \text{ a compact subset of } \beta X \setminus X\}$

with the ordering defined by inclusion. Of course the inclusion map from $C_{co}(\beta X \setminus K)$ into $C_{co}(\beta X \setminus K')$ is continuous whenever K is contained in K'. Since all the spaces considered in (*) are locally convex topological R-algebras, $C_I(X)$ is indeed a Marinescu space as introduced by H. Jarchow in [7]. We leave it to the reader to verify that $C_I(X)$ is a convergence R-algebra [1], meaning that the operations are continuous.

1.2. Completeness of $C_I(X)$

A filter Θ on a commutative convergence group G is called Cauchy if $\Theta - \Theta$ converges to zero, where " − " denotes the difference operation in G. If every Cauchy filter in G converges to some element in G, then the group is said to be complete.

Theorem 1. *For any completely regular topological space* X *, the convergence algebra* $C_I(X)$ *is complete.*

Proof. Let Θ be a Cauchy filter on $C_I(X)$. We must find a function $f \in C_I(X)$ such that Θ converges to f. Here, we remark that a filter Ψ on $C_I(X)$ converges to a function g in $C_I(X)$ if and only if there is a compact $K \subset \beta X \setminus X$ such that $C(\beta X \setminus K)$ contains g and Ψ has a base in $C_{co}(\beta X \setminus K)$ which is a filter convergent to g in this space. Now the filter $\Theta - \Theta$ has a base Φ in $C_{co}(\beta X \setminus K)$ with Φ convergent to zero for some compact $K \subset \beta X \setminus X$. Hence any element A of Φ contains $M - M$ where $M \in \Theta$. We will show that M itself is in $C(\beta X \setminus K')$ for some compact $K' \subset \beta X \setminus X$. Let g be a fixed element in M. For each $f \in M$, the function $f - g$ is in $M - M$, and thus in $C(\beta X \setminus K)$. This means that

$$f^{-1}(\infty) \subset g^{-1}(\infty) \cup K \,.$$

Therefore M is contained in $C(\beta X \setminus K)$ where K' stands for $g^{-1}(\infty) \cup K$. It follows that Θ has a base in $C(\beta X \setminus K')$, call it Θ' . Since

$$C(\beta X \setminus K) \subset C(\beta X \setminus K') \,,$$

the filter $\Theta' - \Theta'$ on $C_{co}(\beta X \setminus K')$ has Φ as a base, and thus Θ' is a Cauchy filter in $C_{co}(\beta X \setminus K')$. The completeness of $C_{co}(\beta X \setminus K')$ implies that Θ' itself converges to some function $t \in C(\beta X \setminus K')$. Hence Θ converges to t in $C_I(X)$ as desired.

1.3. Closed ideals in $C_I(X)$

By an ideal, we mean of course a *proper* ideal. It is evident that for every non-empty subset S of X the ideal

$$I(S) = \{f \in C(X) : f(S) = \{0\}\}$$

is closed in $C_I(X)$. We conjecture that all closed ideals in $C_I(X)$ are precisely of this form.

To prove this, let $J \subset C_I(X)$ be a closed ideal. We call the set of all points $p \in X$ with the property that every function $f \in J$ vanishes on p the null-set of J, and denote this set by $N_X(J)$. It is exactly the intersection of all zero-sets $Z_X(f)$ where f runs through J . By $Z_X(f)$, we mean $\{x \in X: f(x) = 0\}$. Since for any function $f \in J$, there is a bounded function $g \in J$ such that $Z_X(f) = Z_X(g)$, we can represent $N_X(J)$ as

$$\bigcap_{g \in J^o} Z_X(g) ,$$

where J^o denotes the collection af all bounded functions in J . Furthermore, the set J^o is a closed ideal in $C_{co}(\beta X)$, and is therefore of the form $I(N_{\beta X}(J^o))$ where $N_{\beta X}(J^o)$ is a non-empty subset of βX . Evidently the ideal $J \subset I(N_X(J))$. We will show that J is all of $I(N_X(J))$. First, we verify that J^o contains all the bounded functions in $I(N_X(J))$. Since J^o consists of all functions in $C(\beta X)$ vanishing on $N_{\beta X}(J^o)$, it is enough to prove that any bounded element of $I(N_X(J))$ vanishes on $N_{\beta X}(J^o)$. Clearly we are done as soon as we know that $N_{\beta X}(J^o)$ is the closure of $N_X(J)$ in βX . Assume, to the contrary, that $N_{\beta X}(J^o)$ contains $\overline{N_X(J)}$, the closure in βX of $N_X(J)$, as a proper subset. For a point $q \in N_{\beta X}(J^o)$ outside of $\overline{N_X(J)}$, we choose in βX a closed neighborhood U of p disjoint from $\overline{N_X(J)}$. There exists a function $g \in C(\beta X)$ such that $g(q) = 1$ and g vanishes on the complement of U . We assert that $g \in J \cap C(\beta X \setminus K)$, where K denotes the compact set $U \cap N_{\beta X}(J^o)$ contained in $\beta X \setminus X$. Clearly $J \cap C(\beta X \setminus K)$ is a closed ideal in $C_{co}(\beta X \setminus K)$, and therefore consists of all functions vanishing on its null-set. Since the bounded functions in $J \cap C(\beta X \setminus K)$ are precisely the elements of J^o , we conclude that

$N_{\beta X}(J^o) \cap \beta X \setminus K$ is the null-set of $J \cap C(\beta X \setminus K)$. The function g vanishes on $N_{\beta X}(J^o) \cap \beta X \setminus K$, and therefore g is an element of $J \cap C(\beta X \setminus K)$ as claimed. Thus we know $g \in J^o$. On the other hand, g is not an element of $I(N_{\beta X}(J^o))$, which is of course J^o. Because of this contradiction, we conclude that $N_{\beta X}(J^o) = \overline{N_X(J)}$, and thus J^o consists of all bounded functions in $I(N_X(J))$ where $N_X(J)$ is not empty. To complete the proof, let f be an arbitrary element of $I(N_X(J))$. There is a unit u in $C(X)$ such that $f \cdot u$ is bounded. Hence $f \cdot u \in J^o$, and therefore $(f \cdot u) \cdot 1/u \in J$. This implies that $J = I(N_X(J))$.

We now have established

Theorem 2. *An ideal* J *in* $C_I(X)$ *is closed if and only if* $J = I(N_X(J))$.

Corollary 1. *A maximal ideal in* $C_I(X)$ *is closed if and only if it consists of all functions in* $C(X)$ *vanishing at a fixed point in* X.

For every point $p \in X$ there is a continuous $\mathbb{R}$-algebra homomorphism

$$i_X(p): C_I(X) \longrightarrow \mathbb{R},$$

defined by $i_X(p)(f) = f(p)$ for every $f \in C(X)$. Assigning to each point $p \in X$ the homomorphism $i_X(p)$, we obtain a map

$$i_X : X \longrightarrow \mathcal{H}om\ C_I(X) ,$$

where $\mathcal{H}om\ C_I(X)$ denotes the set of all continuous $\mathbb{R}$-algebra homomorphisms from $C_I(X)$ onto $\mathbb{R}$. Since an element of $\mathcal{H}om\ C_I(X)$ is determined by its kernel, a closed maximal ideal in $C_I(X)$, we deduce from corollary 1:

Corollary 2. *The map* i_X *is surjective.*

1.4. The associated locally convex topology of $C_I(X)$

First, let us demonstrate that, in general, $C_I(X)$ is not topological; more precisely:

Theorem 3. $C_I(X)$ *is topological if and only if* X *is locally compact. If* X *is locally compact, then* $C_I(X) = C_{co}(X)$.

Proof. If X is locally compact, then $C(X)$ is of the form $C(\beta X \setminus K)$, where $K = \beta X \setminus X$ is a compact subset of βX. The inclusion map from $C_{co}(\beta X \setminus K')$ into $C_{co}(X)$ is continuous for any compact set $K' \subset \beta X \setminus X$. Thus $C_{co}(X)$ is the finest of all convergence structures making the inclusion maps continuous, i.e., $C_I(X)$ coincides with $C_{co}(X)$ and hence is topological.

Conversely, assume that $C_I(X)$ is topological. Since the neighborhood filter of zero has a base in $C(\beta X \setminus K)$ for some compact $K \subset \beta X \setminus X$ and every neighborhood of zero is absorbent, we have

$$C(X) = C(\beta X \setminus K) .$$

If there were a compact $K' \subset \beta X \setminus X$ strictly containing K, then the neighborhood filter of zero in $C_{co}(\beta X \setminus K')$ would be strictly coarser than the neighborhood filter of zero in $C_{co}(\beta X \setminus K)$. This is apparent since two locally compact spaces Z and Z' are homeomorphic if and only if $C_{co}(Z)$ and $C_{co}(Z')$ are bicontinuously isomorphic (see [3]). Therefore K must be equal to $\beta X \setminus X$ which means X is locally compact.

In view of the fact that $C_I(X)$ is not, in general, topological, we wish to determine the associated locally convex space $C_{\tau I}(X)$ of $C_I(X)$. The topology of $C_{\tau I}(X)$ is generated by all the continuous seminorms on $C_I(X)$.

Let

$$p: C_I(X) \longrightarrow \mathbb{R}$$

be a continuous seminorm. We construct a seminorm $\tilde{p}$ which majorizes p and is more convenient to work with. For a compact set $K \subset \beta X \setminus X$, we denote by p_K the restriction of p to $C(\beta X \setminus K)$. Clearly

$$p_K: C_{co}(\beta X \setminus K) \longrightarrow R$$

is continuous. Therefore we can find a compact set $Q_K \subset \beta X \setminus K$ such that a constant multiple α of the seminorm

$$s_{Q_K} : C_{co}(\beta X \setminus K) \longrightarrow \mathbb{R} ,$$

defined by $s_{Q_K}(f) = \sup_{q \in Q_K} |f(q)|$, majorizes p_K . This implies that for any function $f \in C(\beta X \setminus K)$,

$$\tilde{p}_K(f) = \sup\{p_K(g) : |g| \leq |f| \text{ and } g \in C(\beta X \setminus K)\}$$

is a real number less than or equal to $\alpha s_{Q_K}(f)$. Since for every function $g \in C(X)$ the relation $|g| \leq |f|$ implies that $g \in C(\beta X \setminus K)$, we know that

$$\tilde{p}(f) = \sup\{p(g) : |g| \leq |f| \text{ and } g \in C(X)\}$$

is identical to $\tilde{p}_K(f)$. Of course every function in $C(X)$ is an element of $C(\beta X \setminus K)$ for some compact $K \subset \beta X \setminus X$. It is not difficult to verify that the maps

$$\tilde{p} : C_I(X) \longrightarrow \mathbb{R}$$

and

$$\tilde{p}_K : C_{co}(\beta X \setminus K) \longrightarrow \mathbb{R} \quad \text{for any compact } K \subset \beta X \setminus X ,$$

sending each $f \in C(X)$ to $\tilde{p}(f)$ and each $f \in C(\beta X \setminus K)$ to $\tilde{p}_K(f)$ respectively, are seminorms. Since $\tilde{p}$ restricted to $C(\beta X \setminus K)$ is $\tilde{p}_K$, we conclude that $\tilde{p}$ itself is a continuous seminorm. Furthermore, $\tilde{p}$ has the following properties:

$$\tilde{p}(f) = \tilde{p}(|f|) \quad \text{for all} \quad f \in C(X)$$

and

$$\tilde{p}(f) \leq \tilde{p}(g) \quad \text{for all} \quad f,g \in C(X) \quad \text{with} \quad |f| \leq |g| \ .$$

Lemma 1. *The kernel* P *of* $\tilde{p}$ *, the set of all functions* $f \in C(X)$ *with* $\tilde{p}(f) = 0$ *, is a closed ideal in* $C_I(X)$ *consisting of all elements in* $C(X)$ *vanishing on a compact subset of* X *.*

Proof. P is clearly a linear subspace of $C(X)$. To show it is an ideal, let $g \in P$. For an arbitrary element $f \in C(X)$, we consider

$$((-\underline{n} \vee f) \wedge \underline{n})$$

where $\underline{n}$ denotes the function of constant value $n \in \mathbb{N}$. Now

$$\tilde{p}(g \cdot ((-\underline{n} \vee f) \wedge \underline{n})) \leq \tilde{p}(g \cdot \underline{n}) = n \cdot \tilde{p}(g)$$

and hence $g \cdot ((-n \vee f) \wedge \underline{n}) \in P$. The Fréchet filter generated by the sequence

$$(g \cdot ((-\underline{n} \vee f) \wedge \underline{n}))_{n \in N}$$

converges to $g \cdot f$ in $C_I(X)$. Since P is obviously closed, $g \cdot f$ is an element of P . Thus P is a closed ideal in $C_I(X)$, and therefore consists of all functions in $C(X)$ vanishing on its non-empty null-set $Q \subset X$ (see theorem 2). It only remains to prove that Q is compact. We can express P as the union of the kernels of $\tilde{p}_K$ for all compact $K \subset \beta X \setminus X$. On the other hand, the kernel P_K of $\tilde{p}_K$ contains the kernel H_K of s_{Q_K}. Hence we have

$$N_{\beta X \setminus K}(P_K) \subset N_{\beta X \setminus K}(H_K) \quad .$$

But $N_{\beta X \setminus K}(H_K)$ is nothing else but Q_K. Since Q is contained in the intersection of the null-sets of P_K,

$$Q \subset \bigcap_K Q_K \ ,$$

where K runs through all compact subsets of $\beta X \setminus X$. The fact that $\bigcap_K Q_K$ is a compact subset of X implies that Q is compact.

Next, we will show that $\tilde{p}$ is majorized by a constant multiple of the supremum seminorm s over Q. Let $f \in C(X)$, and consider

$$g = ((-\underline{s(f)} \vee f) \wedge \underline{s(f)}) \ .$$

By the previous lemma, we have

$$\tilde{p}(f - g) = 0 \ .$$

Furthermore,

$$|\tilde{p}(f) - \tilde{p}(g)| \leq \tilde{p}(f - g) ,$$

and hence $\tilde{p}(f) = \tilde{p}(g)$. From the inequality $|g| \leq \underline{s(f)}$, we conclude that

$$\tilde{p}(f) \leq \tilde{p}(\underline{s(f)}) = s(f)\, \tilde{p}(\underline{1}) .$$

Therefore we have proved

Theorem 4. *The associated locally convex space of* $C_I(X)$ *is* $C_{co}(X)$.

The associated locally convex space of $C_I(X)$ coincides with the locally convex inductive limit of the family

$$\{C_{co}(\beta X \setminus K) : K \text{ is a compact subset of } \beta X \setminus X\}.$$

Thus we may state

Corollary 1. *The locally convex inductive limit of the family*

$$\{C_{co}(\beta X \setminus K) : K \text{ is a compact subset of } \beta X \setminus X\}$$

is $C_{co}(X)$.

For any convergence vector space E over R , its dual $\mathcal{L}(E)$ is identical with the dual of the associated locally convex space of E . Therefore:

<u>Corollary</u> 2. $\mathcal{L}(C_I(X)) = \mathcal{L}(C_{co}(X))$.

1.5. <u>Functorial properties of</u> $C_I(X)$

Let X and Y denote completely regular topological spaces. Every continuous map

$$t: X \longrightarrow Y$$

induces a homomorphism

$$t^*: C_I(Y) \longrightarrow C_I(X) ,$$

defined by $t^*(f) = f \circ t$ for every $f \in C(Y)$. To see that t^* is continuous, we consider the restrictions

$$t_K^*: C_{co}(\beta Y \setminus K) \longrightarrow C_I(X)$$

where t_K^* denotes $t^*|C(\beta Y \setminus K)$, and verify that t_K^* is continuous for every compact set $K \subset \beta Y \setminus Y$. To this end, we extend t to a map

$$\overline{t}: \beta X \longrightarrow \beta Y .$$

For each compact $K \subset \beta Y \setminus Y$, we know $\overline{t}^{\,-1}(K)$ is a compact subset of $\beta X \setminus X$. Furthermore, for a compact $K \subset \beta Y \setminus Y$ the map t_K^* is induced by

$$\bar{t}|(\beta X \setminus \bar{t}^{\,-1}(K)) : \beta X \setminus \bar{t}^{\,-1}(K) \longrightarrow \beta Y \setminus K ,$$

which we denote by t_K . That is, $t_K^*(f) = f \circ t_K$ for all $f \in C(\beta Y \setminus K)$. Clearly

$$t_K^*: C_{co}(\beta Y \setminus K) \longrightarrow C_{co}(\beta X \setminus \bar{t}^{\,-1}(K))$$

is continuous for every compact $K \subset \beta Y \setminus Y$, and therefore t^* itself is continuous.

On the other hand, let

$$u: C_I(Y) \longrightarrow C_I(X)$$

be a continuous $\mathbb{R}$-algebra homomorphism sending unity to unity. We will now show that u is of the form t^* where t maps X into Y continuously. The homomorphism u induces a continuous map

$$u^*: \mathcal{H}om_s C_I(X) \longrightarrow \mathcal{H}om_s C_I(Y)$$

defined by $u^*(h) = h \circ u$ for every $h \in \mathcal{H}om\, C_I(X)$. The index s denotes the topology of pointwise convergence. Corollary 2 of theorem 2 implies that the map $i_Z: Z \longrightarrow \mathcal{H}om_s C_I(Z)$ is a homeomorphism for any completely regular topological space Z . Thus we have a continuous map t from X into Y defined by $t = i_Y^{-1} \circ u^* \circ i_X$. Now it is easy to verify that t^* is equal to u.

To summarize these facts, we state:

Theorem 5. *A homomorphism*

$$u: C_I(Y) \longrightarrow C_I(X)$$

taking unity to unity is continuous if and only if there exists a continuous map $t: X \to Y$ *such that* $u = t^*$.

For maps $t: X \to Y$ and $s: Y \to Z$ between completely regular topological spaces, we have the obvious identities:

$$(s \circ t)^* = t^* \circ s^*$$

and

$$id_X^* = id_{C(X)} \quad .$$

1.6. Realcompact spaces

Let X be a completely regular topological space. As before, the zero-set $Z_{\beta X}(f)$ of a function $f \in C(\beta X)$ means the set of all points $p \in \beta X$ where f vanishes.

Here, we consider the collection

$$(**) \qquad \{C_{co}(\beta X \setminus Z_{\beta X}) : Z_{\beta X} \subset \beta X \setminus X \text{ is a zero-set}\}$$

This is a subfamily of the family of all topological algebras $C_{co}(\beta X \setminus K)$ for K a compact subset of $\beta X \setminus X$. As in section 1.1, it is clear that the union of all $C(\beta X \setminus Z_{\beta X})$ for $Z_{\beta X}$ a zero-set outside of X is again $C(X)$. Under the natural ordering (as in section 1.1), the collection (**) is an inductive system, and we denote the inductive limit of this system by $C_{I'}(X)$.

It is easy to see that $C_{I'}(X)$ is actually the finest convergence structure on $C(X)$ obtainable as an inductive limit of a subfamily of the family of all $C_{co}(\beta X\setminus K)$ for K a compact subset of $\beta X\setminus X$. Of course the identity,

$$\text{(I)} \qquad \mathrm{id}: C_{I'}(X) \longrightarrow C_I(X) ,$$

is continuous. Our main concern in this section is to determine under what conditions this identity is a homeomorphism.

If every compact subset of $\beta X\setminus X$ is contained in a zero-set in $\beta X\setminus X$, then clearly the identity (I) is a homeomorphism. Conversely, assume that

$$\mathrm{id}: C_I(X) \longrightarrow C_{I'}(X)$$

is continuous. Therefore we have a continuous injection

$$\mathrm{id}^*: \mathcal{H}om_s C_{I'}(X) \longrightarrow \mathcal{H}om_s C_I(X) ,$$

where $\mathcal{H}om_s C_{I'}(X)$ denotes the set of all continuous $\mathbb{R}$-algebra homomorphism from $C_{I'}(X)$ onto $\mathbb{R}$ together with the topology of pointwise convergence. For both X and its Hewitt realcompactification υX the convergence algebras $C_{I'}(X)$ and $C_{I'}(\upsilon X)$ are identical, since any zero-set contained in $\beta X\setminus X$ is already contained in $\beta X\setminus \upsilon X$ (see [6], p. 118). Thus

$$\mathcal{H}om_s C_{I'}(X) = \mathcal{H}om_s C_{I'}(\upsilon X) .$$

In view of (I), we conclude that the map

$$i_{\upsilon X}\colon \upsilon X \longrightarrow \mathcal{H}om_s C_{I'}(X)$$

is continuous. This tells us that $id^{*} \circ i_{\upsilon X}$ maps υX injectively into $\mathcal{H}om_s C_I(X)$, which is homeomorphic to X . Hence X must be realcompact.

To continue our investigation, without loss of generality we can regard X as a realcompact space. Since by assumption

$$id\colon C_I(X) \longrightarrow C_{I'}(X)$$

is continuous, we know that the inclusion map from $C_{co}(\beta X \setminus K)$ into $C_{I'}(X)$ is continuous for any compact $K \subset \beta X \setminus X$. Thus the neighborhood filter of zero in $C_{co}(\beta X \setminus K)$ has a basis in $C_{co}(\beta X \setminus Z_{\beta X})$ for some zero-set contained in $\beta X \setminus X$. Because every neighborhood of zero in $C_{co}(\beta X \setminus K)$ is absorbent, $C(\beta X \setminus Z_{\beta X}) \supset C(\beta X \setminus K)$ meaning that $Z_{\beta X} \supset K$. To summarize, we have extablished the following:

<u>Theorem</u> 6. *Let X be a realcompact space. $C_I(X)$ is identical to $C_{I'}(X)$ if and only if every compact set in $\beta X \setminus X$ is contained in some zero-set in $\beta X \setminus X$.*

We note that in the case of a realcompact locally compact space X , the convergence algebra $C_I(X)$ coincides with $C_{I'}(X)$ if and only if $\beta X \setminus X$ is a zero-set, i.e., X is σ-compact.

More generally, assume that $C_{I'}(X)$ is topological for a realcompact space X. By arguing as in section 1.4, we conclude that X is of the form $\beta X \setminus Z_{\beta X}$ for some zero-set $Z_{\beta X}$. This means that X is σ-compact and locally compact. Therefore, we can state:

<u>Theorem</u> 7. *Let* X *be a realcompact space. The convergence algebra* $C_{I'}(X)$ *is topological if and only if* X *is locally compact and* σ*-compact.*

As an example of a realcompact space X for which $C_I(X)$ and $C_{I'}(X)$ do not coincide, consider the reals together with the discrete topology.

1.7. <u>Universal representation of</u> $C_I(X)$

For a completely regular topological space X, the homomorphism

$$d\colon C_I(X) \longrightarrow C_c(\mathscr{H}om_c C_I(X)),$$

defined by $d(f)(h) = h(f)$ for all $f \in C(X)$ and all $h \in \mathscr{H}om\, C_I(X)$, is called the universal representation [2] of $C_I(X)$. The subscript c indicates the continuous convergence structure (Limitierung der stetigen Konvergenz [1]) on the sets $\mathscr{H}om\, C_I(X)$ and $C(\mathscr{H}om_c C_I(X))$.

We first investigate the continuous convergence structure on $\mathcal{H}om\ C_I(X)$.

The space $\mathcal{H}om_c C_c(X)$ is homeomorphic to X (see [3]), and thus the continuous convergence structure on $\mathcal{H}om\ C_c(X)$ is the topology of pointwise convergence. Since the evaluation map

$$\omega: C_I(X) \times X \longrightarrow \mathbb{R}$$

(defined by $\omega(f,p) = f(p)$ for all $f \in C(X)$ and all $p \in X$) is continuous, the identity

$$\mathrm{id}: C_I(X) \longrightarrow C_c(X)$$

is continuous. Furthermore, the sets $\mathcal{H}om\ C_I(X)$ and $\mathcal{H}om\ C_c(X)$ are identical (corollary 2 of theorem 2) which means that

$$\mathrm{id}: \mathcal{H}om_c C_c(X) \longrightarrow \mathcal{H}om_c C_I(X)$$

is continuous. On the other hand the identity map from $\mathcal{H}om_c C_I(X)$ into $\mathcal{H}om_s C_I(X)$ is clearly continuous (the subscript s indicates the topology of pointwise convergence). It follows that

$$\mathcal{H}om_c C_I(X) = \mathcal{H}om_s C_I(X) ,$$

which is homeomorphic to X via the map i_X defined earlier. Therefore

$$i_X^*: C_c(\mathcal{H}om_c C_I(X)) \longrightarrow C_c(X)$$

is a bicontinuous isomorphism, and of course $i_X^* \circ d$ is the identity map on $C(X)$.

Our main problem is thus to determine whether $C_I(X)$ and $C_c(X)$ coincide. So far, we can say the following:

Theorem 8. *Let* X *be a completely regular topological space. If there is a point* q *in* X *having a countable base of neighborhoods and no compact neighborhood, then* $C_c(X)$ *can not be an inductive limit of topological vector spaces over* $\mathbb{R}$.

Proof. Any inductive limit of topological vector spaces over $\mathbb{R}$ has the property that for each filter Φ converging to zero, there exists a coarser filter Φ' convergent to zero with

$$\lambda \cdot \Phi' = \Phi'$$

for every real number λ unequal to zero.

Our aim is to show that under the assumption of the theorem, $C_c(X)$ fails to satisfy this condition.

Let $\{Q_m\}_{m \in \mathbb{N}}$ be a countable collection of open sets in X that form a base for the neighborhood filter at q . We define inductively a certain system of nested neighborhoods of q . Let $N_1 = X$ and let $\{O_{1,\alpha}\}$ be an open covering of

X with no finite subcovering. Set

$$U_1 = O_1^q \cap Q_1 ,$$

where O_1^q is a member of $\{O_{1,\alpha}\}$ containing q. Assume that the closed respectively open neighborhoods N_i and U_i are defined. Choose N_{i+1} to be a closed neighborhood of q contained in U_i, and let $\{O_{i+1,\alpha}\}$ be a covering of N_{i+1} by open sets in X having no finite subcovering. We pick U_{i+1} to be an open neighborhood of q contained in

$$O_{i+1}^q \cap Q_{i+1} \cap N_{i+1} ,$$

where O_{i+1}^q is a member of $\{O_{i+1,\alpha}\}$ with $q \in O_{i+1}^q$. With this system of respectively closed and open neighborhoods of q,

$$N_1 \supset U_1 \supset N_2 \supset U_2 \ldots ,$$

we construct a filter Θ that does not satisfy the condition mentioned above. Let

$$T_n = \{f \in C(X): f(N_n) \subset \left[-\frac{1}{n} , \frac{1}{n}\right]\}$$

and let

$$T_x = \{f \in C(X): f(W_x) = \{0\} \}$$

for $x \neq q$, where we choose W_x as follows: Since $x \neq q$, the point x lies in N_r but not in N_{r+1} for some natural number r. Let W_x be a closed neighborhood of x contained in

$$\bigcap_{j=1}^{r} O_j^x \cap N_{r+1} ,$$

where O_j^x is a member of the covering system $\{O_{j,\alpha}\}$ containing x. It is clear that the sets $\{T_n : n \in N\}$ and $\{T_x : x \in X$ and $x \neq q\}$ generates a filter Θ convergent to zero in $C_c(X)$. Assume that there exists a coarser filter Θ' in $C_c(X)$ convergent to zero with

$$\lambda \cdot \Theta' = \Theta'$$

for every real number $\lambda \neq 0$. To the interval $[-1 , 1]$, there is a set $F' \in \Theta'$ and a neighborhood N_k of q such that

$$F'(N_k) = \{f(p) : f \in F' \text{ and } p \in N_k\}$$

is a subset of $[-1 , 1]$. For λ equal to $1/2k$, we have

$$\frac{1}{2k} F'(N_k) \subset \left[\frac{-1}{2k} , \frac{1}{2k}\right] ,$$

and $\frac{1}{2k} F' \in \Theta'$. Thus $\frac{1}{2k} F'$ contains a finite intersection of elements of the form T_n and T_x, say

$$\bigcap_{n \in \tilde{N}} T_n \cap \bigcap_{x \in \tilde{X}} T_x ,$$

where $\tilde{N}$ is a finite subset of N and $\tilde{X}$ is a finite subset of $X \setminus \{q\}$. Now we claim that

$$N_k \not\subset \bigcup_{x \in \tilde{X}} W_x \cup N_{k+1} .$$

Our construction guarantees that for a fixed W_x, either W_x is a subset of the complement of N_k or W_x is contained in an element of the open covering $\{O_{k,\alpha}\}$. Furthermore, N_{k+1} is contained in O_k^q. Since the open covering $\{O_{k,\alpha}\}$ has no finite subcovering, the claim is true. Therefore, we can find a function $g \in C(X)$ vanishing on $\bigcup_{x \in \tilde{X}} W_x \cup N_{k+1}$ with g taking on the value $1/k$ for some point in N_k and $\|g\| \leq \frac{1}{k}$. This function is certainly not in $\frac{1}{2k} F'$ but it is in $\bigcap_{n \in \tilde{N}} T_n \cap \bigcap_{x \in \tilde{X}} T_x$, and this contradiction establishes the theorem.

2.1. <u>Consequences for</u> $C_c(X)$

In this section, we demonstrate consequences of the theory developed in 1.1 to 1.7 in investigating closed ideals in $C_c(Y)$ for a convergence space Y, and in determining both the associated locally convex topological space of $C_c(X)$ and the dual space of $C_c(X)$, where X is a completely regular topological space. The results we obtain can be found in [4] and [5] respectively; however, the proofs given here are simpler than those provided in [4] and [5].

First, we look at closed ideals in $C_c(Y)$.

Let Y be an arbitrary convergence space. To this space we associate a completely regular topological space as follows: Any two points $p, q \in Y$ are said to be equivalent if $f(p) = f(q)$ for all real-valued continuous functions f. As usual, the set of all these functions is denoted by $C(Y)$. The quotient set defined by the above equivalence relation is called Y'. Any function $f \in C(Y)$ defines a function

$$f': Y' \longrightarrow \mathbb{R}$$

by sending each $\bar{p} \in Y'$ to $f(p)$. The initial topology induced by the family

$$\{f': f \in C(Y)\}$$

is, of course, completely regular. The set Y' together with this topology is again denoted by Y'.

The obvious projection

$$\pi: Y \longrightarrow Y'$$

induces an isomorphism (with respect to the usual $\mathbb{R}$-algebra structure)

$$\pi^*: C(Y') \longrightarrow C(Y)$$

defined by $\pi^*(g) = g \circ \pi$ for all $g \in C(Y')$. This isomorphism is continuous if both algebras carry the continuous convergence structure. Hence for any closed ideal J in $C_c(Y)$ (the algebra $C(Y)$ together with the continuous convergence structure), the ideal $\pi^{*-1}(J) \subset C_c(Y')$ is closed. Since the identity map,

$$\text{id}: C_I(Y') \longrightarrow C_c(Y')$$

is continuous, we conclude that $\pi^{*-1}(J)$ is closed in $C_I(Y')$. Therefore, we know by theorem 2 that it is of the form $I(N)$ where $N \subset Y'$ is a closed non-empty subset. It is clear that $I(\pi^{-1}(N)) = J$. Since an ideal of the form $I(M)$ for any non-empty subset of Y is closed in $C_c(Y)$, we have the following result:

Theorem 9. *For any convergence space* Y, *an ideal* J *in* $C_c(Y)$ *is closed if and only if it is of the form* $I(N_Y(J))$.

Another application of the theory developed in chapter 1 is the following theorem:

Theorem 10. *Let* X *be a completely regular topological space. The associated locally convex space of* $C_c(X)$ *is* $C_{co}(X)$.

Proof. Clearly the identity from $C_c(X)$ into the locally convex topological vector space $C_{co}(X)$ is continuous. Since

$$\text{id}: C_I(X) \longrightarrow C_c(X)$$

is also continuous, in view of theorem 4 the proof is complete.

By reasoning as in the proof of the last theorem, we obtain

Theorem 11. *For any completely regular space* X *the spaces* $\mathcal{L}(C_I(X))$, $\mathcal{L}(C_c(X))$, *and* $\mathcal{L}(C_{co}(X))$ *are identical.*

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