On the Solution of Systems of Equations by the ε-Algorithm of Wynn

by

E. Gekeler

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Abstract. The ε-algorithm has been proposed by Wynn several times in order to accelerate the convergence of vector sequences, but one does not know much about the effect it produces on systems of equations. In this paper we prove that the algorithm applied to the Picard sequence $X_{i+1} = F(X_i)$ of an analytic function $F: R^n \supset D \to R^n$ supplies us with a quadratic convergent iterative method and there is no differentiation of F needed. Two examples point out the numerical properties and show that we can get convergence - eventually with a modification of the method - even if F is not contractive near the fixed point.

1. Introduction

The ε-algorithm is a nonlinear method in order to accelerate the convergence of sequences being identical in its simplest form with the δ^2-tranformation of AITKEN [1]. SHANKS [8] and WYNN [9] developed it and WYNN examined it thoroughly together with diverse sequences and series [10-13].For an analytic function $f: \mathbb{C} \supset D \to \mathbb{C}$ we get by the ε-algorithm higher (integer) order iterative methods without differentiation for the computation of a fixed point [2].
Using the generalized matrix inverse following MOORE [5]

and PENROSE [6] the method has been recently applied to sequences of matrices and vectors, the same we get for example in solving linear systems of equations [3, 4, 7, 14, 17, 18, 19]. WYNN points to the fact, that the algorithm supplies us with good results in solving nonlinear systems, too [14, 15, 17, 18]. But there seem to be no statements about convergence up to now. In this paper we examine the behaviour of the ε-algorithm when applied to the Picard sequence of an analytic function $F: \mathbb{R}^n \supset D \to \mathbb{R}^n$ with fixed point Z. With the help of a theorem due to McLEOD [4] we show that the algorithm, in a suitable approach similar to Steffensen's method, is a quadratic convergent iterative method for the computation of Z. The proof of convergence is, as a matter of fact, of local nature and uses the symbols of Landau because of the complicated recursive relations. A short discussion of the numerical properties follows at the end of the paper.

2. The Algorithm and Nonlinear Systems of Equations

The ε-algorithm [9, 18] is a computational procedure in which successive columns of a matrix $(\varepsilon_q^{(p)})_{o\le p,\ o\le q}$ with row index p are obtained by the formula

$$(1) \qquad \varepsilon_{q+1}^{(p)} = \varepsilon_{q-1}^{(p+1)} + (\varepsilon_q^{(p+1)} - \varepsilon_q^{(p)})^{-1}$$

starting from the initial conditions

$$(2) \qquad \varepsilon_{-1}^{(p)} = 0, \quad \varepsilon_o^{(p)} = S_p \qquad (0 \le p).$$

If the inverse of a vector X is defined by [5,6]

(3) $$X^{-1} = (X^*X)^{-1}\,\overline{X},$$

then we can apply the algorithm to sequences $\{S_p\}_{o\le p}$ of vectors and we have here the fundamental theorem [4, 19] we need for our later statements:

Theorem 1. Let $\{S_p\}_{o\le p}$ be a sequence of vectors with complex coefficients which satisfy the irreducible linear recursion

(4) $$\sum_{r=0}^{m} c_r S_{p+r} = \left(\sum_{r=o}^{m} c_r\right) S \qquad (0 \le p),$$

where S is fixed and

(5) $$\sum_{r=o}^{m} c_r \neq 0,\ c_r \in \mathbb{R}.$$

If then the elements of the matrix $(E_q^{(p)})$ are determined by using (1), (2) and (3), and if all $E_q^{(p)}$ with $1 \le q \le 2m$, $0 \le p \le 2m-q$ exist, then

$$E_{2m}^{(o)} = S.$$

Following a conjecture of WYNN [20] and GREVILLE [3] Theorem 1 remains true, if relations (4), (5) hold for complex scalars only, but this has not been yet proved. As a conclusion we get

Corollary. Let Z be the unique solution of the linear system $X = A\,X + C$ with real coefficients and let m be the degree of the minimal polynom of the matrix A for $Y = X_o - Z$. If the ε-algorithm is applied to the Picard sequence $\{X_p;\ X_{p+1} = A\,X_p + C\}_{o\le p}$ and if all $E_q^{(p)}$ with $1 \le q \le 2m$, $0 \le p \le 2m-q$ exist, then

$$E_{2m}^{(o)} = Z.$$

Proof: Let $p(x) = \sum_{r=o}^{m} a_r x^r$ be the minimal polynom of A for Y, then

$$\sum_{r=o}^{m} a_r X_{p+r} = (\sum_{r=o}^{m} a_r)Z + (\sum_{r=o}^{m} a_r A^{p+r})Y = (\sum_{r=o}^{m} a_r)Z$$

because $X_p = Z + A^p Y$ holds. By assumption we have $\sum_{r=o}^{m} a_r \neq 0$, since 1 is not eigenvalue of A, and the Corollary results from Theorem 1.

Let now $\Delta_p E_q^{(p)} = E_q^{(p+1)} - E_q^{(p)}$ and let, as usual, any scalar or vector-valued function be denoted by $O(\|X\|)$, if it is bounded for $\|X\| \to 0$ after division by $\|X\|$.

Lemma 1. Let m be the degree of the minimal polynom of the real matrix A for $0 \neq Y \in \mathbb{R}^n$ and let $\tilde{E}_q^{(p)}$ be the elements obtained from the sequence $\{\tilde{S}_p;\ \tilde{S}_p = Z + A^p Y;\ Z \in \mathbb{R}^n\}_{o \le p}$ by the ε-algorithm. If 1 is not eigenvalue of A and if all $\tilde{E}_q^{(p)}$ with $2 \le q \le 2m$, $0 \le p \le 2m-q$ exists in a neighborhood of Z, then

$$\tilde{E}_q^{(p)} = Z + O(\|Y\|), \qquad q \text{ even},$$

$$\tilde{E}_q^{(p)} = O(\|Y\|^{-1}), \qquad q \text{ odd},$$

for $1 \le q \le 2m$, $0 \le p \le 2m-q$ ($\|..\|$ = Euclid-norm).

Proof: For q = 1 we get $\Delta_p \tilde{E}_o^{(p)} = A^p(A - E)Y = B_p Y \neq 0$, since 1 is not eigenvalue of A. Therefore

$$\tilde{E}_1^{(p)} = (Y^T B_p^T B_p Y)^{-1} B_p Y$$

$$= \frac{1}{\|Y\|} \frac{Y^T Y}{Y^T B_p^T B_p Y} \frac{1}{\|Y\|} B_p Y \le \frac{1}{\|Y\|} \frac{\|B_p\|}{\lambda_{min}},$$

where $0 < \lambda_{min}$ is the smallest eigenvalue of $B_p^T B_p$. Let now $k \in \mathbb{N}$, $k < m$, and let the statement be true for all $q \leq 2k - 1$. By assumption we have $0 \neq \Delta_p \tilde{E}_{2k-1}^{(p)} = O(\|Y\|^{-1})$,

$$(\Delta_p \tilde{E}_{2k-1}^{(p)})^T (\Delta_p \tilde{E}_{2k-1}^{(p)}) = O(\|Y\|^{-2}),$$

and thus

$$\tilde{E}_{2k}^{(p)} = \tilde{E}_{2k-2}^{(p+1)} + ((\Delta_p \tilde{E}_{2k-1}^{(p)})^T (\Delta_p \tilde{E}_{2k-1}^{(p)}))^{-1} \Delta_p \tilde{E}_{2k-1}^{(p)}$$

$$= Z + O(\|Y\|) + O(\|Y\|^2) O(\|Y\|^{-1}) = Z + O(\|Y\|).$$

$\Delta_p \tilde{E}_{2k}^{(p)} \neq 0$, since by assumption, all $\tilde{E}_q^{(p)}$ which contribute to $\tilde{E}_{2m}^{(o)}$ exist. Therefore

$$(\Delta_p \tilde{E}_{2k}^{(p)})^T (\Delta_p \tilde{E}_{2k}^{(p)}) = O(\|Y\|^2),$$

$$\tilde{E}_{2k+1}^{(p)} = \tilde{E}_{2k-1}^{(p+1)} + ((\Delta_p \tilde{E}_{2k}^{(p)})^T (\Delta_p \tilde{E}_{2k}^{(p)}))^{-1} \Delta_p \tilde{E}_{2k}^{(p)}$$

$$= O(\|Y\|^{-1}) + O(\|Y\|^{-2}) O(\|Y\|) = O(\|Y\|^{-1}),$$

and the assertion of the Lemma follows by induction.

Lemma 2. Let m be the degree of the minimal polynom of the real matrix A for $0 \neq Y \in \mathbb{R}^n$ and let $E_q^{(p)}$ be the elements obtained from the real sequence $\{S_p;\ S_p = Z + A^p Y + O(\|Y\|^2);\ Z \in \mathbb{R}^n\}_{o \leq p}$ by the ε-algorithm. If 1 is not eigenvalue of A and if all $E_q^{(p)}$, $\tilde{E}_q^{(p)}$ with $1 \leq q \leq 2m$, $0 \leq p \leq 2m-q$ exist in a neighborhood of Z, then

$$E_q^{(p)} = \tilde{E}_q^{(p)} + O(\|Y\|^2), \qquad q \text{ even}$$

$$E_q^{(p)} = \tilde{E}_q^{(p)} + O(1), \qquad q \text{ odd},$$

for $0 \leq q \leq 2m$, $0 \leq p \leq 2m-q$.

Proof: For $q = 0$ the statement follows by assumption. For $q = 1$ we get $\Delta_p E_o^{(p)} = \Delta_p \tilde{E}_o^{(p)} + O(\|Y\|^2) \neq 0$ and $\Delta_p \tilde{E}_o^{(p)} \neq 0$ according to the proof of Lemma 1. Then

$$(\Delta_p E_o^{(p)})^T(\Delta_p E_o^{(p)}) =$$

$$(\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)}) + 2(\Delta_p \tilde{E}_o^{(p)})^T O(\|Y\|^2) + O(\|Y\|^4)$$

$$= (\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)})$$

$$(1 + 2\,\frac{(\Delta_p \tilde{E}_o^{(p)})^T O(\|Y\|^2)}{(\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)})} + \frac{O(\|Y\|^4)}{(\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)})}).$$

$\Delta_p \tilde{E}_o^{(p)} = O(\|Y\|)$ by Lemma 1 and hence

$$(\Delta_p E_o^{(p)})^T(\Delta_p E_o^{(p)}) = (\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)})(1 + O(\|Y\|)).$$

Since $\Delta_p E_o^{(p)}$ is an analytic function, we get

$$((\Delta_p E_o^{(p)})^T(\Delta_p E_o^{(p)}))^{-1} = ((\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)}))^{-1}(1 + O(\|Y\|))$$

and

$$E_1^{(p)} = \tilde{E}_1^{(p)} + ((\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)}))^{-1} O(\|Y\|)\Delta_p \tilde{E}_o^{(p)}$$

$$+ ((\Delta_p \tilde{E}_o^{(p)})^T(\Delta_p \tilde{E}_o^{(p)}))^{-1}(1 + O(\|Y\|))\; O(\|Y\|^2)$$

$$= \tilde{E}_1^{(p)} + O(1)$$

again by Lemma 1.

Let now $k \in \mathbb{N}$, $k < m$, and let the statement be true for all $q \leq 2k$. By assumption we have $\Delta_p E_{2k}^{(p)} = \Delta_p \tilde{E}_{2k}^{(p)} + O(\|Y\|^2) \neq 0$ and $\Delta_p \tilde{E}_{2k}^{(p)} \neq 0$. According to the proof for $q = 1$ we get

$$((\Delta_p E_{2k}^{(p)})^T(\Delta_p E_{2k}^{(p)}))^{-1}\Delta_p E_{2k}^{(p)} =$$

$$((\Delta_p \tilde{E}_{2k}^{(p)})^T(\Delta_p \tilde{E}_{2k}^{(p)}))^{-1}\Delta_p \tilde{E}_{2k}^{(p)} + O(1)$$

and hence

$$E_{2k+1}^{(p)} = \tilde{E}_{2k+1}^{(p)} + O(1).$$

$\Delta_p E_{2k+1}^{(p)} = \Delta_p \tilde{E}_{2k+1}^{(p)} + O(1)$ and $\Delta_p \tilde{E}_{2k+1}^{(p)}$ are equally supposed to be different from null and therefore we get

$$(\Delta_p E_{2k+1}^{(p)})^T(\Delta_p E_{2k+1}^{(p)}) = (\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)})$$

$$(1 + 2\frac{(\Delta_p \tilde{E}_{2k+1}^{(p)})^T O(1)}{(\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)})} + \frac{O(1)}{(\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)})})$$

$$= (\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)})(1 + O(\|Y\|)).$$

$$E_{2k+2}^{(p)} = \tilde{E}_{2k}^{(p+1)} + O(\|Y\|^2)$$

$$+ ((\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)}))^{-1}(1 + O(\|Y\|))(\Delta_p \tilde{E}_{2k+1}^{(p)} + O(1))$$

$$= \tilde{E}_{2k+2}^{(p+1)} + O(\|Y\|^2) + ((\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)}))^{-1} O(\|Y\|)\Delta_p \tilde{E}_{2k+1}^{(p)}$$

$$+ ((\Delta_p \tilde{E}_{2k+1}^{(p)})^T(\Delta_p \tilde{E}_{2k+1}^{(p)}))^{-1}(1 + O(\|Y\|))O(1)$$

$$= \tilde{E}_{2k+2}^{(p)} + O(\|Y\|^2).$$

As conclusion we have the following result:

Theorem 2. Let $F: \mathbb{R}^n \varepsilon D \to \mathbb{R}^n$ be an analytic function with fixed point $Z \in \overset{o}{D}$ and m the degree of the minimal polynom of $F'(Z)$ for $Y = S_o - Z$. Further, let $E_q^{(p)}$ and $\tilde{E}_q^{(p)}$ be the elements obtained likewise from the sequences $\{S_p;\ S_{p+1} = F(S_p)\}_{o \le p}$ and $\{\tilde{S}_p;\ \tilde{S}_p = Z + (F'(Z))^p Y\}_{o \le p}$ by the ε-algorithm. If 1 is not eigenvalue of $F'(Z)$ and if all $E_q^{(p)}$, $\tilde{E}_q^{(p)}$ with $1 \le q \le 2m$, $0 \le p \le 2m - q$ exist in a neighborhood of Z, then

$$E_{2m}^{(o)} = G(S_o, \ldots , S_{2m}) = H_F(S_o) \tag{6}$$

and the computational procedure

$$X_{i+1} = H_F(X_i) \qquad (0 \le i)$$

is, near Z, a quadratic convergent iterative method for the computation of Z.

Proof: By the Corollary and Lemma 2 we have

$$H_F(X_o) = E_{2m}^{(o)} = Z + O(\|X_o - Z\|^2).$$

3. Numerical results

When a system of equations $X = F(X)$ of order n is to be solved by the ε-algorithm, the way of doing this is normally to put $m = n$. Then we need for each step of iteration $4n^3 + 2n^2$ multiplications, $2n^2 + n$ divisions, $6n^3 - n^2$ additions/subtractions and the computation of $S_p = F(S_{p-1})$ for $1 \le p \le 2n$. The computation of the vectors S_p produces rather quickly a characteristic overflow, if the eigenvalue of the Jacobi matrix $F'(X)$ amount to much more then one near the fixed point. This disadvantage can eventually be eliminated by replacing the Picard sequence $S_p = F(S_{p-1})$ by

$$S_p = F_\alpha(S_{p-1}) = (1 - \alpha)S_{p-1} + \alpha F(S_{p-1}) \qquad (0 \le p)$$

with a suitable α, $0 < \alpha < 1$, by which we obtain a slowdown of the increase. If we have for example $\rho(F'(Z)) = 2$ for the spectral radius ρ of $F'(Z)$, we get $\rho(F'_\alpha(Z)) \le 3/2$ for $\alpha = 1/2$. The eigenvalue λ of $F'(Z)$ with $|\lambda| < 1$ are hereby increased, but they remain smaller than one in absolute value. Besides the convergence is deteriorated, if the eigenvalues of $F'(X)$ approach one near Z.

The rounding errors are partly of great influence. Perhaps it is possible that the numerical properties can be improved, if a modification proposed by WYNN [16] is applied. If the eigenvalues λ of $F'(X)$ with $|\lambda| < 1$ predominate, we can indicate a modification of the method by renouncing on the (theoretic) quadratic convergence, which considerable reduces the amount of work. To achieve this we replace in (6) 2m by $2\left[\frac{m+1}{2}\right]$ and obtain for the basic formula of the algorithm

$$(6^*) \qquad E_n^{(o)} = G(S_o, \ldots , S_n) = H_F^*(S_o)$$

in case $m = n$ even. We need now per step of iteration only $(n^3 + 8n^2 - 4n)/8$ multiplications/divisions, $(6n^3 - 2n^2)/8$ additions/subtractions and the computation of $S_p = F(S_{p-1})$ for $1 \le p \le n$.

Let now

$$U = \frac{1}{2}\cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix},$$

(V = Pascal matrix for n = 4) and let D_1 = (0.9, 0.8, 0.7, 0.6), D_2 = (1.5, 0.8, 0.7, 0.6), D_3 = (2.0, 0.8, 0.7, 0.6) be diagonal matrices, $A_k^T = 0.5\,k\,(1, 1, 1, 1)$ vectors $(0 \le k \le 4)$ and

$$p_1(X) = -(x_1^2 + x_1x_4)/2, \qquad q_1(X) = -x_1^2/4,$$

$$p_2(X) = -x_2^2/2 \qquad , \qquad q_2(X) = -x_2^2/4,$$

$$p_3(X) = -x_3^2/2 \qquad , \qquad q_3(X) = -x_3^2/4,$$

$$p_4(X) = -(x_4x_1 + x_4^2)/2, \qquad q_4(X) = -x_4^2/4.$$

The following tables contain in column i the values $\|X_i - X_{i-1}\|_E$ (compare Theorem 2) with rounded mantissae. Generally spreaking we have found out that in systems of equations like

$$X = Z + F'(Z)(X - Z) + O(\|X - Z\|^2)$$

the algorithm supplies us with better results, if F'(Z) is symmetric. Finally, it should be mentioned that it seems to be impossible at the moment to say more about the error than that it is of quadratic order.

Example 1: $X = A_2 + U\,D_1 U^T(X - A_2) + P(X - A_2)$

l=1	$X_o = A_4$	2.0	$1.2\ 10^{-2}$	$1.0\ 10^{-5}$	$1.8\ 10^{9}$				
l=2	$X_o = A_o$	$7.4\ 10^{-1}$	$6.6\ 10^{-1}$	$4.5\ 10^{-1}$	$1.4\ 10^{-1}$	$6.8\ 10^{-2}$	$8.4\ 10^{-3}$	$7.5\ 10^{-5}$	$2.5\ 10^{-8}$
l=2	$X_o = A_4$	$6.0\ 10^{-1}$	$5.4\ 10^{-5}$	$1.6\ 10^{10}$					
l=3	$X_o = A_4, \alpha = \frac{1}{2}$	1.8	$1.5\ 10^{-1}$	$3.7\ 10^{-2}$	$1.1\ 10^{-2}$	$4.4\ 10^{-3}$	$2.2\ 10^{-3}$	$1.1\ 10^{-3}$	$3.6\ 10^{-4}$

Example 2: as Example 1, using (6*) instead of (6)

l=1	$X_o = A_4$	1.9	$8.6\ 10^{-2}$	$5.5\ 10^{-3}$	$5.0\ 10^{-5}$	$5.2\ 10^{-8}$			
l=2	$X_o = A_o$	$7.9\ 10^{-1}$	$6.5\ 10^{-1}$	$4.3\ 10^{-1}$	$1.3\ 10^{-1}$	$4.6\ 10^{-2}$	$1.6\ 10^{-3}$	$5.1\ 10^{-5}$	$4.8\ 10^{-8}$
l=2	$X_o = A_4$	$6.0\ 10^{-1}$	$6.0\ 10^{-3}$	$4.0\ 10^{-6}$					
l=3	$X_o = A_4, \alpha = \frac{1}{2}$	1.5	$3.2\ 10^{-1}$	$1.1\ 10^{-1}$	$4.6\ 10^{-2}$	$2.1\ 10^{-2}$	$1.0\ 10^{-2}$	$9.5\ 10^{-4}$	$6.5\ 10^{-4}$

Example 3: $X = A_2 + V\,D_1 V^{-1}(X - A_2) + Q(X - A_2)$

l=2	$X_o = A_1$	0.9	$8.2\ 10^{-2}$	$2.7\ 10^{-6}$					
l=2	$X_o = A_3$	2.0	$9.9\ 10^{-1}$	$2.8\ 10^{-6}$					

Example 4: as Example 3, using (6*) instead of (6)

l=2	$X_o = A_1$	$8.9\ 10^{-1}$	$1.1\ 10^{-1}$	$3.2\ 10^{-4}$	$2.4\ 10^{-7}$				
l=2	$X_o = A_3$	$3.8\ 10^{-1}$	$5.1\ 10^{-1}$	$1.1\ 10^{-1}$	$3.8\ 10^{-4}$	$1.1\ 10^{-6}$			

Lehrstuhl IV für Mathematik
der Universität Mannheim (WH)

D-6800 Mannheim, Schloß

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