EQUATIONAL COMPACTNESS IN

UNIVERSAL ALGEBRAS.

von

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It is the aim of this paper to give a unified setting to and report on a new area of research in (universal) algebra: the theory of equationally compact universal algebras.
We attempt in this study to present in a concise fashion all essential results known on the subject matter as far as they have bearing on algebras rather than relational systems. More precisely: We take into account all relevant results in mathematical journals, a few unpublished results if they aid in illuminating the scope of some of the published results and this author's own research. Proofs will not be given in detail if they have appeared in print. Exception to this rule will be made for two reasons only: (1) if new ideas intrude upon a known result, (2) if the readability of this article seems jeopardized by too much omission. The paper will be concluded with a list of open problems.
The terminology follows essentially G. Grätzer (17) and we assume familiarity with the results in that book. Even so, we will collect, mainly for terminological purposes, the basic concepts relevant to this report in § 1. The following is a listing of the various sections of this paper:

§ 5 : Characterization theorems for (weak) atomic compactness of a relational system $\mathfrak{A}$ involving $|A|$.

§ 6 : Connections to topology.

§ 7 : Equational compactness (atomic compactness) in specific universal algebras (relational systems):

(1) Unary Algebras.

(2) W. Taylor's counterexample to a positive answer to Mycielski's problem.

(3) Semilattices.

(4) Lattice-related structures.

(5) Boolean algebras.

(6) Abelian groups.

(7) $\mathcal{R}$- modules and rings.

§ 8 : Compactifications of universal algebras.

§ 9 : Problems.

I am grateful for the opportunity, offered in 1968 by the Carnegie-Mellon University in Pittsburgh / Pa, to give a seminar on the topic. The notes then compiled (48) were an invaluable basis for this work which, in turn, is a condensed version of my "Habilitationsschrift" (51).
Thanks are also due to NRC which has supported this and other projects of mine with generous grants..
In the proof reading I was assisted patiently and critically by my students S. Bulman - Fleming, D. Haley and D. Kelly.
Last not least I thank G. Grätzer for his invitation to write up this paper for publication in the journal "Algebra Universalis"

§ 0 : Introduction.

It is the very question which has given life to the mathematical discipline called "Algebra" that, in a modified form, is underlying our topic: The question about the solvability of certain systems of equations in certain pregiven algebraic domains. However, while the original interest was and is directed toward finding solutions of finite systems of equations (or, at least, toward establishing the existence of such solutions) our interest is of a more relative nature: Given a system of equations over some universal algebra we want to study the conditions under which we can conclude the existence of a solution of that system provided all finite subsystems are solvable. To clarify our point we will initiate our presentation with the aid of a few simple, illustrating examples.

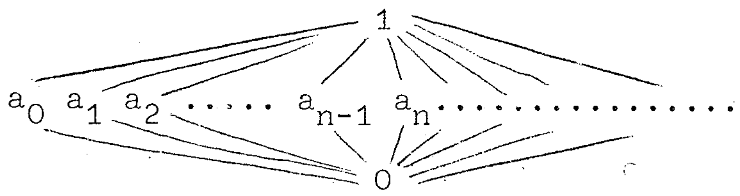
Example 1 : If we consider the cyclic group Z of integers with Addition $+$ then we narrow our attention to the following system Σ of equations:

$$\begin{aligned} 3x_0 + x_1 &= 1 \\ x_1 &= 2x_2 \\ x_2 &= 2x_3 \\ &\vdots \\ x_n &= 2x_{n+1} \\ &\vdots \end{aligned}$$

where n runs through the set N of natural numbers.
Visibly, if $(x_0,x_1,x_2,\ldots\ldots\ldots,x_n,\ldots) \in Z^{\omega_0}$ is a solution of Σ then $0 \neq x_1 = 2^n x_{n+1}$ for every $n \in N$, i.e. 2^n divides $x_1 \neq 0$ for every natural number n. This being impossible, Σ has no solution. On the other hand, if Σ_n denotes the set of the first n equations in Σ, then we choose x_0 and x_n such that $3x_0 + 2^{n-1}x_n = 1$, define $x_{n-1} = 2x_n$, $x_{n-2} = 2x_{n-1},\ldots\ldots$, $x_1 = 2x_2$ and have, thus, a solution $(x_0,x_1,\ldots,x_n)$ of Σ_n. Σ is a system of equations which is not solvable, although every finite subsystem is. This example (that can be found in Mycielski (31)) ought to be contrasted to the next one.

Example 2 : If B is a complete Boolean algebra with join $\vee$, meet $\wedge$, complement ', zero 0 and identity 1, then every system of equations involving variables, constants of B and the above operations is solvable provided every finite subsystem is. We will come back to this result. (See (1) and (44)).

Example 3 : If we replace the Boolean algebra in Example 2 by a complete lattice L with join $\vee$ and meet $\wedge$, then the conclusion is no longer true. To see this, let $L = \{0,a_0,a_1,\ldots\ldots\ldots,a_n,\ldots\ldots,1\}$, $n < \omega_0$, where 0 and 1 are, respectively, the smallest and largest element and the elements a_n are pairwise unrelated:



If S is a set of cardinality $\aleph_1$, then the system $\Sigma = \{x_i \wedge x_j = 0,\ x_i \vee x_j = 1,\ i \neq j \in S\}$ of equations over L is not solvable in L, since $i \neq j$ implies $x_i \neq x_j$ for any solution $(x_s)_{s \in S}$ of Σ, i.e. $\{x_s;\ s \in S\}$ would have cardinality $\aleph_1 > |L|$. On the other hand is it quite evident that every finite subsystem Σ_1 of Σ is solvable.

Example 4 : If B is an arbitrary non-void finite set and F a set of finitary operations on B then any set of equations with an arbitrary number of variables, constants from B and based on finite compositions of the operations in F is solvable in B provided it is finitely solvable. If only a finite number of variables were involved then, of course, the fact that the "Finite Intersection Property" holds for finite sets would trivially yield the result. If infinitely many variables are involved then one could convince oneself of the correctness of our claim by a process of reasoning that mimics the proof of Tychonoff's well-known theorem concerning the product of compact Hausdorff spaces. Since we will relate topological compactness with our algebraic problems in what follows we will skip a redundant argument at this point.

Example 5 : It is the same basic technique we used in example 3 that together with Dirichlet's prime number theorem yields the next peculiar system of $\aleph_1$ equations over the ring of integers Z. Not only is every finite subsystem, but even every countable subsystem of equations solvable; the whole system, however, admits no solution.

If ω_1 is the initial ordinal of $\aleph_1$ then the system

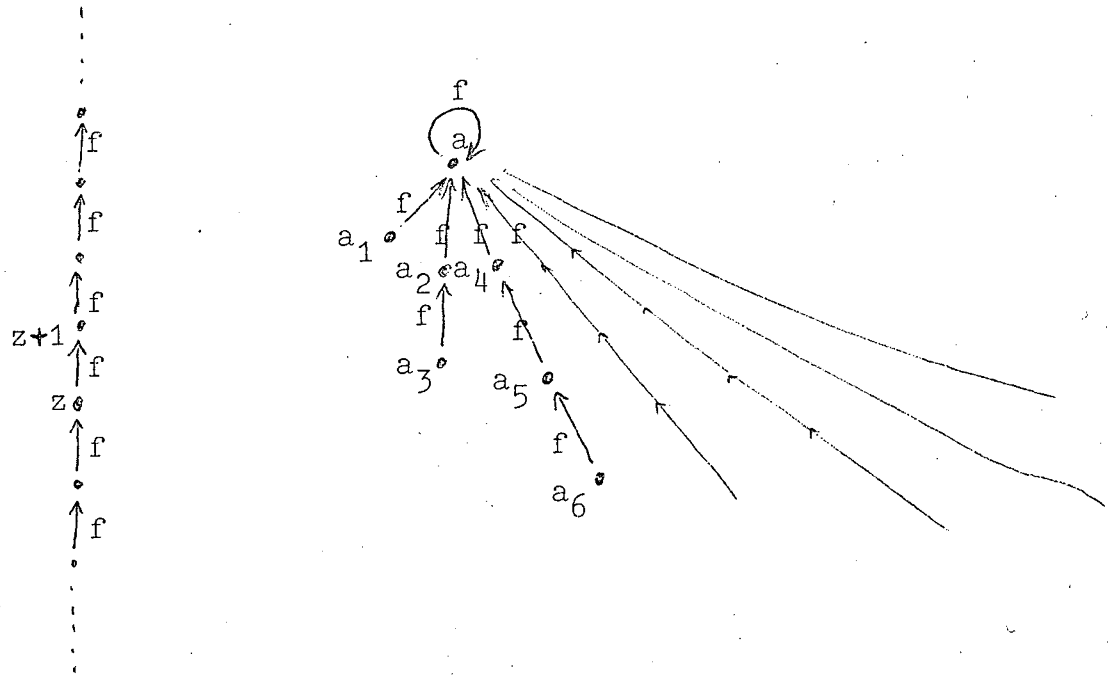
$$\Sigma = \{ x_{\xi,\eta}(m \cdot z_\xi + n) + y_{\xi,\eta}(m \cdot z_\eta + n) = 1 \ ; \ \xi \neq \eta < \omega_1 \}$$

of equations over the ring Z is not solvable if n,m are relatively prime natural numbers such that $\frac{\pm 1 - n}{m} \notin Z$. Mycielski chose m = 5, n = 2.

To see this we realize that for every choice of $z_\xi \in Z$ the integer $m \cdot z_\xi + n$ is different from ± 1; hence, $\xi \neq \eta$ implies $z_\xi \neq z_\eta$ for every solution $(z_\xi)_{\xi < \omega_1}$ of Σ which, of course, is impossible in Z. On the other hand, every countable subsystem Σ' of Σ is solvable in Z. For let $\xi_0, \xi_1, \xi_2, \ldots, \xi_i, \ldots$, $i < \omega_0$, be the indices ξ, η actually occurring in Σ' and choose z_{ξ_i} such that $m \cdot z_{\xi_i} + n = p_i$ is a prime number for every $i \in N$; moreover we make our choice such that $i \neq j$ implies $p_i \neq p_j$. We can do this since $n + m, n + 2m, n + 3m, \ldots\ldots\ldots\ldots$ constitutes an infinite arithmetical progression with initial element n, difference m and (n,m) = 1; thus, Dirichlet's theorem assures an infinite number of prime elements in the progression. Since therefore $(p_i, p_j) = 1$ if $i \neq j$, we can find integers s_{ij} and t_{ij} such that $s_{ij}p_i + t_{ij}p_j = 1$; i.e. Σ' is solvable. Mc Kenzie has recently displayed a system of equations with the above properties over the Abelian group of integers.

<u>Example 6</u> : Let this time Z denote the set of integers with one unary operation f only: $f(z) = z + 1$. Any system of equations involving variables, integers and the operations f^n, $n \in N$, is solvable provided it is finitely solvable. If we add one more

element, say a, and define f(a) = a then the above claim remains still true. Let us, however, continue to extend this algebraic system as follows:



Then the resulting system does no more satisfy that property - not even if we allow no constants. This is seen by the following system of equations:

$y = f(y),\ y = f(x_1),\ x_1 = f(x_2),\ x_2 = f(x_3), \ldots\ldots, x_n = f(x_{n+1}), \ldots$

The author's study of this unary case will be sketched in § 7.

<u>Example 7</u> :

Both divisible Abelian groups G and p-primary Abelian groups G that (being modules over the ring of p-adic integers) (1) have no elements of infinite height and (2) are complete in their p-adic topology do have the property that any system of equations with constants from G and involving only finite compositions of the fundamental operations is solvable provided it is finitely solvable.

The last example was subjected to intensive research by several authors. I. Kaplansky (24), in view of their rôle as direct summands of topologically compact Abelian groups, called Abelian groups with this property of "relative solvability" of equations of the type described in the last example algebraically compact groups. He used two different, although equivalent, definitions in the 1954, resp. 1969 editions of his monograph "Infinite Abelian Groups". One may say that the case of Abelian groups and the interesting structure theory the question of "relative solvability" gave rise to in that case was a strong impetus to further and more general research. In the sequel the concept has attracted the interest of algebraists and logicians alike and has been studied in both a universal algebraic - logical and classical algebraic framework. Structure theorems, decomposition theorems, connections between algebra and topology, model-theoretic and algebraic characterizations within fixed equational classes of algebras were in the foreground of the investigation.

The topic seems to be particularly apt to please those minds that strive after or look for interplay between challenging and deep universal algebraic research with interesting situations that arise once one focuses one's attention on particular "classical structures". And this, indeed, it does if one recalls that this "relative solvability" behaviour forces, e.g., Abelian groups to be direct products of exactly such groups as we displayed in example 7 or Boolean algebras to be injective etc. The interest in classical interpretations of our general problem prevails to this day as, e.g., the 1970 - papers of A. Abian (1), L. Fuchs (12), R.B. Warfield (47) or D. Haley (20) witness.

On the universal algebraic side research has been done mainly by J. Mycielski, B. Weglorz, G. Grätzer, H. Lakser, S. Balcerzyk, J. Łos, W. Taylor, G. Fuhrken, L. Pacholski, Mc Kenzie, the author and others.

W. Taylor has just completed a new preprint with material relevant to our topic (see (42)) which, of course, cannot yet be included in this presentation. Partly the results of that preprint are aimed at relational structures rather than algebras; his nice idea of a "minimum compact algebra" we were able to include here at a fitting place.

We conclude this introductory section with a table of notations that are frequently used throughout this report.

Recurring Notations.

N = set of natural numbers

$N_0 = N \cup \{0\}$

R = set of real numbers

Q = set of rational numbers

H(K) = class of homomorphic images of objects in K

P(K) = class of direct products of objects in K

S(K) = class of subobjects of objects in K

U(K) = class of ultraproducts of objects in K

$U_0(K)$= class of ultrapowers of objects in K

$C(\mathfrak{A})$ = congruence set of $\mathfrak{A}$

2^I = set of subsets of the set I

$|I|$ = cardinality of the set I

$|\alpha|$= cardinality of the ordinal α

$\cup$ and $\cap$ = set-theoretical union and intersection.

$\dot{\cup}$ = disjoint union (If two sets A and B are not disjoint then $A \dot{\cup} B$ denotes a set that results from taking disjoint copies of A and B and forming their union.)

$[a]\Theta$= the block of a under the equivalence Θ

The elements of $\prod(\mathfrak{A}_i; i \in I)$ are denoted by either f or $(if)_{i \in I}$.

$\mathfrak{A} \overset{S}{\leadsto} \mathfrak{L}$: $\mathfrak{L}$ is S-pure with respect to $\mathfrak{A}$

$\mathfrak{A} \overset{S}{\leftrightsquigarrow} \mathfrak{L}$: $\mathfrak{A}$ and $\mathfrak{L}$ are mutually S-pure

$spec_S(\mathfrak{A})$ = spectrum of $\mathfrak{A}$ with constants in S

If f is a polynomial in a universal algebra then we write the arguments behind the symbol: $f(a_1,\ldots,a_n)$. If f is any other function then the argument preceds f: af.

§ 1 : Basic concepts.

A relational system $\mathfrak{A}$ is a triplet (A;F,R) of sets such that the so-called carrier set A is non-empty, F is a set of operation symbols that induce certain concrete finitary operations on A (called "fundamental operations"), R is a set of relation symbols that induce certain concrete finitary relations on A. We generally assume F and R to be well-ordered (more a convenience than a necessity), say $F = \{f_0,\ldots,f_\gamma,\ldots\}_{\gamma<\alpha_1}$ $R = \{r_0,\ldots\ldots,r_\delta,\ldots\ldots\}_{\delta<\alpha_2}$, and we assume f_γ to be an n_γ-ary operation symbol, r_δ to be an m_δ - ary relation symbol (n_γ, $m_\delta \in N_0$). Generally we will (in agreement with widespread usage) use the same notation for the symbols and the concrete operations or relations induced by those symbols. If, however, we need to emphasize what operations or relations we are exactly talking about then we will also write $f_\gamma^{\mathfrak{A}}$ or $r_\delta^{\mathfrak{A}}$, respectively. If $\tau_1 = (n_0,\ldots\ldots,n_\gamma,\ldots\ldots)_{\gamma<\alpha_1}$, $\tau_2 = (m_0,\ldots\ldots,m_\delta,\ldots\ldots)_{\delta<\alpha_2}$, then we call $\tau = (\tau_1, \tau_2)$ the type of $\mathfrak{A}$.

If $R = \emptyset$ then we say that (A;F,R) = (A;F) is a (universal) algebra of type $\tau = \tau_1$. If $F = \emptyset$ then we call (A;F,R) = (A;R) a strict relational system or a <u>relational structure</u> of type $\tau = \tau_2$.

Every universal algebra $\mathfrak{A} = (A;F)$ of type $\tau = (n_0,\ldots,n_\gamma,\ldots)_{\gamma<\alpha}$ is associated with a relational structure $\mathfrak{A}_{rel} = (A;R(F))$ of type $\tau_{rel} = (n_0+1,\ldots\ldots,n_\gamma+1,\ldots\ldots)_{\gamma<\alpha}$ by defining $r_\gamma(a_1,\ldots\ldots,a_{n_\gamma+1})$ to hold true if and only if $f_\gamma(a_1,\ldots\ldots,a_{n_\gamma}) = a_{n_\gamma+1}$. Being interested in algebra rather than model theory we will, however, have only occasional transitions from the one to the other .

If $\mathfrak{A} = (A;F,R)$ and $\mathfrak{B} = (B;F,R)$ are two relational systems in <u>$K(\tau)$</u> , the class of all relational systems of type τ , then a mapping $f: A \dashrightarrow B$ is a <u>homomorphism</u> if firstly $f_\gamma(a_1,\ldots\ldots,a_{n_\gamma})f = f_\gamma(a_1f,\ldots\ldots,a_{n_\gamma}f)$ and secondly $r_\delta(a_1,\ldots\ldots,a_{m_\delta})$ implies $r_\delta(a_1f,\ldots\ldots,a_{m_\delta}f)$ for all $a_i \in A$. If, in addition, $r_\delta(a_1f,\ldots\ldots,a_{m_\delta}f)$ for some $a_i \in A$ always implies the existence of $c_1,\ldots\ldots,c_{m_\delta} \in A$ such that $c_if = a_if$ and $r_\delta(c_1,\ldots\ldots,c_{m_\delta})$ holds true then we speak of a <u>full homomorphism</u>. If, even more, $r_\delta(a_1,\ldots\ldots,a_{m_\delta})$ is always equivalent to $r_\delta(a_1f,\ldots\ldots,a_{m_\delta}f)$ then we speak of a <u>strong homomorphism.</u> Of course, in case of universal algebras the different homomorphism concepts coincide. (See also (17), chapter II, § 13). The concepts of monomorphism, endomorphism, automorphism, epimorphism and isomorphism are then clear (since we talk about set mappings and stay away from category theory). The isomorphism theorems (again see, e.g., (17) or (36)) connect in the usual strong form homomorphisms with <u>congruence relations</u>.

Subsystems and subalgebras are, respectively, relational systems whose carrier sets are subsets of the larger carrier set and whose operations and relations are induced from the larger algebra.

Let us be a little more space-consuming with the construction of the class $P_S(\tau)$ of polynomial symbols of type $\tau = (\tau_1, \tau_2)$ with constants in S. We choose, for every ordinal γ, the symbol e_γ and call the resulting class $\underline{E} = \{ e_\gamma ; \gamma = \text{ordinal} \}$ the class of "projection symbols". $P_S(\tau)$ is defined as a class of symbols as follows:

(1) $E \cup S \subseteq P_S(\tau)$, (2) $p_1, \ldots, p_{n_\gamma} \in P_S(\tau) \Longrightarrow f_\gamma(p_1, \ldots, p_{n_\gamma}) \in P_S(\tau)$ for all $f_\gamma \in F$, (3) $P_S(\tau)$ consists exactly of all elements that can be obtained in a finite number of steps using (1) and (2). Equality is formal equality.

(2) would implicitly define an algebra $(P_S(\tau); F)$ of type τ_1 if $P_S(\tau)$ was a set rather than a class. So if we restrict the projection symbols e_γ by requiring $\gamma < \alpha$ where α is a fixed ordinal, and replace, in (1), E by this set $E^{(\alpha)} = \{ e_\gamma ; \gamma < \alpha \}$ then we obtain the algebra $\mathcal{P}_S^{(\alpha)}(\tau) = (P_S^{(\alpha)}(\tau); F)$ of "α - ary polynomial symbols of type τ with constants in S".

This construction is the obvious modification of the same construction given in (17) for the case S = $\emptyset$. The same remark applies to the following continuation:

Let $K(\tau; S)$ denote the class of all relational systems $\mathfrak{A}$ of type τ with $S \subseteq A$. $P_S^{(\alpha)}(\tau)$ induces on every $\mathfrak{A} \in K(\tau; S)$ an algebra $\mathcal{P}_S^{(\alpha)}(\mathfrak{A})$ of type τ_1, the so-called "algebra of α -ary polynomials on $\mathfrak{A}$" in the following fashion: Every $p \in P_S^{(\alpha)}(\tau)$ induces a mapping $p^{\mathfrak{A}, \alpha} : A^\alpha \longrightarrow A$ according to the agreement that (i) $e_\gamma^{\mathfrak{A}, \alpha}(x_0, \ldots, x_\delta, \ldots)_{\delta < \alpha} = x_\gamma$

and $s^{\mathfrak{A},\alpha}(x_0,\ldots,x_\delta,\ldots)_{\delta<\alpha} = s$ for $s \in S$ and

(ii) if $p_1^{\mathfrak{A},\alpha}, \ldots\ldots\ldots, p_{n_\gamma}^{\mathfrak{A},\alpha}$ are already defined then $f_\gamma(p_1,\ldots,p_{n_\gamma})$ induces $f_\gamma(p_1,\ldots,p_{n_\gamma})^{\mathfrak{A},\alpha}(x_0,\ldots,x_\delta,\ldots)_{\delta<\alpha} = f_\gamma(p_1^{\mathfrak{A},\alpha}(x_0,x_1,\ldots,x_\delta,\ldots)_{\delta<\alpha},\ldots,p_{n_\gamma}^{\mathfrak{A},\alpha}(x_0,x_1,\ldots,x_\delta,\ldots)_{\delta<\alpha})$.

Then $P_S^{(\alpha)}(\mathfrak{A}) =: \{p^{\mathfrak{A},\alpha};\ p \in P_S^{(\alpha)}(\tau)\}$. Moreover, if $f_\gamma \in F$ and $p_1^{\mathfrak{A},\alpha},\ldots\ldots, p_{n_\gamma}^{\mathfrak{A},\alpha} \in P_S^{(\alpha)}(\mathfrak{A})$ then $f_\gamma^{\mathfrak{A}}(p_1^{\mathfrak{A},\alpha},\ldots\ldots, p_{n_\gamma}^{\mathfrak{A},\alpha}) =: (f_\gamma(p_1,\ldots,p_{n_\gamma}))^{\mathfrak{A},\alpha}$ makes f_γ a well-defined operation on $P_S^{(\alpha)}(\mathfrak{A})$.

Again it is a customary habit to drop the various superscripts if confusion cannot arise. Also: $P_\emptyset(\tau) = P(\tau)$, $K(\tau;\emptyset) = K(\tau)$ etc.

If $L_\tau^{(\alpha)}$ denotes the first order language (with identity) with variables $x_0,\ldots,x_\delta,\ldots$, $\delta<\alpha$, then we can proceed as in the case of polynomial symbols and define <u>$L_\tau^{(\alpha)}(S)$</u> for any set S as the set of formulas one obtains by replacing some, none or all of the free variables by elements of S. $L_\tau^{(\alpha)}(S)$ has then its natural interpretation in relational systems $\mathfrak{A} \in K(\tau;S)$. For the remaining notations and concepts of first order logic (as, e.g., atomic formulas, positive formulas, quantifiers, satisfiability of formulas etc) see (17).

If $\mathfrak{A}$ and $\mathfrak{B}$ are relational systems of type τ and $A \subseteq B$, then $\mathfrak{B}$ is called an <u>elementary extension</u> of $\mathfrak{A}$ (notation: $\mathfrak{A} \preceq \mathfrak{B}$) if for each formula $\Psi \in L_\tau^{(\alpha)}$, $\alpha \geq \omega_0$, and each $\bar{a} \in A^\alpha$ the formula Ψ is satisfied by $\bar{a}$ in $\mathfrak{B}$ if and only if it is so in $\mathfrak{A}$. $\mathfrak{A}$ is then called an <u>elementary subsystem</u>. $\mathfrak{A}$ and $\mathfrak{B}$ are called <u>elementarily equivalent</u> if and only if every sentence that holds in one relational system also holds in the other.

The concept "elementary extension" is closely related to the construction of an ultrapower $\mathfrak{A}^I_D$ of a relational system $\mathfrak{A}$ with respect to the index set I and the ultrafilter D of subsets of I. As well-known, every relational system $\mathfrak{A}$ is (via the diagonal embedding) an elementary subsystem of every of its ultrapowers. A. Tarski suggested the study of ultrapowers in connection with the compactness theorem of logic. The theory afforded a new proof for the latter theorem, the meat of which led C. Ryll-Nardzewski to a slight variation that is useful for our purpose:

Remark 1.1 (C. Ryll-Nardzewski (44)): Let $\mathfrak{A}$ be a relational system of type τ. If Σ is a set of formulas in $L_\tau(A)$ each finite subset of which is simultaneously satisfiable in $\mathfrak{A}$ then Σ is satisfiable in an ultrapower of $\mathfrak{A}$.

A relational system $\mathfrak{A}$ with $S \subseteq A$ is called an S-pure subsystem of $\mathfrak{L}$ (or $\mathfrak{L}$ an S-pure extension of $\mathfrak{A}$) if $\mathfrak{A}$ is a subsystem of $\mathfrak{L}$ and any finite set of atomic formulas in $L_\tau(S)$ which is satisfiable in $\mathfrak{L}$ is also satisfiable in $\mathfrak{A}$. If $S = \emptyset$ then we call $\mathfrak{A}$ a weakly pure subsystem of $\mathfrak{L}$. If $S = A$ then $\mathfrak{A}$ is simply a pure subsystem.

Of course, every elementary subsystem is a pure subsystem. We end this section with a few refinements of these concepts and two remarks. (see (44), (45), (50))

If $\mathfrak{A}$, $\mathfrak{L}$ are relational systems and $S \subseteq A, B$ then a homomorphism $g: \mathfrak{A} \dashrightarrow \mathfrak{L}$ is called an S-retraction if $sg = s$ for all $s \in S$. $(Ag;F,R) = \mathfrak{A}g$ is then an S-retract of $\mathfrak{A}$. If every finite set of atomic formulas with constants in S which is satisfiable in $\mathfrak{A}$ is satisfiable in $\mathfrak{L}$ then we call the system $\mathfrak{L}$

S-pure with respect to $\mathfrak{A}$ (notation: $\mathfrak{A} \overset{S}{\rightsquigarrow} \mathfrak{L}$).
If $\mathfrak{A} \overset{S}{\rightsquigarrow} \mathfrak{L}$ and $\mathfrak{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$ hold then $\mathfrak{A}$ and $\mathfrak{L}$ are called mutually S-pure (notation: $\mathfrak{A} \overset{S}{\leftrightsquigarrow} \mathfrak{L}$).

Clearly, $\emptyset$ - retracts are homomorphic images; A-retracts $g: \mathfrak{L} \dashrightarrow \mathfrak{A}$ are simply retracts in the usual sense if $\mathfrak{A} \subseteq \mathfrak{L}$.

Remark 1.2 : Let $\mathfrak{A}$, $\mathfrak{L}$ be relational systems of type τ:

(1) If $\mathfrak{A}$ is an S-retract of the relational system $\mathfrak{L}$ then $\mathfrak{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$.

(2) If $\mathfrak{A}$ is a subsystem of $\mathfrak{L}$ and $\mathfrak{A}$ is a retract of $\mathfrak{L}$ then $\mathfrak{A}$ is a pure subsystem of $\mathfrak{L}$.

(3) If $\mathfrak{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$ and all sets of atomic formulas in $L_\tau(S)$ are satisfiable in $\mathfrak{A}$ provided they are satisfiable in $\mathfrak{L}$ then $\mathfrak{A}$ contains an S-retract of $\mathfrak{L}$.

proof: (1) and (2) are clear. Let us turn to (3): To give that proof we need to introduce a construction that we will use frequently hereafter and therefore introduce via a proof-independent definition:

Definition 1.3 : If $\mathfrak{L}$ is a relational system then let $\underset{\sim}{\mathfrak{L}}$ denote the set of all possible true polynomial equations and relations that hold in $\mathfrak{L}$. (E.g., if $\mathfrak{L} = (Z;+,\leq)$ then $\underset{\sim}{\mathfrak{L}} = \{0+2=2,\ 2+0=2,\ 3+8=11,\ 7+49=56,\ 4\leq 7,\ 8\leq 1002, \ldots \text{etc} \ldots\}$
If $S \subseteq B$ then the spectrum of $\mathfrak{L}$ with respect to S ($spec_S(\mathfrak{L})$) is the system of atomic formulas with constants in S which we get by replacing every $b \in B \setminus S$ in $\underset{\sim}{\mathfrak{L}}$ by the variable x_b.
Back to the proof of (3): $spec_S(\mathfrak{L})$ is satisfiable in $\mathfrak{L}$ by con-

struction. So, by assumption, $\mathrm{spec}_S(\mathcal{L})$ is satisfiable in $\mathfrak{A}$. Let us assume that $(a_b;\ b \in B\setminus S)$ is a solution of $\mathrm{spec}_S(\mathcal{L})$ in $\mathfrak{A}$. Then the mapping $g\colon B \dashrightarrow A$ defined by $bg = a_b$ for $b \in B\setminus S$ and $bg = b$ for $b \in S$ is evidently an S-retraction from $\mathcal{L}$ to $\mathfrak{A}$.

After this proof we are really in a position to strengthen the claim made in 1. 2. (3) as follows:

Remark 1. 4.: If $\mathcal{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$ and $\mathrm{spec}_S(\mathcal{L})$ is satisfiable in $\mathfrak{A}$ then there exists an S-retraction from $\mathcal{L}$ to $\mathfrak{A}$.

§ 2 : Various compactness concepts.

As one checks the literature one finds a variety of algebraic "compactness" concepts that have, in varied forms, one aim in common: the aim to describe a "relative solvability" behaviour of algebraic structures (as discussed in section 0).
We propose a new concept that contains essentially all the others as special cases. Apart from the obvious advantage of unification of concepts this allows us to deduce fundamental results in their due general setting, referring specific results to their proper place as corollaries.

Definition 2.1 : Let $\mathfrak{A} \in K(\tau;S)$ be a relational system, M a fixed subclass of L_τ and $\mathfrak{m}$ a fixed cardinal number.
A relational system $\mathcal{L} \in K(\tau;S)$ is then called $(S,\mathfrak{A},\mathfrak{m})$- M compact if it satisfies the following property:
If $M(S)$ denotes the class of formulas we obtain after having substituted elements from S for some variables in M in all possible ways and Σ is a subset of $M(S)$ with $\leq \mathfrak{m}$ formulas each finite subset of which is satisfiable in $\mathfrak{A}$ (i.e. Σ is "finitely satisfiable" in $\mathfrak{A}$) then Σ is satisfiable in $\mathcal{L}$.
$\underline{c}_M(S,\mathfrak{A},\mathfrak{m})$ denotes the class of all $(S,\mathfrak{A},\mathfrak{m})$ - M compact relational systems. If $\mathcal{L} \in c_M(S,\mathfrak{A},\mathfrak{m})$ for every cardinal number $\mathfrak{m}$ then $\mathcal{L}$ is said to be $(S,\mathfrak{A})$- M compact and the class of all such relational systems is denoted by $\underline{c}_M(S,\mathfrak{A})$.

Via the next examples that (en passant) serve as terminological definitions we will show how the other compactness concepts appearing in print are related to this notion:

Example 2. 2: If $M = M_{at}$ = class of atomic formulas, then $(S,\mathfrak{A},\mathfrak{m})$ - M_{at} compactness is also called *$(S,\mathfrak{A},\mathfrak{m})$-atomic compactness*, $(S,\mathfrak{A})$ - M_{at} compactness also *$(S,\mathfrak{A})$-atomic compactness*. If $R(\tau) = \emptyset$, i.e. if we are dealing with universal algebras, then one speaks of *equational compactness* rather than of atomic compactness under the above circumstances.

A relational system $\mathfrak{A}$ is called *weakly atomic compact* if $\mathfrak{A} \in c_{M_{at}}(\emptyset,\mathfrak{A})$, *atomic compact* if $\mathfrak{A} \in c_{M_{at}}(A,\mathfrak{A})$. Similarly, $\mathfrak{A}$ is called *$\mathfrak{m}$- weakly atomic compact* if $\mathfrak{A} \in c_{M_{at}}(\emptyset,\mathfrak{A},\mathfrak{m})$ and *$\mathfrak{m}$- atomic compact* if $\mathfrak{A} \in c_{M_{at}}(A,\mathfrak{A},\mathfrak{m})$. Again we use the phrase "*equationally compact*" or "*weakly equationally compact*" under the above circumstances if $\mathfrak{A}$ is a universal algebra.

If we are dealing with M_{at} then we will simply drop the index and write *$c(\emptyset,\mathfrak{A})$* rather than $c_{M_{at}}(\emptyset,\mathfrak{A})$ *etc*.

Example 2. 3: If $M = M_{pos}$ is the class of all positive formulas, i.e. of all those formulas that are generated from the atomic formulas via the application of $(\exists x_\gamma)$, $(\forall x_\gamma)$, $\wedge$, $\vee$, then $(S,\mathfrak{A},\mathfrak{m})$ - M_{pos} compactness is called *$(S,\mathfrak{A},\mathfrak{m})$- positive compactness etc*. $\mathfrak{A}$ is *positively compact* if $\mathfrak{A} \in c_{M_{pos}}(A,\mathfrak{A})$, *weakly positively compact* if $\mathfrak{A} \in c_{M_{pos}}(\emptyset,\mathfrak{A})$. The concepts of *$\mathfrak{m}$-positive compactness* etc are then formulated as above.

(see , e.g., (31), (44), (50)).

Example 2. 4: If $M = L_\tau$ then $(S, \mathfrak{A}, \mathfrak{m})$-$L_\tau$ compactness is also called $(S, \mathfrak{A}, \mathfrak{m})$ - elementary compactness. $\mathfrak{A}$ is called ($\mathfrak{m}$-) elementarily compact if $\mathfrak{A} \in c_{L_\tau}(A, \mathfrak{A})$ ($c_{L_\tau}(A, \mathfrak{A}, \mathfrak{m})$) and, accordingly, weakly ($\mathfrak{m}$-) elementarily compact if $\mathfrak{A} \in c_{L_\tau}(\emptyset, \mathfrak{A})$ ($c_{L_\tau}(\emptyset, \mathfrak{A}, \mathfrak{m})$).

Although elementary compactness is now introduced, it is unfortunately, an uninteresting concept. For the class of weakly elementarily compact relational systems coincides with the class of elementarily compact relational systems of the same type which, in turn, coincides with the class of all finite relational systems of the given type. To see this we look at the system of formulas $\Sigma(I) = \{x_i \neq x_j ; i \neq j \in I\}$ for an arbitrary index set I. Surely, these systems of formulas force a weakly elementarily compact relational system to be finite. Vice versa, if $\mathfrak{A}$ is a finite relational system and Σ a finitely satisfiable set of formulas in $L_\tau(A)$ then Σ is satisfiable in some ultrapower of $\mathfrak{A}$ by 1.1. However, every such ultrapower is isomorphic to $\mathfrak{A}$. Thus, Σ is satisfiable in $\mathfrak{A}$.

The same claim of lacking interest holds no longer true for $\mathfrak{m}$- elementary compactness as a result of Mycielski and Ryll-Nardzewski (32) shows.

Example 2. 5: A relational system $\mathfrak{L}$ containing the subsystem $\mathfrak{A}$ such that $\mathfrak{L} \in c(A, \mathfrak{A})$ is called a closure of $\mathfrak{A}$ in (46) and a quasi-compactification of $\mathfrak{A}$ in (48).

(41),

Example 2. 6: In one of his preliminary reports, W. Taylor calls a closure of a relational system $\mathfrak{A}$ what, in our terminology is a relational system $\mathcal{L}$ containing $\mathfrak{A}$ such that $\mathcal{L} \in c(A, \mathcal{L})$. In a later preprint W. Taylor changes the above terminology and calls such a relational system a "strong closure". However, we stick to the first term.

Example 2. 7: In the light of the above terminology we may re-read some of the examples in §0.

Example 0.1 states that the cyclic group (Z;+) is not equationally compact, not even $\aleph_0$ - equationally compact. In a later section we will see that that is not an accidental phenomenon. It is generally true that $|A|$ - atomic compactness of a relational system implies its atomic compactness.

Example 0.2 states that complete Boolean algebras are equationally compact.

The lattice in example 0.3 is not weakly equationally compact if we adjoin the elements 0,1 as nullary operations. It also cannot be embedded in any weakly equationally compact lattice of the same type (nor can it be embedded in any equationally compact universal algebra of the same type, as a quick theoretical argument will show).

Example 0.4 states that finite relational systems are always atomic compact. A proof of this fact, independent of Tychonoff's theorem, is implicit in 2.4.

Example 0.5 illustrates a broadening of our basic question. Rather than requiring that all finite subsystems of a given system of equations are satisfiable, we insist on the solvability of certain distinguished subsystems that are not necessarily finite. Here is surely an area of mathematics wide open for research in which not too much has been done for the time being. However, it is not within the defined limits of this work.

Example 0.6 gives examples of both equationally compact and not weakly equationally compact unary algebras (A;f).

Example 0.7, finally, shows that divisible Abelian groups and p-primary Abelian groups that have no elements of infinite height and are complete in their p-adic topology are examples of equationally compact Abelian groups.

This fact will be included in the section on special structures.

We close this section with a remark relating $\mathfrak{A}$ and $\mathfrak{A}_{rel}$ in case of universal algebras and characterizing $\mathfrak{m}$- atomic compactness of relational systems in an important instance via the number of variables rather than the number of formulas that can be used.

Remark 2.8 :(1) The universal algebra $\mathfrak{A}$ is (weakly) equationally compact if and only if the relational system $\mathfrak{A}_{rel}$ is (weakly) atomic compact.

(2) Let $\mathfrak{A}$ be a relational system and $\mathfrak{m}$ a cardinal number with $\mathfrak{m} \geq |A|^{+}$. Then $\mathfrak{A}$ is (weakly) $\mathfrak{m}$ - atomic compact if and only if every finitely satisfiable system of atomic formulas with (without) constants in A is satisfiable in $\mathfrak{A}$ provided it involves $\leq \mathfrak{m}$ variables.

proof: (1) is clear. (2) can, e.g., be found in (14) where the result is claimed for $\mathfrak{m} \geq |A|$. However, the idea of the proof suggested there only works for $\mathfrak{m} \geq |A|^{+}$ as the weakly atomic compact case clearly demonstrates. We give that proof for $\mathfrak{m} \geq |A|^{+} \aleph_0$ which fully suffices for our needs:

Let $\Sigma = \Sigma(x_0, x_1, x_2, \ldots\ldots\ldots\ldots, x_\gamma, \ldots\ldots\ldots)_{\gamma<\mu}$ be a system of atomic formulas involving the variables $x_\gamma, \gamma<\mu$, (with or without constants in A) where μ is the initial ordinal of $\mathfrak{m}$.

If $\sigma = \sigma(x_{\nu_1}, \ldots\ldots\ldots, x_{\nu_m})$ is a formula in Σ then the set of elements in A^μ that satisfy σ in $\mathfrak{A}$, in short: the solution set Sol(σ) of σ, is of the form $B \times A^{\mu - \{\nu_1, \ldots, \nu_m\}}$ where $B \subseteq A^m$ (one copy of A for each ν_i). There are at the most $2^{|A|} \cdot |A| \leq \mathfrak{m}$ such solution sets and, hence, there is a subsystem $\Sigma_1 \subseteq \Sigma$ involving only $\leq \mathfrak{m}$ formulas such that the solution sets of Σ and of Σ_1 coincide. The proof is complete.

A final word: G. Grätzer has pointed out to me that the concept of $(S, \mathfrak{A}, \mathfrak{m})$ - M compactness does not take into account as parameter the number of variables that may be used. This, in general, is admittedly so; but I feel that a concept extended in that direction would no more guarantee the simplicity and generality in the formulation of the fundamental results to be derived in the next section. Besides, we will encounter no inconvenience in describing the few results that, to this day, involve the number of variables in the systems of formulas to be used. Finally, the last remark shows that, in studying properties of a relational system $\mathfrak{A}$ ($|A| \geq \aleph_0$) that result from the requirement that systems of atomic formulas be satisfiable provided they are finitely satisfiable <u>and</u> involve only $\mathfrak{m}$ variables, we do not go beyond our terminology if $\mathfrak{m} \geq |A|^+$. It might be an interesting problem to study other instances of that phenomenon (e.g., in specific classes of relational systems or universal algebras).

§ 3 : A Fundamental Characterization Theorem and its Consequences.

In (44) B. Weglorz proved a basic result characterizing atomic compactness of a relational system in various ways. In the same paper and in (45) he finds himself exposed to rather similar situations involving different compactness concepts and is confronted with the task of stating a new modified characterization theorem in each case. In accordance with our stated aim (see also the author's paper (50)) we give these characterization theorems a unified setting in the following fundamental theorem:

Theorem 3.1: Let S be a subset of the carrier set A of the relational system $\mathfrak{A}$. Then the following statements are equivalent for a relational system $\mathcal{B}$ of the same type:

(1) $\mathcal{B} \in c(S, \mathfrak{A})$, i.e. $\mathcal{B}$ is $(S, \mathfrak{A})$ - atomic compact.

(2) $\mathcal{B}$ contains an S-retract of every $\mathcal{L}$ such that $\mathcal{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$.

(3) $\mathcal{B}$ contains an S-retract of every S-pure extension of $\mathfrak{A}$.

(4) $\mathcal{B}$ contains an S-retract of every pure (i.e. A-pure) extension of $\mathfrak{A}$.

(5) $\mathcal{B}$ contains an S-retract of every elementary extension of $\mathfrak{A}$.

(6) $\mathcal{B}$ contains an S-retract of every ultrapower of $\mathfrak{A}$.

proof: (1) implies (2): If $\mathcal{B} \in c(S, \mathfrak{A})$ and $\mathcal{L} \overset{S}{\rightsquigarrow} \mathfrak{A}$ then $\mathrm{spec}_S(\mathcal{L})$ is finitely satisfiable in $\mathfrak{A}$, hence completely satisfiable in $\mathcal{B}$. By theorem 1.4 , $\mathcal{B}$ contains an S-retract of $\mathcal{L}$.

(2) implies (3), (3) implies (4), (4) implies (5), (5) implies (6)

since
an ultrapower is an elementary extension, an elementary extension is a pure extension, a pure extension is an S-pure extension and for an S-pure extension $\mathcal{L}$ of $\mathfrak{A}$ obviously $\mathcal{L} \xrightarrow{S} \mathfrak{A}$ holds.

(6) implies (1) : By 1.1 , if Σ is a system of atomic formulas with constants in S which is finitely satisfiable in $\mathfrak{A}$ then there is an ultrapower $\mathfrak{A}^I_D$ of $\mathfrak{A}$ in which Σ is satisfiable. An S-retraction will, of course, preserve the solutions of Σ . This proves the theorem.

The theorem immediately yields as corollaries the cited theorems of B. Weglorz (the meat of whose proofs is, of course, retained in the above proof).

Corollary 3.2. ((44)): Let $\mathfrak{A}$ be a relational system then the following conditions are equivalent:

(1) $\mathfrak{A}$ is positively compact.

(2) $\mathfrak{A}$ is atomic compact.

(3) $\mathfrak{A}$ is a retract of every pure extension.

(4) $\mathfrak{A}$ is a retract of every elementary extension.

(5) $\mathfrak{A}$ is a retract of every ultrapower.

proof: Taking S = A and $\mathcal{L} = \mathfrak{A}$ in theorem 2.2.1 yields the equivalence of (2),(3), (4) and (5). (1) implies obviously (2) and (5) implies (1) since positive formulas are preserved if one passes to homomorphic images while the constants of A remain unchanged under A-retracts.

Corollary 3.3 ((44)): Let $\mathfrak{A}$ be a relational system then the following conditions are equivalent:

(1) $\mathfrak{A}$ is weakly atomic compact.

(2) $\mathfrak{A}$ contains a homomorphic image of every weakly pure extension.

(3) $\mathfrak{A}$ contains a homomorphic image of every pure extension.

(4) $\mathfrak{A}$ contains a homomorphic image of every elementary extension.

(5) $\mathfrak{A}$ contains a homomorphic image of every ultrapower.

proof: Let $S = \emptyset$ and $\mathfrak{A} = \mathcal{L}$ in theorem 3.1.

One might wonder for a moment why weak positive compactness does not appear in 3.3 while positive compactness appeared in 3.2. The reason is that a retract made $\mathfrak{A}$ a homomorphic image of, say, $\mathfrak{A}^I_D$, while a homomorphism only embeds a homomorphic image into $\mathfrak{A}$. If one, in addition, reflects on the meaning of universal quantifiers one is bound to stop wondering.The following explicit counterexample shows that, indeed, we could not have included the additional condition on weak positive compactness.

Example 3.4: The ring $\mathfrak{Z} = (Z;+,\cdot)$ of integers is weakly equationally compact since every system of equations without constants has the solution $(0,0,0,\ldots\ldots)$. However, $\mathfrak{Z}$ is not weakly positively compact inasmuch as (see in this context example 1 in § 0) the following system is finitely satisfiable but not satisfiable in $\mathfrak{Z}$:

$$\{(\forall x_{\omega_0+1})(x_{\omega_0}\cdot x_{\omega_0+1} = x_{\omega_0+1})\}\cup\{3x_0+x_1 = x_{\omega_0},\ x_1 = 2x_2,\ x_2 = 2x_3,\ x_3 = 2x_4,\ldots\ldots,\ x_n = 2x_{n+1},\ldots;n\in N\}.$$

Another counterexample was given by W. Taylor and G. Fuhrken in (14). As a matter of fact, W. Taylor pointed my attention to an erroneous claim of mine in (48) concerning exactly the question just dealt with.

<u>Corollary 3.5</u>((45)): Let $\mathfrak{A}$, $\mathcal{L}$ be relational systems such that $\mathfrak{A}$ is a subsystem of $\mathcal{L}$. Then the following conditions are equivalent:

(1) $\mathcal{L}$ is a quasi-compactification of $\mathfrak{A}$ (i.e. $\mathcal{L} \in c(A,\mathfrak{A})$)

(2) $\mathcal{L}$ contains an A-retract of every pure extension of $\mathfrak{A}$.

(3) $\mathcal{L}$ contains an A-retract of every elementary extension of $\mathfrak{A}$.

(4) $\mathcal{L}$ contains an A-retract of every ultrapower of $\mathfrak{A}$.

<u>proof</u>: Let S = A in theorem 3.1.

Let us conclude this series of immediate consequences with a result of B. Weglorz(44).

<u>Corollary 3.6</u>: If K is a class of relational systems of type τ closed under the formation of ultrapowers (i.e. $U_0(K) \subseteq K$) then every absolute S-retract $\mathfrak{A} \in K$ ($S \subseteq A$) is $(S,\mathfrak{A})$- atomic compact. (We recall: $\mathfrak{A}$ is an "absolute S-retract" in K if it contains an S- retract of every extension.)

Let us go one, as we believe, useful step further by adopting the next definition.

<u>Definition 3.7</u>: $\mathfrak{A} \in K(\tau)$ is called a <u>"pure injective"</u> relational system provided every homomorphism g: $\mathfrak{D} \dashrightarrow \mathfrak{A}$ can be extended to every pure extension of $\mathfrak{D}$ in $K(\tau)$ that is within the equational class generated by $\mathfrak{A}$.

The last corollary (obviously weakly stated) suggests an immediate strengthening which we find useful to record, since we have not yet seen it in print. Warfield (47) attributes the result to Weglorz(44). That claim, however, is not correct.

Theorem 3.8: A relational system $\mathfrak{A}$ of type τ is atomic compact if and only if it is pure injective.

proof: Obviously, if $\mathfrak{A}$ is pure injective then it is atomic compact since every ultrapower of $\mathfrak{A}$ is a pure extension within the equational class generated by $\mathfrak{A}$. So let, conversely, $\mathfrak{A}$ be atomic compact, assume $\mathfrak{D}$ to be a pure subsystem of $\mathfrak{L}$ ($\mathfrak{D}, \mathfrak{L} \in HSP(\mathfrak{A})$) and let $g: \mathfrak{D} \dashrightarrow \mathfrak{A}$ be a homomorphism:

$$\begin{tikzcd} \mathfrak{A} & \\ \mathfrak{D} \arrow[u, "g"] \arrow[r, phantom, "\subseteq"] \arrow[r, squiggly, bend right, "D"'] & \mathfrak{L} \arrow[ul, dashed, "g'"'] \end{tikzcd}$$

Then $\mathrm{spec}_D(\mathfrak{L})$ is finitely satisfiable in $\mathfrak{D}$, hence $\mathrm{spec}_{Dg}(\mathfrak{L})$ is finitely satisfiable in $\mathfrak{A}$ and therefore satisfiable in $\mathfrak{A}$ (Here $\mathrm{spec}_{Dg}(\mathfrak{L})$ is the system of formulas obtained from $\mathrm{spec}_D(\mathfrak{L})$ by replacing d by dg for all $d \in D$). We define $g': \mathfrak{L} \dashrightarrow \mathfrak{A}$ by $cg' = cg$ for $c \in D$ and $cg' = a_c$ if $c \in C \setminus D$ and $(a_c)_{c \in C \setminus D}$ is a sequence in A satisfying $\mathrm{spec}_{Dg}(\mathfrak{L})$. Then g' is a homomorphism extending g.

We would like to end this section with a theorem that restricts the size of the ultrapowers to be used in 3.2 for testing a relational system with respect to atomic compactness or a universal algebra with respect to equational compactness. To this end, however, we need general criteria that interrelate these concepts with $|A|$ .... criteria that we will derive in the next but one section. To derive them we prefer to first discuss some results on weakly atomic compact relational systems of general interest.

§ 4 . Connections between Weak Equational Compactness and Equational Compactness in Universal Algebras.

Lemma 4.1: (1)If $\mathfrak{A} \subseteq \mathfrak{L}$ are given relational systems and $g: \mathfrak{L} \dashrightarrow \mathfrak{A}$ is an S-retraction then the (S,X)-atomic compactness of any $X \in \{\mathfrak{A}, \mathfrak{L}, \mathfrak{L}_g\}$ implies the (S,Y)-atomic compactness for the remaining two Y's.

(2) If $S \subseteq A, B$ and $\mathfrak{L} \in c(S, \mathfrak{L})$ then $\mathfrak{A} \overset{S}{\rightsquigarrow} \mathfrak{L}$ holds if and only if there is an S-retraction $g: \mathfrak{A} \dashrightarrow \mathfrak{L}$.

(3) If $S \subseteq A_i, i=1,2,$ $\mathfrak{A}_i \in c(S, \mathfrak{A}_i)$ then $\mathfrak{A}_1 \overset{S}{\leftrightsquigarrow} \mathfrak{A}_2$ holds if and only if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are S-coretractive.

proof (see (45) and (50)): Let us just hint at the proof of (1):
If, e.g., $\mathfrak{L}g$ is $(S, \mathfrak{L}g)$ - atomic compact then so is $\mathfrak{L}$: For take a system Σ of atomic formulas with constants in S which is finitely satisfiable in $\mathfrak{L}$. Then (in view of the existence of an S-retraction) it is finitely satisfiable in $\mathfrak{L}g$, hence (since $\mathfrak{L}g \subseteq \mathfrak{L}$) in $\mathfrak{L}$. The remaining cases are handled similarly.

Corollary 4.2 : If a relational system $\mathfrak{A}$ has a 1-element subsystem then $\mathfrak{A}$ is weakly atomic compact.

Many classical algebras have 1-element subalgebras and are therefore weakly equationally compact; e.g., lattices, semilattices, groups and rings. If one is after any interesting information concerning weak atomic compactness one finds oneself bound, therefore, to occasionally introduce constants as nullary operations in order to get away from 1-element subalgebras. The following ideas were developed independently by W. Taylor (41) and the author (50) for different reasons. While W. Taylor was after a theory of minimum compact models or, as it turns out, equivalently of minimal retracts of weakly atomic compact relational systems, we attempted to solve a very specific (and still open) problem posed by B. Weglorz (45), namely whether or not the existence of $\mathcal{L} \in c(A, \mathfrak{A})$, $\mathcal{L} \geq \mathfrak{A}$, implies the existence of $\mathcal{L} \in c(C, \mathcal{L})$, $\mathcal{L} \supseteq \mathfrak{A}$, i.e. whether or not a relational system has a quasi-compactification if and only if it has an atomic compactification. The latter question being referred to § 8 on compactifications (where we will give an affirmative answer to a weaker version of that problem) we will here use our methods to derive a relationship between weak atomic compactness and atomic compactness in simple universal algebras and the algebraic setting of an interesting theorem due to W. Taylor whose approach, however, is rather model-theoretic in nature. To this end, let us define with W. Taylor what one understands by a "minimum compact model" (41).

<u>Definition 4.3</u>: A universal algebra $\mathfrak{A}$ is <u>minimum compact</u> if <u>(i)</u> $\mathfrak{A}$ is weakly equationally compact, <u>(ii)</u> whenever $g: \mathfrak{A} \dashrightarrow \mathcal{L}$ is a homomorphism where $\mathcal{L}$ is an algebra mutually $\emptyset$ - pure with $\mathfrak{A}$ then g is a monomorphism onto a subalgebra of $\mathcal{L}$.

To be precise: W. Taylor did define the above concept for relational systems replacing "equational" by "atomic", "algebra" by "relational system" and "subalgebra" by "subsystem". In remark 1.6 of (41) he asks whether or not in case of an algebra $\mathfrak{A}$ it is possible to replace the requirement that $\mathcal{L}$ be a relational system by letting $\mathcal{L}$ range over algebras only. Since the answer, in view of the crucial (crucial, at least, for our approach) next theorem (see also (50)), turns out to be affirmative, we have not altered Taylor's original definition with 4.3.

Theorem 4.4(50): Let $\mathfrak{A}$ be a weakly equationally compact universal algebra all of whose endomorphisms are monic. Then all endomorphisms are already automorphisms. (The same, for that matter, holds for relational systems.)

proof (We include a full proof since we will use the theorem in at least two essential instances): Assume that there exists a proper monomorphism $g: \mathfrak{A} \dashrightarrow \mathfrak{A}$ with $Ag \subsetneq A$. Then we can, in the natural fashion, build up an increasing chain of mutually isomorphic algebras:

$$(*) \quad \mathfrak{A} = \mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \mathfrak{A}_3 \subsetneq \cdots\cdots\cdots \mathfrak{A}_n \subsetneq \cdots\cdots$$

Clearly, $\mathfrak{A}_{\omega_0} = \bigcup(\mathfrak{A}_n;\ n = 0,1,2,\ldots\ldots)$ is a weakly pure extension of $\mathfrak{A}$. Hence, by lemma 2.3.1, there is a homomorphism $r: \mathfrak{A}_{\omega_0} \dashrightarrow \mathfrak{A}$ and $\mathfrak{A}_{\omega_0} \xrightarrow{\emptyset} \mathfrak{A}$.

If p was a non-monic endomorphism of $\mathfrak{A}_{\omega_0}$ then we would have the non-monic homomorphism $p_1 = p|_{A_n}$ for suitable $n \in N$. Then $p_1 \circ r: \mathfrak{A}_n \dashrightarrow \mathfrak{A}_n$ was a non-monic endomorphism of $\mathfrak{A}_n$ which is a contradiction to $\mathfrak{A}_n \cong \mathfrak{A}_0 = \mathfrak{A}$.

So $\mathfrak{A}_{\omega_0}$ has the same properties that $\mathfrak{A}$ had (i.e. it is weakly equationally compact and all endomorphisms are monic) and, like all $\mathfrak{A}_n$, $n < \omega_0$, it possesses a monomorphism into $\mathfrak{A}_0 = \mathfrak{A}$. So we can now build a chain as in (*) beginning with $\mathfrak{A}$ but including $\mathfrak{A}_{\omega_0}$ and going on with $\mathfrak{A}_{\omega_0+1}$ etc:

$$\mathfrak{A} = \mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \mathfrak{A}_3 \subsetneq \;\ldots\ldots \subsetneq \mathfrak{A}_n \subsetneq \ldots \subsetneq \mathfrak{A}_{\omega_0} \subsetneq \mathfrak{A}_{\omega_0+1} \subsetneq \ldots\ldots$$

In short: we have all the ingredients to build, by transfinite induction, an increasing chain of arbitrary pregiven length:

$$\mathfrak{A} = \mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \;\ldots\ldots \subsetneq \mathfrak{A}_{\omega_0} \subsetneq \mathfrak{A}_{\omega_0+1} \subsetneq \ldots\ldots \subsetneq \mathfrak{A}_\gamma \subsetneq \ldots\ldots \quad , \gamma < \alpha ,$$

such that each $\mathfrak{A}_\gamma$ is weakly equationally compact, has only monic endomorphisms and (this is what matters to conclude the proof) possesses a monomorphism into $\mathfrak{A}_0 = \mathfrak{A}$. The last property, of course, yields an immediate cardinality contradiction since $\mathfrak{A}$ cannot contain isomorphic copies of each member in this chain.

As a corollary we obtain an interesting theorem(see (42) or (50))

Corollary 4.5: A simple algebra (or relational system) without 1-element subalgebras is weakly equationally compact if and only if it is equationally compact.

proof: Let $\mathfrak{A}$ be weakly equationally compact. Then it contains a homomorphic image of every ultrapower $\mathfrak{A}^I_D$. If $g: \mathfrak{A}^I_D \dashrightarrow \mathfrak{A}$ is such a homomorphism then it is a monomorphism on $\mathfrak{A}$. By 4.4, $g|_A$ is an automorphism and, hence, $\mathfrak{A}$ is a retract of $\mathfrak{A}^I_D$. This settles the claim in view of 3.2.

The above result can also be used to give a short algebraic proof for a basic result of W. Taylor. We give this new proof, while we refer to (41) or (42) for Taylor's original proof.

Theorem 4.6 (W. Taylor (41) and (42)):

If $\mathcal{L}$ is weakly equationally compact then there exists a unique (up to isomorphism) minimum compact algebra $\mathfrak{A}$ which is mutually $\emptyset$ - pure with $\mathcal{L}$.

Every such $\mathfrak{A}$ has, in addition, the following three properties:

(1) $\mathfrak{A}$ is equationally compact.

(2) $\mathfrak{A}$ is retract of every weakly equationally compact algebra $\mathcal{L}$ which is mutually $\emptyset$-pure with $\mathfrak{A}$ (hence with $\mathcal{L}$).

(3) Every endomorphism of $\mathfrak{A}$ is an automorphism.

proof: Let $\mathcal{F} = \{\Theta ; \Theta \in C(\mathcal{L}) ; \mathcal{L}/\Theta \overset{\emptyset}{\rightsquigarrow} \mathcal{L}\}$. Clearly, $\mathcal{F}$ has a maximal element Θ_0 and therefore $\mathfrak{A} = \mathcal{L}/\Theta_0 \overset{\emptyset}{\rightsquigarrow} \mathcal{L}$.

Since $\mathcal{L}$ is weakly equationally compact, there is a homomorphism g: $\mathcal{L}/\Theta_0 \dashrightarrow \mathcal{L}$ which is a monomorphism (otherwise g', the congruence relation on $\mathcal{L}$ which is induced by the kernel of g, would be in $\mathcal{F}$ and g' $\gneq \Theta_0$).

Thus, $\mathcal{L}/\Theta_0 \subseteq \mathcal{L}$, $\mathcal{L}/\Theta_0$ is weakly equationally compact and every endomorphism of $\mathcal{L}/\Theta_0$ is a monomorphism, i.e., by 4.4, an automorphism. This implies, as in the proof of 4.5, that $\mathcal{L}/\Theta_0$ is minimum compact and equationally compact (minimum compact since every homomorphism into a mutually $\emptyset$-pure algebra needs to be a monomorphism by construction). (2) is also clear since $\mathfrak{A} = \mathcal{L}/\Theta_0$ contains a homomorphic image of $\mathcal{L}$. However, due to weak equational compactness of $\mathcal{L}$ we can embed $\mathfrak{A}$ in $\mathcal{L}$ via a monomorphism. Since the restriction of the homomorphism from $\mathcal{L}$ to $\mathfrak{A}$ to $\mathfrak{A}$ is again a monomorphism, hence an automorphism we are done. (3) and the uniqueness up to isomorphism are also clear.

Appendix: (1) In 4.6 we did not need to assume that $\mathcal{I}$ be weakly equationally compact if we, instead, assumed $\mathcal{I} \supseteq \mathfrak{A}$.

(2) W. Taylor did, indeed, show a little more, namely that $|A| \leq 2^{m}$ where m is the power of the first order language of $\mathcal{L}$, i.e. $m = \aleph_0 + |R| + |F|$. We have not derived that result. However, for our present interests, it seems quite sufficient to know that $|B|$ is an upper bound for $|A|$ (particularly since, e.g., for groups, rings, lattices etc. the cardinality of A in the last theorem is exactly 1).

(3) The minimum compact algebras $\mathfrak{A}$ within $\mathcal{L}$ in the last theorem are exactly the minimal retracts of $\mathcal{L}$. So the theorem contains the interesting information that all weakly equationally compact algebras do have such minimal retracts.

(4) The theorem can be proved for relational systems (as Taylor indeed did).

(5) Assume we would, in definition 4.3, allow that $\mathcal{L}$ ranges over all relational systems of the associated type. Then, using (4), we know that such a stronger minimum compact algebra is within any given one of ours as retract. However the only possible retracts are the identity and we are done with showing that the two concepts coincide. W. Taylor had realized that in (42).

(6) The paper from which we took the above theorem contains much new material that we want to recommend ((42)). Much of it is aimed at relational systems. We will, of course, not report on it, since it is a paper whose preprint is hardly out. We made an exception with theorem 4.6 since it served well to illustrate our method of 4.4 and, thus, enabled us to give a different proof as in (41).

§ 5. Characterization Theorems for (Weak) Atomic Compactness of a Relational System $\mathfrak{A}$ Involving $|A|$.

The two main results in this section are due to J. Mycielski (32) and W. Taylor & G. Fuhrken (14), respectively. For the second theorem, however, we give a new proof which allows us to bypass a few model-theoretic prerequisites that are implicit in the proof of Taylor and Fuhrken.
Before doing so we derive a cute and useful auxiliary theorem of J. Mycielski and C. Ryll-Nardzewski, thus following the path of (32).

Theorem 5.1: If $\mathfrak{A}$ is a relational system and $\mathfrak{m}$ is an infinite cardinal number then the following conditions are equivalent:

(1) $\mathfrak{A}$ is $\mathfrak{m}$ - atomic compact

(2) Let $\Sigma \subseteq L_\tau(A)$, $|\Sigma| \leq \mathfrak{m}$ be such that all formulas in Σ have exactly one and the same free variable x_0 and are of the form $\Psi_j = (\exists x_1)\ldots\ldots(\exists x_{m_j})(\Phi_1^j \wedge \ldots\ldots \wedge \Phi_{n_j}^j)$ where $m_j, n_j \in N$ and the Φ_i^j are atomic formulas with constants in A.
If Σ is finitely satisfiable in $\mathfrak{A}$ then Σ is satisfiable in $\mathfrak{A}$.

The proof goes along the following lines (details see (32)):
(1) implies (2): If (1) holds then we switch from Σ to a system Σ' of atomic formulas with variables in A by replacing the bound variables in each Φ_i^j by new ones such that none of these new variables appears simultaneously in Φ_i^j and Φ_l^k unless $j = k$. If we denote the new formulas corresponding to Φ_i^j by $\Phi_i^{j}{}'$ then $\Sigma' = \{\Phi_i^{j}{}';\ \Phi_i^j$ appears in the matrix of a formula in $\Sigma\}$ is a system of atomic formulas finitely satisfiable

in $\mathfrak{A}$. Thus, the proof is accomplished.

(2) implies (1): If Σ is a system of atomic formulas with constants in A such that $|\Sigma| \leqq m$, Σ is finitely satisfiable in $\mathfrak{A}$ and Σ involves the variables x_γ, $\gamma < \alpha$, then the authors construct, by transfinite induction on $\tau \leqq \alpha$, elements $a_\gamma(\tau)$ ($\gamma < \tau$) such that (1) the system Σ_τ that results from replacing x_γ by $a_\gamma(\tau)$, $\gamma<\tau$, is still finitely satisfiable in $\mathfrak{A}$ and (2) $\tau_1 \leqq \tau_2$ implies $a_\gamma(\tau_1) = a_\gamma(\tau_2)$ for $\gamma < \tau_1$. ~~If we succeed to do so then~~ The case $\tau = \alpha$ will yield a finitely satisfiable system Σ_α in which there are no more variables left: thus, $(a_\gamma(\alpha))_{\gamma<\alpha}$ is a solution of Σ.

Remark 5.2: In theorem 5.1 we can drop the cardinal number m in both conditions.

Theorem 5.3: For every relational system $\mathfrak{A}$ the following conditions are equivalent:

(1) $\mathfrak{A}$ is atomic compact.

(2) $\mathfrak{A}$ is $|A|$ - atomic compact.

proof: Clearly, (1) implies (2). To show the converse take a system Σ of formulas as in theorem 5.1. (2) without $|\Sigma| \leqq m$, not satisfiable in $\mathfrak{A}$. This means that for every $a_0 \in A$ there exists $\sigma(a_0) \in \Sigma$ such that $\sigma(a_0)$ is not satisfiable after substitution of x_0 by a_0. Then $\Sigma(A) = \{\sigma(a_0); a_0 \in A\}$ is a subsystem of Σ such that $|\Sigma(A)| \leqq |A|$ and $\Sigma(A)$ is not satisfiable in $\mathfrak{A}$. In view of 5.1 we obtain that either Σ is not finitely satisfiable or Σ is satisfiable in $\mathfrak{A}$. In short, $\mathfrak{A}$ is atomic compact.

While reflecting upon the theorem one may recall example 3 in §0 in which we showed that the lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ with $L = \{a_0, a_1, \ldots\ldots, a_n, \ldots\ldots\} \cup \{0,1\}$, $a_i \vee a_j = 1$, $a_i \wedge a_j = 0$ for $i \neq j$ and the nullary operations 0,1 is not weakly equationally compact. We did this by displaying a class (not even a set) of suitable systems of equations whose very characteristic was the fact that it contained systems of equations with an arbitrary high number of equations. By theorem 5.3 , however, we ought to be able to show the non-equational compactness of $\mathcal{L}$ by displaying a suitable system of equations with constants in L of cardinality $\aleph_0$. And, indeed, the system $\Sigma = \{a_i \vee x = 1 \; ; \; i = 0,1,\ldots\} \cup \{a_i \wedge x = 0; \; i = 0,1,\ldots\ldots\}$ is finitely solvable but not solvable.

However, while our systems of equations in example 3 of $\S$ 0 only involved the constants 0,1, Σ involves, in addition, all the a_i , $i = 0,1,2,\ldots\ldots$ So Σ no more proves the fact that $\mathcal{L}$ is not weakly equationally compact. And this is not surprising in view of the fact that $\mathcal{L}$ is indeed $\aleph_0$ - weakly equationally compact as J. Mycielski mentioned in (31) without proof. However, an inductive proof can be easily devised as in 5.1.

Example 5.4: The lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ with the two nullary operations 0,1 that is defined by $L = \{a_0, a_1, \ldots, a_n, \ldots\} \cup \{0,1\}$ and $a_i \vee a_j = 1$, $a_i \wedge a_j = 0$ for $i \neq j$ is not weakly equationally compact, but it is $\aleph_0$ - weakly equationally compact.

In the last example we got acquainted with an $\aleph_0$ - weakly equationally compact universal algebra that was not weakly equationally compact. This,therefore, makes an immediate generalization of theorem 5.3 to the weakly atomic case impossible. The best one can hope for is then a theorem that claims that weak atomic compactness is implied by $|A|^+$-weak atomic compactness in case of a relational system $\mathfrak{A}$. This result was conjectured by J. Mycielski in (31) and proved for the first time by W. Taylor and G. Fuhrken in (14). As mentioned before, we have found a <u>new</u> proof that we want to include in this report.

<u>Theorem 5.5</u>: For every relational system $\mathfrak{A}$ the following conditions are equivalent:

(1) $\mathfrak{A}$ is weakly atomic compact.

(2) $\mathfrak{A}$ is $|A|^+$- weakly atomic compact.

Before we proceed to the proof we use Mycielski's last theorem in order to cut down the number of ultrapowers we have to use in order to test equational or atomic compactness. With this we do what we desired to do at the end of section 2. The only ultrapower that really matters enters in the proof of 1.1 and is studied in (13). We define it in our context:

Definition 5.6: If $\mathfrak{A}$ is a relational system and $\mathfrak{m}$ an infinite cardinal number then the ultrapower $\underline{\mathfrak{A}(\mathfrak{m})}$ of $\mathfrak{A}$ is defined as follows:

Let J be a set of cardinality $\mathfrak{m}$ and let $I = 2^{(J)}$ be the ideal of all finite subsets of J. For arbitrary $x \in J$ define $I_x = \{T;\ T \in I,\ x \in T\}$. Then $\{I_x; x \in J\}$ generates a filter D_1 of subsets of I which has the finite intersection property. If D is an arbitrary ultrafilter of subsets of I containing D_1 (and such exist, as we know!) then $\mathfrak{A}(\mathfrak{m})$ is defined to be $\mathfrak{A}^I_D$. (Obviously, $\mathfrak{A}(\mathfrak{m})$ depends on the choice of D. However, for our needs it fully suffices if we fix a D for every $\mathfrak{m}$, as we agree to have done.)

Theorem 5.7: Let $\mathfrak{A}, \mathfrak{B}$ be relational systems, $S \subseteq A,B$: If $\mathfrak{B}$ contains an S-retract of the ultrapower $\mathcal{L}(\mathfrak{m})$ for some infinite cardinal number $\mathfrak{m}$ and some extension $\mathcal{L}$ of $\mathfrak{A}$ then $\mathfrak{B}$ is $(S, \mathfrak{A}, \mathfrak{m})$- atomic compact.

proof: If Σ is a system of atomic formulas with constants in S such that $|\Sigma| \leq \mathfrak{m}$ and Σ is finitely satisfiable in $\mathfrak{A}$ then let $J = \Sigma$ in definition 5.6 of $\mathcal{L}(\mathfrak{m})$. The remainder follows as in the proof of 1.1. (See also (13), (32) and (44)).

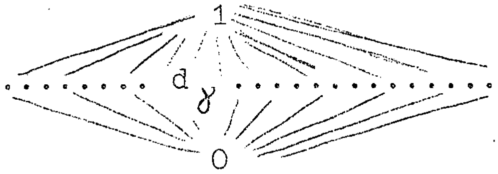
Corollary 5.8: The relational system $\mathfrak{A}$ is atomic compact if and only if $\mathfrak{A}$ is retract of $\mathfrak{A}(|A|)$.

proof: This follows in view of 5.3 and 5.7.

The "two ways statement" in 5.8, in connection with 5.7, has to be considered with caution. To illustrate this point let us look at the next example.

Example 5.9: In example 5.4 we dealt with a lattice $\mathcal{L}$ that was $\aleph_0$ - weakly equationally compact. Does this imply (in the spirit of 5.8, strengthening 5.7) that $\mathcal{L}$ contains a homomorphic image of $\mathcal{L}(|L|)$? Obviously not! For $\mathcal{L}(|L|)$ is both an elementary extension of $\mathcal{L}$ and, as we know from (13), of cardinality $\mathcal{C}^{\vee} = |\omega_1|$:

$\mathcal{L}(|L|)$:



, $\gamma < \omega_1$

Here $\gamma_1 \neq \gamma_2$ implies $d_{\gamma_1} \vee d_{\gamma_2} = 1$ and $d_{\gamma_1} \wedge d_{\gamma_2} = 0$.

A homomorphism $g: \mathcal{L}(|L|) \dashrightarrow \mathcal{L}$ must take 0 to 0, 1 to 1 and different d_γ's to different a_{ν_γ}'s ...... and an obvious cardinality contradiction arises.

We can now return to our proof of 5.5: Our proof is based on ideas of the last section which, as a matter of fact, lend themselves to a nice application of Mycielski's theorem, (For the original proof of Fuhrken and Taylor see (14).) 5.3. So let $\mathfrak{A}$ be an $|A|^+$- weakly atomic compact system. $\mathcal{F} = \{\Theta ; \Theta \in C(\mathfrak{A}); \mathfrak{A}/\Theta \overset{\emptyset}{\leftrightsquigarrow} \mathfrak{A}\}$ has, by Zorn's lemma, a maximal element Θ_0. Then $\mathfrak{A}/\Theta_0 \overset{\emptyset}{\leftrightsquigarrow} \mathfrak{A}$ and every homomorphism $g: \mathfrak{A}/\Theta_0 \dashrightarrow \mathfrak{A}$ is monic. Since $\mathrm{spec}_\emptyset(\mathfrak{A}/\Theta_0)$ has exactly $|A/\Theta_0| \leq |A|$ variables we can (by assumption on $\mathfrak{A}$ and remark 2.8) satisfy $\mathrm{spec}_\emptyset(\mathfrak{A}/\Theta_0)$ in $\mathfrak{A}$ and have, therefore, an embedding $g: \mathfrak{A}/\Theta_0 \dashrightarrow \mathfrak{A}$. If $\mathfrak{A}/\Theta_0 g$ is not yet a subsystem of $\mathfrak{A}$ then the reason for this can only be that R acts a little poorly on $\mathfrak{A}/\Theta_0$. So we correct this by agreeing that $\underline{\mathcal{L}}$ is the enriched $\mathfrak{A}/\Theta_0 g$, i.e.

we define the effect of the relations $r_\delta \in R$ on $A/\Theta_0 g$ by $r_\delta^{\mathcal{L}} = (A/\Theta_0 g)^{m_\delta} \cap r_\delta^{\mathfrak{A}}$ such as to make it indeed a subsystem. This leads at the worst (or best ?) to an enrichment of $\mathfrak{A}/\Theta_0 g$ by a few valid relationships.

As in theorem 4.4 we prove that $\mathcal{L}$ has only endomorphisms that are already automorphisms (the crucial argument there goes still through, since we can stop at $\mathcal{L}_\gamma$ with $|C_\gamma| = |A|^+$ and get a monomorphism $r: \mathcal{L}_\gamma \dashrightarrow \mathfrak{A}$ due to the fact that spec($\mathcal{L}_\gamma$) involves $|A|^+$ variables and is therefore satisfiable in $\mathfrak{A}$).

If we can find a retraction $h: \mathcal{L}(|C|) \dashrightarrow \mathcal{L}$ then, by corollary 5.8, $\mathcal{L}$ is atomic compact and, hence, by lemma 4.1.(1), $\mathfrak{A}$ is weakly atomic compact and we are through. However, spec($\mathcal{L}(|C|)$) is a system of atomic formulas with $|C|^{|C|}$ variables and is, hence, satisfiable in $\mathfrak{A}$ since $|C|^{|C|} \leq |A|^+$. Thus, we have a homomorphism $h': \mathcal{L}(|C|) \dashrightarrow \mathfrak{A}$. If p is the canonical projection from $\mathfrak{A}$ onto $\mathfrak{A}/\Theta_0$ then $h' \circ p : \mathcal{L}(|C|) \dashrightarrow \mathcal{L}$ is a homomorphism which, restricted to $\mathcal{L}$ is an automorphism. Thus, there is a retraction $h: \mathcal{L}(|C|) \dashrightarrow \mathcal{L}$, and we are through.

§ 6 . Connections To Topology.

Although the use of the word "compactness" in the preceding definitions is sufficiently justified by the very definitions, there is a much stronger reason coming from topology as was observed by J. Mycielski in (31).

Let us recall that in our dealings with algebraic structures such as semigroups or groups or rings or modules we are not just interested in arbitrary topologies that we might be able to inflict on the carrier sets of those algebras. In general, we want the fundamental operations f_γ to be continuous functions $f_\gamma : A^{n_\gamma} \dashrightarrow A$ if A^{n_γ} is endowed with the Tychonoff product topology and we want to be able to have "small enough" neighborhoods to separate points, i.e. we insist on Hausdorff topologies. This is summed up in the next definition.

Definition 6.1: If $\mathfrak{A} \in K(\tau)$ is an algebra then $(\mathfrak{A}, \mathcal{T})$ shortly $\mathfrak{A}$, is a topological algebra of type τ if $\mathcal{T}$ is a Hausdorff topology on A which makes the operations continuous. $(\mathfrak{A}, \mathcal{T})$ is a topologically compact algebra if the topology is compact, i.e. every cover of A with open subsets of A has a finite subcover.

If $\mathfrak{A} = (A;F,R)$ is a relational system then we call $(\mathfrak{A}, \mathcal{T})$ a topological relational system if $((A;F), \mathcal{T})$ is a topological algebra and the relations induced by R are closed subsets of the respective powers of A.

This is all we need to derive Mycielski's fundamental observation.

<u>Theorem 6.2</u> : Every retract $\mathfrak{A}$ of a topologically compact universal algebra is equationally compact.

<u>proof</u>: Since retracts of equationally compact algebras are evidently again equationally compact, we may assume that $\mathfrak{A}$ itself is topologically compact.

Let Σ be a non- empty set of polynomial equations with constants in A which is finitely solvable. We well-order the finite subsystems: $\Sigma_0, \Sigma_1, \ldots, \Sigma_\gamma, \ldots, \gamma<\beta$, of Σ and denote their (non-empty) solution sets in A^α by $S_0, S_1, \ldots, S_\gamma, \ldots$ respectively. Since A is endowed with a compact topology, Tychonoff's theorem assures that A^α carries a compact topology. Since $S_\gamma = S_\gamma^1 \cap \ldots \cap S_\gamma^{t(\gamma)}$ where S_γ^ν, $\nu = 1, \ldots, t(\gamma)$, are the solution sets of the different single equations in Σ_γ and since the sets S_γ^ν are closed, we conclude that each S_γ is a closed set and the family $\{S_\gamma; \gamma<\beta\}$ has obviously the finite intersection property. Thus, since A^α is compact, $\bigcap(S_\gamma; \gamma<\beta) \neq \emptyset$. Since the latter set is the solution set of the whole system Σ we are done.

In exactly the same way one can, of course, prove the same result for relational systems. However, we will care less and less since, unfortunately, it is not true that an algebra $\mathfrak{A}$ is topological if and only if the associated relational system $\mathfrak{A}_{rel}$ is. Thus, one has to be very careful every time one is passing from the one to the other and tries to carry specific results along. However, we mention the more general form of this fundamental observation in a separate remark:

Remark 6.3: Theorem 6.2 remains valid if we replace the universal algebra $\mathfrak{A}$ by a relational system.

Or, even more generally:

Remark 6.4: Every S-retract of a topologically compact relational system $\mathfrak{A}$ that is a subsystem of $\mathfrak{A}$ is $(S,\mathfrak{A})$- atomic compact.

A strong impetus for investigations concerning the interplay between the algebraic properties and the topological behaviour in case of atomic compact relational systems, respectively equationally compact universal algebras, arose from J. Mycielski's question whether the converse of theorem 6.2 (respectively, remark 6.3) holds true. For a long time the answer turned out to be positive in specific classes of algebras: Unary algebras (A;f), Boolean algebras, Abelian groups, vectorspaces or, more generally, R-modules, commutative Noetherian rings with 1 etc. It was W. Taylor who via an (as it strikes us) very cunning counterexample defeated a positive answer to Mycielski's question in the general setting of universal algebras (see (39)). We will present this counterexample in the next section. It remains, however, of independent interest to find an answer to Mycielski's question in any given equational class of algebras whose equationally compact members are accessible to characterization. We will refer to this problem whenever it arises as Mycielski's problem.

Definition 6.5: Given a class K of universal algebras of a given type τ, Mycielski's problem consists of the task to decide whether or not every equationally compact algebra in K is retract of a topologically compact algebra in K (or, at least in $K(\tau)$).

For the time being, there is no fully satisfactory connection found between topological compactness and equational compactness of universal algebras. Possibly algebras that are limit spaces could provide a link of some (possibly weaker) sort. However, a systematic investigation of such "convergence algebras" is still to be carried out. W. Taylor's first attempt in that direction is quite dominated by his interest in relational structures (see (40)) as one of his results clearly demonstrates In attempting to answer Mycielski's problem for the class of semilattices (in which the equationally compact ones were characterized by G. Grätzer and H. Lakser) his methods only allow him to conclude that, if $\mathcal{L}$ is an equationally compact semilattice, then $\mathcal{L}_{rel}$ is indeed retract of a topologically compact relational system. And from there it seems to be a long way to a final decision of Mycielski's problem for semilattices .

The fact that, to this day, there is no fully satisfactory interplay between equational compactness and topological compactness in the general setting of universal algebras is not altered by the fact that W. Taylor (39) turned theorem 5.1, respectively the succeeding remark, into a topological language. Let us look at that description:

<u>Remark 6.6:</u> Let $\mathfrak{A}$ be a relational system and let $\underline{E}$ be the class of formulas of the type $\Phi(x_0)(\exists x_1)\ldots(\exists x_m)(\Phi_1 \wedge \ldots \wedge \Phi_n)$ $m,n \in N$, where all Φ_i are atomic formulas with constants in A and every formula Φ_i has exactly the one free variable x_0.
Then may $\mathcal{T}$ denote the topology on A which has the subbase of closed sets $\{\{x \in A; \mathfrak{A} \Rightarrow \Phi(x)\}; \Phi \in E\}$.
Then $\mathfrak{A}$ is atomic compact if and only if $\mathcal{T}$ is a compact topology.

The proof, of course, consists of re-reading 5.1 or 5.2.
The definite shortcoming (for algebraic purposes), however, is the fact that $\mathcal{T}$ is neither Hausdorff nor does it necessarily make the fundamental operations continuous if we are dealing with algebras.

The fact that atomic compactness or equational compactness are characteristics of a relational system that allow an algebraic treatment of basically topological properties is, however, illuminated by both the frequent positive answer to Mycielski's problem , the last remark and, for that matter, by the very definition of the concepts "atomic compactness" or "equational compactness". To end this discussion with a concrete example we refer to the following extension of a lemma of Numakura due to A. Hulanicki and B. Weglorz (see (45)).

<u>Theorem 6.7:</u> Every equationally compact semigroup with cancellation law is a group.

The proof is published in detail in (45). A rough indication of the method is the following: One proves the result first for Abelian semigroups using the system $\Sigma = \{x = s\cdot x_s;\ s\in S\}$. The solvability of Σ implies the existence of 1 and inverses. For arbitrary semigroups $\mathfrak{S}$ one uses this result together with the fact that one can cover $\mathfrak{S}$ by Abelian subsemigroups to derive the result.

<u>Corollary 6.8:</u> (Lemma of Numakura): Every topologically compact semigroup with cancellation law is a group.

<u>Remark 6.9:</u> The only equationally compact skewfields are the finite fields.

To prove this one can use the result that every ultrapower $\mathfrak{K}(\mathfrak{m})$ of the skewfield $\mathfrak{K}$ is again a skewfield. Thus, all retractions g: $\mathfrak{K}(\mathfrak{m}) \dashrightarrow \mathfrak{K}$ are embeddings which yields a cardinality contradiction unless $\mathfrak{K}$ is finite. Mycielski, who observed this fact first, has given a different reason in (31): he considers the system $\{x\cdot(a-x_a) = 1;\ a\in K\}$ which is finitely solvable but not solvable in infinite fields.

An interesting situation arises in the case of equationally compact integral domains. According to 6.7 they ought to be skewfields, hence finite....... if the multiplicative semigroup of non-zero elements was equationally compact all by itself. So non-finiteness ought to give rise to some sort of a neighborhoodsystem of (0) which might lead to a usable characterization. <u>Every</u> integral domain is equationally compact with respect to systems of equations with one variable (as D. Haley observed).

§ 7 : Equational Compactness and Atomic Compactness in Specific Relational Systems.

In the various subsections of this section we will investigate our concepts in algebraic structures of special interest.

(1) Unary algebras.

The results concerning mono-unary algebras are due to this author (see (49)). The cited paper contains a characterization of equationally compact unary algebras $(A;f)$ of type (1) and an affirmative answer to Mycielski's problem in that case.

We recall: $\mathfrak{A}$ is connected if for any two $a,b \in A$ there exist $n,m \in N_0$ such that $f^n(a) = f^m(b)$. A few new concepts facilitate our task:

Definition 7.1: If $\mathfrak{A} = (A;f)$ is a mono-unary algebra then $a \in A$ is called a stagnant element if $f(a) = a$. $st(\mathfrak{A})$ is the set of all stagnant elements in A. If $n \in N$ and $a \in A$ then the n-periphery $n_{\mathfrak{A}}(a)$ is the set of all elements $b \in A$ such that $f^n(b) = a$ and $f^{n-1}(b) \neq a$. $b \in A$ is called a minimal element if it is not in the image of f.

If $a \in A$ satisfies $f^m(a) \neq a$ for all $m \in N$ and $n_{\mathfrak{A}}(a)$ contains a minimal element then a is said to have order n in $\mathfrak{A}$. Let $o_{\mathfrak{A}}(a)$ denote the set of all orders of an element a.

We add to $o_{\mathfrak{A}}(a)$ the symbol ∞ if $f^m(a) \neq a$ for all $m \in N$ and there exists an infinite sequence $a_0, a_1, \ldots\ldots, a_m, \ldots.$ such that $a = a_0$, $a_i \neq a_j$ for $i \neq j$ and $a_m = f(a_{m+1})$.

Example 7.2: The algebra $\mathcal{L}_n = (C_n;f)$ with n elements, $C_n = \{a_0, \ldots\ldots, a_{n-1}\}$, whose indices are determined modulo n and whose operation f is defined by $f(a_k) = a_{k+1}$ is called the cyclic algebra with n elements. It has no stagnant elements unless $n = 1$. None of its elements has an order (be it finite or infinite).

Example 7.3: The algebra $\mathcal{J} = (J;f)$ whose carrier set is, say, the set of integers and whose operation f is defined by $f(n) = n+1$ has neither stagnant nor minimal elements. Every element has exactly infinite order.

We are now in a position to state the characterization theorem for equationally compact unary algebras $\mathfrak{A} = (A;f)$.

Theorem 7.4: The mono-unary algebra $\mathfrak{A} = (A;f)$ is equationally compact if and only if the following conditions hold:

(1) Every element whose finite orders approach infinity is of infinite order.

(2) $\mathfrak{A}$ contains either some subalgebra $\mathcal{L}_n$ $(n \in N)$ or the subalgebra $\mathcal{J}$.

In order to attack the naturally next question, namely Mycielski's problem in the class K((1)), we recall that the Stone-Čech compactification βA of a completely regular topological Hausdorff space A is the (up to homeomorphism unique) compact Hausdorff space that contains A as a dense subspace and to which every continuous mapping from A into a compact Hausdorff space B can be continuously extended. So if $\mathfrak{A} = (A;f)$ is a unary algebra then we can endow A with the discrete topology under which f becomes a continuous mapping. If we now extend $f: A \dashrightarrow A$ to $f: \beta A \dashrightarrow \beta A$ then $\beta\mathfrak{A} = (\beta A;f)$ becomes a unary algebra.

Definition 7.5: $\beta\mathfrak{A}$, as just defined, is the Stone-Čech compactification of the mono-unary algebra $\mathfrak{A} = (A;f)$.

($\beta\mathfrak{A}$ can, of course, be defined for arbitrary, not necessarily mono-unary, unary algebras in exactly the same fashion).

Thus, surely every unary algebra can be embedded in a topologically compact unary algebra, namely its Stone-Čech compactification.

The fact that equationally compact mono-unary algebras are even retracts of their Stone-Čech compactification depends essentially on the following lemma:

Lemma 7.6: If $\mathfrak{A} = (A;f)$ is a mono-unary algebra and $\mathcal{L}_m$ is a subalgebra of $\beta\mathfrak{A}$ then there exists a subalgebra $\mathcal{L}_n$ of $\mathfrak{A}$ such that n divides m.

Theorem 7.7: If $\mathfrak{A} = (A;f)$ is an equationally compact mono-unary algebra then it is retract of its Stone-Čech compactification.

Here a remark seems in order: In the proof of the above results we have referred to (35) in which paper a theorem is stated which claims certain Stone-Čech compactifications to be elementary extensions of the underlying unary algebras. Since the arguments used there seem questionable, we want to point out that it is quite implicit in (49) and (51) that $\beta\mathfrak{A}$ is always a pure extension of $\mathfrak{A}$ and that is all we need for our purpose. Whether or not $\beta\mathfrak{A}$ is an elementary extension of $\mathfrak{A}$ in case $\mathfrak{A}$ has no cyclic subalgebras (as is indeed claimed in (35))seems to us an open question for the time being. In general, there are no suitable criteria to decide whether or not a given mono-unary algebra is an elementary extension of a given mono-unary algebra $\mathfrak{A}$. It would be interesting to find such.

There is not much we know about equationally compact unary algebras with more than one operation. The fact that W. Taylor defeated a positive answer to Mycielski's problem by an algebra with only two unary operations indicates how quickly things get out of hand in that situation. It also underlines that our last theorem has the strongest possible setting. There is an occasional isolated remark as, e.g., the following due to J. Mycielski:

Remark 7.8: ((31)) Let $\mathfrak{A} = (A;\{g;g\in G\})$ be an algebra such that G is a group of permutations of A such that the family $\mathcal{F}$ of fixed point sets $F_g=\{x\in A;\ g(x)=x\}$, $g\in G$, has the following property:

"Every subfamily of $\mathcal{F}$ which satisfies the finite intersection property has a non-empty intersection". Then $\mathfrak{A}$ is equationally compact.

For a moment one might wonder whether a unary algebra $(A;F)$ of type $\tau = (1,1,1,1,......)$ is equationally compact if and only if every mono-unary algebra $(A;p)$, $p \in P^{(1)}(\tau)$, is equationally compact. This conjecture, however, is quickly defeated by the following final example:

Example 7.9 : Let $\mathfrak{A} = (A;f,g)$ be of type $(1,1)$ with $A = Z \dot{\cup} \{a_n; n \in N\}$. We define f by $f(z) = z+1$, $f(a_n) = a_{n+1}$ and g by $g(z) = z+1$, $g(a_n) = a_n$. Then theorem 7.4 implies immediately that $(A;p)$ is equationally compact for all $p \in P^{(1)}((1,1))$. However, $\mathfrak{A}$ is not equationally compact as the following system of equations shows:

$\{x = f(x_1),\ x_1 = f(x_2),\ ....,\ x_n = f(x_{n+1}),.....;\ n<\omega_0\} \cup \{x = g(x)\}$. $\mathfrak{A}$ is not even weakly equationally compact!

It is clear from (at the latest) this example that eventual criteria for equational compactness have to take into account that subalgebras of an equationally compact algebra that are distinguishable by a set of identities need again to be equationally compact.

(2) W. Taylor's Counterexample to a Positive Answer to Mycielski's Problem. (see (39).

As mentioned before, W. Taylor has constructed an algebra $(A;f_1,f_2)$ of type (1,1) which is equationally compact but not a retract of any topologically compact algebra. The meat of the matter is a highly interesting graph-theoretical construction via which Taylor settles in the negative Mycielski's problem in case of relational systems. This relational system (a graph) then serves as basis for the definition of an algebra of type (1,1) which is equationally compact but not a retract of a topologically compact algebra. As far as the algebra is concerned, W. Taylor did so far not publish the resulting example but rather one of type (2) similarly constructed. As W. Taylor has pointed out himself, the published example does not do what it is supposed to do. His unpublished example of type (1,1), however, is correct.

Definition 7.10: A relational structure $\mathfrak{A} = (A;r)$ of type (2) is called a graph if $r^{\mathfrak{A}}$ is a symmetric and anti-reflexive binary relation. Let $k \geq 2$: A circuit of length k is a k-tuple of mutually different elements of A such that any two successive ones are a pair in $r^{\mathfrak{A}}$ as is the pair consisting of the last and the first entry. If $\mathfrak{n}$ is a cardinal number then a colouring of $\mathfrak{A}$ in $\mathfrak{n}$ colours is a map $c: A \dashrightarrow C$ such that $|C| = \mathfrak{n}$ and $(x,y) \in r^{\mathfrak{A}}$ implies $xc \neq yc$. The chromatic number $\chi(\mathfrak{A})$ of the graph $\mathfrak{A}$ is defined as the least cardinal number $\mathfrak{n}$ such that a colouring of $\mathfrak{A}$ in $\mathfrak{n}$ colours exists. $\mathfrak{A}_n = (A_n;r)$ is the complete graph of n vertices if $|A_n| = n$ and any two different elements of A_n are in relation under r. Finally: A subset S of A is called independent in the graph $\mathfrak{A}$ provided $(s,t) \notin r^{\mathfrak{A}}$ for any two elements $s,t \in S$.

W. Taylor constructed an atomic compact graph $(G;r)$ which has infinite chromatic number. Since graphs that are retracts of topologically compact relational systems have always finite chromatic number, this puts at our disposal the desired counter-example.

Theorem 7.11: If the graph $\mathcal{O} = (A;r)$ is retract of the topologicall compact graph (or even relational system) $\mathcal{L} = (B;r)$ then $\chi(\mathcal{G}) < \aleph_0$.

Theorem 7.12: If $\mathcal{G}_n$ is a finite graph with $\chi(\mathcal{G}_n) \geq n$ which has no odd circuits of length $\leq n$ for every $n \in N$ then $\mathcal{G} = \mathcal{G}_3 \mathbin{\dot\cup} \mathcal{G}_4 \mathbin{\dot\cup} \mathcal{G}_5 \mathbin{\dot\cup} \ldots\ldots\ldots\ldots \mathbin{\dot\cup} \ldots\ldots$ is an atomic compact graph with $\chi(\mathcal{G}) = \aleph_0$.

We sketch the proof of this crucial observation:

proof: Of course, all we need to show is atomic compactness. So let $\mathcal{E}$ be an elementary extension of $\mathcal{G}$ and let us construct a retraction $g: \mathcal{E} \dashrightarrow \mathcal{G}$.

The following sentence with constants in G is then true in $\mathcal{E}$ if and only if it is true in $\mathcal{G}$:

$$\Psi_{n,g} = (\forall x)((g,x) \in r \longrightarrow (x=g_1 \vee \ldots\ldots \vee x=g_m))$$

where $G_n = \{g_1, \ldots, g_m\}$, $g \in G$ and $n \in N$.

In other words: No element in $E \setminus G$ is related to any element in G: $\mathcal{E} = \mathcal{G} \mathbin{\dot\cup} (\mathcal{E} \setminus \mathcal{G})$ where $\mathcal{E} \setminus \mathcal{G} = (E \setminus G;r)$.

Thus, to construct a retraction $g: \mathcal{E} \dashrightarrow \mathcal{G}$ we need simply to construct a homomorphism $h: \mathcal{E} \setminus \mathcal{G} \dashrightarrow \mathcal{G}$.

This, however, is simple. For $\mathcal{G}$ contains a <u>fixed</u> finite number t_k of odd circuits for any given length $2k+1$ $(k \in N)$. This fact can be pinned down by a first order sentence. Thus, $\mathcal{E} \setminus \mathcal{G}$ contains <u>no</u> circuits of odd length, i.e. $\chi(\mathcal{E} \setminus \mathcal{G}) \leq 2$. Then, however, we simply map $E \setminus G$ via h on an edge of $\mathcal{G}$ and we are done.

The last theorem reduces the announced task to finding a graph $\mathcal{G}_n$ with $\chi(\mathcal{G}_n) \geq n$ and without circuits of odd length $\leq n$. This is achieved (up to finiteness of $\mathcal{G}_n$) with the next remark whose proof relies on elementary Euclidean geometry and measure theory.

Remark 7.13: Let $n \in N$ and $e \in R$ such that $2n \leq \frac{2-e}{\sqrt{4e-e^2}}$: Then the graph $\mathcal{G}(e,k) = (G(e,k);r)$ has no circuits of odd length $\leq 2n+1$ for all $k \in N$ if $G(e,k)$ is the unit disc in R^k and $(x,y) \in r$ holds if and only if the (usual Euclidean) distance $d(x,y)$ is larger than $2-e$. Apart from that we can find, for ever $t \in N$, some $k_t \in N$ such that $\chi(\mathcal{G}(e,k)) \geq t$ for all $k \geq k_t$.

This remark surely provides us with graphs $\mathcal{G}_n'$ without circuits of odd length $\leq n$ and chromatic number $\geq n$. However, the graphs we obtain are not finite. This last obstacle we surmount with a theorem of De Bruijn and Erdös for which W. Taylor has given a new and interesting proof. This latter proof we can still abbreviate by rewriting it in the language of spectra (as D. Haley observed):

Theorem 7.14: (De Bruijn & Erdös): A graph $\mathcal{O}\!\mathcal{l}$ can be coloured in k colours ($k \in N$) if and only if every finite subgraph can be coloured in k colours.

proof (W. Taylor; slightly modified by D. Haley): If we can construct a graph homomorphism $g: \mathcal{O}\!\mathcal{l} \dashrightarrow \mathcal{O}\!\mathcal{l}_k$ then we are obviously done. Since (due to our hypothesis) $\mathrm{spec}_\emptyset(\mathcal{O}\!\mathcal{l})$ is finitely solvable in $\mathcal{O}\!\mathcal{l}_k$ it is solvable in $\mathcal{O}\!\mathcal{l}_k$ and we are done.

Putting together the preceding pieces leads to the next result:

Theorem 7.15: There exists an atomic compact graph which is not retract of any topologically compact relational system.

Theorem 7.16: There is an algebra $\mathfrak{A} = (A;f_1,f_2)$ of type (1,1) which is equationally compact but not a retract of any topologically compact universal algebra.

The necessary construction is as follows: We take the graph that led to theorem 7.15: For every $n \in N$, we take a finite graph $\mathcal{G}_n$ with $\chi(\mathcal{G}_n) \geq n$ and without circuits of odd length $\leq n$. Then that graph was $\mathcal{G} = \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \ldots\ldots\ldots\ldots\ldots \cup \mathcal{G}_n \cup \ldots\ldots\ldots$

We identify once more r with $r^{\mathcal{G}}$ and have $r \subseteq G \times G$. Let us take two new elements e_1, e_2 and $A = r \cup G \cup \{e_1,e_2\}$. We then define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} g \text{ if } x = (g,h) \in r \\ e_1 \quad \text{otherwise} \end{cases} \qquad f_2(x) = \begin{cases} h \text{ if } x = (g,h) \in r \\ e_2 \quad \text{otherwise .} \end{cases}$$

Then $\mathfrak{A} = (A;f_1,f_2)$ is the equationally compact algebra which is not a retract of any topologically compact algebra.

We might ask whether, at least, every equationally compact algebra is such that its associated relational system is retract of of a topologically compact relational system . The algebra of type (1,1) just constructed defeats a positive answer to that question as well. This was observed and proved by D. Haley in his Master's thesis, and we find this a fitting place to present it (see (19)).

Remark 7.17: (Taylor's) bi-unary algebra just constructed is such that its associated relational system is not the retract of any topologically compact relational system.

proof: Let $\mathfrak{A} = (A;f_1,f_2)$ be the algebra of theorem ~~[illegible]~~ 7.16

Suppose that $\mathcal{L} = (B;r_1,r_2)$ is a topologically compact relational structure of type (2,2) that retracts upon $\mathfrak{A}_{rel}$.

We define s and s' as follows : $s \subseteq A \times A$ and $s' \subseteq B \times B$ and

$(x,y) \in s$ if and only if $(z,x) \in r_1^{\mathfrak{A}}$ and $(z,y) \in r_2^{\mathfrak{A}}$ for some $z \in A$,

$(x,y) \in s'$ if and only if $(z,x) \in r_1^{\mathcal{L}}$ and $(z,y) \in r_2^{\mathcal{L}}$ for some $z \in B$.

Since s defines a graph structure on A we derive a contradiction by showing that $(A;s)$ is a retract of $(B;s')$ and $(B;s')$ is a topologically compact relational system . The same mapping that retracts $(B;r_1,r_2)$ onto $(A;r_1,r_2) = \mathfrak{A}_{rel}$ turns out to be a retraction of $(B;s')$ onto $(A;s)$. It remains therefore to be checked that s' is closed in B^2:

Let $W = \{(a,b,a,c);\ a,b,c \in B\} \subseteq B^4$. W is closed in B^4 because the topology on B is Hausdorff. Moreover, $r_1^{\mathcal{L}} \times r_2^{\mathcal{L}}$ is closed in B^4 because, by assumption, $r_1^{\mathcal{L}}$ and $r_2^{\mathcal{L}}$ are closed in B^2. Thus, $U = W \cap (r_1^{\mathcal{L}} \times r_2^{\mathcal{L}})$ is closed in B^4. If π is the projection from B^4 onto B^2 defined by $(a,b,c,d)\pi = (b,d)$ then s' is the image of U under π . Thus, s' is closed because π is a closed map.

We conclude this section with posing a <u>problem</u> whose solution we are very interested in:

If an algebra $\mathfrak{A} = (A;F)$ is such that the associated relational system $\mathfrak{A}_{rel}$ is retract of a topologically compact relational system, is then $\mathfrak{A}$ retract of a topologically compact algebra ?

<u>(3) Semilattices.</u>

Hardly anything is known about equationally compact lattices. The closest result so far is the characterization of equationally compact semilattices due to G. Grätzer and H. Lakser (see (18)).

<u>Theorem 7.18:</u> The semilattice $\mathfrak{S} = (S;\vee)$ is equationally compact if and only if the following three conditions are satisfied:

(1) Every subset $T \subseteq S$ has a least upper bound $\bigvee(t;t \in T) = \bigvee T$.

(2) Every chain $C \subseteq S$ has a greatest lower bound $\bigwedge(c;c \in C) = \bigwedge C$.

(3) If $a \in S$ and C is a chain in $\mathfrak{S}$ then $a \vee (\bigwedge C) = \bigwedge(a \vee c;c \in C)$.

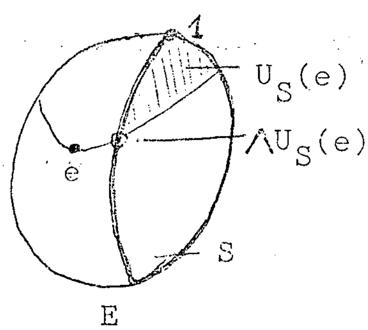
It is easy to establish that the conditions are necessary. The sufficiency was established by G. Grätzer and H. Lakser via the following two lemmas:

Lemma 7.19: Let $s \vee x_{i_0} \vee \dots\dots\dots \vee x_{i_{n-1}} = r \vee x_{j_0} \vee \dots\dots\dots \vee x_{j_{m-1}}$ be an equation and K a set of solutions which is downward directed (in the product order). Then $t = \bigwedge(k;k \in K)$ is also a solution of that equation.

Lemma 7.20: Let $p(x) = s \vee x_{i_0} \vee \dots\dots \vee x_{i_{n-1}}$, $q(x) = r \vee x_{j_0} \vee \dots\dots \vee x_{j_{m-1}}$ and let K be an arbitrary set of solutions of the equation $p(x) = q(x)$. Then $t = \bigvee(k;k \in K)$ is a solution for $p(x) = q(x)$.

We found a new proof based on a different idea which we want to include at this place :

We are concerned with proving that (1),(2),(3) imply the equational compactness of the semilattice $\mathcal{S}$. So let $\mathcal{S}$ be a complete semilattice in which chains have always greatest lower bounds and, if C is such a chain, $(\bigwedge C) \vee a = \bigwedge(c \vee a; c \in C)$. Thus, if C,D are downward directed sets then $(\bigwedge C) \vee (\bigwedge D) = \bigwedge(c \vee d; (c,d) \in C \times D)$ and the latter greatest lower bound exists. Let $\mathcal{E}$ be an elementary extension of $\mathcal{S}$ and $e \in E \setminus S$. Then the set $U_S(e)$ of upper bounds of e that are also in S is not empty since $1 \in U_S(e)$. Let $U_S(e) = \{d_i; i \in I\}$, choose an arbitrary



finite subset I' of I and consider the finite set of polynomial equations with constants in S, $\sum(I')=\{x_{I'}\vee d_i=d_i;i\in I'\}$. Since $e\in E$ solves these equations there is an element $s\in S$ which solves $\sum(I')$. Since $\mathcal{T}$ is complete there is even a greatest lower bound of $\{d_i;i\in I'\}$ in $\mathcal{T}$, say $e_{I'}$. $e_{I'}$, however, is itself in $U_S(e)$, for the substitution $x=e_{I'}, y=e_{I'}$, $y_i=d_i, i\in I'$, satisfies the formula $(\forall x)(x\leq y_i, i\in I', \rightarrow x\leq y)$ in $\mathcal{T}$ and, hence, in $\mathcal{E}$. In other words: The sentence with constants in S, $(\forall x)(x\leq d_i, i\in I', \rightarrow x\leq e_{I'})$, holds in $\mathcal{E}$; hence, $e\leq e_{I'}$. Surely $I'\subseteq I''$ implies $e_{I'}\geq e_{I''}$; hence, $U_S(e)$ is downward directed and has, therefore, a greatest lower bound $\bigwedge U_S(e)$ in $\mathcal{T}$. Since, for $e\in S$, we surely have $\bigwedge U_S(e)=e$ we have the desired retraction $g: \mathcal{E}\dashrightarrow\mathcal{T}$ by $eg =: \bigwedge U_S(e)$ provided the latter mapping is indeed a homomorphism. That property of g, however, is easily checked:

Let $e_1,e_2\in E$. Then $U_S(e_i)\supseteq U_S(e_1\vee e_2)$, $i=1,2$, and, thus, $(\bigwedge U_S(e_1))\vee(\bigwedge U_S(e_2))\leq\bigwedge U_S(e_1\vee e_2)$, i.e. $e_1g\vee e_2g\leq(e_1\vee e_2)g$.

Vice versa: $e_1g\vee e_2g = (\bigwedge U_S(e_1))\vee(\bigwedge U_S(e_2))$ = (by hypothesis) $\bigwedge(c_1\vee c_2;\ c_i\in U_S(e_i))$. If $(c_1,c_2)\in U_S(e_1)\times U_S(e_2)$ then $c_1\vee c_2\in U_S(e_1\vee e_2)$. Thus, $\bigwedge U_S(e_1\vee e_2)\leq(\bigwedge U_S(e_1))\vee(\bigwedge U_S(e_2))$, i.e. $g(e_1\vee e_2)\leq g(e_1)\vee g(e_2)$.

Thus, $(e_1\vee e_2)g = e_1g\vee e_2g$. $\mathcal{T}$ is therefore equationally compact.

The problem of Mycielski is open to this day in the class of semilattices. W. Taylor's result (see (40)) that every equationally compact semilattice is retract of a topologically compact relational system does not answer the real question. It would, if we had an answer to the problem we posed at the end of the last section.

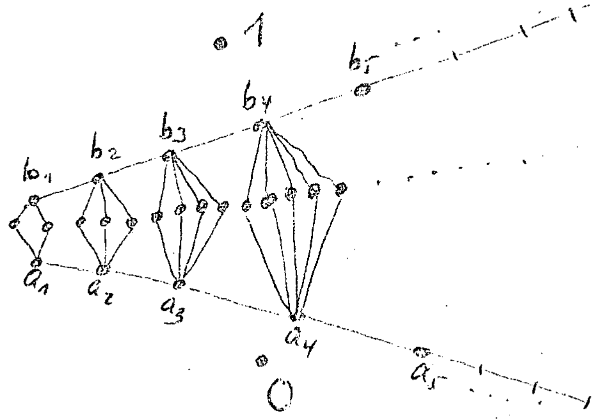
Corollary 7.21: If $\mathcal{L} = (L; \vee, \wedge)$ is an equationally compact lattice then

(1) $\mathcal{L}$ is complete, (2) If $D \subseteq L$ is a downward (upward) directed set then $a \vee (\bigwedge D) = \bigwedge (a \vee d; d \in D)$ $(a \wedge (\bigvee D) = \bigvee (a \wedge d; d \in D))$.

(3) If $a > b$ are two elements in L and $\{c_i; i \in I\}$ is a set of elements in L such that $i \neq j$ implies $c_i \vee c_j = a$, $c_i \wedge c_j = b$, then $|I| < \aleph_0$.

The fact that these conditions are not sufficient is, e.g., seen by the following example:

Example 7.22 (D. Kelly) :



This lattice is not equationally compact as the following system of equations shows:

$$\{x_i \vee x_j = z_1\} \cup \{x_1 \wedge x_j = z_0\}$$
$$\{z_1 \vee b_i = z_1\} \cup \{z_0 \wedge a_i = z_0\}$$

where $i, j \in \mathbb{N}$, $i \neq j$.

The original proof of 7.18 in (18) brings to light the truth of the following remark:

Remark 7.23: A semilattice is equationally compact if and only if every system of polynomial equations in one variable with constants is solvable provided it is finitely solvable.

(4) Lattice-related structures.

If $\mathbb{L}$ denotes the class of all lattices and $\mathcal{L} \in \mathbb{L}$ then $\mathcal{L}(\leq)$ denotes the associated poset $(L; \leq)$ and $\mathcal{L}_0^1$ denotes the smallest lattice with smallest element 0 and largest element 1 which contains $\mathcal{L}$. If $\mathbb{L}(\leq)$ denotes the class of all posets $\mathcal{L}(\leq)$ then $\mathbb{L}(\leq) \subsetneq \mathbb{O}$ where $\mathbb{O}$ is the class of all posets. $\mathbb{L}(\leq)$ consists of the so-called lattice-induced posets. Weglorz (46) proved the following basic results on these:

Theorem 7.24: If $\mathcal{L} = (L; \vee, \wedge)$ is a lattice then the following statements are equivalent:

(1) $\mathcal{L}$ $(\leq)$ is atomic compact.

(2) $\mathcal{L}$ is complete.

(3) $\mathcal{L}$ $(\leq)$ is injective in $\mathbb{O}$.

(4) $\mathcal{L}$ $(\leq)$ is injective in $\mathbb{L}$ $(\leq)$.

(5) $\mathcal{L}$ $(\leq)$ is an absolute retract in $\mathbb{L}$ $(\leq)$.

(6) $\mathcal{L}$ $(\leq)$ is an absolute retract in $\mathbb{O}$.

(5) Boolean Algebras.

Although we do not know much about lattices in general, the equational class of Boolean algebras $\mathcal{B} = (B; \vee, \wedge, {}', 0, 1)$ is quite accessible to our questions. The following two theorems contain the crucial information (see B. Weglorz (44)).

Theorem 7.25: If $\mathcal{B} = (B; \vee, \wedge, {}', 0, 1)$ is a Boolean algebra then the following conditions are equivalent:

(1) $\mathcal{B}$ is complete.

(2) $\mathcal{B}$ is injective in the class $\mathbb{B}$ of all Boolean algebras.

(3) $\mathcal{B}$ is an absolute retract in the class $\mathbb{B}$ of all Boolean algebras

(4) $\mathcal{B}$ is equationally compact.

Theorem 7.26: The Boolean algebra is equationally compact if and only if it is retract of a topologically compact Boolean algebra.

While the first of the two theorems follows quickly with the aid of Sikorski's theorem stating that completeness and injectivity coincide in Boolean algebras, the second uses, in addition, the fact that every Boolean algebra can be embedded in a direct product of 2-element lattices.

Both theorems were recently proved by different (partly ring-theoretical) methods in (1). A. Abian achieves in that paper even a slight strenghtening of the first theorem:

Theorem 7.27(1) A Boolean algebra $\mathcal{L} = (B; \vee, \wedge, ', 0, 1)$ is complete if and only if every system of polynomial equations in one unknown each of whose equations is of the form $a \wedge x = b$, $a, b \in B$, is solvable provided every subsystem with two equations has a solution.

As far as I know it was W. Taylor who observed first the following remark (that follows easily from the fact that finitely generated Boolean algebras are finite):

Remark 7.28: Every subalgebra of any given Boolean algebra is a pure subalgebra.

(6). Abelian Groups.

In general, equationally compact $\mathcal{R}$ - modules are not yet characterized, and such a characterization will depend strongly on $\mathcal{R}$. The situation is quite different if we confine our attention to particular rings $\mathcal{R}$ and, more specifically, to the interesting case of unital modules over the ring $\mathcal{Z}$ of integers, alias Abelian groups. Both the algebraic structure of equationally compact Abelian groups $\mathcal{G} = (G; +, -, 0)$ has been extensively studied and Mycielski's problem has been answered affirmatively in the class A of Abelian groups. The relevant results are scattered through I. Kaplansky (24), J. Łos (27),(28), S. Balcerzyk (2) and S. Gacsalyi (15). We will attempt in this section to give an account of the relevant results in A .

Gacsalyi proved in (15) that pure subgroups (in the group-theoretical sense) and pure subsystems (in our sense) of Abelian groups coincide. This yields immediately the following theorem:

Theorem 7.29: An Abelian group is equationally compact if and only if it is a direct summand of every extending Abelian group in which it is pure.

An affirmative answer to Mycielski's problem in A was given in (2) by S. Balcerzyk. His proof uses essentially the following refinement of Birkhoff's subdirect representation theorem due to J. Łos (27):

Theorem 7.30: If $\mathcal{Z}_{p^n}$ denotes the cyclic group with p^n elements (p = prime integer) and $\mathcal{Z}_{p^\infty}$ is the Prüfer group over the prime number p then each Abelian group $\mathcal{G}$ can be embedded in a direct product $\mathfrak{Z}$ of groups of the type $\mathcal{Z}_{p^\alpha}$ $(\alpha = 1,2,\ldots,\infty)$ such that (1) $\mathcal{G}$ is a pure subgroup of $\mathfrak{Z}$ and (2) $\mathcal{G}$ is a subdirect product of the groups $\mathcal{Z}_{p^\alpha}$ occurring in $\mathfrak{Z}$.

Theorem 7.31: (2): An Abelian group is equationally compact if and only if it is a direct summand of a topologically compact Abelian group.

In (2) Balcerzyk links up equationally compact Abelian groups with ~~an~~ another important ~~other~~ class of Abelian groups: the algebraically compact Abelian groups (in the sense of (24), 1954-edition). Let us combine this result with the others and we obtain the following result:

Theorem 7.32: If $\mathcal{G}$ is an Abelian group then the following statements are equivalent:

(1) $\mathcal{G}$ is equationally compact.

(2) $\mathcal{G}$ is direct summand of every extending Abelian group which contains $\mathcal{G}$ as pure subgroup.

(3) $\mathcal{G}$ is direct summand of some topologically compact Abelian group.

(4) $\mathcal{G}$ is algebraically compact.

(5) $\mathcal{G}$ is the direct sum of a divisible Abelian group and an Abelian group which is complete and Hausdorff in its $\mathcal{Z}$-topology.

Of course, other characterizations have been given, but the ones listed in 7.32 seem to us the most interesting ones. There is one exception to this remark. J. Łos (28') proved the following theorem:

Theorem 7.33: An Abelian group is equationally compact if and only if there is a generalized λ -limit for each ordinal λ .

A"generalized λ - limit" is a linear mapping Lim: $G^{\omega_\lambda} \dashrightarrow G$ which satisfies, in addition, the following laws: (1) Lim((g)) = g and (2) $\mathrm{Lim}((x_\alpha)_{\alpha<\omega_\lambda}) = \mathrm{Lim}((y_\alpha)_{\alpha<\omega_\lambda})$ if there exists $\xi<\omega_\lambda$ such that $x_\alpha = y_\alpha$ for all $\alpha \geq \xi$.

Since Łos gave his proof when algebraic compactness rather than equational compactness was emphasized, it appears to be a worthwhile problem to attempt a generalization of the last theorem to (at least, certain) classes of equationally compact $\mathcal{R}$-modules.

For examples of equationally compact Abelian groups we must refer to (24).

(7) $\mathcal{R}$ - Modules and Rings.

Let $\mathcal{R}$ = $(R;+,-,0,\cdot,1)$ be a ring with identity 1 and $\mathcal{M}$ = $(M;+,-,0,\{f_r;r\in R\})$ a unital left $\mathcal{R}$-module. J. Mycielski and C. Ryll-Nardzewski proved in (32) an interesting generalization of a result of Balcerzyk (3):

Theorem 7.34: The $\mathcal{R}$ -module $\mathcal{M}$ is equationally compact if and only if it is $(|R|+\aleph_0)$ - equationally compact. (Balcerzyk had proved this result for Abelian groups only).

Of course, every $\mathcal{R}$-module can be embedded in an equationally compact one (namely, its injective hull). This result has been essentially improved by R.B. Warfield in (47) using Bohr-compactifications. Since this approach allows an affirmative answer to Mycielski's problem, we will follow it up.

If $\mathcal{G}$ is an Abelian group then we recall the Bohr-compactification $B(\mathcal{G})$ of $\mathcal{G}$ to be the Abelian group $\mathrm{Hom}(\mathrm{Hom}(\mathcal{G},\mathcal{L}),\mathcal{L})$ where $\mathcal{L}$ is the circle group in the complex plane.
Thereby we consider $\mathcal{G}$ a discrete group as we do $\mathrm{Hom}(\mathcal{G},\mathcal{L})$.
Then $\mathrm{Hom}(\mathrm{Hom}(\mathcal{G},\mathcal{L}),\mathcal{L})$ can, as set, be considered a closed subgroup of $\mathcal{L}^{\mathrm{Hom}(\mathcal{G},\mathcal{L})}$ which, by Tychonoff's theorem, is a compact Hausdorff space. Then $B(\mathcal{G})$ is a compact Abelian group which contains $\mathcal{G}$ as dense subgroup (if we identify $g \in G$ with the "evaluation map" e_g in $B(\mathcal{G})$ mapping χ to $g\chi$).
To put it via a universal property (see,e.g., (21) and (22)):
$B(\mathcal{G})$ is the topologically compact Abelian group which (up to homeomorphism and isomorphism) is determined uniquely by the following two properties: (i) $\mathcal{G}$ is a dense subgroup of $B(\mathcal{G})$, (ii) if $\mathcal{D}$ is a topologically compact Abelian group and $\chi : \mathcal{G} \dashrightarrow \mathcal{D}$ is a homomorphism then there exists a (unique) continuous extension $\hat{\chi}: B(\mathcal{G}) \dashrightarrow \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
B(\mathcal{G}) \arrow[dr, dashed, "\hat{\chi}"] & \\
\mathcal{G} \arrow[u, phantom, "\cup"] \arrow[r, dashed, "\chi"'] & \mathcal{D}
\end{tikzcd}$$

If $\mathcal{M}$ is an $\mathcal{R}$-module then we denote by $B(\mathcal{M})$ the Bohr-compactification of the Abelian group $\mathcal{M}'$ underlying $\mathcal{M}$. Since the scalar multiplications $f_r: M' \dashrightarrow M'$ are continuous we can extend them to $f_r: B(\mathcal{M}') \dashrightarrow B(\mathcal{M}')$ and obtain thereby the compact $\mathcal{R}$-module $B(\mathcal{M})$ containing $\mathcal{M}$ as dense submodule, again called the Bohr-compactification of $\mathcal{M}$.

Theorem 7.25 (4.7) If $\mathcal{M}$ is an $\mathcal{R}$-module and $B(\mathcal{M})$ its Bohr-compactification then $\mathcal{M}$ is a pure submodule of $B(\mathcal{M})$ (i.e. every finite system of linear equations with constants in M is solvable in $\mathcal{M}$ if and only if it is solvable in $B(\mathcal{M})$).

We obtain as an immediate consequence an affirmative answer to Mycielski's problem.

Theorem 7.36: If $\mathfrak{M}$ is an equationally compact $\mathcal{R}$- module then $\mathfrak{M}$ is a retract of its Bohr-compactification.

We sum up the different characterizations of equationally compact $\mathcal{R}$ - modules in the next theorem:

Theorem 7.37 (47): If $\mathfrak{M}$ is an $\mathcal{R}$ -module then the following conditions are equivalent:

(1) $\mathfrak{M}$ is pure-injective.

(2) $\mathfrak{M}$ is a direct summand of $B(\mathfrak{M})$.

(3) $\mathfrak{M}$ is equationally compact.

As far as a characterization of equationally compact $\mathcal{R}$ -modules is concerned only partial results have been achieved: partial in the sense that they depend strongly on restrictions upon the ring $\mathcal{R}$. So R.B. Warfield has shown (in (47)) that equationally compact $\mathcal{R}$ -modules must in general satisfy very strong completeness-conditions ($\mathfrak{Z}$ -modules, we recall, were equationally compact if and only if they were a direct sum of a divisible group and a group complete and Hausdorff in the $\mathfrak{Z}$ - topology). By specializing the ring (as,e.g., Warfield does for Prüfer rings) one can indeed achieve situations where these completeness conditions turn out to be essentially sufficient. But we only refer the interested reader to these rather ring theoretical results. We refer to I. Kaplansky (24) and D. Fieldhouse (11) for further discussions of the module-theoretic aspects of pure injectives. Of interest sounds a paper announced by L. Fuchs (12) on "Algebraically compact modules over Noetherian rings" which we have not yet seen; interesting since (as Warfield (47) mentions) it is still an open question whether any topologically complete module over a Noetherian ring is necessarily equationally compact.

Commutative Noetherian rings with 1 are a natural first choice if one attempts to characterize equationally compact rings $\mathcal{R}$. There are two reasons for this: Firstly one can, for every ideal $\mathfrak{a} = (a_1, \ldots, a_n)$ of $\mathcal{R}$, express the statement " $r \in \mathfrak{a}$ " by requiring the satisfiability of $x_1 \cdot a_1 + \ldots\ldots + x_n \cdot a_n = r$ in $\mathcal{R}$ and gets therefore control over the ideal lattice of $\mathcal{R}$ via polynomial equations (which is important in view of Warner's results(1) on topologically compact Noetherian rings with 1). Secondly one has at one's disposal the classical ideal theory in such rings.

D. Haley, a student of mine mentioned before, has investigated these rings in his Master's thesis and has come up with a complete answer which has just appeared in the "Mathematische Annalen" (see (20)).

<u>Theorem 7.38</u>: If $\mathcal{R}$ is a commutative Noetherian ring with 1 then the following conditions are equivalent:

(1) $\mathcal{R}$ is a topologically compact ring.

(2) $\mathcal{R}$ is an equationally compact ring.

(3) $\mathcal{R} \cong \bigoplus(\mathcal{R}_i; i=1,\ldots\ldots,s)$ where each $\mathcal{R}_i$ has exactly one maximal ideal $\mathfrak{m}_i$, $\mathcal{R}_i/\mathfrak{m}_i$ is finite and $\mathcal{R}_i$ is complete and Hausdorff in the $\mathfrak{m}_i$-adic topology.

The following are the two essential lemmata used in the proof:

<u>Lemma 7.39:</u> Quotientrings of equationally compact Noetherian rings with 1 are again equationally compact.

<u>Lemma 7.40:</u> If $\mathcal{R}$ is an equationally compact commutative Noetherian ring with 1 such that the set of zero divisors is contained in a proper ideal, then $\mathcal{R}$ has a unique maximal ideal $\mathfrak{m}$, $\mathcal{R}/\mathfrak{m}$ is a finite ring and $\mathcal{R}$ is complete and Hausdorff in the $\mathfrak{m}$-topology (defined via the neighborhood system $\{\mathfrak{m}^n; n \in N\}$ of O).

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(1) locally cited: S. Warner, Compact Noetherian Rings, Math. Annalen 178, 1968, pp. 53-61 (MR 38, No 5864)

§ 8 : Compactifications of Universal Algebras.

(1). Comp($\mathfrak{A}$), $\mathrm{Comp}^{W}(\mathfrak{A})$, c($\mathfrak{A}$), cl($\mathfrak{A}$).

Definition 8.1: Let $\mathfrak{A}$ be a universal algebra and $\mathfrak{m}$ an infinite cardinal number: The algebra $\mathfrak{L}$ is called a $\mathfrak{m}$-(weak) equational compactification of $\mathfrak{A}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{L}$ and $\mathfrak{L}$ is $\mathfrak{m}$-(weakly) equationally compact. The class of all $\mathfrak{m}$-(weak) equational compactifications of $\mathfrak{A}$ is denoted by $\mathrm{Comp}_{\mathfrak{m}}(\mathfrak{A})$ ($\mathrm{Comp}^{W}_{\mathfrak{m}}(\mathfrak{A})$). We then define $\mathrm{Comp}(\mathfrak{A})$ ($\mathrm{Comp}^{W}(\mathfrak{A})$) as the intersection of all $\mathrm{Comp}_{\mathfrak{m}}(\mathfrak{A})$ ($\mathrm{Comp}^{W}_{\mathfrak{m}}(\mathfrak{A})$) where $\mathfrak{m}$ runs through the cardinals. The elements of $\mathrm{Comp}(\mathfrak{A})$ ($\mathrm{Comp}^{W}(\mathfrak{A})$) are called equational compactifications (weak equational compactifications) of $\mathfrak{A}$. $\mathfrak{L}$ is called a $\mathfrak{m}$-quasi-compactification of $\mathfrak{A}$ if $\mathfrak{L}$ contains $\mathfrak{A}$ as subalgebra and $\mathfrak{L} \in c(A, \mathfrak{A}, \mathfrak{m})$. The class $c_{\mathfrak{m}}(\mathfrak{A})$ contains all $\mathfrak{m}$-quasi-compactifications of $\mathfrak{A}$. Again $c(\mathfrak{A}) = \bigcap(c_{\mathfrak{m}}(\mathfrak{A}); \mathfrak{m} = \text{cardinal})$ is the class of all quasi-compactifications of $\mathfrak{A}$. If $\mathfrak{L}$ contains the subalgebra $\mathfrak{A}$ and $\mathfrak{L} \in c(A, \mathfrak{L}, \mathfrak{m})$ then $\mathfrak{L}$ is called an $\mathfrak{m}$-closure of $\mathfrak{A}$ and $\mathrm{cl}_{\mathfrak{m}}(\mathfrak{A})$ denotes the class of all $\mathfrak{m}$- closures of $\mathfrak{A}$. Again $\mathrm{cl}(\mathfrak{A}) = \bigcap(\mathrm{cl}_{\mathfrak{m}}(\mathfrak{A}); \mathfrak{m} = \text{cardinal})$ is the class of all closures of $\mathfrak{A}$ (see also (41)).

Corollary 8.2: If $\mathfrak{A}$ is a universal algebra and $\mathfrak{m}$ is an infinite cardinal number then $\mathrm{Comp}_{\mathfrak{m}}(\mathfrak{A}) \subseteq \mathrm{cl}_{\mathfrak{m}}(\mathfrak{A}) \subseteq c_{\mathfrak{m}}(\mathfrak{A})$ and, accordingly, $\mathrm{Comp}(\mathfrak{A}) \subseteq \mathrm{cl}(\mathfrak{A}) \subseteq c(\mathfrak{A})$.

In this terminology we can now restate Weglorz's (frequently quoted) problem as follows:

Problem 8.3:(Weglorz (45)): Does $\mathrm{Comp}(\mathfrak{A}) = \emptyset$ imply that

$$c(\mathfrak{A}) = \emptyset \ ?$$

The problem has, so far, resisted our attempts to settle it. We expect a negative answer, although we were able to give a positive answer to the weaker version of the problem that arises when one replaces in the above problem c by cl.

It is quite clear that not every universal algebra has an equational or even only a weakly equational compactification. As example of such an algebra we could take the lattice in example 3 of § 0 if we add 0,1 as nullary operations.

It is, however, true that every algebra has an $\mathfrak{m}$- equational compactification. Much stronger than that earlier result of Weglorz is the following result of J. Mycielski and C. Ryll-Nardzewski (32) which subsumes the above and other similar results.

Theorem 8.4: (32) If $\mathfrak{A}$ is an arbitrary infinite universal algebra and $\mathfrak{m}$ is an infinite cardinal number then there exists always an algebra $\mathfrak{L}$ such that (1) $\mathfrak{L}$ is an elementary extension of $\mathfrak{A}$, (2) $|B| = |A|^{\mathfrak{m}}$, (3) $\mathfrak{L}$ is $\mathfrak{m}$- elementarily compact.

proof: Let μ be the initial ordinal of $\mathfrak{m}^+$ which is a regular ordinal number. We define $\mathfrak{A}_\gamma$, $\gamma<\mu$, by transfinite induction as follows:

$\mathfrak{A}_0 = \mathfrak{A}$. If $\gamma<\mu$ then $\mathfrak{A}_{\gamma+1} = \mathfrak{A}_\gamma(\mathfrak{m})$ (see 5.6).

If $\delta\leq\mu$ is a limit ordinal then $\mathfrak{A}_\delta = \bigcup(\mathfrak{A}_\gamma; \gamma<\delta)$.

We claim that $\mathfrak{L} = \mathfrak{A}_\mu$ satisfies all required conditions: Surely, (1) is true. (2) is seen as follows: If, for $\gamma<\mu$, $|A_\gamma| = |A|^{\mathfrak{m}}$ then $|A_{\gamma+1}| = |A_\gamma|^{\mathfrak{m}} = |A|^{\mathfrak{m}^2} = |A|^{\mathfrak{m}}$. If $\delta\leq\mu$ is a limit ordinal and $|A_\gamma| = |A|^{\mathfrak{m}}$ for all $\gamma<\delta$ then $|A_\delta|\leq|\delta|\cdot|A|\leq\mathfrak{m}^+\cdot|A|^{\mathfrak{m}} = |A|^{\mathfrak{m}}$. Thus, indeed $|A_\mu| = |B| = |A|^{\mathfrak{m}}$. (3) is clear: If Σ is a system of $\leq\mathfrak{m}$ formulas with constants in B which is finitely solvable in $\mathfrak{L} = \mathfrak{A}_\mu$ then (due to the regularity of μ) all the constants of Σ are already in $\mathfrak{A}_\xi$ with $\xi<\mu$. Since $\mathfrak{A}_\xi$ is an elementary subalgebra of $\mathfrak{A}_{\mu}$, Σ is finitely satisfiable in $\mathfrak{A}_{\xi}$, hence satisfiable in $\mathfrak{A}_{\xi+1}$. Thus, Σ is satisfiable in $\mathfrak{L}$.

Corollary 8.5: If $\mathfrak{A}$ is an arbitrary universal algebra and $\mathfrak{m}$ is an infinite cardinal number then

$\mathrm{Comp}^{W}_{\mathfrak{m}}(\mathfrak{A}) \neq \emptyset$, $\mathrm{Comp}_{\mathfrak{m}}(\mathfrak{A}) \neq \emptyset$, $\mathrm{cl}_{\mathfrak{m}}(\mathfrak{A}) \neq \emptyset$ and $c_{\mathfrak{m}}(\mathfrak{A}) \neq \emptyset$.

The lattice in example 3, $\S 0$, suffices to prove all of the following possibilities:

$\mathrm{Comp}^{W}(\mathfrak{A}) = \emptyset$, $\mathrm{Comp}(\mathfrak{A}) = \emptyset$, $\mathrm{cl}(\mathfrak{A}) = \emptyset$ and $c(\mathfrak{A}) = \emptyset$.

We want to give one more example of an algebra $\mathfrak{G}$ such that $\mathrm{Comp}(\mathfrak{G})$ is empty. The example is the group of all transformations $ax+b$, $0 \neq a,b \in Q$. Freudenthal showed that $\mathfrak{G}$ cannot be embedded into any topologically compact group. Mycielski and Ryll-Nardzewski went one step further and showed that $\mathfrak{G}$ cannot be equationally compactified.

We do not know of any group that can be equationally compactified but has no topological compactification.

Theorem 8.6 (32) The group $\mathfrak{G}$ of all transformations $ax+b, 0\neq a,b\in Q$, cannot be equationally compactified, not even quasi-compactified.

We close this subsection with three simple observations. The first two were made by B. Weglorz in (44) and (45). The last (8.9) is equally simple but has never been remarked before.

Remark 8.7: Let K be the smallest universal class containing a finite number of weakly equationally compact algebras.

Then $\mathrm{Comp}^{W}(\mathfrak{A}) \cap K \neq \emptyset$ for every $\mathfrak{A} \in K$.

proof: By a theorem of Łos one knows that $K = SU\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ if $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are the finitely many algebras in question.

Since $U\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\} \subseteq HP\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ we know that all algebras in $U\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ are weakly equationally compact.

Remark 8.8: For each algebra $\mathfrak{A}$ and each infinite cardinal number $\mathfrak{m}$ there is a set $\Sigma_{\mathfrak{m},\mathfrak{A}}$ of polynomial equations with constants in A which is finitely solvable and has the property that $\mathfrak{L} \in c_{\mathfrak{m}}(\mathfrak{A})$ holds if and only if $\mathfrak{L} \supseteq \mathfrak{A}$ and $\Sigma_{\mathfrak{m},\mathfrak{A}}$ is solvable in $\mathfrak{L}$.

proof: Take all possible systems of polynomial equations with constants in A with $\leq m$ equations involving the variables $\{x_0, x_1, \ldots\ldots, x_\gamma, \ldots\ldots, \gamma < \mu\}$, μ = initial ordinal of m ; say, $\Sigma_0, \Sigma_1, \ldots\ldots\ldots, \Sigma_\gamma, \ldots\ldots\ldots, \gamma < \xi$.

Next we exchange the variables in the various Σ_γ such that $\gamma \neq \delta$ implies now that Σ_γ and Σ_δ have no variables in common. If we denote the resulting systems by $\Sigma_0', \Sigma_1', \ldots\ldots\ldots, \Sigma_\gamma', \ldots\ldots\ldots, \gamma < \xi$, then $\Sigma = \bigcup(\Sigma_\gamma'; \gamma < \xi)$ is obviously the desired system.

Remark 8.9 : (1) If $\mathfrak{A}$ is a universal algebra then there exists a (unique) smallest congruence relation $\Theta_{\mathfrak{A}}$ on $\mathfrak{A}$ such that $\mathfrak{A}/\Theta_{\mathfrak{A}}$ is equationally compactifiable.

(2) If K is an equational class of universal algebras then every $\mathfrak{A} \in$ K can be equationally (topologically) compactified if and only if every subdirectly irreducible algebra in K can be equationally (topologically) compactified.

Proof: (1) Let $\mathcal{F} = \{\Theta ; \Theta \in C(\mathfrak{A}), \mathfrak{A}/\Theta$ is equationally compactifiable$\}$. Then ι (=universal congruence) $\in \mathcal{F}$. Let $\Theta_0 = \bigcap(\Theta ; \Theta \in \mathcal{F})$. Then $\mathfrak{A}/\Theta_0$ is a subdirect product of all algebras $\mathfrak{A}/\Theta$, $\Theta \in \mathcal{F}$. If $\mathcal{L}_\Theta \in$ Comp$(\mathfrak{A}/\Theta)$ then $\mathfrak{A}/\Theta_0 \leq \prod(\mathcal{L}_\Theta ; \Theta \in \mathcal{F})$, and the latter algebra is equationally compact. Thus, $\Theta_0 \in \mathcal{F}$ and we are done. The same proof applies, of course, for the topological case. (2) is clear by Birkhoff's subdirect representation theorem.

It seems to us an interesting problem to determine the congruence $\Theta_{\mathfrak{A}}$ specified in the last remark for every $\mathfrak{A}$ with Comp$(\mathfrak{A})$ = $\emptyset$. (2), of course, yields topological compactifications for Abelian groups, modules etc even without the Bohr-compactification. The implication for Boolean algebras and distributive lattices is also clear.

(2) Compactifications preserving positive sentences.

Let Pos($\mathfrak{A}$) denote the class of all universal algebras of the type of $\mathfrak{A}$ that satisfy every positive sentence with constants in A that is satisfied in $\mathfrak{A}$.

Theorem 8.10: (50) If $\mathfrak{A}$ is an algebra and $\mathfrak{L} \in$ Comp($\mathfrak{A}$) then there exists $\mathfrak{J} \in$ Comp($\mathfrak{A}$) $\cap$ Pos($\mathfrak{A}$). Such is every extension $\mathfrak{J}$ of $\mathfrak{A}$ which is maximal with respect to being a subsystem of $\mathfrak{L}$ and an element of Pos($\mathfrak{A}$).

If we replace Pos by HSP then this result was stated by Weglorz in (45). The same holds true for the next theorem; however, our proof is quite different and in the spirit of 3.1.

Theorem 8.11: If $\mathfrak{A}$ is an algebra and $\mathfrak{L} \in c(\mathfrak{A})$ then there exists $\mathfrak{J} \in c(\mathfrak{A}) \cap$ Pos($\mathfrak{A}$) such that $\mathfrak{A} \subseteq \mathfrak{J} \subseteq \mathfrak{L}$.

proof: By 3.1, $\mathfrak{L}$ contains an A-retract of every ultrapower of $\mathfrak{A}$. In particular: $\mathfrak{L}$ contains an A-retract $\mathfrak{A}_{\mathfrak{m}}$ of $\mathfrak{A}(\mathfrak{m})$ for every $\mathfrak{m}$. Thus, $\mathfrak{A}_{\mathfrak{m}}$ is $(A, \mathfrak{A}, \mathfrak{m})$-equationally compact and, of course, in Pos($\mathfrak{A}$). Thus, $\mathfrak{L}$ contains an $(A, \mathfrak{A}, \mathfrak{m})$-equationally compact algebra $\mathfrak{A}_{\mathfrak{m}}$ with $\mathfrak{A} \subseteq \mathfrak{A}_{\mathfrak{m}}$ for every $\mathfrak{m}$. One of them must then be $(A, \mathfrak{A})$-equationally compact; otherwise there exists $\mathfrak{n}(\mathfrak{A}_{\mathfrak{m}}) > \mathfrak{m}$ for each $\mathfrak{A}_{\mathfrak{m}}$ such that $\mathfrak{A}_{\mathfrak{m}} \notin c(A, \mathfrak{A}, \mathfrak{n})$ for all $\mathfrak{n} \geq \mathfrak{n}(\mathfrak{A}_{\mathfrak{m}})$. If we have σ different subalgebras of the form $\mathfrak{A}_{\mathfrak{m}}$ in $\mathfrak{L}$, say $\mathfrak{A}^{(0)}, \ldots\ldots, \mathfrak{A}^{(\gamma)}, \ldots\ldots, \gamma < \sigma$, then $\mathfrak{A}_{\Sigma(\mathfrak{n}(\mathfrak{A}^{(\gamma)}); \gamma < \sigma)}$ would not be amongst them. This is a contradiction, and the proof is complete.

Theorem 8.12: If $\mathfrak{A}$ is an algebra and $\mathfrak{L} \in cl(\mathfrak{A})$ then there exists $\mathfrak{J} \in cl(\mathfrak{A}) \cap$ Pos($\mathfrak{A}$) such that $\mathfrak{A} \subseteq \mathfrak{J} \subseteq \mathfrak{L}$.

proof : The essence of the proof is contained in theorem 4.4. Let $\mathcal{F} = \{\Theta ; \Theta \in C(\mathcal{L}) ; \Theta$ separates A and $\mathcal{L}/\Theta \xrightarrow{A} \mathcal{L}\}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element Θ_0 and, therefore, $\mathcal{I} = \mathcal{L}/\Theta_0 \xrightarrow{A} \mathcal{L}$. Since $\mathcal{L} \in cl(\mathfrak{A})$ there exists an A-retraction $g: \mathcal{I} \dashrightarrow \mathcal{L}$ which is evidently a monomorphism. Also we have that all A-retractions $h: \mathcal{I} \dashrightarrow \mathcal{I}$ are monomorphisms. By theorem 4.4 we conclude that all A-retractions $h: \mathcal{I} \dashrightarrow \mathcal{I}$ are automorphisms. Clearly, $\mathcal{I} \in cl(\mathfrak{A})$. By theorem 3.1, there exists an A-retraction $r: \mathcal{I}(|C|) \dashrightarrow \mathcal{I}$. Since its restriction to $\mathcal{I}$ needs to be an automorphism, we are done.

As mentioned before, a different (and still unpublished) proof was given by W. Taylor in (41).

In view of the preceding results the following theorem is somewhat surprising. It gives a negative answer to a question raised twice by B. Weglorz ((44) and (45)) without having received an answer so far.

Theorem 8.13: (50): It is, in general, not true that the existence of a weak equational compactification for an algebra $\mathfrak{A} = (A;F)$ implies the existence of a weak equational compactification within the same equational class.

To see this one may take the algebra $\mathfrak{A} = (A; \vee, \wedge, ')$ of type (2,2,1) with $A = \{a_1, a_2, \ldots\ldots, a_n, \ldots\} \cup \{0,1\}$ such that $(A; \vee, \wedge)$ is the lattice of example 3 in section 0, $a_i' = a_{i+1}$, $0' = 1$, $1' = 0$.

§ 9 : Problems.

This last section contains a list of problems (in arbitrary order) that are naturally connected to this paper:

(1) Given an equationally compact partial algebra $\mathfrak{A}$:
Can we (possibly after suitable extension of the carrier set) make the partial operations full operations such that the resulting universal algebra is equationally compact? Is there some "canonical" such full algebra associated with $\mathfrak{A}$?

(2) (a) Study equationally compact integral domains. Are they always topolgically compact rings?
(b) Characterize equational compactness in suitable classes of rings.

(3) The only equationally compact skewfields are the finite ones. How does that situation change if we allow only systems of equations with finitely many (>1) variables ?

(4) Given an equational class K of universal algebras in which equational compactness has been characterized: If m is a fixed cardinal number, then characterize those algebras that are not equationally compact but are "$((m))$-equationally compact" (i.e. any system of polynomial equations with constants is solvable provided it is finitely solvable and involves $< m$ variables).

(5) Consider the following property P_n ($n \in N$) of a universal algebra: $\mathfrak{A}$ has property P_n if and only if $\mathfrak{A}$ is ((n))-equationally compact but not ((n+1))-equationally compact (see problem (4)).
(a) Construct algebras with property P_n.
(b) Are there, for any given n, unary algebras with property P_n?
(c) Let $\mathfrak{A}$ be of type (2) and $n_0 \in N$: Can one find a set Σ of

identities which makes n_0 minimal with respect to the following property:
"If $\mathfrak{A}$ is in the model class of Σ then $((n_0))$ - equational compactness of $\mathfrak{A}$ implies equational compactness of $\mathfrak{A}$." ?
(For $n_0 = 2$ it was shown in (18) that $\{$ idempotency, associativity, commutativity $\}$ is such a set of identities).

(6) Let λ be a limit ordinal: Find and study algebras which are $((|\lambda|))$-equationally compact, but not $((|\lambda|^+))$-equationally compact. (see problem (4)).

(7) Develop a theory of equationally compact lattices. More specifically:

(a) Is a distributive lattice equationally compact if and only if it is complete and fully distributive ?(This conjecture came up in a conversation with G. Grätzer)

(b) Is there a connection between the interval topology and equational compactness in distributive lattices ?

(c) Is there a lattice which is ((2))-equationally compact but not equationally compact ? (see problem 4).

(8) Study compact Hausdorff topologies on semilattices. Has Mycielski's problem an affirmative answer in the class of semilattices?

(9) (Weglorz(45)): Does the existence of a quasi-compactification of an algebra $\mathfrak{A}$ always imply the existence of an equational compactification of that algebra ? (We expect a negative answer.)

(10) Is there a universal algebra $\mathfrak{A}$ such that $\mathfrak{A}_{rel}$ is retract of a topologically compact relational system, but $\mathfrak{A}$ is not retract of a topologically compact algebra ?

(11) (a) Is there a group that can be equationally compactified but has no embedding into a topologically compact algebra?

(b) (Mycielski (31)): Study equational compactness in the class of non-Abelian, connected, locally compact topological groups.

(12) (Warfield (47)): Are there any topologically complete $\mathcal{R}$-modules over a Noetherian ring which are not equationally compact?

(13) (a) Study equationally compact unary algebras with more than one operation.

(b) Study pure (elementary) extensions of mono-unary algebras.

(c) Let $\mathfrak{A}$ be a unary algebra: Is $\beta\mathfrak{A}$ always an elementary extension of $\mathfrak{A}$ provided it is a pure extension? (see (35) where a questionable proof for a special instance is given.)

(d) If $\mathfrak{A}$ is a mono-unary algebra and $\beta\mathfrak{A}$ contains $\mathcal{L}_n$ does then $\mathfrak{A}$ also contain some $\mathcal{L}_n$? (A positive answer to (c) would, of course, imply a positive answer to (d). Compare lemma 7.6.)

(14) Develop a theory of convergence algebras (i.e. universal algebras with a compatible limit space structure). Relate it to questions concerning equational compactness in universal algebras

(15) Study the relationship between the existence of generalized λ- limits and equational compactness in $\mathcal{R}$-modules where $\mathcal{R}$ is not the ring of integers. (see 7.33).

(16) Let $\mathcal{R}$ be a ring and $\sum(x_i, y_j, z_k)$, $i \in I$, $j \in J$, $k \in K$, a system of $\mathcal{R}$-module polynomial equations involving the mutually disjoint sets of variables $\{x_i; i \in I\}$ and $\{y_j; j \in J\}$ and the constants $z_k; k \in K$, from R:

Are there rings other than the integers where one can find one such system $\sum = \sum(x_i, y_j, z_k)$ of cardinality $|R| + \aleph_0$ with the following property:" The $\mathcal{R}$ - module $\mathfrak{M}$ is equationally compact if and only if every substitution of y_j by module elements that makes $\sum$ (now a system with variables x_i) finitely solvable makes $\sum$ solvable in $\mathfrak{M}$."? (In (3) Balcerzyk finds the system $\{x_0 - y_n = n!x_n; n \in N\}$ for Abelian groups.)

(17) (a). Let α be a limit ordinal with $\aleph_0 \leq |\alpha| \leq |A|$:
If $\mathfrak{A}$ is not equationally compact, can one always find a system Σ of polynomial equations with constants in A such that every subsystem with less than $|\alpha|$ equations is solvable, while Σ is not solvable in $\mathfrak{A}$?

(b) Let $\mathfrak{A}$ be a universal algebra and define the class $S(\mathfrak{A})$ as follows:

$S(\mathfrak{A}) = \{\alpha$; α = limit ordinal and every system of polynomial equations with constants in A is solvable if every subsystem Σ_+ with $|\Sigma_+| < |\alpha|$ is solvable$\}$.

Is there anything that can be said about the classes $S(\mathfrak{A})$ that occur that way?

(18) Is it possible to develop a theory of "canonical" equational compactifications of universal algebras in certain equational classes of algebras (possibly via non-monic representations of the algebras as it is quite usual in categorical treatments of topological compactifications. See also 8.9) ?

(19) Characterize $(\mathfrak{m}, \mathcal{R})$ - equational compactness of (commutative) Noetherian rings $\mathcal{R}$ (with 1) if $\mathfrak{m}$ is an ideal of $\mathcal{R}$.

(20) (D. Haley): Is equational compactness of a universal algebra $\mathfrak{A}$ implied by $((|o(\tau)| + \aleph_0)^+)$ - equational compactness of $\mathfrak{A}$? ($|o(\tau)|$ = cardinality of the type τ of $\mathfrak{A}$ = $|F|$.) (see problem 4).

D. Haley has some interesting results on problem 5 and related questions.

We refer to (42) for another interesting source of problems in a slightly different direction.

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