CONTINUOUS CONVERGENCE IN A GELFAND THEORY FOR TOPOLOGICAL ALGEBRAS

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Nr. 10 (1971)

Contents.

Introduction.

This thesis deals with convergence spaces,particularly with c-embedded ones,and applies that theory to the problem of representing commutative topological algebras as algebras of continuous functions.

The first chapter (chapter 0) is purely introductory, summarising our notation and those results already known, which we need. Covering systems (which are a generalisation to convergence spaces of ' open cover ' in topological spaces) are defined and discussed in the next chapter.With these,many topological concepts can be extended to convergence spaces - for instance,compactness can be characterised by covering systems; we define local compactness,and could define paracompactness,axioms of countability,and so on. This line is investigated more thoroughly in the thesis of Feldman [15], for example.

Next,in chapter 2, we develop a sort of Stone-Weierstrass theorem,extending those of Binz [7, Theorem 5] and Feldman [15].

Classical Gelfand representation theory for commutative Banach algebras is extended to commutative topological algebras.We shew that for each topological algebra A there is a locally compact c-embedded <u>convergence space</u> $Hom_c A$, which acts as carrier in the following sense:

The Gelfand map

$$d : A \longrightarrow C_c Hom_c A$$

is continuous (see [5], where it is called the universal representation of A), and $C_c Hom_c A$ is furthermore a complete topological algebra, carrying the topology of uniform convergence on the compact subsets of $Hom_c A$. Conditions on A are given (coinciding with those known for Banach algebras, when A itself is a Banach algebra), establishing when d is actually an embedding or a homeomorphism.

Last, these results are used to obtain anew the characterisation first given by Binz [8] of compact c-embedded convergence spaces, and extend it to locally compact spaces. We then give two examples of c-embedded locally compact spaces which are, though, not topological. These examples have other properties, enabling us to disprove certain conjectures. The thesis ends by looking at the relationship between a completely regular topological space X, and the $\mathbb{R}$-algebra $C\tilde{X}$ of all continuous $\mathbb{R}$-valued functions on the one point compactification $\tilde{X}$ of X. On the way, we run across examples shewing that our Stone-Weierstrass theorem, proved in chapter 2, is a genuine extension of those of Binz and Feldman.

0 General background.

0.1 Convergence spaces and continuous maps.

The idea of a convergence space (whose utility as a generalisation of topological spaces is,we hope,already justified or will become so) is central to this thesis. To make it reasonably self-contained,we give here the definition and some simple properties of convergence spaces. More details can be found in [1],[2] or [4].

A pair (X,Λ), where X is a set and Λ a function with domain X assigning to each point x of X a set Λx of filters on X, is called a <u>convergence space</u> iff for each x in X the following conditions hold:

i) $\dot{x} \in \Lambda x$,

ii) $\phi \in \Lambda x$ and $\phi' \geqq \phi$ imply $\phi' \in \Lambda x$, and

iii) $\phi, \phi' \in \Lambda x$ imply $\phi \wedge \phi' \in \Lambda x$.

Note: Our notation throughout is as above; namely,

i) $\dot{x}$ denotes the trivial ultrafilter at x in X,

ii) $\phi \geqq \phi'$ means that ϕ and ϕ' are filters on the same set,and that ϕ is finer than ϕ', and

iii) the greatest lower bound (with respect to this ordering) of sets Φ or $\{\phi_1,\ldots,\phi_n\}$ of filters on a set X is denoted by $\wedge\Phi$, and $\overset{n}{\underset{1}{\wedge}}\phi_i$ or $\phi_1\wedge\ldots\wedge\phi_n$ respectively.We use analogous formulae for the union,intersection and product of families of sets as well.

With this definition come a number of 'shorthand' notations: for instance,the convergence space (X,Λ) is often referred to as the space X_Λ, or simply X when no ambiguity is possible. Further, we write ' $\phi \to x$ in X ' and say ' ϕ converges to x in X ', or ' $x\phi$ is a pair from X ' iff $\phi \in \Lambda x$. The set X is called the <u>set underlying the space</u> X_Λ, and Λ is called its <u>(convergence) structure</u>.

For all points x of X, the filter $\wedge \Lambda x$ is denoted by ϕ_x. Using these filters we construct a space cX, the <u>principal space associated with</u> X, as follows:

$$\phi \to x \text{ in } cX \text{ iff } \phi \geqq \phi_x.$$

A space X is called principal iff ϕ_x converges to x in X, for every point x. Equivalently, X and cX are the same space.

Every topological space will be regarded as a principal convergence space, in which a filter converges to a point iff it is finer than the neighbourhood filter at that point.

Any space X possesses an adherence operator a_X, defined by

$$a_X(A) = \{x \in X \mid \text{there is } \phi \in \Lambda x \text{ such that } \phi \cap A\},$$

for each subset A of X. (We use the symbol $\phi \cap A$ for two purposes: first, to mean that the filter ϕ has a trace on A, that is, $F \cap A$ is nonvoid for all $F \in \phi$, and second, to denote the resulting filter $\{F \cap A \mid F \in \phi\}$ on A. It will always be clear what is meant.) One checks easily that this operator satisfies three conditions, namely:

i) $a_X(\emptyset) = \emptyset$, (the symbol $\emptyset$ denoting the empty set)

ii) $A \subseteq a_X(A)$, for all subsets A of X, and

iii) $a_X(A \cup A') = a_X(A) \cup a_X(A')$, for all $A, A' \subseteq X$.

Thus a_X is a closure operator,as soon as one knows that it is idempotent,which it need not always be.Moreover,since

$$a_X(A) = \{x \in X \mid \phi_X \cap A\}$$

for every subset A of X, the operators a_X and a_{cX} coincide.This means simply that convergence spaces are not uniquely determined by their adherence operators,as non-principal spaces do exist.However,there is a one-to-one correspondence between the principal structures on a set,and the adherence operators (that is,any operator satisfying the requirements given above). This fact,whose proof is a not unpleasant calculation,appeared first in [1,Satz 5].

A subset A of a convergence space X is said to be <u>closed</u> iff $A = a_X(A)$, and <u>open</u> iff the complement of A in X (denoted by $X \setminus A$) is closed in X. Obviously, A is open in X iff $A \in \phi$ whenever $x \in A$ and $\phi \to x$ in X, and also,the collection of open subsets of X forms a topology on X. The set X together with this topology is written tX, and called the <u>topological space associated with</u> X. Clearly tX and tcX are the same topological space,since the topology is given purely in terms of the adherence operator.

An obvious question: when is a convergence space topological? One can now see readily that a space X is topological iff it is principal and its adherence operator is idempotent.

This characterization is also due to Kowalsky [1].

Given a convergence space X and a filter ϕ on X, the cluster set $cl_X(\phi)$ of ϕ is defined in the same way as in topology, by

$$cl_X(\phi) = \{x \in X \mid \text{there is } \phi' \to x \text{ in } X \text{ with } \phi' \geq \phi\}.$$

The relation

$$cl_X(\phi) \subseteq cl_{cX}(\phi) = \bigcap\{a_X(F) \mid F \in \phi\}$$

is easily proved, along with the fact that the left inclusion may be proper, for non-principal spaces.

For any convergence spaces X and Y, a map $f \in Y^X$ is said to be <u>continuous</u> iff $f(\phi) \to f(x)$ in Y, whenever $x\phi$ is a pair from X. (The convention we follow here is that $f(\phi)$ denotes that filter on Y having $\{f(F) \mid F \in \phi\}$ as base.) It is to be noted that this definition and the usual one are equivalent, when X and Y are both topological spaces. Constant maps are clearly continuous, as is the composite of continuous maps, when defined. Furthermore, any continuous map $f: X \longrightarrow Y$ has the following pair of properties:

i) $f(a_X(A)) \subseteq a_Y(f(A))$, for all subsets A of X, and

ii) when B is closed (or open) in Y, the set $f^{-1}(B)$ is closed (open) in X.

As usual, the symbol $C(X,Y)$ means the set of all continuous members of Y^X. When Y is the real or complex field with the normal metric topology, we often slim this to CX, using also C^0X for the set of bounded continuous functions on X.

At this point we should remark that the empty set $\emptyset$ is vacuously a convergence space, and that $C\emptyset$ consists solely of the void function. Clearly $C\emptyset$ is a perfectly well-defined algebra, whose operations are defined pointwise - we shall always consider CX as an algebra with respect to the pointwise defined operations, for all spaces X. Later however, it will be convenient to demand that our algebras be nontrivial. For this reason we assume from now on that <u>all spaces are nonvoid</u>.

The associated principal and topological spaces of a convergence space are neatly characterized by means of an universal property (which phrase we often shorten to UP).

<u>Proposition</u> 0.1 : *Let* X *be any convergence space,* C *any principal convergence space, and* T *any topological space. Then*

i) a map $f:X \to C$ *is continuous iff* $f:cX \to C$ *is continuous, and*

ii) a map $g:X \to T$ *is continuous iff* $g:tX \to T$ *is continuous.*

An immediate consequence of this proposition is that the identity maps

$$X \longrightarrow cX \longrightarrow tX$$

are continuous, for all spaces X.

0.2 Induced structures and universal properties.

In this section we look briefly at initial and final induced structures for convergence spaces, comparing them with corresponding structures already known in the theory of topological spaces. Then we introduce that convergence structure on $C(X,Y)$ with which we are most directly concerned in this thesis, the structure of continuous convergence.

We start with initial structures. Let X be a non-void set. A collection F of mappings is called an <u>initial system of mappings on</u> X iff each map $f \in F$ has X as domain, and range X_f, say. When each X_f is a convergence space, the <u>initial structure on</u> X <u>induced by</u> F is obtained as follows:

$$\phi \to x \text{ in } X \text{ iff } f(\phi) \to f(x) \text{ in } X_f, \text{ for all } f \in F.$$

The usual UP holds, and furthermore, when the spaces X_f are all topological, this structure is also topological, being in fact the initial topology induced by F. More formally, we have

<u>Proposition</u> 0.2 : *Let X carry the initial structure induced by an initial family of mappings, say, F. Then*

i) for any convergence space X′, a map $g : X' \longrightarrow X$ *is continuous iff each map* $f \circ g \in C(X', X_f)$. *Further,*

ii) when each space X_f *is topological (or principal) X is also topological (principal), the topology coinciding with the initial topology induced by F.*

The proof is omitted, as it is straightforward calculation for the first part, and diagramme-chasing for the second.

A subset A of a space X, and the product $\Pi\mathcal{X}$ of a collection $\mathcal{X}$ of spaces are given the initial structures induced by the inclusion j_A, and the family $\{\pi_X \mid X \in \mathcal{X}\}$ of coordinate projections respectively. We define now the term <u>embedding</u>, as a homeomorphism onto a subspace, the term homeomorphism needing no explanation.

We need later to know (at least for subspaces) what cX and tX are, when X carries the structure induced by an initial family $\mathcal{F}$. To this end, we set up the commuting diagramme given below, for each $f \in \mathcal{F}$:

$$\begin{array}{ccccc} X & \xrightarrow{\ id\ } & cX & \xrightarrow{\ id\ } & tX \\ f\downarrow & & f\,\downarrow & & f\,\downarrow \\ X_f & \xrightarrow{\ id\ } & cX_f & \xrightarrow{\ id\ } & tX_f \end{array}$$

Our convention is that those maps whose continuity a diagramme purports to prove are marked with broken arrows. Here the desired continuity is proved by means of the UP for cX and tX.

Next let $c'X$ denote that space obtained from the initial family $\{f: X \longrightarrow cX_f \mid f \in \mathcal{F}\}$, and $t'X$ be defined similarly. By using the above diagrammes and the UP for $c'X$ and $t'X$, we see that

$$id: cX \longrightarrow c'X \ , \text{ and } \ id: tX \longrightarrow t'X$$

are both continuous. For subspaces we can do better. To do so, we must use a formula given in

Lemma 0.3 : *Let X be a space, and* $B \subseteq A \subseteq X$. *Then*

$$a_A(B) = a_X(B) \cap A.$$

Proof: Since the inclusion j_A is continuous, we have the inequality $a_A(B) \subseteq a_X(B)$, and hence

$$a_A(B) \subseteq a_X(B) \cap A.$$

On the other hand, if $x \in a_X(B) \cap A$ there is by definition a filter ϕ on X, with $x\phi$ a pair from X and $\phi \cap B$. In particular, ϕ has a trace on A, and the filter $\phi \cap A$ converges to x in A. However $\phi \cap A$ has a trace on B, which fact verifies the inequality $a_A(B) \supseteq a_X(B) \cap A$.

A simple corollary of the formula just proved is

Lemma 0.4 : *i) The adherence operator of a convergence space is idempotent iff it is idempotent for each subspace.*

ii) For each convergence space X, and each subset A of X, the inclusion map $j_A : cA \longrightarrow cX$ *is an embedding. That is, the spaces* cA *and* c´A *are the same.*

In proving the second of these claims one must know, as remarked before, that principal convergence structures are uniquely determined by their adherence operators.

The associated topology of a subspace is not quite so amenable to treatment, as our next lemma shews.

Lemma 0.5 : *Let* X *be a convergence space, and* $A = V \cap F$, *whereby* V *is an open subset of* X, *and* F *is closed in* X. *Then* tA *is a subspace of* tX, *that is*, $tA = t'A$.

Proof: First, A is open in F, for if $x\phi$ is a pair from F with $x \in A$, then $j(\phi) \to x$ in X (here j denotes the inclusion map of F into X). But V is open in X, so that $V \in j(\phi)$. Hence $V \cap F = A \in \phi$, as required.

Next we shew that $tF = t'F$. We know already that tF carries a finer topology than $t'F$. So, let C be closed in tF. Then C is closed in F, by definition, and thus closed in X, by Lemma 0.3. Accordingly, C is closed in tX, and even in $t'F$, this latter being a subspace of tX, by definition of $t'F$. Hence $tF = t'F$, as claimed.

A similar proof, using open sets, shews that the topology on tV is exactly that inherited from tX. In particular, tA is a subspace of tF, since A is open in F. By combining these facts, we have our lemma.

We turn now to final convergence structures. A collection G of mappings is called a final system of mappings in (a non-void set) X iff each map $g \in G$ has range X, and domain X_g, say. When each X_g is a convergence space, the final structure on X induced by G is defined below:

For all points x of X, let

$$\Xi x = \{\dot{x}\} \cup \{g(\psi) \mid g \in G \text{ and } y\psi \text{ is a pair from } X_g \text{ with } g(y) = x\}.$$

Now we demand that $\phi \to x$ in X iff there is a finite subset Ξ' of Ξx, such that $\phi \geq \wedge \Xi'$ (that is, ϕ belongs to the $\wedge$-ideal generated by Ξx in the semi-lattice of all filters on X). The expected UP holds - again we do not prove it - and is stated below.

<u>Proposition</u> 0.6 : *For each convergence space* X', *a map* $f: X \longrightarrow X'$ *is continuous iff for each* $g \in G$ *the map* $f \circ g: X_g \longrightarrow X'$ *is continuous.*

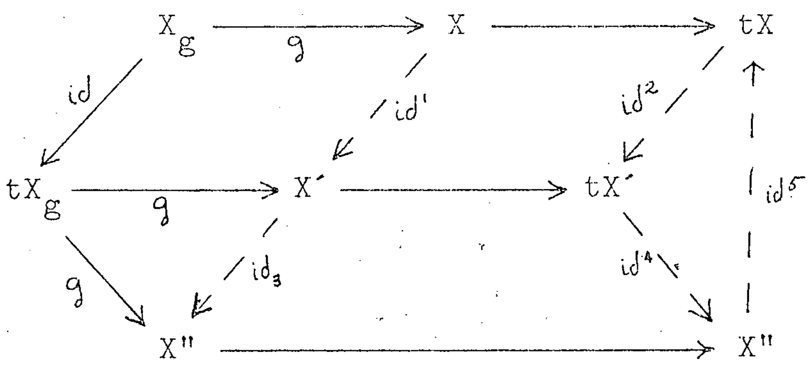
It is not necessarily true that X is principal or topological when the spaces X_g are all principal or topological.Our Examples 4.5 and 4.6 will shew this,among other things.In particular,the final topology and the final structure are not generally homeomorphic;the exact relation is given now.

Let X carry the final structure induced by a family G of mappings, X' that induced from the family

$$G' = \{g: tX_g \longrightarrow X \mid g \in G\},$$

and X'' carry the final <u>topology</u> induced by the family G'. Then tX, tX', and X'' are all the same topological space.

Proof: Consider,for all $g \in G$, the commuting diagramme drawn on the next page - in these diagrammes,any map not marked 'g' is the identity map on X.



The map id^1 is continuous,from the UP for X' . Hence id^2 is continuous (UP for tX).

The UP for X' is used next ,to shew that id^3 is continuous.This,in turn,shews the continuity of id^4.

Last,the maps $g:X_g \longrightarrow tX$ are all continuous,and thus $id^5 \circ g:tX_g \longrightarrow tX$ are also all continuous.This,though,is exactly what is needed to prove the continuity of id^5, by the UP for X''. With this,our claim is proven.

When X and Y are topological spaces,there are many topologies with which C(X,Y), and particularly CX, can be invested,some of these having been extensively studied.There are two main reasons for introducing on C(X,Y) the structure of continuous convergence [3], which is often not even a topology.First,it is defined in a very natural way,and one might hope for correspondingly 'natural' results - this hope is justified - see [4], for example.Second,it reflects to some extent the properties of both X and Y, the question being,how far.This question,asked of CX, has been in theory totally answered for a wide class of spaces,including the completely regular topological. However, the details of this relationship are still being worked out, and there are many open problems.

We now set out the definitions. First, for any subset H of Y^X, where X and Y are sets, the <u>evaluation map</u> ω is defined by

$$\omega : H \times X \longrightarrow Y$$
$$(f,x) \longmapsto f(x).$$

(We should append to ω the indices H, X, and Y; we do not do so, since it will always be clear what is meant. In fact for all 'standardised' objects, such as adherence operators a, identity maps id, inclusion maps j, and others to come later, we omit indices as much as possible while avoiding ambiguity.)

Given next convergence spaces X and Y, and a non-void subset $\mathcal{H}$ of $C(X,Y)$, we give the <u>structure of continuous convergence on</u> $\mathcal{H}$ (obtaining the space $\mathcal{H}_c$), in which

$$\theta \to f \text{ in } \mathcal{H}_c \quad \text{iff} \quad \omega(\theta \times \phi) \to f(x) \text{ in } Y,$$

for each pair $x\phi$ from X. The UP of this structure follows.

<u>Proposition</u> 0.7 : *For any convergence space* Z, *a map* $g : Z \longrightarrow \mathcal{H}_c$ *is continuous iff the map* $\tilde{g}$, *made up from the diagramme*

$$\begin{array}{ccc} Z \times X & \xrightarrow{g \times id} & \mathcal{H}_c \times X \\ & \searrow^{\tilde{g}} & \downarrow \omega \\ & & Y \end{array} ,$$

is continuous.

Consequently, $\mathcal{H}_c$ is a subspace of $C_c(X,Y)$. Further, each point x of X induces a continuous map

$$\hat{x} : \mathcal{H}_c \longrightarrow Y, \quad \text{with} \quad \hat{x}(f) = f(x),$$

called the <u>point evaluation at</u> x. The resulting map

$$i^{\mathcal{H}}_{X,Y} : X \longrightarrow \hat{X}_c \subseteq C_c(\mathcal{H}_c, Y)$$

is also continuous, by the UP. We reserve the letter i for these maps, sloughing as many of the indices as we can.

However, $\mathcal{H}$ can receive the <u>structure of pointwise convergence</u>, the initial structure induced by the family $\hat{X}$. We denote the space so obtained by $\mathcal{H}_s$, and observe that

$$id : \mathcal{H}_c \longrightarrow \mathcal{H}_s$$

is continuous. Further, we shall mostly use this structure when Y is topological, in which case it is a topology too. Finally, nothing in its definition makes use of the convergence structure on X; thus H_s is well-defined, as above, whenever X is a set, $H \subseteq Y^X$, and Y is a convergence space.

Returning to the subject at hand, we suppose that X, X' and Y are all spaces, and that $g : X \longrightarrow X'$ is continuous. In the natural way, g induces a map

$$g^\circ : C_c(X',Y) \longrightarrow C_c(X,Y)$$
$$f \longmapsto f \circ g$$

which is actually continuous also.

Our introduction to the structure of continuous convergence ends here; those interested are referred to [4], in which its general properties are thoroughly studied.

0.3 Algebras and homomorphisms.

Here we set out our algebraic conventions, giving a few remarks of a purely algebraic nature, before turning to convergence algebras. Their carrier spaces and universal representations are defined as in [5]. As the theory is explained, it is applied to C_cX, and later again in chapter 3 in our study of topological algebras.

We use the more-or-less standard symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of natural numbers, rationals, strictly positive reals, reals and complex numbers, all carrying the usual metric topology. Also, $\mathbb{F}$ means $\mathbb{R}$ or $\mathbb{C}$, and Δ the unit ball in $\mathbb{F}$.

Throughout, the word ' algebra ' is taken to mean an associative, commutative $\mathbb{F}$-algebra. Further, unless otherwise stated, every algebra has a multiplicative identity element 1 unequal to the additive identity 0. The term homomorphism is reserved for $\mathbb{F}$-algebra homomorphisms, taking 1 to 1, whenever both the range and domain algebras have 1.

If A is an $\mathbb{R}$- ($\mathbb{C}$-) algebra, we denote by Hom A the set of all $\mathbb{R}$- ($\mathbb{C}$-) valued homomorphisms of A. This set can be equipped with the topology of pointwise convergence, described on the previous page. Thus defined, $\mathrm{Hom}_s A$ is a completely regular topological space, in fact, realcompact.

Our purely algebraic remarks will be mainly concerned with $\mathbb{C}$-algebras with involution, and the complexification of real algebras.

If A is a $\mathbb{C}$-algebra, a map $^\circ : A \longrightarrow A$ is called an <u>involution</u> iff it is self-inverse, satisfying also

$$(\lambda x + \mu y)^{\circ} = \lambda^{-} x^{\circ} + \mu^{-} y^{\circ}, \text{ and } (xy)^{\circ} = x^{\circ} y^{\circ},$$

for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. Let $\mathrm{Hom}^{\circ} A$ be the set of all $^{\circ}$-true members of $\mathrm{Hom}\ A$; that is,

$$h \in \mathrm{Hom}^{\circ} A \text{ iff } h \in \mathrm{Hom}\ A, \text{ and}$$
$$h(a^{\circ}) = h(a)^{-}, \text{ for all } a \in A.$$

The complex conjugation in $\mathbb{C}$ is always written $^{-}$ in this thesis, as above.

To each $h \in \mathrm{Hom}\ A$ corresponds a conjugate homomorphism $h^{*} \in \mathrm{Hom}\ A$, with $h^{*} : a \longmapsto h(a^{\circ})^{-}$, for all $a \in A$. Clearly $(h^{*})^{*} = h$, for all $h \in \mathrm{Hom}\ A$.

The algebra with involution $(A, ^{\circ})$ is called <u>fully symmetric</u> iff every member of $\mathrm{Hom}\ A$ is self-conjugate, or equivalently, iff $\mathrm{Hom}^{\circ} A = \mathrm{Hom}\ A$, since the members of $\mathrm{Hom}^{\circ} A$ are exactly the self-conjugate homomorphisms.

An element $a \in A$ is <u>Hermitian</u> iff $a = a^{\circ}$, and just as in $\mathbb{C}$, each point $x \in A$ admits an unique decomposition into Hermitian components

$$\tfrac{1}{2}(x + x^{\circ}) \quad \text{and} \quad -\tfrac{1}{2}\iota(x - x^{\circ}),$$

whereby $\iota \in \mathbb{C}$, with $\iota^2 = -1$. The set $A^{\#}$ of all Hermitian elements of A forms an $\mathbb{R}$-algebra containing 1. One can see that a homomorphism $h \in \mathrm{Hom}\ A$ is self-conjugate iff $h(A^{\#}) = \mathbb{R}$.

These observations are collected in the next proposition. To do this neatly, we make another definition, applicable to any $\mathbb{F}$-algebra whatsoever. An ideal M of an $\mathbb{F}$-algebra A is an <u>$\mathbb{R}$-ideal</u> (or a <u>$\mathbb{C}$-ideal</u>) iff the algebra A/M is isomorphic to $\mathbb{R}$ (respectively, to $\mathbb{C}$), and an <u>$\mathbb{F}$-ideal</u> iff it is either an $\mathbb{R}$-ideal or a $\mathbb{C}$-ideal. Obviously all $\mathbb{F}$-ideals are maximal, and no $\mathbb{C}$-algebra possesses $\mathbb{R}$-ideals. However, an $\mathbb{R}$-algebra may indeed have $\mathbb{C}$-ideals - for example, $\mathbb{C}$ regarded as an $\mathbb{R}$-algebra.

<u>Proposition</u> 0.8 : *Let* A *be an algebra with involution* $^\circ$.

i) The topological spaces $\mathrm{Hom}_s^\circ A$ *and* $\mathrm{Hom}_s A^{\#}$ *are in a natural way homeomorphic.*

ii) A *is fully symmetric iff* $A^{\#}$ *has no* $\mathbb{C}$*-ideals.*

iii) A *is fully symmetric when* $1 + a^2$ *is a multiplicative unit of* A*, for all Hermitian elements* $a \in A$.

iv) $\mathrm{Hom}^\circ A$ *is a closed subset of* $\mathrm{Hom}_s A$.

Proof: Once the set-theoretic content of i has been shewn, its topological part, and iv also, follow easily with standard methods.

It is enough, then, to write down a map $\alpha : \mathrm{Hom}\, A^{\#} \longrightarrow \mathrm{Hom}^\circ A$ and a map $\beta : \mathrm{Hom}^\circ A \longrightarrow \mathrm{Hom}\, A^{\#}$, and point out that they are mutually inverse. For any $x \in A$, with Hermitian components a and a', and any homomorphism $h_0 \in \mathrm{Hom}\, A^{\#}$, we define

$$\alpha(h_0)(x) = h_0(a) + \imath h_0(a').$$

To verify that $\alpha(h_0)$ is a homomorphism is not hard, and left to the reader. Equally easily one sees that $\alpha(h_0)$ is

$^\circ$-true. On the other hand, the map β is more simply given; for $h \in \mathrm{Hom}^\circ A$, we define $\beta(h)$ to be the restriction of h to $A^{\#}$. The details of shewing that α and β are inverse are left out.

Claim ii is proven, when one observes that to each $\mathbb{C}$-ideal of $A^{\#}$ correspond two distinct mutually conjugate homomorphisms in $\mathrm{Hom}\, A \setminus \mathrm{Hom}^\circ A$, and conversely, for each $h \in \mathrm{Hom}\, A \setminus \mathrm{Hom}^\circ A$, the ideals

$$A^{\#} \cap \mathrm{Ker}\, h \quad \text{and} \quad A^{\#} \cap \mathrm{Ker}\, h^{*}$$

coincide, being actually $\mathbb{C}$-ideals of $A^{\#}$.

Last, if A is not fully symmetric, there is a homomorphism $h \in \mathrm{Hom}\, A$ for which $h(A^{\#}) = \mathbb{C}$. Thus there is $a \in A$ with $h(a) = \imath$, and $h(1 + a^2) = 0$. This is not possible, if $1 + a^2$ is a unit of A, and $h(1) = 1$. With this, the proof is at least completely sketched.

The algebra $C(X,\mathbb{R})$ is the set of all Hermitian elements of $C(X,\mathbb{C})$, for any convergence space X. Furthermore $1 + f^2$ is invertible, whenever f is a real-valued continuous function on X. Hence we have

<u>Corollary</u> 0.9 : *Given any space X, the algebra $C(X,\mathbb{C})$ is fully symmetric (with respect to the involution induced from conjugation in $\mathbb{C}$), and there is a canonical homeomorphism between $\mathrm{Hom}_s C(X,\mathbb{R})$ and $\mathrm{Hom}_s C(X,\mathbb{C})$.*

A very similar theory can be built up for $\mathbb{R}$-algebras, with the help of their complexification; the <u>complexification</u> A^2 of an $\mathbb{R}$-algebra A is the set $A \times A$, in which addition is defined componentwise, multiplication by the formula

$$(a,a')(b,b') = (ab - a'b', ab' + a'b),$$

and scalar multiplication similarly. The resulting $\mathbb{C}$-algebra whose multiplicative identity is $(1,0)$, possesses an involution

$$^\circ : (a,a') \longmapsto (a,-a').$$

The image of the injective $\mathbb{R}$-homomorphism

$$\sigma : A \longrightarrow A^2, \text{ defined by } \sigma(a) = (a,0),$$

is exactly the set of Hermitian elements of A^2. Similarly a $^\circ$-true $\mathbb{C}$-isomorphism can be defined between any algebra with involution $(A,^\circ)$, and the complexification of its Hermitian subalgebra, under which

$$x \longmapsto \left(\tfrac{1}{2}(x + x^\circ) , -\tfrac{1}{2}\imath(x - x^\circ) \right).$$

In particular, the algebras $C(X,\mathbb{C})$ and $C(X,\mathbb{R})^2$ are $\mathbb{C}$-isomorphic, for all spaces X.

Accordingly it makes no difference from a purely algebraic standpoint, whether one studies algebras with involution or $\mathbb{R}$-algebras and their complexifications. We shall usually take the latter point of view.

Convergence structures appear again now - we shall be rather informal, and define only the term ' convergence algebra '. Other such terms used here have analogous definitions.

A <u>convergence algebra</u> A_Λ is a $\mathbb{F}$-algebra A, together with a convergence structure Λ, such that the addition and multiplication maps: $A_\Lambda \times A_\Lambda \longrightarrow A_\Lambda$, and the scalar multiplication map: $\mathbb{F} \times A_\Lambda \longrightarrow A_\Lambda$ are continuous.

We could actually discuss universal algebras (with or without external operations), and obtain the following straightforward results:

1) Subalgebras of a convergence universal algebra (CUA) are also CUAs, as subspaces of their parent.

2) The product of a family of CUAs all of the same type is again a CUA.

3) The methods of [4, section 2] can be used to show that whenever X is a convergence space, and A a CUA, then $C_c(X,A)$ is a CUA of the same type, with respect to the operations induced pointwise from A.

4) If A is a CUA without external operations, whose internal operations are all finitary, and if $\Sigma : A \longrightarrow A'$ is any surjective homomorphism, then A' is a CUA under the final structure induced by Σ.

As a particular case of the third remark above, we have

<u>Theorem</u> 0.10 : *The algebra* $C_c X$ *is a locally convex convergence algebra, for all spaces* X. *Furthermore, the mod function* $: f \longmapsto |f|$ *is continuous.*

All that remains to be explained is the term ' local convexity '. A convergence vector space E_Λ over $\mathbb{F}$ is called <u>locally convex</u> iff whenever $\phi \to 0$ in E_Λ, there is a coarser filter ϕ' on E, which also converges to 0 in E_Λ, and has a base of absolutely convex sets. (This definition clearly coincides with the usual one, for topological vector spaces.)

The local convexity of $C_c X$ is easily verified directly; in fact though, for $\theta \to 0$ in $C_c X$, the filter θ_0 to be given in Lemma 0.17 has a base of absolutely convex sets.

Local convexity has the same permanence properties as in the topological case; namely, subspaces and products of locally convex convergence vector spaces are also locally convex, as is the complexification of a locally convex convergence vector space over $\mathbb{R}$.

Later the duality between a Banach algebra A and its carrier set Hom A is extended to topological algebras. In doing this, one should consider only those homomorphisms over which one has some control - the continuous ones. This selection principle is vacuous for Banach algebras, since each $\mathbb{F}$ -valued homomorphism is continuous for these algebras.

With this in mind, for any convergence algebra A_Λ, we call the set

$$\mathit{Hom}\ A_\Lambda = \{h \in \text{Hom } A \mid h \text{ is continuous}\}$$

the <u>carrier set</u> of A_Λ. There is no reason a priori for knowing if this set is vacant - the set Hom A itself may be empty (for example, Hom $\mathbb{C} = \emptyset$, when $\mathbb{C}$ is considered as $\mathbb{R}$-algebra). We assume from now on that <u>whenever</u> $\mathit{Hom}\ A_\Lambda$ <u>is</u>

mentioned, it is non-void. This assumption is of the same character as that banishing the empty convergence space: it is not necessary, but convenient.

Since *Hom* $A_\Lambda \subseteq CA_\Lambda$, it makes sense to consider the convergence space $Hom_c A_\Lambda$, which is called the carrier space of A_Λ.

If A_Λ, and $A'_{\Lambda'}$ are convergence algebras, and $u: A_\Lambda \longrightarrow A'_{\Lambda'}$ a continuous homomorphism, we get a map

$$u^\circ : Hom_c A'_{\Lambda'} \longrightarrow Hom_c A_\Lambda, \text{ with } \quad h \longmapsto h \circ u.$$

By using the UP of the structure of continuous convergence it is easy to shew that u° is continuous. Obviously, u° is a homeomorphism, when u is a bicontinuous isomorphism.

Our next result concerns the complexification of a convergence $\mathbb{R}$-algebra A_Λ, and is a ' continuous ' version of Proposition 0.8.

Proposition 0.11 : *Let* A_Λ *be a convergence* $\mathbb{R}$*-algebra. Then*

i) its complexification $A_\Lambda{}^2$ *(together with the product convergence structure) is a convergence* $\mathbb{C}$*-algebra with continuous involution, and*

ii) the spaces $Hom_c A_\Lambda$ *and* $Hom^\circ_c A_\Lambda{}^2$ *are homeomorphic.*

Proof: We shall first shew that the multiplication map

$$m : A_\Lambda{}^2 \times A_\Lambda{}^2 \longrightarrow A_\Lambda{}^2$$

is continuous. Let the real and imaginary projections of $A_\Lambda{}^2$ on A_Λ be π_r and π_i respectively. Thus, to prove the continuity of m, it is enough to shew that of $\pi_r \circ m$, and

$\pi_i \circ m$. To this end, consider the commuting diagramme

$$\begin{array}{ccc} A_\Lambda{}^2 \times A_\Lambda{}^2 & \overset{m}{\dashrightarrow} & A_\Lambda{}^2 \\ \downarrow{\cdot\times\cdot} & & \downarrow{\pi_r} \\ A_\Lambda \times A_\Lambda & \xrightarrow{-} & A_\Lambda \end{array} \quad \text{, with} \quad \begin{array}{ccc} (a,a';b,b') & \longmapsto & (ab - a'b', ab' + a'b) \\ \downarrow & & \downarrow \\ (ab, a'b') & \longmapsto & ab - a'b' \end{array}$$

and its imaginary counterpart.

As multiplication is continuous in A_Λ, and the product of continuous mappings is continuous, the lower path is continuous. Hence $\pi_r \circ m$ is also continuous, as claimed. Similar diagrammes and arguments prove the rest of the first part.

Turning now to the second part, we point out without proof that when $h_0 \in Hom\ A_\Lambda$, the homomorphism $\alpha(h_0)$ taking (a,a') in $A_\Lambda{}^2$ to $h_0(a) + \imath h_0(a')$ in $\mathbb{C}$ is also continuous. On the other hand, the map $\sigma : A_\Lambda \longrightarrow A_\Lambda{}^2$ given earlier, with $\sigma(a) = (a,0)$, is continuous. Accordingly, $\alpha^{-1}(h) = h \circ \sigma$ is continuous, for all $h \in Hom^\circ A_\Lambda{}^2$. Thus

$$\alpha : Hom\ A_\Lambda \longrightarrow Hom^\circ A_\Lambda{}^2$$

is a bijection.

We shall have shewn the continuity of α as soon as we know that

$$\tilde{\alpha} : Hom_c A_\Lambda \times A_\Lambda{}^2 \longrightarrow \mathbb{C}$$

is continuous. That this is so, derives immediately from the commuting diagramme given below, and its imaginary companion.

$$\begin{array}{ccccc}
Hom_c A_\Lambda \times A_\Lambda{}^2 & \xrightarrow{\tilde{\alpha}} & \mathbb{C}, & (h_0,a,a') \longmapsto & h_0(a) + \imath h_0(a') \\
\big\downarrow \text{projection} & & \big\downarrow \pi_r & \big\downarrow & \swarrow \\
Hom_c A_\Lambda \times A_\Lambda & \xrightarrow{\omega} & \mathbb{R} & (h_0,a) \longmapsto & h_0(a)
\end{array}$$

Similarly the commuting diagramme

$$\begin{array}{ccccc}
Hom^\circ_c A_\Lambda{}^2 \times A_\Lambda & \xrightarrow{\widetilde{\alpha^{-1}}} & \mathbb{R}, & (h,a) \longmapsto & h((a,0)) \\
\big\downarrow id \times \sigma & & \big\uparrow \pi_r & \big\downarrow & \big\uparrow \\
Hom^\circ_c A_\Lambda{}^2 \times A_\Lambda{}^2 & \xrightarrow{\omega} & \mathbb{C} & (h,a,0) \longmapsto & h((a,0))
\end{array}$$

establishes the continuity of α^{-1}, and completes the proof of the proposition.

It is not hard to verify that if $(A_\Lambda,{}^\circ)$ is an algebra with continuous involution and operations,then the natural $\mathbb{C}$-isomorphism

$$\begin{aligned} \rho : A_\Lambda &\longrightarrow (A^{\#}_\Lambda)^2 \\ x &\longmapsto \left(\tfrac{1}{2}(x + x^\circ), -\tfrac{1}{2}\imath(x - x^\circ) \right) \end{aligned}$$

is a homeomorphism.On applying this to $C_c X$, we have

<u>Proposition</u> 0.12 : *There is a bicontinuous isomorphism between the* $\mathbb{C}$*-algebras* $C_c(X,\mathbb{C})$ *and* $C_c(X,\mathbb{R})^2$*. Further the spaces* $Hom_c C_c(X,\mathbb{C})$ *and* $Hom_c C_c(X,\mathbb{R})$ *are canonically homeomorphic ,for all spaces* X.

The proof is omitted - it consists solely of applying the preceding three results.

In [5] Binz defined the <u>universal representation</u> (also known as the Gelfand map, as in Banach algebra theory) of a convergence $\mathbb{R}$-algebra A_Λ to be that map

$$d : A_\Lambda \longrightarrow C_c \mathit{Hom}_c A_\Lambda ,$$

given by $d(a)(h) = h(a)$, for all $h \in \mathit{Hom}\, A_\Lambda$ and $a \in A$. The definition is equally applicable to $\mathbb{C}$-algebras, of course.

<u>Theorem</u> 0.13 : *The universal representation of any convergence algebra is a continuous homomorphism.*

We omit the straightforward proof, which is to be found in [5].

Two concepts, symmetry and quasi-symmetry, are useful when dealing with convergence $\mathbb{C}$-algebras, for they allow us to apply Stone-Weierstrass theorems. A convergence algebra A_Λ is said to be <u>quasi-symmetric</u> iff the subalgebra $d(A)$ of $C\mathit{Hom}_c A_\Lambda$ is closed under complex conjugation. When A_Λ has an involution $^\circ$, which need not be continuous, it is called <u>symmetric</u> iff $\mathit{Hom}\, A_\Lambda = \mathit{Hom}^\circ A_\Lambda$. Equivalently, d is $^\circ$-true, that is, $d(a^\circ) = d(a)^-$, for all $a \in A$. In particular, symmetric algebras are quasi-symmetric.

In the last part of this section we restrict our attention to subalgebras of $C_c(X,\mathbb{R})$, for any convergence space X, proving that for some of these, including $C(X,\mathbb{R})$ itself, the carrier set consists exactly of all the point evaluations. This result was first given, for $C_c(X,\mathbb{R})$, in [6, Satz 1].

First, however, we fix our terminology: each subalgebra A of $C(X,\mathbb{R})$ inherits the pointwise-defined ordering. When A is also a sublattice of $C(X,\mathbb{R})$, it is called a sublattice-algebra. In this case, for any $h \in \operatorname{Hom} A$, we have

$$(h(f))^2 = h(f^2) = h(|f|^2) = (h(|f|))^2$$

and hence

$$h(|f|) = \pm|h(f)| ,$$

for all $f \in A$. We call A <u>monotone</u> iff it is a sublattice-algebra, and each homomorphism $h \in \operatorname{Hom} A$ is also a lattice homomorphism. Two simple sufficient conditions for this are given below without proof - the first is due to Isbell.

<u>Lemma</u> 0.14 : *i) A sublatticealgebra* A *of* $C(X,\mathbb{R})$ *is monotone if* $1 + |f|$ *is a unit in* A*, for all* $f \in A$.

ii) A subalgebra A *of* $C(X,\mathbb{R})$ *is monotone, if for each* $f \in A$*, the function* $\sqrt{|f|} \in A$ *also.*

We are now able to state our result formally:

<u>Proposition</u> 0.15 : *The continuous real-valued homomorphisms of a monotone subalgebra* A_c *of* $C_c(X,\mathbb{R})$ *are exactly the point evaluations.*

Proof: We have seen earlier that each point evaluation $\hat{x}: A_c \longrightarrow \mathbb{R}$ is continuous - it is clearly a homomorphism. Let now $h \in \operatorname{Hom} A \setminus i(X)$. There is then for each $x \in X$ a function $f_x \in A$, with $f_x(x) \neq h(f_x)$. One can obviously arrange that $h(f_x) = 0$ and that $f_x(x) = 1$, for all $x \in X$.

If there is a filter θ on A with a trace on Ker h, such that $\theta \to 1$ in A_c, then h can not be continuous, as the filter $h(\theta)$ does not converge to $h(1) = 1$ in $\mathbb{R}$. To complete the proof, we need only construct the filter θ, as follows -

For any $\varepsilon \in \mathbb{R}^+$, and any pair $x\phi$ from X, the set $B_{\varepsilon x} = f_x^{-1}(1 + \varepsilon\Delta)$ belongs to ϕ, by the continuity of f_x. (Recall that Δ denotes the unit ball in the scalar field.) Furthermore, the set

$$D_{\varepsilon x} = \{g \mid g \in A \text{ and } g(B_{\varepsilon x}) \subseteq 1 + \varepsilon\Delta\}$$

contains 1, and f_x. Thus

$$\mathcal{D} = \{D_{\varepsilon x} \mid \varepsilon \in \mathbb{R}^+ \text{ and } x \in X\}$$

generates a filter θ, converging to 1 in A_c. We must next verify that $\theta \cap$ Ker h.

To each set $D \in \theta$, there are $D_1, \ldots, D_\tau \in \mathcal{D}$ and corresponding indices ε_α and x_α, such that $D \supseteq \bigcap_{\alpha=1}^{\tau} D_\alpha$. However the function

$$f_D = 1 \wedge \left(\bigvee_{\alpha=1}^{\tau} f_{x_\alpha}\right)$$

clearly belongs to each set D_α, since A is a sublattice of $C(X,\mathbb{R})$. In addition, $h(f_D) = 0$, as A is even monotone. Our proof is now finished, since $f_D \in D \cap$ Ker h, as required.

This proof scarcely needs alteration, in shewing that each c-continuous linear lattice homomorphism of a vector sublattice of $C(X,\mathbb{R})$ into $\mathbb{R}$ is a positive multiple of a point evaluation.

By working a little harder, one can extend the result of Proposition 0.15 to locally bounded subalgebras of $C(X,\mathbb{R})$, this term being defined in paragraph 2.1 later. What happens for other, more general, subalgebras is still an open problem.

Corollary 0.16 : *For each convergence space* X, *the map*

$$i : X \longrightarrow Hom_c C_c(X,\mathbb{R})$$

is continuous and surjective.

Proof: Obvious, thanks to Lemma 0.14.

0.4 On c-embedded spaces.

This paragraph is shorter, and rather a mixed bag - we introduce on any convergence space X the initial topology induced by the family CX, and then a technical lemma, which will be used several times later. Finally c-embedded convergence spaces are defined as in [5] or [6], and some of their properties given. Much more information, and proof, is provided in these two references.

Let X be a space, and H a subset of CX. Then the family i(X) of point evaluations induces the s-<u>topology</u> on H, also known as the topology of pointwise convergence, or the weak topology. The resulting topological space H_s is in fact a subspace of C_sX. Because point evaluations are continuous on H_c, the identity map $id: H_c \longrightarrow H_s$ is also continuous.

For each space X, the initial structures induced on X by the families $C(X,\mathbb{R})$ and $C(X,\mathbb{C})$ are the same, and actually topologies. The set X together with this topology is denoted by wX. From the UPs for tX and wX, it is clear that $id: tX \longrightarrow wX$ is continuous. It is by no means necessary that wX be a Hausdorff topological space: it is Hausdorff iff CX separates the points of X.

The next result is the useful technicality mentioned above, in which the situation is the followng; X is a convergence space, and H a non-void subset of CX, and closure in the space H_s is denoted by $^{-}$.

Lemma 0.17 : *If* $f\theta$ *is a pair from* H_c, *then so is* $f\theta^-$, *where* θ^- *is the filter on* H *generated by the collection* $\{ B^- \mid B \in \theta \}$ *of s-closed sets.*

Proof: For each pair $x\phi$ from X and each positive real number λ, there are sets $B_{\lambda x\phi} \in \theta$ and $C_{\lambda x\phi} \in \phi$ with

$$\omega(B_{\lambda x\phi} \times C_{\lambda x\phi}) \subseteq f(x) + \lambda\Delta.$$

The collection

$$\{E(C_{\lambda x\phi}) \mid \lambda \in \mathbb{R}^+ \text{ and } x\phi \text{ is a pair from } X\},$$

whereby

$$E(C_{\lambda x\phi}) = \{g \in H \mid g(C_{\lambda x\phi}) \subseteq f(x) + \lambda\Delta\},$$

consists purely of s-closed sets, and generates a filter θ_0 on H. Obviously $\theta \geqq \theta^- \geqq \theta_0$, and $\theta_0 \to f$ in H_c.

Now, with the lemma proven, we note in particular that when $\theta \to 0$ in C_cX, the filter θ_0 constructed above has a base of absolutely convex sets. This verifies our earlier claim, that C_cX is locally convex.

It is shewn in [6, Satz 1] that the continuous map

$$i : X \longrightarrow Hom_c C_c(X, \mathbb{R})$$

is surjective, for all spaces X. We call X c-<u>embedded</u> iff this map is a homeomorphism. Equivalently, X is c-embedded iff $i : X \longrightarrow Hom_c C_c(X, \mathbb{C})$ is a homeomorphism, as one readily verifies with the help of Proposition 0.12 and a commuting diagramme. The results from [5] and [6] concerning c-embedded spaces are collected below.

Theorem 0.18 : *i) C_cX is c-embedded, for all spaces X.*

ii) Subspaces and products of c-embedded spaces are also c-embedded.

iii) Completely regular topological spaces are c-embedded.

iv) The universal representation

$$d: C_cX \longrightarrow C_c Hom_c C_c X$$

is a homeomorphism, for all spaces X.

Another property of c-embedded spaces is a variant of Lemma 0.17 . From the UPs of wX and $Hom_s C_c X$, we see that i: $wX \longrightarrow Hom_s C_c X$ is continuous, for all spaces X. Similarly, when i is injective, it is a homeomorphism. From this follows

Lemma 0.19 : *If X is a c-embedded space, and $x\phi$ is a pair from X, then so is $x\phi^-$, where $^-$ represents closure in wX.*

Proof: We know from Lemma 0.17 that $i(\phi)^- \to x$ in $Hom_c C_c X$, Hence, using the homeomorphism between wX and $Hom_s C_c X$, we deduce that

$$\phi^- = i^{-1}(i(\phi)^-) \to x \text{ in } X,$$

as required.

1 Covering systems.

1.1 Definition and apology.

In passing from topological to convergence spaces not only structure,but also language is lost: in this chapter is given a generalisation to convergence spaces of the topological term ' covering system of neighbourhoods '. This allows us to extend to convergence spaces many ideas and results in topology,which are expressed in such terms.For example, local compactness,paracompactness,axioms of countability, and so on.These are not idle extensions: in chapter 3 we use local compactness; in his thesis [15], Feldman characterises among other things, second countable completely regular topological spaces (that is, second countable metric) as those completely regular spaces X for which C_cX is also second countable.

To come to the point (at last),let X be a convergence space,and $\mathcal{S}$ a nonvoid collection of nonvoid subsets of X. The final family of inclusions $\{j_S \mid S \in \mathcal{S}\}$ - with each S considered as a subspace of X - induces a structure on X, creating thereby a space written $X_{\mathcal{S}}$. It holds always that id : $X_{\mathcal{S}} \longrightarrow X$ is continuous. We answer the obvious question in the next lemma,after two definitions.

A collection $\mathcal{S}$ of subsets of a space X is

1) a <u>quilt</u> <u>for</u> X iff there is for each pair $x\phi$ from X a finite subset $\mathcal{S}_{x\phi}$ of $\mathcal{S}$ such that

α) $x \in \bigcap \mathcal{S}_{x\phi}$, β) $\bigcup \mathcal{S}_{x\phi} \in \phi$, and

γ) $\phi \cap S$, for each $S \in \mathcal{S}_{x\phi}$;

2) a <u>covering system for</u> X iff for each pair $x\phi$ from X there is $S_{x\phi} \in \mathcal{S}$ such that $x \in S_{x\phi} \in \phi$.

Note: 1) Every covering system is a quilt. A quilt $\mathcal{Q}$ which is directed upwards (meaning that to any $S, S' \in \mathcal{Q}$ there is $S'' \in \mathcal{Q}$ with $S \cup S' \subseteq S''$) is already a covering system.

2) We can always assume that $X = \bigcup \mathcal{S}$, without affecting the space $X_{\mathcal{S}}$; precisely, $X_{\mathcal{S}}$ and $X_{\mathcal{S}'}$ coincide for all collections $\mathcal{S}$, when $\mathcal{S}' = \mathcal{S} \cup \{\{x\} \mid x \in X\}$. With this assumption, which is always made from now on, $\mathcal{S}$ is a quilt for $X_{\mathcal{S}}$.

We next shew that from nothing less than a quilt can the parent space be wholly reconstructed.

<u>Lemma</u> 1.1 : *Let* X *be a convergence space, and* $\mathcal{S}$ *a collection of subsets of* X *(with* $X = \bigcup \mathcal{S}$ *). Then* X *and* $X_{\mathcal{S}}$ *are the same space iff* $\mathcal{S}$ *is a quilt for* X.

Proof: First, let $\mathcal{S}$ be a quilt for X. We need only shew that every pair $x\phi$ from X is also a pair from $X_{\mathcal{S}}$. Now for each $S \in \mathcal{S}_{x\phi}$, we know that the filter $\phi \cap S$ converges to x in S. However, conditions β and γ guarantee that

$$\phi = \wedge \{j_S(\phi \cap S) \mid S \in \mathcal{S}_{x\phi}\},$$

and hence converges to x in $X_{\mathcal{S}}$.

On the other hand, let $X = \cup S$, and $X = X_S$. This means that for any pair $x\phi$ from X there are sets $S_1, \ldots, S_\rho \in S$ and filters $\phi_1, \ldots, \phi_\rho$ on X such that

i) $\phi_\alpha \cap S_\alpha$ and $x \in S_\alpha$, for each $\alpha = 1, \ldots, \rho$,

ii) $\phi_\alpha \cap S_\alpha \to x$ in S_α, for all indices, and

iii) $\phi \geqq \bigwedge_{\alpha=1}^{\rho} j_{S_\alpha}(\phi_\alpha \cap S_\alpha)$.

The third condition shews in particular that $S_1 \cup \ldots \cup S_\rho \in \phi$. Now define $S_{x\phi} = \{S_\alpha \mid \phi \cap S_\alpha\}$. It is not possible for this set to be empty - as if it were, there would be for each index α a set $B_\alpha \in \phi$, with $B_\alpha \cap S_\alpha = \emptyset$. This would imply

$$\emptyset = (\bigcup_{\alpha=1}^{\rho} S_\alpha) \cap \bigcap_{\alpha=1}^{\rho} B_\alpha \in \phi,$$

which is not allowable. A similar argument shews that the set $\bigcup S_{x\phi} \in \phi$, completing the proof.

When dealing with c-embedded spaces, it is enough to consider only those covering systems which are made up from w-closed sets. For if X is c-embedded, and S is a cover for X (naturally, the phrase ' covering system ' is often shortened to ' cover ') then we can give a cover T for X refining S, and composed entirely of w-closed subsets of X, in the following way:

For any pair $x\phi$ from X, the filter ϕ^- also converges to x in X. There is then a set $S_{x\phi^-} \in S$, with $x \in S_{x\phi^-} \in \phi^-$. This means, though, there is a w-closed set $T_{x\phi^-} \in \phi^-$, for which $x \in T_{x\phi^-} \subseteq S_{x\phi^-}$. Clearly the collection $\{T_{x\phi^-} \mid x\phi \text{ is a pair from } X\}$ is a cover, satisfying the claim.

The construction of the space X_S can be usefully dualised, as we now describe. Given a space X and a collection S of subsets of X, we get an initial system of mappings

$$\{j_S^{"} : CX \longrightarrow C_c S \mid S \in S\}$$

on CX, obtaining from it the space $C_S X$. It is easy to check that id: $C_c X \longrightarrow C_S X$ is always continuous.

However, we shall shew that $C_c X$ and $C_S X$ coincide, when S is a quilt or, in particular, a cover for X. This means, in theory at least, that one can more easily see how the properties of $C_c X$ depend on the local properties of X. Possibly the simplest example of its use in this way is given in Proposition 1.11, where we show that for each locally compact space X, the space $C_c X$ is topological, carrying in fact the topology of uniform convergence on the compacta of X.

A converse question, whose answer could allow us to deduce local properties of X from properties of $C_c X$, is only partly answered: namely, under certain restrictions on S, when $C_c X$ and $C_S X$ coincide, S is a quilt for X.

Note: An analogous space $C_S(X,Y)$ can be given, for any space Y, but to investigate these would be beyond the scope of this thesis.

Next it is shewn that the algebraic operations on $C_S X$ are continuous, and then a sequence of calculations is given, at the end of which we shall have proven the claims just made.

<u>Proposition</u> 1.2 : *For all spaces* X, *and non-void collections* $\mathcal{S}$ *of subsets of* X, $C_{\mathcal{S}}X$ *is a convergence algebra, and further, the mod function:* $C_{\mathcal{S}}X \longrightarrow C_{\mathcal{S}}X$ *(under which* $f \longmapsto |f|$ *) is continuous as well.*

Proof: It will be enough to do this for the addition map ad; the other proofs are all similar to this.

The diagrammes

$$\begin{array}{ccc} C_{\mathcal{S}}X \times C_{\mathcal{S}}X & \xrightarrow{\;j_S{}^{\circ}\circ \text{ad}\;} & C_cS \\ {\scriptstyle j_S{}^{\circ}\times j_S{}^{\circ}}\searrow & & \nearrow{\scriptstyle \text{ad}} \\ & C_cS \times C_cS & \end{array}$$

commute, shewing the continuity of $j_S{}^{\circ}\circ \text{ad}$, for all $S \in \mathcal{S}$. That of ad itself now follows from the UP for $C_{\mathcal{S}}X$.

Our sequence of calculations starts here, ending at Theorem 1.4.

<u>Lemma</u> 1.3 : *For any space* X, *if* $\mathcal{S}$ *is a quilt, then* C_cX *and* $C_{\mathcal{S}}X$ *are the same space.*

Proof: It is enough to shew that $\theta \to 0$ in C_cX, whenever 0θ is a pair from $C_{\mathcal{S}}X$. So, let $x\phi$ be a pair from X. Then

$$\omega_S(j_S{}^{\circ}(\theta) \times \phi \dot{\cap} S) \to 0 \text{ in } C_cS$$

for each $S \in \mathcal{S}_{x\phi}$, where ω_S is the corresponding evaluation map. Hence for any positive real number ε, there is $A_S \in \theta$ and $B_S \in \phi$, such that

$$\omega(A_S \times (B_S \cap S)) \subseteq \varepsilon \triangle,$$

for each $S \in S_{x\phi}$. It follows that $\omega(A \times B) \subseteq \varepsilon\Delta$, where

$$A = \bigcap\{A_S \mid S \in S_{x\phi}\} \in \theta, \text{ and}$$

$$B = (\bigcup S_{x\phi}) \cap \bigcap\{B_S \mid S \in S_{x\phi}\} \in \phi.$$

This shews that $\theta \to 0$ in $C_c X$, and proves the lemma.

<u>Lemma</u> α: *Let* X *be any convergence space, and* S *a collection of* <u>*w-closed*</u> *subsets of* X, *such that* $C_c X$ *and* $C_S X$ *are the same. Then* S *is a quilt for* X.

Proof: We suppose that S is not a quilt for X, that is, there is a pair $x\phi$ from X at which S fails to satisfy requirements. For each $y \in X \setminus \{x\}^-$, we can find a w-closed neighbourhood B_y of y in wX, which misses x. We set

$$A_Y = \{ f \in CX \mid f(B_y) \subseteq \Delta, \text{ for all } y \in Y\},$$

whenever Y is a finite subset of $X \setminus \{x\}^-$.

Next let D be a fixed w-closed neighbourhood of x in wX, and $B_T = D \cap \bigcup T$, for any finite subset T of S. Now we put

$$A_T = \{ f \in CX \mid f(B_T) \subseteq \Delta\}.$$

Clearly the collection

$$\{ \lambda A_Y \mid \lambda \in \mathbb{R}^+ \text{ and } Y \text{ is a finite subset of } X \setminus \{x\}^-\}$$

$$\cup \{ \mu A_T \mid \mu \in \mathbb{R}^+ \text{ and } T \subseteq S \text{ is finite } \}$$

has the finite intersection property (when B_T is void, A_T is naturally CX itself), generating a filter θ on CX.

Since whenever $x \in S \in \mathcal{S}$, the set $D \cap S$ is a neighbourhood of x in tS, the filter θ converges to $\mathcal{O}$ in $C_S X$. However it can not converge in $C_c X$. To see this, we argue three cases, corresponding to non-fulfilment of the three conditions in the definition of a quilt.

Case 1: $x \in X \setminus \bigcup \mathcal{S}$. For any indices Y and $\mathcal{T}$, the w-closed set $B_{\mathcal{T}} \cap \bigcup\{B_y \mid y \in Y\}$ does not contain x. Thus there is $f \in CX$, vanishing on this set and taking the value 1 at x. We deduce from this that the filter $\omega(\theta \times \dot{x})$ is the indiscrete filter $\{\mathbb{F}\}$ on $\mathbb{F}$, and in particular, does not converge. Hence θ also does not converge, in $C_c X$.

Case 2: for all $S \in \mathcal{S}$, either $x \notin S$ or ϕ has no trace on S. Using again the set $B_{\mathcal{T}} \cap \bigcup\{B_y \mid y \in Y\}$ given above, we see that its complement in X is an open set; so for any set $C \in \phi$, there is a point x' of C, lying outside that closed set. As before, we can conclude that the filter $\omega(\theta \times \phi)$ is indiscrete, and that θ does not converge in $C_c X$. The last remaining possibility, in which we suppose $\mathcal{S}_0$ to be that non-void subset of $\mathcal{S}$, consisting of sets containing x, on which ϕ has a trace, is that whenever $\mathcal{T}$ is a finite subset of $\mathcal{S}_0$, then $\bigcup \mathcal{T} \notin \phi$. However it too is easily disposed of in the same way as above.

With this, the proof of the lemma is complete.

Assembling these lemmas, we obtain

Theorem 1.4 : *Let* X *be a convergence space, and* S *a collection of* w-*closed subsets of* X. *Then* $C_S X$ *and* $C_c X$ *coincide iff* S *is a quilt.*

That this theorem is not as general as it might be, is shewn by a simple example: when $X = \mathbb{R}$, and $S = \{\mathbb{Q}\}$, the homomorphism $j_{\mathbb{Q}}^{\bullet} : C_c\mathbb{R} \to C_c\mathbb{Q}$ is an embedding, as can easily be seen. Thus when S contains sets which are not w-closed, it may fail to be a quilt, even if $C_c X$ and $C_S X$ are the same.

This example, and the need to prove an analogue of Lemma 1.3 guide us to the following definition:

A collection S of subsets of a space X has property ($\bullet$) iff for each pair $x\phi$ from X there is a finite subset $S_{x\phi}$ of S such that $S_{x\phi}^{-} = \{S^{-} | S \in S_{x\phi}\}$ has the properties demanded in the definition of a quilt, and further, for each $B \in \phi$ and $S \in S_{x\phi}$, there is $B_S \in \phi$, with $B_S \subseteq B$, and

$$(B_S \cap S)^{-} = B_S^{-} \cap S^{-}.$$

This enables us to state two lemmas, the first proved in the same way as Lemma 1.3, the second a corollary of Lemma α.

Lemma β: *Let* X *be a convergence space, and* S *have property* ($\bullet$). *Then* $C_S X$ *and* $C_c X$ *are equal.*

Lemma γ: *If* $C_c X = C_S X$, *then* $S^{-} = \{S^{-} | S \in S\}$ *is a quilt for* X.

1.2 Applications to compactness.

In this section we shew by example that it is often possible with covering systems to use arguments well-known in topology. In particular, compact convergence spaces admit a characterisation by covering systems, mirroring the original Borel-Lebesgue definition. The one point compactification is then introduced for convergence spaces, where it is shewn to retain (most of) its usual properties. Finally, locally compact convergence spaces are defined and briefly studied.

First, the definitions needed: a space X is said to be T_2 iff each convergent filter on X has exactly one limit (equivalently, the cluster set of every convergent filter contains exactly one point), and compact iff every ultrafilter on X converges.

Note: We do not include T_2 in our definition of compactness.

A list of the properties of compact spaces follows.

Theorem 1.5 : 1) *Tychonov's theorem holds.*

2) *Closed subspaces of compact spaces are compact.*

3) *A compact subspace of a T_2 space is closed.*

4) *The continuous image of a compact space is compact.*

5) *Any finite union of compact subsets of a space is again compact.*

6) *The following statements are equivalent:*

i) X is compact.

ii) The cluster set of every filter on X is non-void.

iii) For each covering system S for X there is a finite subset S_0 of S with $\bigcup S_0 = X$.

iv) For each quilt Q for X there is a finite subset Q_0 of Q with $\bigcup Q_0 = X$.

Proof: Statements 1 to 5 are easy to prove, either directly or with the help of 6. Fischer [2] observed the equivalence of i and ii in 6, which we now prove in its entirety.

That iv implies iii is obvious. Now let us consider ' iii implies i '. Suppose there is a non-convergent ultrafilter ψ on X. This means that ψ is not finer than $\phi \wedge \dot{x}$, for all pairs $x\phi$ from X. There is thus a set $A_{x\phi} \in \phi \wedge \dot{x}$, with $X \setminus A_{x\phi} \in \psi$. Now the family

$$\{ A_{x\phi} \mid x\phi \text{ is a pair from } X \}$$

is a covering system for X, not satisfying iii.

Next comes ' i implies ii '. Let X be compact, and ξ a filter on X. Zorn's lemma provides us with an ultrafilter χ on X, finer than ξ, and which converges, thanks to compactness. Hence the cluster set of ξ is non-void.

Last, ' ii implies iv '. Suppose Q is a quilt for X not satisfying iv. Then $Q' = \{X \setminus Q \mid Q \in Q\}$ has the finite intersection property, and generates a filter τ. We claim that the cluster set of τ is empty. To see this, we note that for each pair $x\phi$ from X, the set $\bigcup Q_{x\phi} \in \phi$, whereas $X \setminus \bigcup Q_{x\phi} \in \tau$. Consequently ϕ is not finer than τ, which proves the claim, and completes the proof of the theorem.

With the help of this, a straightforward ' classical ' proof of a result of Binz's can be given [8]; namely that C_cX carries the topology of uniform convergence on X, when X is a compact convergence space. (For any space X, we denote this topology, and the topology of uniform convergence on the compact subsets of X, by C_nX and C_kX respectively) Formally, the result is

Proposition 1.6 : *For each convergence space X, the identity map*

$$\mathrm{id}: C_nX \longrightarrow C_cX$$

is continuous. When X is compact, it is a homeomorphism.

Before proving this, we need a technicality, of the same ilk as Lemma 0.17.

Lemma 1.7 : *Let X be a convergence space, H a non-empty subset of CX, and fθ a pair from H_c. Then to each positive real number λ, there is a cover S(λ,θ) for X by sets closed in the initial topology induced on X by H.*

Proof of lemma: There are sets $B_{x\phi} \in \theta$ and $C_{x\phi} \in \dot{x} \wedge \phi$ such that

$$\omega(B_{x\phi} \times C_{x\phi}) \subseteq f(x) + \lambda\Delta$$

holds, for each pair $x\phi$ from X. After defining

$$E(\lambda, x\phi) = \{y \mid y \in X \text{ and } |g(y) - f(x)| \leq \lambda, \text{ for all } g \in B_{x\phi}\},$$

we see first that $E(\lambda, x\phi) \supseteq C_{x\phi}$ and hence that

$$\{ E(\lambda, x\phi) \mid x\phi \text{ is a pair from } X \}$$

is a cover for X, and second that all the sets $E(\lambda, x\phi)$ are closed in the initial topology generated by H.

This done, we can prove our proposition as well.

Proof of proposition 1.6: It is shewn in [3] in a much more general framework that

$$id: C_n X \longrightarrow C_c X$$

is always continuous: it is, however, easily verified directly. In any case, the proof is left out.

Now let X be compact. To shew that id^{-1} is continuous in this case, it is enough to shew that $\theta \to 0$ in $C_n X$, whenever $\theta \to 0$ in $C_c X$. For each positive number λ, the cover $S(\lambda, \theta)$ to be seen in the lemma above has a finite ' subcover ', say $\{E(\lambda, x_\kappa \phi_\kappa) \mid \kappa = 1, \dots, \zeta\}$. Hence

$$\omega\Big(\bigcap_{\kappa=1}^{\zeta} B_{x_\kappa \phi_\kappa} \times X\Big) \subseteq \lambda\Delta,$$

where the $B_{x\phi}$'s correspond to the $E(\lambda, x\phi)$'s in the lemma. This means, though, the convergence of θ to 0 in $C_n X$, as claimed.

It is perhaps worth pointing out that $C_n X$ and $C_c X$ may well be the same, without X's being compact - see Example 4.4 . This can not happen when X is c-embedded, as was shewn in [8], and will be shewn in another way here in chapter 4 . Even now the reader has enough information to shew the compactness of X, when X is c-embedded and the spaces $C_n X$ and $C_c X$ coincide, but not enough to shew that X is also topological.

Some more definitions are given next, the last for this section. The <u>one point compactification</u> (OPC) of a convergence space X (denoted by $\tilde{X}$) has $X \cup \{\infty\}$ as underlying set, for any point $\infty \notin X$. The convergence structure on X is specified by requiring

i) that for points x in X, the pair $x\phi$ is from $\tilde{X}$ iff $\phi \cap X \to x$ in X, and that

ii) $\phi \to \infty$ in $\tilde{X}$ iff $\phi \geq \phi_\infty$, where ϕ_∞ is that filter on X having $\{\tilde{X} \setminus K \mid K \text{ is a compact subset of } X\}$ as base. Clearly X is embedded in $\tilde{X}$, just as usual. Again as usual, the name ' one point compactification ' is misleading, for when X is compact already, it is not dense in $\tilde{X}$ - see the following theorem, in which we have assembled the simpler properties of OPC's.

A point x of a space X is said to be <u>locally compact</u> iff ϕ contains a compact set, whenever $x\phi$ is a pair from X. The set of all locally compact points of X is written X_1, and X itself is called <u>locally compact</u> iff it admits a cover composed entirely of compact subsets, or equivalently, iff each point is locally compact.

These definitions are tied together in

<u>Theorem</u> 1.8 : *1) $\tilde{X}$ is compact, for each space X.*

2) X is dense in $\tilde{X}$ iff X is not compact.

3) The sets $\tilde{X} \setminus X_1$ and $cl_{\tilde{X}}(\phi_\infty)$ are equal, for all spaces.

4) $\tilde{X}$ is T_2 iff X is locally compact and T_2.

Proof: We begin with 1, aiming to shew that every ultrafilter ψ on $\tilde{X}$ converges in $\tilde{X}$. Two possibilities arise: either ψ has a trace on K, for some compact subset K of X, or not. In the first case, there is a point $k \in K$ such that $\psi \cap K \to k$ in K, and also, we have $K \in \psi$. Then if j_K is the inclusion map of K in $\tilde{X}$, it follows that

$$\psi = j_K(\psi \cap K) \to k \text{ in } \tilde{X}.$$

In the other case, ψ is clearly finer than ϕ_∞, and thus converges to ∞ in $\tilde{X}$.

The second claim is just as easy, for if X is compact, then $\phi_\infty = \dot{\infty}$. Hence ϕ_∞ does not have a trace on X, shewing that X is closed in $\tilde{X}$.

Conversely, when X is not compact, ϕ_∞ has a trace on X, and accordingly, $a_{\tilde{X}}(X) = \tilde{X}$. In other words, X is dense in $\tilde{X}$.

To prove our third claim, we need a purely set-theoretic result, whose proof is omitted, as it is standard Zorn:

Let Y be a set, $\mathcal{K}$ a family of subsets of Y closed under finite unions, and ϕ'' a filter on Y disjoint from $\mathcal{K}$. Then the set of all filters on Y finer than ϕ'' and disjoint from $\mathcal{K}$ possesses maximal elements (with respect to the ordering $\geqq$), and for any such maximal element ψ,

$$\psi \supseteq \{Y \setminus K \mid K \in \mathcal{K}\}.$$

We start with the inequality $\tilde{X} \setminus X_1 \subseteq cl_{\tilde{X}}(\phi_\infty)$. Clearly $\infty \in cl_{\tilde{X}}(\phi_\infty)$. Also, for each $x \in X \setminus X_1$, there is a filter ϕ'

on X such that $x\phi'$ is a pair from X, and no member of ϕ' is compact. Applying the result quoted above here, with $Y = \tilde{X}$, and K the collection of all compact subsets of X, and for ϕ'' the filter $j(\phi')$ on $\tilde{X}$, we get a filter ψ on $\tilde{X}$, such that $\psi \to x$ in $\tilde{X}$, and $\tilde{X} \setminus K \in \psi$, for all compact subsets K of X. Hence $\psi \geqq \phi_\infty$, and so $x \in cl_{\tilde{X}}(\phi_\infty)$.

On the other hand, when $x \in cl_{\tilde{X}}(\phi_\infty) \cap X$, there is by definition a filter ϕ on $\tilde{X}$ with $\phi \geqq \phi_\infty$, which converges to x in $\tilde{X}$. This means that $\phi \cap X \to x$ in X. But obviously the filter $\phi \cap X \ni X \setminus K$, for all compact subsets of X, shewing that x is not a locally compact point of X.

We have thus proven the theorem, since the fourth part of it is a direct application of the part just shewn.

Locally compact spaces as such come now into question, some of their properties being given in the next theorem, in part 2 of which we get less than the usual results (mainly because in the case often discussed the space is locally compact Hausdorff topological, and one has rather more to work with). However new results appear in part 3, this fact being due to the difference between topological and convergence space final structures.

Before stating the theorem, let us recall our notation: for each g in a final family G of mappings into X, the domain of g is a set (convergence space) X_g. Now let K_0 be the set of all finite subsets of X, and K_1 the set of all images of compact sets: that is

$$\mathcal{K}_1 = \{ g(K) \mid K \text{ is a compact subset of } X_g, \text{ and } g \in G \}.$$

Finally, let $\mathcal{K}'$ be the closure of $\mathcal{K}_0 \cup \mathcal{K}_1$ under finite unions. Clearly the members of $\mathcal{K}'$ are all compact in X. In the third part of this theorem, we shew how far $\mathcal{K}'$ represents the compact subsets of X.

Theorem 1.9 : *1) If $\mathcal{X}$ is a family of convergence spaces, then the space $\Pi\mathcal{X}$ is locally compact iff all bar a finite number of the members of $\mathcal{X}$ are compact, the exceptions being locally compact.*

2) Closed subspaces of locally compact spaces are again locally compact.

3) Let $\mathcal{G}$ be a final family of mappings into a set X. Then i) if K is compact in X, there are finite subsets $\mathcal{G}_0$ of $\mathcal{G}$, and K_0 of X, such that

$$K \subseteq K_0 \cup \bigcup \{ g(X_g) \mid g \in \mathcal{G}_0 \}, \text{ and}$$

ii) when each domain space is locally compact, then so is X, and $\mathcal{K}'$ is a cover for X. If K is a compact subset of X, there is K′ in $\mathcal{K}'$ containing K. When X is T_2, then K itself belongs to $\mathcal{K}'$.

Proof: We prove only part 3, since there is nothing non-standard involved in the first two.

First, we note that covering systems can be cut down to subspaces in the same way as in topology, meaning that if Y is a convergence space, and A a subspace of Y, then each cover $\mathcal{S}$ for Y yields a cover $\mathcal{S} \cap A = \{ S \cap A \mid S \in \mathcal{S} \}$ for A. Quilts can be cut down in this way, too.

To see 3i, we need only observe that $\mathcal{K}_0 \cup \{g(X_g) \mid g \in G\}$ forms a quilt for X, and then use Theorem 1.5iv.

Now when every space X_g is locally compact, it follows immediately from the definition of the final structure that $\mathcal{K}'$ is a cover for X. However $\mathcal{K}'$ is composed wholly of compact sets, as remarked before, which shews the local compactness of X.

Since $\mathcal{K}'$ is a cover for X closed under finite unions, when K is compact in X, there is $K' \in \mathcal{K}'$ containing K, which was our next claim. More precisely, there is a finite subset K_0 of X, and a finite subset G_0 of G, with compact subspaces K_g of X_g for each $g \in G_0$, such that

$$K \subseteq K_0 \cup \bigcup\{ g(K_g) \mid g \in G_0 \}.$$

Last, if X is T_2 then K is closed in X, and hence the set $K_g \cap g^{-1}(K)$ is compact in X_g, for all $g \in G_0$. It is now clear how to change the inequality given above into an equality, verifying our last assertion.

It does not seem possible to improve these results much, as the following example shews.

Example 1.10 : Let $\underline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and ϕ be a non-trivial ultrafilter on $\underline{\mathbb{N}}$ (such exist in profusion, as [13, Theorem 9.2] demonstrates). Further, let the set of all other non-trivial ultrafilters on $\underline{\mathbb{N}}$ be denoted by Φ.

Two structures (actually topologies) are now defined on $\underline{\mathbb{N}}$, and we obtain the usual OPC of $\mathbb{N}$ as a final structure. (This is one occasion on which the topological and convergence space final structures do coincide.) However there are compact subsets of $\tilde{\mathbb{N}}$, such as $\tilde{\mathbb{N}}$ itself, which can not be realised in the way described in Theorem 1.9 .

The space $\underline{\mathbb{N}}_1$ is defined as follows: when ψ is a filter on $\underline{\mathbb{N}}$, we require that $\psi \to n$ in $\underline{\mathbb{N}}_1$ iff $\psi = \dot{n}$, and $\psi \to \infty$ in $\underline{\mathbb{N}}_1$ iff $\psi \in \{\phi, \dot{\infty}, \phi \wedge \dot{\infty}\}$. That is, $\underline{\mathbb{N}}_1$ is principal and its structure differs from the discrete topology at only one point, ∞. It is easy to check that such spaces are always topological.

The space $\underline{\mathbb{N}}_2$ is also topological, being defined in the same way as $\underline{\mathbb{N}}_1$, except at ∞, where we require

$$\psi \to \infty \text{ in } \underline{\mathbb{N}}_2 \text{ iff } \psi \geq \dot{\infty} \wedge (\wedge \Phi)$$

It is clear that $\tilde{\mathbb{N}}$ is obtained as the final structure on $\underline{\mathbb{N}}$ induced by the identity mappings $id: \underline{\mathbb{N}}_1 \longrightarrow \underline{\mathbb{N}}$, and $id: \underline{\mathbb{N}}_2 \longrightarrow \underline{\mathbb{N}}$.

It is also easy to see that the compact subsets of $\underline{\mathbb{N}}_1$ are exactly the finite ones, using a cardinality argument. The compact sets in $\underline{\mathbb{N}}_2$ are just the compact sets of $\tilde{\mathbb{N}}$, except for those which are members of ϕ. Since every member of ϕ has infinite cardinality, the compact space $\tilde{\mathbb{N}}$ can not be expressed as $\tilde{\mathbb{N}} = M_1 \cup M_2$, where M_1 is compact in $\underline{\mathbb{N}}_1$, and M_2 compact in $\underline{\mathbb{N}}_2$. Thus the claims of

Theorem 1.9,3ii apply only to families of locally compact spaces,in general.

Our last result in this section is an analogue for locally compact spaces of Proposition 1.6 , stating that $C_c X$ and $C_k X$ coincide,when X is locally compact.

Proposition 1.11 : *The identity map*

$$id : C_c X \longrightarrow C_k X$$

is continuous, for all spaces X. When X is locally compact, it is a homeomorphism.

Proof: Let $\mathcal{K}$ be the collection of all compact subsets of X. Then,as was shewn in section 1.1 , the identity map $id : C_c X \longrightarrow C_{\mathcal{K}} X$ is continuous. Also $C_n K$ and $C_c K$ are the same,by Proposition 1.6 , for all $K \in \mathcal{K}$. Thirdly, $C_k X$ carries the initial topology induced by the family

$$\{ j_K^{\circ} : CX \longrightarrow C_n K \mid K \in \mathcal{K} \},$$

and hence $id : C_k X \longrightarrow C_{\mathcal{K}} X$ is a homeomorphism,for all spaces X. Putting these facts together,we have our first claim. The spaces $C_{\mathcal{K}} X$ and $C_c X$ coincide,as soon as $\mathcal{K}$ is a covering system for X, that is,as soon as X is locally compact. With this,the proof is complete.

2 A Stone-Weierstrass theorem.

Stone-Weierstrass theorems tend to be very useful, where-ever they exist. In particular, the case of $C_k X$ (the k indicating that CX carries the topology of uniform convergence on compacta) is worked out, for topological spaces X. The situation is very different for $C_c X$, though. Here we have the SWT (our abbreviation for ' Stone-Weierstrass theorem ' is SWT) in the same form, for locally compact topological spaces X, since for these $C_c X$ and $C_k X$ are the same space, and the classical SWT applies. Then Binz [7, Theorem 5] proved the following:

Let X be a completely regular topological space, and A a closed subalgebra of $C_c(X,\mathbb{R})$, containing the constant functions. If also A generates the topology of X (meaning that A and CX generate the same initial topology on X) then $A = CX$.

Questions are - how far can one get away from topology generating? how far from closed? also what happens when the algebra does not separate the points of X - for in the classical case, the SWT can be stated in such a way as to cope with this possibility [11, Chapter 17] ? Feldman [15] in his thesis shews that it is enough in Binz's result to demand that A be an algebra of bounded functions, satisfying Binz's conditions, apart from closedness.

In this chapter, we work towards a theorem relaxing both the topology-generating and the closedness requirements, and which is moreover stated for algebras of $\mathbb{R}$- or $\mathbb{C}$-valued functions,which need not separate the points of X, or contain the constant functions. In chapter 4 we give an example shewing that our escape from topology-generation requirements is genuine. We should point out,though,that there is no reason at all to expect our result to be in any sense ' best possible ' ; it just shews there was room for improvement,and leaves the hope that there are better SWTs yet to be found.

2.1 Preliminary results.

Here are given a sequence of rather disconnected technical lemmas,to be used in the next section in the proof of our SWT. First we introduce the symbol $\overline{\mathbb{R}}$, meaning the two point compactification of $\mathbb{R}$.

Let now X be a convergence space,and $\mathcal{S}$ a non-void family of subsets of X. This defines on CX and its subsets the topology $n(\mathcal{S})$ of uniform convergence on the members of $\mathcal{S}$. (The topologies of pointwise convergence,of uniform convergence on X, and of uniform convergence on the compacta of X are particular cases of this topology.) As indicated in the proof of Proposition 1.11 , this topology is generated in two ways; as the initial topology induced by

$$\{ j_S^{\circ} : CX \longrightarrow C_n S \mid S \in \mathcal{S} \},$$

or by the improper seminorms (improper,meaning $\overline{\mathbb{R}}$-valued)

$$n_S : f \longmapsto \sup\{ |f(x)| \mid x \in S \}$$

Further, let $A(\mathcal{S})$ be the set of all functions in CX which are bounded on each set $S \in \mathcal{S}$; namely,

$$A(\mathcal{S}) = \{f \in CX \mid f(S) \text{ is bounded in } \mathbb{F}, \text{for each } S \in \mathcal{S}\}.$$

A subset A of CX is called $\mathcal{S}$-<u>bounded</u> exactly when it is contained in $A(\mathcal{S})$, and <u>locally bounded</u> when it is $\mathcal{S}$-bounded for some <u>covering system</u> $\mathcal{S}$ for X. (It is the locally bounded subalgebras which we shall be able to deal with in the SWT.)

It would perhaps be helpful to shew where $(CX, n(\mathcal{S}))$ lies in relation to C_cX and C_SX, and we do this here.

Remark: 1) For all collections $\mathcal{S}$, the identity map $id : (CX, n(\mathcal{S})) \longrightarrow C_SX$ is continuous. When it is a homeomorphism, and X is c-embedded, there is for each set $S \in \mathcal{S}$ a compact set K in X, which contains S.

2) The identity map $id : (CX, n(\mathcal{S})) \longrightarrow C_cX$ is continuous iff $\mathcal{S}^- = \{S^- \mid S \in \mathcal{S}\}$ is a quilt for X - where $^-$ denotes the closure operator in wX.

3) When each $S \in \mathcal{S}$ is contained in some compact subset of X (which may depend on S), then

$$id : C_cX \longrightarrow (CX, n(\mathcal{S}))$$

is continuous. On the other hand, if it is continuous, and X is c-embedded, then each set S lies in some compact subset K_S of X.

The proof is a straightforward calculation - except perhaps at those parts involving c-embeddedness, where an as yet unproven result, Lemma 3.4, is helpful - and is omitted.

Note: When X is c-embedded, and $C_c X$ and $(CX, n(S))$ coincide, for some collection S of subsets of X, it follows from parts 2 and 3 of the above remark that X is locally compact, and that S^- is a quilt for X.

This last fact is actually a particular case of our Theorem 4.3.

Another easily proved fact is that the composition map

$$\circ : (A(S), n(S)) \times C_k\mathbb{R} \longrightarrow (A(S), n(S))$$

is continuous. Similarly, given any spaces X, Y and Z,

$$\circ : C_c(X,Y) \times C_c(Y,Z) \longrightarrow C_c(X,Z)$$

is also continuous [4, Satz 6], and in particular,

$$\circ : C_c X \times C_k\mathbb{R} \longrightarrow C_c X$$

is continuous, for all spaces X. With these facts, we can prove our next result, the third part of which generalises one of Binz's [7, Theorem 3]. Moreover, the proof is shorter.

Proposition 2.1 : *Let X be a convergence space, and S a non-empty family of subsets of X.*

i) Whenever $A \subseteq CX$ is a subgroup (subring), the addition (and multiplication) is continuous, with respect *to the* $n(S)$*-topology. The scalar multiplication (if defined at all) need not be continuous. In fact, it is continuous iff* A *is a vector subspace of* $A(S)$*, the algebra of all* S*-bounded functions on* X*. (Compare with* [13, exercise 2M]*.)*

ii) The $n(S)$*-closure of any subset of* CX *is* S*-bounded exactly when the subset is itself* S*-bounded.*

If A *is an* S*-bounded subalgebra of* CX*, then its* $n(S)$*-closure* A^- *is also a subalgebra of* CX*, which is furthermore monotone.*

iii) The adherence of any subalgebra of C_cX *is a monotone subalgebra of* CX*.*

Proof: The first two parts are well-known - we shall not repeat their proofs (although it will become clear that the second half of ii can be proven in the same way as iii).

So suppose A to be a subalgebra of CX. Since the operations in C_cX are continuous, $a(A)$ is clearly also a subalgebra of CX. Thanks to Lemma 0.14ii , to shew that $a(A)$ is monotone, it is sufficient to shew that $\sqrt{|f|} \in a(A)$ whenever $f \in a(A)$.

To this end, we take a sequence of polynomials with real coefficients and no constant term, say, (g_n) , such that $(g_n) \to g$ in $C_k\mathbb{R}$, where we define $g(\alpha) = \sqrt{|\alpha|}$, for all $\alpha \in \mathbb{R}$. That this can be done, is due to the classical Weierstrass approximation theorem.

Next, since $f \in a(A)$, there is a filter θ on CX with a trace on A, and converging to f in C_cX. Hence the filter

$$(g_n) \circ \theta \to g \circ f = \sqrt{|f|} \quad \text{in} \quad C_c X,$$

and further, has a trace on A, since A is an algebra. This, though, completes our proof.

Remark: 1) Our proof is valid, whether or not the subalgebra A contains the constant functions.

2) At the cost of retreating into calculations, one can get a stronger result:

When A is an (S-bounded) subalgebra of CX, and $f \in a(A)$ (or $f \in A^-$), then the function $1/f$ belongs to $a(A)$ (or to A^-) as well, provided f is a unit of CX.

3) Finally in this connection, one defines in the obvious way the <u>sequential</u> <u>adherence</u> <u>operator</u> α_Y of a convergence space Y (these operators also satisfying the same three properties that adherence operators satisfy, as one easily can check). In particular, for any subalgebra of $C_c X$, its sequential adherence is a monotone subalgebra of CX, closed under inversion in the above sense.

The next proposition shews that when computing the adherence of subsets of $C_c X$, it is enough to do it for the $n(S)$-closed ones, where S is this time a <u>cover</u> for X.

<u>Proposition</u> 2.2 : *For any cover S for X, and each subset H of CX, we have $a(H) = a(H^-)$, where $^-$ denotes the $n(S)$-closure operator.*

Proof: Clearly, $a(H) \subseteq a(H^-)$, since $H \subseteq H^-$. Conversely, suppose that $f\theta$ is a pair from C_cX, and that θ has a trace on H^-. By definition, for each positive real number ε and each pair $x\phi$ from X, there is $S_{x\phi} \in S$, $C_{\varepsilon x\phi} \in \phi$ and $B_{\varepsilon x\phi} \in \theta$ with $C_{\varepsilon x\phi} \subseteq S_{x\phi}$, and

$$\omega(B_{\varepsilon x\phi} \times C_{\varepsilon x\phi}) \subseteq f(x) + \varepsilon\Delta.$$

Now let $D_{x\phi}$ be the set of all functions in CX assuming values between -1 and 1 on $S_{x\phi}$, and consider the set

$$\{ B_{\varepsilon x\phi} + \varepsilon D_{x\phi} \mid \varepsilon \in \mathbb{R}^+ \text{ and } x\phi \text{ is a pair from } X \}.$$

It obviously has the finite intersection property, generating a filter θ' coarser than θ. Nevertheless, θ' still converges to f in C_cX.

We now shew that θ' has a trace on H. Let us take any finite collection of triples, say $\{\varepsilon_\kappa x_\kappa \phi_\kappa \mid \kappa = 1,\ldots,\tau\}$. By assumption there is a function

$$f' \in \bigcap_{\kappa=1}^{\tau} B_{\varepsilon_\kappa x_\kappa \phi_\kappa} \cap H^-,$$

and by definition of H^-, there is a function $f'' \in H$, differing from f' by less than ε_κ on $S_{x_\kappa \phi_\kappa}$, for each index κ. Hence

$$f'' \in H \cap \bigcap_{\kappa=1}^{\tau} (B_{\varepsilon_\kappa x_\kappa \phi_\kappa} + \varepsilon_\kappa D_{x_\kappa \phi_\kappa}),$$

as required.

It is convenient now to give some more notation - this will allow us to state some results more succinctly, and we hope, more clearly.

For each subset A of CX, we get a subset N_A of X, the <u>null-set of</u> A, by defining

$$N_A = \{ x \in X \mid f(x) = 0 \text{ for all } f \in A \}.$$

The set N_A is closed in X, being in fact closed in the initial topology induced on X by A.

On the other hand, for any subset B of X, we define an ideal $I(B)$ of CX by

$$I(B) = \{ f \in CX \mid f(B) \subseteq \{0\} \}.$$

That we take $I(\emptyset)$ to be CX is consistent with this definition, and with the set theory that we use.

For a discussion of the relationship between the operators N and I, see [9]. The only results that we need therefrom, are that the ideals $I(B)$ are always closed in C_cX, and that $A \subseteq I(N_A)$, for all $A \subseteq CX$. (These facts are easily verified directly, without consulting [9].) From this it follows that the adherence in C_cX of each subset A of CX is also contained in $I(N_A)$.

The second new notation will allow us to deal with subalgebras of CX which do not separate the points of X.

Any equivalence relation E on the convergence space X defines a subalgebra $A(E)$ of CX; namely, the set of all functions in CX which are constant on E-classes. It is again easy to see that subalgebras obtained in this way are always closed in C_cX; more generally, if A is any subset of $A(E)$, then $a(A) \subseteq A(E)$.

On the other hand, any subalgebra A (or even subset) of CX induces an equivalence relation E_A on X, whose equivalence classes are given by

$$E_A x = (i_X^A)^{-1}(\hat{x}),$$

for each $x \in X$. (The symbols $\hat{x}$ and i_X^A are those explained in section 0.2, being the point evaluation at x, and the map : $x \longmapsto \hat{x}$ respectively.)

For the rest of this section, X denotes always a completely regular topological space, and CX means $C(X,\mathbb{R})$, exclusively.

The universal property of the Stone-Čech compactification βX of X provides for each function $f \in CX$, a continuous mapping

$$\underline{f} : \beta X \longrightarrow \overline{\mathbb{R}},$$

coinciding with f on X. Each subset A of CX yields in this way a corresponding subset $\underline{A}$ of $C(X,\overline{\mathbb{R}})$, which partitions βX, the equivalence classes being of the form

$$E_A^\beta x = (i_{\beta X,\overline{\mathbb{R}}}^{\underline{A}})^{-1}(\hat{x}).$$

It is not hard to see that E_A is the restriction of E_A^β to X, that is, $E_A^\beta x \cap X = E_A x$, for all $x \in X$.

Let ∇ denote the projection of βX onto $Y = \beta X / E_A^\beta$, and let Y carry the final topology induced by ∇. For each $f \in A$, the mapping

$$\underline{\underline{f}} : Y \longrightarrow \overline{\mathbb{R}},$$

defined by the equation $\underline{f} = \underline{\underline{f}} \circ \nabla$, is continuous, thanks to the UP for the topological quotient structure. Further, the topologies induced on Y by ∇, and by the initial family $\underline{\underline{A}} = \{ \underline{\underline{f}} \mid f \in A \}$ are the same, since the former is compact, and the latter Hausdorff and coarser than the first.

Using the notation just developed, we state the classical SWT (mainly to see how it looks in this form), and then the last two lemmas to be given in this section.

<u>The classical SWT</u> : *Let Z be a completely regular topological space, and A a subalgebra of $C(Z,\mathbb{F})$, closed under complex conjugation in the complex case. Then the closure of A in $C_k(Z,\mathbb{F})$ is exactly the set*

$$A(E_A) \cap I(N_A).$$

(That is, either the algebra $A(E_A)$ itself, or the maximal ideal of that algebra consisting of functions vanishing on the set N_A.)

<u>Lemma</u> 2.3 : *Suppose that x is a point of X, and C a closed subset of βX. If $\nabla(x) \notin \nabla(C)$, there is a function $g \in A$ such that*

$$\underline{g}^{-1}(g(x) + \Delta) \subsetneq \beta X \setminus C.$$

Further, if $1 \in A$, then g can be chosen with $g(x) = 0$.

This lemma needs no proof, beyond observing three things: first, that the set $\nabla(C)$ is closed in Y; second, that the set $\underline{\underline{A}}$ induces the topology on Y; and last, A is an algebra.

Lemma 2.4 : *For an S-bounded and n(S)-closed subalgebra A of CX, the algebra $A \cap C^0X$ is canonically isomorphic to either CY, or to a maximal ideal of CY.*

Proof: We know from Proposition 2.1 ii , that A is a monotone subalgebra of CX, and in particular, a sublattice. Hence $\underline{\underline{A \cap C^0X}}$ separates the points of Y, at least in the weak sense. Moreover, $A \cap C^0X$ itself is n_X-closed (recall our notation for sup-norms), and it is obvious that

$$n_X(f) = n_Y(\underline{\underline{f}}),$$

for all $f \in A$. It follows that the map

$$t : A \cap C^0X \longrightarrow CY$$

in which $f \longmapsto \underline{\underline{f}}$, is actually an injective, sup-norm preserving homomorphism, whose image $\underline{\underline{A \cap C^0X}}$ is thus a closed point separating subalgebra of CY. The classical SWT now says that it is either CY itself, or a maximal ideal of CY. This establishes our lemma.

We add only the remark that the homomorphism

$$\nabla^{\circ} : CY \longrightarrow CX$$

is injective, and a left inverse for t. To see this, one simply looks at the definition of ∇°, and sees that $\nabla^{\circ}(\underline{\underline{f}}) = \underline{\underline{f}} \circ \nabla = f$, for all $f \in A \cap C^0X$.

2.2 The theorem.

We give here our main SWT, a corollary, and then a different SWT, of use in chapter 3.

Theorem 2.5 : *Let* X *be a completely regular topological space, and* A *a locally bounded subalgebra (not necessarily containing* 1 *) of* $C(X,\mathbb{F})$, *closed under complex conjugation in the complex case.*

Further, suppose there is a subset G *of* X *such that*

1) $G \supseteq \{ x \in X \mid E^{\beta}_{A}x \setminus X$ is nonvoid $\}$, *and*

2) G *is compact in the initial topology induced on* X *by the algebra* $A(E_A)$.

Then the adherence in $C_c(X,\mathbb{F})$ *of* A *is exactly the algebra* $A(E_A) \cap I(N_A)$.

Remark: The algebras discussed by Feldman and Binz were topology generating; this corresponds in the statement above to point separating, and being able always to choose G to be empty.

Proof of the theorem: One inequality we already know -

$$a(A) \subseteq A(E_A) \cap I(N_A).$$

The other, in which we now engage, is somewhat harder. We confine ourselves until further notice to the real case only. We assume also, as Proposition 2.2 allows us to, that A is S-bounded and $n(S)$-closed, for some covering system S for X. (For if A is S-bounded as we have assumed, its $n(S)$-closure A^- is also S-bounded, with $E_A = E_{A^-}$, and

$N_A = N_{A^-}$, so that A^- satisfies the conditions of the theorem when A does.)

First we shew that for each function $f \in A(E_A) \cap I(N_A)$ there is a function $f' \in A$ agreeing with f on G.

Let X' denote the set X, together with the initial topology induced by the family $A(E_A)$. Then the subset $\nabla(G \cup N_A)$ of Y is closed, and inherits the same compact topology from Y as it does from X' via ∇. Hence the restriction of f to $G \cup N_A$ induces a continuous function on $\nabla(G \cup N_A)$, which has by Tietze's theorem a continuous extension $\underline{\underline{f}}'$ to all of Y. Lemma 2.4 now furnishes a function $f' = \nabla^*(\underline{f}')$ in A agreeing with f on G.

Consequently, it is enough to shew that any function in $A(E_A) \cap I(N_A \cup G)$ also belongs to $a(A)$. Let now f be such a function, that is, $f \in A(E_A) \cap I(N_A \cup G)$, and let X'' be the set $X \setminus f^{-1}(0)$.

For each $x \in X''$ and positive real number ε, we define

$$\varepsilon(x) = \min\{\varepsilon, \tfrac{1}{2}|f(x)|\},$$

and

$$W(\varepsilon,x) = \{y \in \beta X \mid |\underline{f}(y) - \underline{f}(x)| \leq \varepsilon(x)\} \setminus \nabla^{-1}(\nabla(G)).$$

We have assumed that $G \cup N_A \subseteq f^{-1}(0)$. Accordingly, the equivalence class $\nabla(x) = E_A^{\beta}x$ lies entirely in X, and so does not meet the closed set $\beta X \setminus W(\varepsilon,x)$. We get then from Lemma 2.3 a function $u_{\varepsilon x} \in A$ such that

$$\underline{u_{\varepsilon x}}^{-1}(u_{\varepsilon x}(x) + \triangle) \subseteq W(\varepsilon,x).$$

Let $M(\varepsilon,x) = \{y \in \beta X \text{ with } |\underline{u}_{\varepsilon x}(y) - \underline{u}_{\varepsilon x}(x)| \leq 1\}$,

$N(\varepsilon,x) = \{y \in \beta X \text{ with } |\underline{u}_{\varepsilon x}(y) - \underline{u}_{\varepsilon x}(x)| \leq \frac{1}{3}\}$,

$U(\varepsilon,x) = \{y \in \beta X \text{ with } |\underline{u}_{\varepsilon x}(y) - \underline{u}_{\varepsilon x}(x)| < \frac{2}{3}\}$,

and

$$D(\varepsilon,x) = \{g \in A \cap C^0 X \text{ with } |\underline{g}(y) - \underline{f}(y)| < \varepsilon(x), \text{ for all } y \in M(\varepsilon,x)\}.$$

Since $M(\varepsilon,x)$ is saturated with respect to E_A^β, meaning that it is a union of E_A^β-classes, the closed sets $\nabla(M(\varepsilon,x))$ and $\nabla(\beta X \setminus W(\varepsilon,x))$ are disjoint in Y. Thus there is $\underline{g} \in CY$, constant zero on the latter set, and one on the former. As before, the function $g = \nabla^*(\underline{g})$ belongs to $A \cap C^0 X$, and a multiple of g, namely, $f(x)g$, belongs to $D(\varepsilon,x)$. We note further that the inequality

$$|\underline{g}(y)| \leq 2|\underline{f}(y)|$$

holds for all $y \in M(\varepsilon,x)$ and $g \in D(\varepsilon,x)$, thanks to our choice of $\varepsilon(x)$.

We claim that the set

$$\mathcal{D} = \{ D(\varepsilon,x) \mid \varepsilon \in \mathbb{R}^+ \text{ and } x \in X'' \}$$

has the finite intersection property, and prove this in the same way as Binz [7, Theorem 5].

Suppose $\{(\varepsilon_\nu, x_\nu) \mid \nu = 1,\dots,n\}$ is an arbitrarily chosen finite collection of indices. It is no restriction to require

$$\varepsilon_1(x_1) \leq \dots \leq \varepsilon_n(x_n).$$

Now let

$$M_\upsilon = \bigcup_{\kappa=1}^{\upsilon} M(\varepsilon_\kappa, x_\kappa), \text{ and}$$

$$D_\upsilon = \bigcap_{\kappa=1}^{\upsilon} D(\varepsilon_\kappa, x_\kappa)$$

for $\upsilon = 1,\dots,\eta$. We know that D_1 is non-void, and shall complete our proof by shewing that $D_{\upsilon+1}$ is non-void, when D_υ is. Suppose then that $g \in D_\upsilon$.

Two possibilities arise: if $M_\upsilon = M_{\upsilon+1}$, then $g \in D_{\upsilon+1}$ automatically. Otherwise, let E, F be any two $\mathcal{E}_A^\beta$-classes both lying in $M_{\upsilon+1} \setminus M_\upsilon$. We next define a function

$$f_{EF} : \nabla(M_\upsilon \cup E \cup F \cup N_A) \longrightarrow \mathbb{R}$$

assuming the values 0 at the point $\nabla(N_A)$, and $f(x_{\upsilon+1})$ at the points $\nabla(E)$ and $\nabla(F)$, and agreeing with g on $\nabla(M_\upsilon)$. That this function is well-defined, is because all the sets concerned are mutually disjoint, and $\mathcal{E}_A^\beta$-saturated. It is clearly continuous, since the three (or two, when $E = F$) extra points are isolated from $\nabla(M_\upsilon)$..

Now by using Tietze's theorem and Lemma 2.4, we get a bounded function $g_{EF} \in A$, such that the restriction of the function $\underline{\underline{g_{EF}}}$ to the set $\nabla(M_\upsilon \cup E \cup F)$ coincides with f_{EF}. Hence the set

$$U_{EF} = \{y \in \beta X \mid \underline{g_{EF}}(y) > \underline{f}(y) - \varepsilon_{\upsilon+1}(x_{\upsilon+1})\}$$

is a well-defined subset of βX, containing $M_\upsilon \cup E \cup F$. Thus for fixed F, the collection

$$\{ U_{EF} \mid E \subseteq M_{\upsilon+1} \setminus M_{\upsilon} \text{ is a } \mathcal{E}_A^{\beta}\text{-class} \}$$

forms an open cover for the compact set $M_{\upsilon+1}$, from which we extract a finite subcover

$$\{ U_{E_1F}, \ldots, U_{E_\chi F} \}.$$

This shews that the function

$$g_F = \bigvee_{\kappa=1}^{\chi} g_{E_\kappa F} \in A \cap C^0X,$$

which agrees with $\underline{g}$ on M_υ, is greater than $\underline{f} - \varepsilon_{\upsilon+1}(x_{\upsilon+1})$ on $M_{\upsilon+1}$. Similarly, the sets

$$U_F = \{ y \in \beta X \mid \underline{g_F}(y) < \underline{f}(y) + \varepsilon_{\upsilon+1}(x_{\upsilon+1}) \}$$

are open, and contain $M_\upsilon \cup F$. The resulting cover of $M_{\upsilon+1}$ also admits a finite subcover $\{U_{F_1}, \ldots, U_{F_\xi}\}$. Now clearly,

$$g' = \bigwedge_{\kappa=1}^{\xi} g_{F_\kappa} \in A \cap C^0X,$$

coincides with $\underline{g}$ on M_υ, and even satisfies the inequality

$$|\underline{g}' - \underline{f}| < \varepsilon_{\upsilon+1}(x_{\upsilon+1})$$

on $M_{\upsilon+1}$. With these properties, $g' \in D_{\upsilon+1}$, completing the induction step, and so verifying our claim, that $\mathcal{D}$ had the finite intersection property.

We note in passing that for any $g \in D_\upsilon$, the inequality $|\underline{g}| \leq 2|\underline{f}|$ holds on M_υ. Further, if there are no overlapping equivalence classes, that is, if we can take $G = \emptyset$, then the filter generated by $\mathcal{D}$ itself converges to f in C_cX, shewing that $f \in a(A)$.

Otherwise, and in fact, in any case, let us define

$$F(\varepsilon,x) = \{ g \in A \text{ with } |\underline{g}(y) - \underline{f}(y)| < \varepsilon(x) \text{ on } N(\varepsilon,x), \text{ and } |g| \leq 2|f| \text{ on } X\}$$

and

$$\mathcal{F} = \{ F(\varepsilon,x) \mid \varepsilon \in \mathbb{R}^+ \text{ and } x \in X'' \}.$$

We claim (again) that $\mathcal{F}$ has the finite intersection property. Consider then a typical finite collection of indices, say, $\{(\varepsilon_1,x_1),\ldots\ldots,(\varepsilon_\eta,x_\eta)\}$. The sets

$$\bigcup_{\kappa=1}^{\eta} N(\varepsilon_\kappa,x_\kappa) \quad \text{and} \quad \beta X \setminus \bigcup_{\kappa=1}^{\eta} U(\varepsilon_\kappa,x_\kappa)$$

defined at the top of page 64 are closed in βX, disjoint, and E_A^β-saturated. Hence we can find a function $h \in A$, such that $0 \leq h \leq 1$, being constant one on the first-named set, and constant zero on the other.

Now for any function $g \in \bigcap_{\kappa=1}^{\eta} D(\varepsilon_\kappa,x_\kappa)$, it is clear that $|hg| \leq 2|f|$ on X, and further, since $hg = g$ on $\bigcup_{\kappa=1}^{\eta} N(\varepsilon_\kappa,x_\kappa)$ we have $hg \in F(\varepsilon_\kappa,x_\kappa)$, for each κ.

The filter θ so generated has a trace in A, by construction. We prove next that $f\theta$ is a pair from C_cX, which is not hard to do.

For $x \in X''$, the filter $\omega(\theta \times \mathcal{U}_x)$ converges to $f(x)$ in $\mathbb{R}$, since for any positive number ε, the set $N(\varepsilon,x) \cap X$ is a neighbourhood of x in X, with

$$\omega(\ F(\varepsilon,x) \times N(\varepsilon,x)\cap X\ ) \subseteq f(x) + 2\varepsilon\Delta$$

(recall that $|\underline{f}(y) - f(x)| < \varepsilon(x) \leq \varepsilon$ on $N(\varepsilon,x)$). Next, suppose that $f(x) = 0$. Since f is continuous, for any positive number ε, there is a neighbourhood V of x in X with

$$\omega(\ \{f\} \times V\ ) \subseteq \varepsilon\Delta.$$

However, for any $F \in \mathcal{F}$, we have $|g| \leq 2|f|$, whenever $g \in F$. Hence

$$\omega(\ F \times V\ ) \subseteq 2\varepsilon\Delta,$$

proving the convergence of $\omega(\theta \times \mathcal{U}_x)$ to 0 in $\mathbb{R}$.

In this fashion every point of X has been accounted for, as only the possibilities $x \in X''$ or $f(x) = 0$ can occur, and the proof of the theorem is complete, for $\mathbb{R}$.

The usual arguments allow us to extend the result to subalgebras of $C(X,\mathbb{C})$ which are closed under complex conjugation.

From now on, we return to our convention, under which CX means $C(X,\mathbb{F})$.

When A is a subalgebra of CX such that at most finitely many E_A^β-classes meet both X and $\beta X\setminus X$, the set $\{\ y \in X \mid E_A^\beta y$ meets $\beta X \setminus X\ \}$ is itself compact, in the initial topology induced on X by the family $A(E_A)$. Accordingly, we can state

<u>Corollary</u> 2.6 : *When* A *is a locally bounded subalgebra of* CX*, closed under conjugation in the complex case, such that at most finitely many* E_A^{β}*-classes meet both* X *and* $\beta X \setminus X$*, then*

$$a(A) = A(E_A) \cap I(N_A).$$

The section is ended with yet another SWT, applying whenever X is a convergence space,making C_cX topological.There are no other restrictions on X at all.The result does depend on another (Theorem 3.3) which we prove later,but without using ' circular logic '.

<u>Theorem</u> 2.7 : *Let* X *be a convergence space such that* C_cX *is topological, and* A *be a subalgebra of* CX*, closed under conjugation in the complex case. Then*

$$a(A) = A(E_A) \cap I(N_A).$$

Proof: The classical SWT stated earlier is used here,in our proof. Since each compact subset of Hom_cC_cX remains compact in Hom_sC_cX, the identity map

$$id : C_kHom_sC_cX \longrightarrow C_kHom_cC_cX$$

is continuous.Further,

$$d : C_cX \longrightarrow C_cHom_cC_cX$$

is a linear homeomorphism.Last, C_cX is topological,and now Remark 3.4 and Theorem 3.3 are summoned,to shew that Hom_cC_cX is a locally compact convergence space,and that

$$\mathrm{id} : C_c Hom_c C_c X \longrightarrow C_k Hom_c C_c X$$

is also a homeomorphism. Putting these together, we see that

$$d^{-1} : C_k Hom_s C_c X \longrightarrow C_c X$$

is continuous. Also, the subalgebra $d(A)$ of $C_k Hom_s C_c X$ satisfies the conditions of the usual SWT - the closure under conjugation is easily verified, as each member of $Hom C_c X$ is a point evaluation - and

$$d(A(E_A)) = A(E_{d(A)}).$$

Our theorem follows directly from these facts.

Remark 2.8 : We have already noted what it means for a subalgebra (or even subset) A of CX to generate the topology on X - an equivalent version reads:

A subset A of CX is topology-generating on X iff no E_A^β-equivalence class meets both βX X and X, and A separates the points of X.

We shall need this form of the statement later, in section 4.2 . Its proof, if not already known to the reader, is a straightforward calculation.

3 Topological algebras.

3.1 Generalities and Banach algebras.

Here we restate our blanket assumptions, that our algebras are commutative $\mathbb{F}$-algebras, from now on always possessing a multiplicative identity element 1. A topological algebra A_T is not always required to be Hausdorff; however, the multiplication should be (jointly) continuous. Further, Hom A and *Hom* A_T denote the set of all $\mathbb{F}$-valued, and all continuous $\mathbb{F}$-valued homomorphisms of the topological $\mathbb{F}$-algebra A_T. Other symbols to be seen are A_p and A_P, which mean that A carries the topology generated by a seminorm p defined on A, and that by a non-empty set P of seminorms on A respectively.

Those properties of normed algebras required as background are summarised below - proofs can be found in Rickart [10], for example.

1) Any normed algebra A_p can be given an equivalent norm q, which is submultiplicative, and normalised so that $q(1) = 1$. We assume this to have been done in future.

2) The complexification $A_p{}^2$ of any normed algebra A_p over $\mathbb{R}$ can be given a norm such that the injection

$$\sigma : A_p \longrightarrow A_p{}^2$$

is norm-preserving [10, Theorem 1.3.2].

3) Every maximal ideal of a Banach algebra is an $\mathbb{F}$-ideal, that is, the quotient field is $\mathbb{F}$. When the algebra is over $\mathbb{C}$, only $\mathbb{C}$-ideals can appear as maximal ideals.

4) For any complex Banach algebra A_p, the topological space $\mathrm{Hom}_s A$ is compact and Hausdorff, and the inequality

$$|h(a)| \leq p(a)$$

holds, for all homomorphisms $h \in \mathrm{Hom}\, A$ and points $a \in A$. Hence, every member of $\mathrm{Hom}\, A$ is norm-continuous, and the Gelfand map

$$d : A_p \longrightarrow C_n(\mathrm{Hom}_s A, \mathbb{C})$$

is also continuous.

5) The results quoted in 4 are also true of real Banach algebras as well, as one may easily check, using complexifications.

Suppose A_p to be a real Banach algebra. We have seen in section 0.3 that $\mathrm{Hom}_s A$ is (homeomorphic to) a closed subspace of $\mathrm{Hom}_s A^2$, which is compact, by 4 and 2. Thus can $\mathrm{Hom}_s A$ be proved compact.

Now let $h_0 \in \mathrm{Hom}\, A$ and $a \in A$. As in section 0.3, we have $h \in \mathrm{Hom}\, A$, taking (a'', a') to $h_0(a'') + \imath h_0(a')$. Hence

$$\begin{aligned} h_0(a) &= h((a,0)) \\ &\leq p'((a,0)) = p(a) \end{aligned}$$

the inequality stemming from 4, and the subsequent equality from 2, if p' is that norm whose existence 2 furnishes.

It follows that every real-valued homomorphism of A is norm-continuous, as is the Gelfand map

$$d : A_p \longrightarrow C_n(\mathrm{Hom}_s A, \mathbb{R}).$$

6) In any complex Banach algebra A_p, an element $a \in A$ is invertible iff $h(a)$ is non-zero, for all $h \in \mathrm{Hom}\, A$.

7) For any real Banach algebra A_p, the following conditions are equivalent (see [10, Theorem 3.1.21]) :

i) Every maximal ideal of A is an $\mathbb{R}$-ideal.

ii) $1 + a^2$ is invertible, for all $a \in A$.

iii) The complexification of A_p is symmetric.

Proof: Proposition 0.8 shews that ii implies iii, and also that i implies iii.

We next prove ' iii $\Rightarrow$ i '. Since each member of $\mathrm{Hom}\, A_p{}^2$ is continuous, by 4, the symmetry of $A_p{}^2$ is equivalent to its full symmetry. But from 3, every maximal ideal of A is an $\mathbb{F}$-ideal, and by the full symmetry of A^2, even an $\mathbb{R}$-ideal. Accordingly, i holds.

Last, we prove that i and iii together imply ii, which will be enough to prove the equivalence. For the reason given above, we assume without loss of generality that A^2 is fully symmetric. Hence

$$h((1 + a^2, 0)) = 1 + h((a,0))^2 \geq 1,$$

for all $h \in \mathrm{Hom}\, A^2$, and $a \in A$. This shews that $(1 + a^2, 0)$ is invertible in A^2, by 6, and consequently $1 + a^2$ is a unit in A.

The preliminaries being over, we consider a Banach $\mathbb{F}$-algebra A_p, remembering that the sets *Hom* A_p and Hom A are equal, and that

$$id : Hom_c A_p \longrightarrow Hom_s A$$

is continuous (section 0.2). On the other hand, since $Hom_s A$ is a compact space, the evaluation map

$$\omega : C_n Hom_s A \times Hom_s A \longrightarrow \mathbb{F}$$

is also continuous. It follows that

$$\omega \circ (d \times id) : A_p \times Hom_s A_p \longrightarrow \mathbb{F}$$
$$(a,h) \longmapsto h(a)$$

(which is none other than the evaluation map) is continuous, and so, so is

$$id : Hom_s A_p \longrightarrow Hom_c A_p$$

by the UP for the structure of continuous convergence. We have just proved

<u>Proposition</u> 3.1 : *The spaces* $Hom_s A$ *and* $Hom_c A_p$ *are identical, for any Banach algebra* A_p*. That is, the carrier space (in our sense) of any Banach algebra is a compact Hausdorff topological space.*

Further, for any normed linear algebra A_p*, the spaces* $Hom_s A_p$ *and* $Hom_c A_p$ *coincide, and are compact topological.*

3.2 The carrier space of a topological algebra.

We turn our attention to topological algebras in general, and shall shew that the carrier space of any (commutative) topological algebra (with identity) is a locally compact c-embedded convergence space. Then the first of a sequence of results on the universal representation of these algebras is given.

To start with, we know that $C^0_n X$ is a Banach algebra, for each convergence space X, and that

$$j_0 : C^0_n X \longrightarrow C_c X$$

is continuous. In particular, if A_T is a topological algebra with topology T, and E is a non-void subset of $Hom_c A_T$, the maps

$$E \xrightarrow{i_E} Hom_c C_c E \xrightarrow{j_0^{\bullet}} Hom_c C^0_n E \overset{id}{\rightleftarrows} Hom_s C^0 E$$

are all continuous - the last homeomorphism thanks to Proposition 3.1. Using this terminology, we state

<u>Lemma</u> 3.2 : *Let* E *be closed in* $Hom_c A_T$*, and suppose there is a continuous* $\mathbb{F}$*-algebra homomorphism*

$$h_E : A_T \longrightarrow C^0_n E$$

such that the diagramme

$$\begin{array}{ccc} E & \xrightarrow{j_E} & Hom_c A_T \\ {\scriptstyle j_0^{\bullet} \circ i_E} \searrow & & \uparrow {\scriptstyle h_E^{\bullet}} \\ & Hom_s C^0 E \cong Hom_c C^0_n E & \end{array}$$

commutes. Then E *is compact.*

To prove the lemma, it is enough to remark that E is a closed subset of the compact subspace $h_E(\mathrm{Hom}_s C^0 E)$ of $Hom_c A_T$, and thus itself compact.

Theorem 3.3 : *The space $Hom_c A_T$ is locally compact and c-embedded, for each topological algebra A_T.*

During the proof, we shall need the technical lemma given below; it is of the same sort as Lemma 1.7 .

Lemma : *If X is a convergence space, H a non-void subset of CX, and $x\phi$ a pair from X, there is for each positive number λ a covering system $S(\lambda,\phi)$ for H_c consisting entirely of s-closed sets.*

Proof of lemma : Suppose $\theta \to f$ in H_c. By definition there are sets $B_{f\theta} \in \dot{f} \wedge \theta$ and $C_{f\theta} \in \phi$ with

$$\omega(B_{f\theta} \times C_{f\theta}) \subseteq f(x) + \lambda\Delta.$$

Now we put

$$E(f\theta) = \{ g \in H \mid g(C_{f\theta}) \subseteq f(x) + \lambda\Delta \}$$

and note that as $f\theta$ ranges over the pairs from H_c, we get a cover with the required properties.

Proof of Theorem 3.3 : Let θ be the neighbourhood filter at 0 in A_T. The lemma just above yields a cover, say $S(1,\theta)$, of $Hom_c A_T$, such that for each pair $h\phi$ from $Hom_c A_T$ we have

$$\omega(E(h\phi) \times U_{h\phi}) \subseteq \Delta, \quad . \ . \ . \ . \ . \ . \ . \ (*)$$

for some suitable neighbourhood $U_{h\phi}$ of 0.

If $j_{h\phi}$ denotes the inclusion map of $E(h\phi)$ in the space $Hom_c A_T$, the homomorphism

$$j_{h\phi}^{\bullet} \circ d : A_T \longrightarrow C_c E(h\phi)$$

is continuous,as always.However,the set $E(h\phi)(a)$ is bounded in $\mathbb{F}$, for any $a \in A$, since $U_{h\phi}$ is absorbent. Thus

$$j_{h\phi}^{\bullet} \circ d(A) \subseteq C^0 E(h\phi).$$

Next, (*) shews that the reduced map

$$j_{h\phi}^{\bullet} \circ d : A_T \longrightarrow C_n^0 E(h\phi)$$

is continuous as well. Last,for each $h'' \in E(h\phi)$ and $a \in A$,

$$\begin{aligned} \left[j_{h\phi}^{\bullet} \circ d\right]^{\bullet} \circ j_0^{\bullet} \circ i_{E(h\phi)}(h'')(a) &= \\ & j_0^{\bullet} \circ i_{E(h\phi)}(h'')\left(j_{h\phi}^{\bullet} \circ d(a)\right) \\ &= i_{E(h\phi)}(h'') \circ j_0\left(j_{h\phi}^{\bullet} \circ d(a)\right) \\ &= j_{h\phi}^{\bullet} \circ d(a)(h'') \\ &= d(a)\left(j_{h\phi}(h'')\right) \quad = \quad h''(a). \end{aligned}$$

It follows that $j_{h\phi}^{\bullet} \circ d$ satisfies the condition of Lemma 3.2, allowing us to conclude that $E(h\phi)$ is compact, for all pairs $h\phi$ from $Hom_c A_T$. In other words, $Hom_c A_T$ is locally compact; it is c-embedded, being a subspace of the c-embedded space $C_c A_T$, and the theorem stands proven.

Remark 3.4 : *It is now clear that* $C_c Hom_c A_T$ *is a locally convex topological algebra whenever* A_T *is a topological algebra, and in fact that*

$$id : C_c Hom_c A_T \longrightarrow C_k Hom_c A_T \quad .$$

is a homeomorphism.

This result is needed in the proof of our second SWT, Theorem 2.7 .It is clear,we hope,that nowhere in proving this remark have we used any type of SWT; the next proposition follows immediately from Theorem 2.7, though.

Proposition 3:5 : *If* A_T *is a topological* $\mathbb{R}$*-algebra,or a quasi-symmetric topological* $\mathbb{C}$*-algebra,then the algebra* d(A) *is dense in* $C_c Hom_c A_T$.

The universal representation of any topological algebra is,as mentioned earlier,always continuous,but need not be an embedding,even when it is injective.It cannot be,if A is not locally convex,for example.

In the coming sections we investigate this question and obtain results,characterising those topological algebras for which the universal representation is actually an embedding.

3.3 A special case, the submultiplicative seminorm.

The nature of the universal representation is investigated here rather more closely, for a certain class of locally convex topological algebras, and results parallel to, and including, those well-known for commutative Banach algebras [12] are obtained.

First, a few pertinent definitions. A seminorm p on an $\mathbb{F}$-algebra A is called <u>weakly submultiplicative</u> iff there is a positive real number β_p such that

$$p(aa') \leq \beta_p p(a)p(a'),$$

for all $a, a' \in A$, and <u>submultiplicative</u> iff we can take $\beta_p = 1$. For each seminorm p on A, the <u>kernel of</u> p (written $\operatorname{Ker} p$) is exactly the set of all members of A at which p vanishes. Obviously, $\operatorname{Ker} p$ is a subspace of A, being actually an ideal, if p is weakly submultiplicative.

In this case, the quotient algebra $A/\operatorname{Ker} p$ can be normed, since p is constant on equivalence classes. We shall denote the completion of this normed algebra by $\bar{A}_{\bar{p}}$, creating in this way a Banach algebra, whose norm $\bar{p}$ is weakly submultiplicative, and even submultiplicative, if p is. We call the natural homomorphism from A into $\bar{A}_{\bar{p}}$, under which $a \longmapsto [a]$, π_p. That is, $[a] = \pi_p(a)$.

If now A_T is a topological algebra, and p a continuous weakly submultiplicative seminorm on A, then

$$\pi_p : A_T \longrightarrow \bar{A}_{\bar{p}}$$

is continuous and the map

$$\pi_p^{\bullet} : \mathrm{Hom}_s\bar{A}_{\bar{p}} \longrightarrow \mathit{Hom}_c A_T$$

is well-defined,continuous and injective.(The first two facts derive from Proposition 3.1 , the third from the density of A/Ker p in $\bar{A}_{\bar{p}}$.) Further,the set $\pi_p^{\bullet}(\mathrm{Hom}_s\bar{A}_{\bar{p}})$ is compact in $\mathit{Hom}_c A_T$, being the continuous image of a compact space.

Using the notation set out above,we give now sufficient conditions on A_T for its universal representation to be an embedding - these conditions being actually necessary also,in the case described below.

Proposition 3.6 : *Suppose* A_T *is a topological algebra such that*

i) d *is injective,and*

ii) for each filter (or net) θ *on* A, *if* $d(\theta)$ *converges in* $C_c\mathit{Hom}_c A_T$, *then* θ *converges in* A_T.

In this case, d *is an embedding. When also* A_T *is complete,the image algebra* d(A) *is closed in* $C_c\mathit{Hom}_c A_T$, *being in fact the whole algebra if* A_T *is an* $\mathbb{R}$*-algebra or a quasi-symmetric* $\mathbb{C}$*-algebra.*

Proof : Conditions i and ii say exactly that d is an embedding,since we know already that it is continuous.That now the completeness of A_T implies that of d(A) in $C_c\mathit{Hom}_c A_T$ is just as clear,since d is linear.Accordingly d(A) is closed in $C_c\mathit{Hom}_c A_T$, the last claim following now directly from Proposition 3.5 .

To any submultiplicative seminorm p on an algebra A corresponds a seminorm ν_p on A, defined by

$$\nu_p(a) = \lim_n \left(p(a^n)\right)^{1/n}$$

for all $a \in A$. Its properties are described in [10, Theorem 1.4.1] ; one which we need is that

$$\nu_p(a) \leq p(a),$$

on A. If in addition, there is $\gamma \geq 1$ with

$$p(a)^2 \leq \gamma p(a^2)$$

on A, then

$$p \leq \gamma \nu_p .$$

Such seminorms are here called γ-<u>seminorms</u>.

The Banach algebra $\bar{A}_{\bar{p}}$ shares any such properties with A_p, as

$$\begin{aligned} \bar{p}(\bar{a})^2 &= \lim \bar{p}([a_n])^2 \\ &= \lim p(a_n)^2 \\ &\leq \gamma \lim p(a_n{}^2) = \gamma \bar{p}(\bar{a}^2), \end{aligned}$$

whenever $\bar{a} \in \bar{A}_{\bar{p}}$, and the sequence $([a_n]) \to \bar{a}$ in $\bar{A}_{\bar{p}}$. Further, [10, Corollary 3.1.7] states that

$$\nu_{\bar{p}}(\bar{a}) \geq \sup\{|h(\bar{a})| \; | h \in \operatorname{Hom} \bar{A}_{\bar{p}} \},$$

for any $\bar{a} \in \bar{A}_{\bar{p}}$. Equality occurs, if $\bar{A}_{\bar{p}}$ is a complex algebra, or a real one whose every maximal ideal is an $\mathbb{R}$-ideal - conditions equivalent to this are given in paragraph 7 of section 3.1 . Thus

$$(o) \;.\;.\;.\; \nu_p(a) \geq \sup\{\, |h(a)|, \text{ with } h \in \pi_p^{\bullet}(\mathrm{Hom}\, \bar{A}_{\bar{p}}) \,\},$$

for all $a \in A$, since $p(a) = \bar{p}([a])$.

A real algebra A is said to be <u>good</u> <u>wo</u> P, where P is a collection of submultiplicative seminorms on A, iff equality holds in (o), for all $p \in P$. It is clearly sufficient for this to hold, if every closed $\mathbb{F}$-ideal of A_P - this symbol denoting A together with the topology induced by P - is actually an $\mathbb{R}$-ideal. The preamble now over, we can state our next results.

<u>Theorem</u> 3.9 : *Let A_T be a Hausdorff topological algebra whose topology is generated by a family P of γ-seminorms. In the real case we demand in addition that A be good wo P. Then the universal representation of A_T is an embedding.*

Conversely, if it is an embedding, the topology on A is generated by a family of sup-seminorms (which are in particular γ-seminorms).

Proof : The converse follows without further comment from the fact that $C_c \mathrm{Hom}_c A_T$ carries the topology of uniform convergence on the compact subsets of the (locally compact) space $\mathrm{Hom}_c A_T$.

To prove the first part, we aim to apply Proposition 3.6 . Since $A_{\mathcal{T}}$ is Hausdorff, each non-zero element a of A is 'seen' by some seminorm $p \in \mathcal{P}$, that is, $p(a) \neq 0$. From our assumptions, though, it follows that

$$p(a) \leq \gamma_p \nu_p(a)$$

for some positive number γ_p, and hence $h(a) \neq 0$, for some homomorphism $h \in \pi_p^{\bullet}(\mathrm{Hom}\,\bar{A}_{\bar{p}})$. This shews the injectivity of the universal representation.

Next, we have also seen that

$$\mathcal{K}' = \{ \pi_p^{\bullet}(\mathrm{Hom}\,\bar{A}_{\bar{p}}) \mid p \in \mathcal{P} \}$$

is a collection of compact subsets of $\mathit{Hom}_c A_{\mathcal{T}}$. Accordingly,

$$\mathrm{id} : d(A)_k \longrightarrow (d(A), n(\mathcal{K}')),$$

and

$$d^{-1} : (d(A), n(\mathcal{K}')) \longrightarrow A_{\mathcal{Q}}$$

are both continuous, where k and $n(\mathcal{K}')$ denote the topologies of uniform convergence on the compacta in $\mathit{Hom}_c A_{\mathcal{T}}$, and on the family $\mathcal{K}'$ respectively, and $\mathcal{Q}$ is the family $\{ \nu_p \mid p \in \mathcal{P} \}$ of seminorms on A. (That d is an isometry of seminorms guarantees us the continuity of d^{-1}.) Hence

$$d : A_{\mathcal{T}} \longrightarrow d(A)_c$$

is a homeomorphism, since at one end $A_{\mathcal{T}}$ and $A_{\mathcal{Q}}$ coincide, as topological spaces, and at the other, $d(A)_c$ and $d(A)_k$ also coincide, and in between, the continuous maps d and d^{-1}.

help to complete the proof of our claim.

Corollary 3.8 : *Suppose* A_T *to be as in the previous Proposition. Then*

i) the formula

$$p(a) = \sup\{\ |h(a)|\ ,\ \text{with}\ h \in \pi_p^{\bullet}(\text{Hom}\ \bar{A}_{\bar{p}})\ \}$$

applies to each seminorm $p \in P$ *with* $\gamma_p = 1$,

ii) when A_T *is complete,* $d(A)$ *is closed in* $C_c Hom_c A_T$ *and then also*

iii) in the real and quasi-symmetric complex cases, the universal representation of A_T *is a homeomorphism.*

The proof being clear, we move on to the preparations to our last result of this sort, dealing with topological algebras with a (not necessarily continuous) involution.

We recall that a set P of seminorms on an algebra A is directed iff for any $p, p' \in P$ there is $p'' \in P$ with $p'' \geq p \vee p'$.

Now let $(A, ^\circ)$ be an algebra with involution. A family P of submultiplicative seminorms on A satisfies condition

(α) iff $p(aa^\circ) = p(a)p(a^\circ)$, ⎫ for all $a \in A$

(β) iff $p(a) = 0 \Rightarrow p(a^\circ) = 0$, ⎭ and $p \in P$.

(γ) iff the conjugate homomorphism h^* of h belongs to *Hom* A_p, whenever $h \in$ *Hom* A_p for some $p \in P$.

(δ) iff $p(a^\circ) = p(a)$, ⎫

(ε) iff $p(aa^\circ) = p(a)^2$, ⎭ for all $a \in A$ and $p \in P$.

Remark : 1) Any Banach algebra with involution satisfies conditions (β) and (γ) automatically.

2) Condition (δ) implies conditions (β) and (γ), and (ε) implies all the others. Plainly in both these cases the involution is continuous.

We are now able to give a lemma, strongly resembling one in [12, section 1.7.8] in form and proof.

<u>Lemma</u> 3.9 : *If* $(A,{}^{\circ})$ *is an algebra with involution, and* P *is a family of submultiplicative seminorms on* A, *then* A_P *is symmetric if any of the following statements holds:*

1) P *is directed, and* (α) *and* (γ) *hold.*

2) (α) *and* (δ) *hold.*

3) (ε) *holds.*

Proof : Suppose the claim false. This means there is a homomorphism $h \in \mathit{Hom}\ A_P$, for which $h(b\ ) \neq \overline{h(b)}$ for some point b of A. In fact we can find an Hermitian element $a \in A$, with $h(a) = \imath$, as noted in section 0.3 . Clearly

$$h(a + \imath n1) = \imath(1 + n)$$

for all natural numbers $n \in \mathbb{N}$.

Digressing slightly, we point out that for any submultiplicative seminorm p and any $h \in \mathrm{Hom}\ A$,

$$h(a) \leq p(a)$$

for all $a \in A$ iff h is p-continuous.

At this point, the proofs for the different cases start to diverge from each other. The first case is dealt with first. Since h is $\mathcal{P}$-continuous and $\mathcal{P}$ is directed, there is a seminorm $p \in \mathcal{P}$, such that $h \in Hom\, A_p$. This means the conjugate homomorphism h^* is also p-continuous. Thus

$$\begin{aligned} h^*(a - n\imath 1) &= h((a - n\imath 1)^\circ)^- \\ &= h(a + n\imath 1)^- = -\imath(1 + n), \end{aligned}$$

for all $n \in \mathbb{N}$. However, by the p-continuity of h and h^*,

$$p(a + n\imath 1) \geq |h(a + n\imath 1)| = 1 + n$$

and

$$p((a + n\imath 1)^\circ) \geq |h^*(a - n\imath 1)| = 1 + n,$$

and so

$$\begin{aligned} (1 + n)^2 &\leq p(a + n\imath 1)p((a + n\imath 1)^\circ) \\ &= p(a^2 + n^2 1), \quad \text{by } (\alpha) \\ &\leq p(a^2) + n^2, \end{aligned}$$

for all $n \in \mathbb{N}$, an impossibility. Thus case 1 is proved.

Case 3 being a particular instance of 2, we are finished when case 2 is proven. With this as our intention, we remark next that

$$p(x + \imath y) \geq \max\{ p(x), p(y)\} ,$$

when x and y are Hermitian in A, and p is any seminorm on A satisfying (δ). (For then

$$p(x + \imath y) = p(x - \imath y),$$

and hence

$$p(x) \leq \tfrac{1}{2}(\, p(x + \imath y) + p(x - \imath y)\,) = p(x + \imath y)$$

by the triangle inequality.By symmetry,the other inequality needed is also true.)

As before,we have $h \in Hom\ A_P$, and an Hermitian element $a \in A$ with $h(a) = \imath$. The first fact shews the existence of a finite subset P' of P, such that h is P'-continuous. Let $p = \max P'$ (that is,

$$p(b) = \max\{\, q(b) \mid q \in P' \,\}$$

for all $b \in A$.) Then p is a submultiplicative seminorm, not necessarily satisfying (δ). Nevertheless,

$$1 + n = |h(a + n\imath 1)| \leq p(a + n\imath 1),$$

for all natural numbers,and furthermore there is a seminorm $p_n \in P'$ depending on n, for which

$$p(a + n\imath 1) = p_n(a + n\imath 1).$$

However, $p_n(a - n\imath 1) \geq n$, as remarked above.Consequently,

$$\begin{aligned} n(n+1) &\leq p_n(a + n\imath 1)p_n(a - n\imath 1) \\ &= p_n(a^2 + n^2 1) \\ &\leq p(a^2 + n^2 1) \leq p(a^2) + n^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Again this is impossible,and the proof of the lemma is in this way completed.

Note : The last inequality derives from the triangle inequality, together with the fact that $p(1) = 1$. Now for any seminorm q satisfying (α), we have

$$q(1) = q(11^{\circ}) = q(1)q(1^{\circ}) = q(1)^2,$$

since $1 = 1^{\circ}$. Hence, in the lemma, $p(1) = \max\{q(1) \mid q \in P'\}$ and so $p(1) = 1$, as required.

<u>Proposition</u> 3.10 : *Suppose A_T is an Hausdorff topological $\mathbb{C}$-algebra with involution, whose topology is generated by a family P of submultiplicative seminorms.*

If in addition conditions (α) and (β) are satisfied, then the universal representation is an embedding, and for all $a \in A$ and $p \in P$,

$$p(a) = \sup\{ |h(a)| \mid h \in \pi_p^{\bullet}(\mathrm{Hom}\ \bar{A}_{\bar{p}}) \}.$$

Proof : It is only necessary to shew that $p(a^2)$ and $p(a)^2$ are always equal, and then appeal to Theorem 3.7 and Corollary 3.8 i .

Accordingly, let $a \in A$ and $p \in P$ be arbitrarily chosen. Then

$$\begin{aligned} p(a^2)\,p((a^2)^{\circ}) &= p(a^2 a^{2\circ}) \\ &= p((aa^{\circ})(aa^{\circ})^{\circ}) \\ &= p(aa^{\circ})^2 \\ &= p(a)^2 p(a^{\circ})^2, \end{aligned}$$

by applying (α) several times. If $p(a^2) < p(a)^2$, this equality can be preserved only if $p(a^\circ) = 0$, and with this, also $p(a) = 0$, using (β). The ensuing contradiction establishes the claimed equality, and the theorem.

Corollary 3.11 : *Let* $(A_T, {}^\circ)$ *and* P *be as above. Then the universal representation is a homeomorphism if*

1) A_T *is complete and quasi-symmetric, and conditions* (α) *and* (β) *hold, or*

2) A_T *is complete,* P *is directed, and all of* (α), (β) *and* (γ) *hold, or*

3) A_T *is complete, and both* (α), (δ) *hold, or*

4) A_T *is complete, and* (ε) *holds.*

4 Applications and examples.

4.1 Compactness and local compactness.

We apply here the results of the previous chapter to get a different proof of Binz's characterisation [8, Satz 9] of compact c-embedded spaces in terms of C_cX, and then extend it as far as possible to locally compact c-embedded spaces. Two examples follow, shewing that little better can be done, for non-compact spaces. These examples have a number of other properties of interest, some of which we point out.

For any convergence space X, we say that C_cX is a Banach algebra, when it is topological (and automatically complete - see [7] or [3]), and the topology is normable. Proposition 3.1 is now adapted to our present ends.

<u>Lemma</u> 4.1 : *When C_cX is a Banach algebra, the space Hom_cC_cX is a compact Hausdorff topological space, and C_cX carries the topology of uniform convergence on X.*

Proof : Suppose p to be that norm on CX generating the topology on C_cX. Then by 3.1 we have

$$\mathrm{Hom}_s CX \xrightarrow{id} Hom_c C_p X \xrightarrow{id} Hom_c C_c X$$

are both homeomorphisms. Next,

$$C_c X \xrightarrow{d} C_c Hom_c C_c X \xrightarrow{id} C_n \mathrm{Hom}_s CX$$

are also homeomorphisms, thanks to Theorem 0.18 iv and Proposition 1.6 . However, for each function $f \in CX$,

$$f(X) = d(f)(Hom\ C_c X),$$

and so

$$C_n X \xrightarrow{d} C_n Hom_c C_c X$$

is a norm isometry, for all spaces X. These observations, when put together, verify the lemma.

Theorem (Binz) 4.2 : *For any c-embedded convergence space, the following statements are equivalent:*

i) X is compact.

ii) X is compact and topological.

iii) $C_c X$ is a Banach algebra (in this case necessarily carrying the topology of uniform convergence on X).

Proof : Clearly ii implies i, and we have already seen in Proposition 1.6 that i implies iii. Now supposing iii, we note that c-embeddedness and Lemma 4.1 together imply that X is compact and topological.

In order that ' iii ⇒ ii ' or even ' iii ⇒ i ' should be true, c-embeddedness or something very close is needed - in example 4.4 a space X is constructed, satisfying iii, and for which

$$i : X \longrightarrow Hom_c C_c X$$

is a bijection, but X is not compact.

The extension to locally compact spaces follows just as directly out of the apparatus we have set up.

Theorem 4.3 : *For any c-embedded space X, the statements given below are equivalent:*

i) X is locally compact.

ii) X is locally compact and locally topological.

iii) C_cX is a topological algebra (whose topology is necessarily that of uniform convergence on the compact subsets of X).

Proof : The term ' locally topological ' is to be taken as meaning that the space under discussion has a covering system whose members all inherit a topological convergence structure from the parent space.

Trivially, ' ii $\Rightarrow$ i ' , and Proposition 1.11 shews that ' i $\Rightarrow$ iii ' . Last,Theorem 3.3 and c-embeddedness imply the local compactness of X. However,each compact subset of X is topological,by Theorem 4.2, so that X is also shewn to be locally topological.

Two examples (4.5 and 4.6) of c-embedded locally compact convergence spaces are presented,one not principal, the other principal but not topological.These shew,inter alia,it is not possible to obtain a result as sharp as Binz's in the locally compact case - namely,a c-embedded locally compact space need not be topological.

Example 4.4 : Throughout these examples,we are concerned with real-valued functions only,and use CX to mean $C(X,\mathbb{R})$ exclusively. Now let Z be that convergence space whose underlying set is the interval $[0,1]$, and whose structure is the usual topology everywhere except at 0. There the structure is to be principal, with a base for the filter ϕ_0 being the set

$$\{ [0,\tfrac{1}{n}] \cap \mathbb{Q} \mid n \in \mathbb{N} \}$$

It is easy to check that $CZ = C[0,1]$, and also that $C_c Z$ carries the topology of uniform convergence. However, Z is not compact, since there is no finite ' subcover ' for the covering system

$$\{ [0,1] \cap \mathbb{Q}, (\tfrac{1}{2},1] \} \cup \{ (\tfrac{1}{n+2},\tfrac{1}{n}) \mid n \in \mathbb{N} \}.$$

For the next two examples, let D be the right open half plane in $\mathbb{C}$, and X be the set $D \cup \{0\}$, with the natural topology, in which the neighbourhood filter at a point $x \in X$ is denoted T_x.

<u>Example</u> 4.5 : Let (A_n) be a sequence of closed discs all contained in X and all containing the point 0. More, we suppose the sequence nested (that is, increasing strictly) and that $X = \bigcup_{n=1}^{\infty} A_n$. For each $n \in \mathbb{N}$, let

$$\phi_n = T_0 \cap A_n.$$

The space X' has underlying set X, and carries the final structure induced by the family of injections

$$\{ j_n : A_n \longrightarrow X \mid n \in \mathbb{N} \}.$$

Clearly if $x \in D$ then $x\phi$ is a pair from X' iff the filter ϕ is finer than T_x, and $\phi \to 0$ in X' iff

$$\phi \geq j_n(\phi_n)$$

for some $n \in \mathbb{N}$.

We shall prove that X' has the following properties:

1) It is a non-principal locally compact c-embedded convergence space.

2) There are continuous functions on X' which are not continuous on X.

3) The spaces cX', tX' and wX' all coincide,and are not locally compact.

Proof of 1 : Evidently X' is not a principal space.Part 3 of Theorem 1.9 tells us that X' is locally compact, being a final convergence structure derived from a family of locally compact spaces.Since

$$id : X' \longrightarrow X$$

is continuous,as one sees from the definition of X', it follows that $CX \subseteq CX'$. One deduces from this that

$$i : X' \longrightarrow \mathit{Hom}_c C_c X'$$

is injective,allowing us to identify the sets X and $\mathit{Hom}\, C_c X'$.

The last thing to check in this group of properties is the c-embeddedness of X'. Let then θ' denote the neighbourhood filter at 0 in $C_c X'$ (which exists, since X' is locally compact). One can now readily see from the definition of the structure of continuous convergence that $x\phi$ is a pair from $Hom_c C_c X'$ iff

$$\omega(\theta' \times \phi) \to 0 \quad \text{in } \mathbb{R},$$

and

$$f(\phi) \to f(x) \quad \text{in } \mathbb{R}$$

for all $f \in CX'$. The second of these requirements is true in particular for functions from CX, and so, for all $x \in X$,

$$\phi \to x \quad \text{in } Hom_c C_c X'$$

implies that ϕ is finer than T_x.

When x is not zero, this alone suffices for $x\phi$ to be a pair from X'. On the other hand, if $\phi \to 0$ in $Hom_c C_c X'$, there are $B \in \theta'$ and $C \in \phi$ with

$$\omega(B \times C) \subseteq [-1,1].$$

Thus there is a positive number λ and a compact subset K of X' for which

$$B \supseteq \{ f \in CX' \mid f(K) \subseteq [-\lambda,\lambda] \}.$$

However, the inequalities $\lambda \leq 1$ and $C \subseteq K$ must hold true, otherwise even the functions in CX yield a contradiction. Theorem 1.9 is used again, to shew that K, as a compact subset of X', lies in A_n for some $n \in \mathbb{N}$. This shews that ϕ has a base on A_n, and hence that $\phi \geq j_n(\phi_n)$. With this,we have proven also that X' is c-embedded.

Proof of 2 : Let (x_n) be a sequence in X, converging to 0 in X, and such that $x_n \in A_{n+1} \setminus A_n$, for all $n \in \mathbb{N}$. With the help of Tietze's extension theorem,we can construct a function f on X, satisfying

i) $f(A_1) = \{0\}$,

ii) $f(x_n) = 1$, for all $n \in \mathbb{N}$, and

iii) the restriction of f to A_n is continuous,for all $n \in \mathbb{N}$.

It is clear that $f \in CX'$, by the universal property of final structures,and as well, f cannot be continuous on X.

Proof of 3 : The identity maps

$$cX' \longrightarrow tX' \longrightarrow wX'$$

are both continuous,as pointed out on page 7. Thus to shew the identity of all three spaces,it is enough to shew that

$$id : wX' \longrightarrow cX'$$

is continuous too.

From the definition of wX', a filter ϕ converges to x in wX' iff $f(\phi) \to f(x)$ in $\mathbb{R}$, for all $f \in CX'$. In particular, $\phi \geq T_x$, which reduces our task to those filters suspected of converging to 0.

Thus in proving that each member of the filter $\bigwedge_{n=1}^{\infty} j_n(\phi_n)$ contains a wX'-neighbourhood of 0, we shall have proven all our claims.

To each $A \in \bigwedge_{n=1}^{\infty} j_n(\phi_n)$ we associate a monotone decreasing sequence (a_n) in the following fashion:

There is an open disc $N(0,a_1)$ centred at 0 and of radius a_1, such that

$$A_1 \cap N(0,a_1) \subseteq A_1 \cap A.$$

For $n > 1$, we choose a_n with $0 < a_n \leq a_{n-1}$, and

$$A_n \cap N(0,a_n) \subseteq A_n \cap A.$$

The boundary of the disc $N(0,a_n)$ and of A_{n-1} meet at points u_n, v_n, for all $n > 1$. The points u_n are to be all above the real axis,and the v_n's all below. Since the sequence (a_n) is monotone decreasing,we can join u_2 to v_2, u_n to u_{n+1} and v_n to v_{n+1} by straight lines, for all $n \geq 2$, and obtain in this way a continuous curve enclosing a portion U of X, with $0 \in U$.

By construction, $U \subseteq A$. We now shew that U is a neighbourhood of 0 in wX'. To see this,we construct a function $f \in CX'$ with $U = \{ x \in X \mid f(x) \in (-1,1) \}$.

The set $A_1 \setminus U$ is closed in A_1, and does not meet 0. That means that

$$g_1(0) = 0, \text{ and } g_1(A_1 \setminus U) = \{1\},$$

and also $0 \leq g_1 \leq 1$, for some $g_1 \in CA_1$, since A_1 is a completely regular space. Arguing by induction, we suppose we have $g_n \in CA_n$, and that

$$g_n | A_{n-1} = g_{n-1},$$

$$g_n(A_n \setminus U) = \{1\}, \text{ and}$$

$$0 \leq g_n \leq 1.$$

Along the common boundary of A_n and $A_{n+1} \setminus (A_n \cup U)$, the function g_n is constant, 1. Consequently, we can define a continuous function h_n on $A_n \cup (A_{n+1} \setminus (A_n \cup U))$, extending g_n, and constant 1 on the remaining part. Now since the domain of h_n is closed in A_{n+1}, Tietze's theorem gives us $g_{n+1} \in CA_{n+1}$ extending h_n, bounded between 0 and 1, and constant 1 on $A_{n+1} \setminus U$.

The desired map f is exactly that, whose restriction to each A_n is g_n, for f is continuous on X', and

$$f^{-1}(-1,1) = U.$$

Last, one can see that the (identical) spaces cX', tX' and wX' are not locally compact, by noting that if any neighbourhood of 0 in wX' is compact, there is one of the form $f^{-1}[-1,1]$ for f in CX' with $f(0) = 0$.

Such subsets are not compact, for a cover exists, without finite subcover. The set $F = f^{-1}[-1,1] \setminus f^{-1}(-\frac{1}{2},\frac{1}{2})$ lies entirely in D, and is not compact as a subset of $\mathbb{C}$, since

0 is an accumulation point of F in the usual topology. Let S be a cover of F by open subsets of D, admitting no finite subcover.Then $S \cup \{f^{-1}(-\frac{1}{2},\frac{1}{2})\}$ is a cover for $f^{-1}[-1,1]$ also admitting no finite subcover.

Thus none of these sets $f^{-1}[-1,1]$ is a compact neighbourhood of 0 in wX', and hence wX' is not locally compact.Next comes the second example of this sort.

Example 4.6 : Let X" be that space whose underlying set is X, and whose structure is the final structure induced by

$$j : D \longrightarrow X, \quad \text{and}$$

$$j'' : [0,\infty) \longrightarrow X.$$

We put $\phi'' = j''(T_0 \cap [0,\infty))$, observing that for $x \in D$,

$$\phi \to x \text{ in } X'' \text{ iff } \phi \geq T_x,$$

and 0ϕ is a pair from X" iff $\phi \geq \phi''$. Further,

$$id : X'' \longrightarrow X'$$

is continuous,so that $CX'' \supseteq CX'$.

Some properties of X" of interest to us are:

1) there are functions in CX" which fail to be continuous on X',

2) X" is a principal c-embedded locally compact convergence space which is not topological,and

3) the spaces tX'' and wX'' are identical, and not locally compact.

Perhaps we should explain why these claims are of interest, before going any further. First, that non-topological locally compact convergence spaces exist, is itself interesting, particularly in view of our extension of parts of Gelfand theory to topological algebras, involving local compactness through Theorem 3.3 . Second, it would be interesting to know exactly when c-embedded locally compact spaces are topological. Our example shews that to require that they be principal is not enough. Last, we suspect that if Z is locally compact c-embedded, and if $id : Z \longrightarrow Z' \longrightarrow wZ$ are both continuous, then Z' is also locally compact only if $Z = Z'$. Our examples at least partially bear this out.

Turning now to the proofs of our claims, we observe that the second may be proved in the same way as its counterpart in the previous example, the details being accordingly omitted.

Proof of 1 : The function $f : X \longrightarrow \mathbb{R}$ defined for all points $x \in D$ by

$$f(x) = \mathcal{R}\left[\frac{x}{|x|}\right],$$

and by $f(0) = 1$, is continuous on X'', but not on X' as one readily verifies.

Proof of 3 : We know already that $id : tX'' \longrightarrow wX''$ is continuous, being actually a homeomorphism when restricted to D. Hence, as in the previous example, it is enough to shew that every tX"-neighbourhood of 0 is even a wX"-neighbourhood.

With this in mind, we introduce the idea of a <u>standard triangle</u>, which is nothing more nor less than the set of points in the open triangle with vertices 0, u, and v, together with the point 0, whereby u and v are complex numbers such that $u + v$ is real and strictly positive. Each standard triangle is a wX"-neighbourhood of 0, as can be seen with the help of the function f constructed in the first part of this example.

It will be shewn that each tX"-neighbourhood of 0 contains a homeomorphic image of a standard triangle, and this will be then the proof that wX" and tX" are the same.

Digressing momentarily, we recall that the function $d(\ ,B) : Y \longrightarrow \mathbb{R}$ given by the formula

$$d(y,B) = \inf\{ d(y,b) \mid b \in B \}$$

for $y \in Y$, is continuous , for each subset B of the metric space Y. In particular, if A is a tX"-neighbourhood of 0, the sets $A \cap [0,\infty)$ and $A \cap D$ are open in $[0,\infty)$ and D respectively, when A itself is tX"-open.

For each $x \in X$, let

$$\lambda_x = d(x, X \setminus A),$$

and note that the open disc $N(x,\lambda_x) \subseteq A \cap D$, for each $x \in A \cap D$. Now suppose that $[0,a_1] \subseteq A$, and $0 < \alpha < \alpha' \leq a_1$. It follows from the foregoing digression that λ_x attains a positive minimum on this interval. Let λ_1 denote the lesser of $\frac{1}{2}a_1$ and the minimum of λ_x, computed on the interval $\left[\frac{1}{2}a_1, a_1\right]$, then let n_1 be the smallest natural number such that $\frac{1}{2}a_1 \leq (n_1 - 1)\lambda_1$. Clearly then, for $i = 1, \ldots, n_1 - 1$ the open discs $N(x_i, \lambda_1)$ all lie in $A \cap D$, where x_i is the point $a_1 - (i - 1)\lambda_1$.

We repeat this construction, for the interval $\left[\frac{1}{2}a_2, a_2\right]$, for $a_2 = x_{n_1}$. And then again - and so on. We should, of course, do it formally with induction; the procedure is clear.

In this way the interval $(0, a_1]$ has been covered by an infinite sequence of open discs $N(x_\varkappa, \upsilon_\varkappa)$ of decreasing radius lying in $A \cap D$. Further, the points $u_\varkappa$, $v_\varkappa$ of intersection of the boundary of the $\varkappa$-th with the $\varkappa+1$-th disc allow us to draw a curve (by joining u_1 to v_1, and $u_\varkappa$ to $u_{\varkappa+1}$, and $v_\varkappa$ to $v_{\varkappa+1}$ as in the earlier example) enclosing a [illegible] region U, with $U \subsetneq A$. Naturally, we have included 0 in U as well.

Next we define a continuous bijection $g : X'' \longrightarrow X''$ with the property that the set $g(U)$ is a standard set, in fact that with vertices 0, u_1, and v_1. This will establish the result, for then U is a wX''-neighbourhood of 0, contained in A.

The mapping g is constructed simply by vertical dilation, as follows: if $u = \mathcal{R}[u_1]$, there is a continuous function $\beta : (0, u] \longrightarrow \mathbb{R}^+$, whose graph is that portion of the curve bounding U above the real axis. Now g is defined by

$$g(\mu + \iota\nu) = \mu + \iota\nu, \text{ if } \mu \geq \mathcal{R}[u],$$

$$g(\mu + \iota\nu) = \mu + \iota\nu\mu\mathcal{J}[u]/\mathcal{R}[u]\,\beta(\mu) \text{ otherwise.}$$

except for $g(0) = 0$, for all $\mu \in \mathbb{R}^+$ and $\nu \in \mathbb{R}$; the continuity of g is easy to check, since g leaves the positive real numbers unchanged, and the restriction of g to D is also continuous, as $\beta(\mu)$ is never zero. One can then verify that g is indeed a homeomorphism of X'' onto itself, and of wX'' onto itself also.

<u>Remark</u> 4.7 : The space X'' just given enables us to controvert two conjectures; it is <u>not</u> true that

i) the adherence operator in C_cY is idempotent, for all spaces Y, and

ii) the algebraic operations are continuous in the principal space associated to C_cY, for all spaces Y.

Proof : We have seen that the adherence operator in X'', and hence in $\mathit{Hom}_c C_c X''$, is not idempotent. The latter is, however, a subspace of $C_c C_c X''$, whose adherence operator therefore can not be idempotent (Lemma 0.4 i).

Next, were the operations in cC_cC_cX'' continuous, this principal space would already be topological (since principal convergence groups are topological [2, section 3.3, Satz 5]). This would contradict the first part of this remark, as then the adherence operator in C_cC_cX'' would be idempotent.

4.2 The subalgebra A of CX.

Here a particular, already well-known, subalgebra A of CX is introduced, and its relation to X, and particularly to the subset of locally compact points of X, briefly indicated - at least for completely regular topological spaces X. Naturally we fit A_c into this relationship.

On the way, we run across a large class of spaces, for which A separates the points of X, but does not generate the topology on X. Our Stone-Weierstrass theorem applies, (namely, Corollary 2.6), whereas those of Binz and Feldman do not.

Throughout this section, X is a non-compact convergence space, later restricted to be also completely regular topological. Thus X is always dense in its one point compactification $\tilde{X}$; another symbol we recall - the coarsest filter converging to ∞ in $\tilde{X}$ is ϕ_∞. We now define

$$A = \{ f \in CX \mid f(\phi_\infty \cap X) \text{ is a Cauchy filter in } \mathbb{F} \}, \text{ and}$$

$$\underline{A} = \{ f \in CX \mid f(\phi_\infty \cap X) \text{ is a Cauchy filter in } \tilde{\mathbb{F}} \}.$$

It is an easy calculation to shew that

$$j^\bullet : C_n\tilde{X} \longrightarrow A_n \subseteq C_nX$$

(induced by the inclusion map $j : X \longrightarrow \tilde{X}$) is a norm-preserving isomorphism, allowing us in what follows to use A_n and $C_n\tilde{X}$ interchangeably.

Proposition 1.6 shews that $C_c\tilde{X}$ is a Banach algebra, under the uniform-norm topology, and hence (Proposition 3.1)

$$id : Hom_c C_c\tilde{X} \longrightarrow Hom_s C\tilde{X}$$

is a homeomorphism, and $Hom_s A$ (that is, $Hom_s C\tilde{X}$ under another name) is a compact Hausdorff topological space. Further,

$$i_{\tilde{X}} : \tilde{X} \longrightarrow Hom_s A$$

is surjective, and as a result, $Hom\ A \setminus i_{\tilde{X}}(X)$ can contain at most one point, $\hat{\infty}$. In fact, the sets $Hom\ A$ and $i_{\tilde{X}}(X)$ are equal if X is not locally compact. Clearly the one point compactification of the space $Hom_s A \setminus \{\hat{\infty}\}$ is exactly $Hom_s A$ again.

The sets A and $\underline{A}$ are given in a different, but equivalent, way when X is completely regular:

$$A = \{ f \in C^0X \mid f \text{ is constant on } \beta X \setminus X \},$$

and

$$\underline{A} = \{ f \in CX \mid \tilde{f} : \beta X \longrightarrow \tilde{\mathbb{F}} \text{ is constant on } \beta X \setminus X \},$$

with βX the Stone-Čech compactification of X, and $\tilde{f}$ the Stone extension of $f \in CX$. From now on, we consider completely regular spaces only.

The set X_1 of all points in X possessing compact neighbourhoods is open in X, and locally compact as a subspace of X. Moreover, X_1 and $Hom_s A \setminus \{\hat{\infty}\}$ are obviously homeomorphic, and so A generates the topology on X_1.

The algebra A can be regarded as a subalgebra of C_cX_1, in which it is dense, by the classical SWT. Thus A is closed in C_cX_1 iff it is the whole algebra CX_1. However, from the foregoing remarks, we know that $A = C\widetilde{X_1}$. This means simply that the Stone-Čech and one point compactifications of X_1 coincide, shewing that X_1 is almost compact [13, page 95]. Conversely, if X_1 is almost compact, then $A = CX_1$, being in particular closed in C_cX_1.

Let X_t denote the set $\beta X \setminus X_1$. The equivalence relation generated on βX by A, for which the symbol E_A^β was used in chapter 2, has equivalence classes

$$E_A^\beta x = \{x\}, \text{ if } x \in X_1, \text{ and}$$

$$E_A^\beta x = X_t \text{ otherwise.}$$

Further, since each function in A is bounded on X, the conditions of Corollary 2.6 are fulfilled, and we conclude

$$a(A) = \{ f \in CX \mid f \text{ is constant on } X_t \cap X \}.$$

(When $X_t \cap X = \emptyset$, this is to be interpreted as no restriction at all; that is, if X is locally compact, then A is dense in $C_cX = C_kX$, as is well-known.)

Two possibilities arise, when A is closed in C_cX. First, if $X_t \cap X$ is empty, then $A = CX$, which shews that X is almost compact. Second, when X is not locally compact, the set $X_t \cap X$ of points without compact neighbourhoods in X is dense in X_t, for otherwise there would be functions in CX constant on $X_t \cap X$ but not on $\beta X \setminus X$, contradicting our assumption that A is closed in C_cX.

In both cases the converse is clearly true.

Example 4.7 : A completely regular topological space is called almost locally compact iff it has at most one point without compact neighbourhoods. When X is almost locally compact but not locally compact, A separates the points of X but does not generate the topology thereon. Nevertheless A is dense in C_cX, by Corollary 2.6 .

An example of such a space is the space X of Examples 4.5 and 4.6 . Thus our relaxation of topology-generation requirements in our SWT is genuine, though small.

To summarise, we state

Theorem 4.8 : *Suppose X is a completely regular topological space. Then the following statements hold:*

1) A is closed in C_cX $\Leftrightarrow$ (either X is almost compact or the set $(\beta X \setminus X)^{-} \cap X$ is dense in $(\beta X \setminus X)^{-}$.)

2) X is almost locally compact $\Leftrightarrow$ A separates the points of X $\Leftrightarrow$ A is dense in C_cX.

3) X is locally compact $\Leftrightarrow$ A generates the topology of X $\Leftrightarrow$ A_c has a (necessarily unique) dense maximal ideal $\Leftrightarrow$ C_cX is topological.

4) X is almost compact $\Leftrightarrow$ $A = CX$ $\Leftrightarrow$ A separates the points of X and is closed in C_cX.

5) X is locally compact and σ-compact $\Leftrightarrow$ the sets A and $\underline{A}$ are unequal $\Leftrightarrow$ C_cX is topological, the topology stemming from a translation-invariant metric.

Proof : The proofs of 1 and 2 have already been sketched. That of 3 we give now.

When X is a locally compact Hausdorff topological space, $\tilde{X}$ is a compact Hausdorff topological space, whose topology is generated by $C\tilde{X}$. Thus A generates the topology of X.

Next, let A be topology-generating, and $y \in \beta X \setminus X$. Then $\hat{y} \in \text{Hom}\, A$. However, A is a monotone subalgebra of CX and so $\hat{y} : A_c \longrightarrow \mathbb{F}$ is not continuous (Proposition 0.15). Hence the kernel of $\hat{y}$ is a non-closed maximal ideal in A_c, and accordingly dense (since the adherence of any ideal is an ideal as well).

The maximal ideals of A are in one-to-one correspondence with the set $\text{Hom}\, A$, as is known from Banach algebra theory. If one of these is dense in A_c, it can only be that corresponding to $\hat{\infty}$, since $\hat{\infty}$ is the only homomorphism with any chance of being in $\text{Hom}\, A \setminus i_{\tilde{X}}(X)$. This means that X is locally compact - for otherwise $\hat{\infty}$ coincides with some point evaluation on X, as noted earlier in this section.

Last, the equivalence of the first and last statements comes immediately from Theorem 4.3, and with this, the proof of claim 3 is complete.

We have already dealt with 4 in the observations preceding this theorem, and so only 5 remains to be proved. The equivalence of the first and third conditions therein is (modulo Theorem 4.3) a well-known result from the theory of the topology of uniform convergence on compacta - see [14, Theorem 2], for example.

Now let X be a Hausdorff, σ-compact, locally compact but not compact topological space. Then one can arrange that $X = \bigcup_{\xi=1}^{\infty} A_\xi$, where $A_{\xi+1}$ is a neighbourhood of A_ξ, $A_{\xi+1} \setminus A_\xi$ is non-void, and A_ξ is compact, for all $\xi \in \mathbb{N}$. Using these sets, it is straightforward to construct a function $f \in CX$ such that $f(\phi_\infty \cap X) \to \infty$ in $\widetilde{\mathbb{F}}$. In fact, for each positive integer ζ, there is $f_\zeta \in CX$, with $0 \leq f_\zeta \leq 1$,

$$f_\zeta(A_\zeta) = \{0\}, \text{ and } f_\zeta((X \setminus A_\zeta)^-) = \{1\}.$$

(This derives from the complete regularity of X, and the compactness of A_ζ.) Now the function

$$f = \sum_{\zeta=1}^{\infty} f_\zeta : X \longrightarrow \mathbb{F}$$

is well defined and continuous, and further,

$$f(\phi_\infty \cap X) \to \infty \text{ in } \widetilde{\mathbb{F}}.$$

Hence $f \in \underline{A} \setminus A$.

On the other hand, if $\underline{A} \setminus A$ is non-empty, containing a function f, say, then for each $\rho \in \mathbb{N}$, the set

$$K_\rho = \{ x \in X \text{ with } |f(x)| \leq \rho \}$$

is compact (being a closed subset of the compact space $\tilde{X}$). However the collection $\{ K_\rho \mid \rho \in \mathbb{N} \}$ clearly forms a covering system for X, which is thus shewn to be locally compact and σ-compact, as required.

(Note that in the above paragraph, all we use is that $\underline{A} \setminus A$ is non-void, and general properties of the one point compactification.)

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