AN APPROXIMATION THEORETIC ALTERNATIVE TO ASYMPTOTIC EXPANSIONS FOR SPECIAL FUNCTIONS

Günter Meinardus

Nr. 61

März 1985

Erscheint in:

COMPUTERS AND MATHEMATICS WITH APPLICATIONS

Prof. Dr. G. Meinardus, Lehrstuhl Mathematik IV,
Universität Mannheim,
Seminargebäude A 5, B 123,
6800 Mannheim 1
West-Germany

Abstract of the paper: An Approximation Theoretic Alternative To Asymptotic Expansions For Special Functions.

Abstract: Some classes of functions, which are solutions of ordinary linear homogeneous differential equations of second order with an irregular singularity at infinity possess asymptotic expansions with respect to a real positive variable at infinity. In the case of non-oscillatory behavior of such functions these asymptotic expansions can be replaced by near-best relative approximations by polynomials of the reciprocal variable and by approximations with rational functions, using the socalled Carathéodory-Féjèr method. The investigations include Kummer functions resp. Whittaker functions (confluent hypergeometric functions) with this behaviour. A large class of special functions can be considered as Kummer functions resp. Whittaker functions. Two examples concerning the incomplete Gamma function and the transformed Gaussian probability function are given in some detail.

AN APPROXIMATION THEORETIC ALTERNATIVE TO ASYMPTOTIC EXPANSIONS FOR SPECIAL FUNCTIONS

by

Günter Meinardus
Lehrstuhl Mathematik IV
Universität Mannheim
Seminargebäude A 5, B 123
6800 Mannheim 1
West-Germany

Many special functions, mostly solutions of ordinary linear homogeneous differential equations of second order, possess asymptotic expansions at the irregular singular point infinity. These expansions do not represent good approximations to the function, in general. Frequently it is not even possible to compute values of the functions with prescribed accuracy from those expansions. On the other hand, as we will show here, these expansions can be replaced by near-best polynomial approximations in the reciprocal variable or by near-best rational approximations.

We here consider special differential equations of second order, which include the Kummer resp. the Whittaker functions (confluent hypergeometric functions) of real exponential behavior at infinity. In the first step we investigate the existence, uniqueness and estimation of solutions of a given differential equation, which tend to 1 for $x \to \infty$. These results are used to construct explicitly polynomial approximations and to discuss these approximations with respect to their near-best property. Here only informations directly available from the differential equation enter in the construction. We will get upper and lower bounds for the minimal deviation in the uniform approximation. Then to polynomial approximations of sufficient accuracy we apply the socalled Carathéodory-Féjèr method to construct near-best rational approximations with the same degree of the numerator and the denominator polynomial.

At the end of the paper two examples are treated in detail, i.e. we compute corresponding approximations and give ranges for the errors for the incomplete Gamma function and for the Gaussian probability function.

This paper can be considered as a continuation and an improvement of [5]. It is planned to investigate the oscillatory case as well as approximation problems in sectors of the complex plane in a forthcoming paper.

1. Notations and estimations. Let $q(x)$ be a real and continuous function on $(0,\infty)$, given by

$$q(x) = q_0 + \frac{q_1}{x} + \frac{p(x)}{x^2} \tag{1}$$

with real constants q_0, q_1, $q_0 < 0$, and bounded function p:

$$|p(x)| \leq \gamma \quad \text{for all} \quad x \in (0,\infty) . \tag{2}$$

We are interested in solutions of the differential equation

$$y'' + q(x)y = 0 \qquad \text{on} \quad [T,\infty), \tag{3}$$

where T is positive and sufficiently large. Let

$$\tau = +\sqrt{-q_0} \tag{4}$$

and

$$\rho = -\frac{q_1}{2\tau} \tag{5}$$

be two associate numbers. The transformation

$$z(x) = e^{\tau x} x^{\rho} y(x) \tag{6}$$

leads to the differential equation

$$Lz = 0$$

with

$$Lz := z'' - 2\left(\tau + \frac{\rho}{x}\right)z' + \frac{\rho(\rho+1) + p(x)}{x^2}\, z \quad . \tag{7}$$

Let T be a suitable positive number and $\tilde{C}[T,\infty)$ the space of all real, continuous and bounded functions f on $[T,\infty)$. We will make use of the norm

$$\| f \|_T = \sup(|f(x)|, x \in [T,\infty)) \quad .$$

For any given $r \in \tilde{C}[T,\infty)$ we are looking for such solutions z of the differential equation

$$Lz = \frac{r(x)}{x^2} \qquad \text{on} \qquad [T,\infty) \quad , \tag{8}$$

which have the properties

$$\lim_{x\to\infty} z(x) = 1 \qquad \text{and} \qquad \lim_{x\to\infty} z'(x) = 0 \; . \tag{9}$$

This problem is equivalent to the following: We are looking for functions $z \in \tilde{C}[T,\infty)$ which satisfy the integral equation

$$z(x) = 1 - \int_x^\infty G(x,t)\left\{(\rho(\rho+1)+p(t))z(t) - r(t)\right\}\frac{dt}{t^2} \quad . \tag{10}$$

Here $G(x,t)$ denotes the non-negative kernel

$$G(x,t) = \int_x^t e^{-2\tau(t-v)} \left(\frac{v}{t}\right)^{2\rho} dv \quad , \quad 0 < x \le t < \infty, \quad (11)$$

which is bounded with respect to t, uniformly in x, provided $x \ge x_0 > 0$.

Lemma 1: *For arbitrary* $T > 0$ *the inequality*

$$\left\| \int_x^\infty G(x,t) \frac{dt}{t^2} \right\|_T \le \frac{\kappa(T)}{T} \quad (12)$$

is valid with

$$\kappa(T) = \begin{cases} \frac{1}{2\tau} & \text{for } \rho \ge -1 \\ \sum_{\nu=0}^{-[2\rho]-2} \frac{\nu! \binom{-2\rho-2}{\nu}}{(\nu+1)(2\tau)^{\nu+1}} \frac{1}{T^\nu} & \text{for } \rho < -1. \end{cases} \quad (13)$$

Here $[2\rho]$ *denotes the largest integer less than or equal to the negative number* 2ρ.

Proof: From (11) we get

$$\int_x^\infty G(x,t) \frac{dt}{t^2} = \int_x^\infty \int_v^\infty e^{-2\tau(t-v)} \left(\frac{v}{t}\right)^{2\rho} \frac{dt}{t^2} dv \quad .$$

Let $\rho \ge -1$. Then, using integration by parts,

$$\int_v^\infty e^{-2\tau(t-v)} \left(\frac{v}{t}\right)^{2\rho} \frac{dt}{t^2} = \frac{1}{2\tau v^2} - \frac{\rho+1}{\tau} \int_v^\infty e^{-2\tau(t-v)} \left(\frac{v}{t}\right)^{2\rho} \frac{dt}{t^3} \le \frac{1}{2\tau v^2} ,$$

hence

$$\int_x^\infty G(x,t)\,\frac{dt}{t^2} \le \frac{1}{2\tau T} \quad .$$

For $\rho < -1$ we have to apply the process of integration by parts repeatedly until we get a negative sign in front of the last integral. So one gets

$$\int_v^\infty e^{-2\tau(t-v)}\left(\frac{v}{t}\right)^{2\rho}\frac{dt}{t^2} \le \frac{1}{2\tau v^2}\sum_{\nu=0}^{-[2\rho]-2}\frac{\nu!\binom{-2\rho-2}{\nu}}{(2\tau v)^\nu} \quad . \qquad (14)$$

If we integrate the inequality (14) from x to infinity with respect to v, we are lead directly to the estimation (12) again.

Theorem 1 (cp. [3]): *Let* $\kappa(T)$ *be defined as in* (13). *Furthermore let* T *satisfy*

$$\omega := \frac{\kappa(T)}{T}(|\rho(\rho+1)| + \gamma) < 1 \quad . \qquad (15)$$

Then the integral equation (10) *has one and only one solution* $z \in \tilde{C}[T,\infty)$.
For $r_1, r_2 \in \tilde{C}[T,\infty)$ *let* z_1 *resp.* z_2 *be the corresponding solutions of the integral equation* (10), *provided* (15) *is satisfied. Then the estimation*

$$\| z_1 - z_2 \|_T \le \frac{\kappa(T)}{T(1-\omega)} \| r_1 - r_2 \|_T \qquad (16)$$

is valid.

Remark: The solution of (10) can be continued uniquely to the interval $(0,\infty)$. Only estimations like the one given in (16) are not available in this simple form.

Proof of theorem 1: The right hand side of (10) defines an inner mapping of the Banach space $\tilde{C}[T,\infty)$. According to (12) and (15) this mapping is a contraction. Therefore for given $r \in \tilde{C}[T,\infty)$ there exists one and only one solution z of (10) in $\tilde{C}[T,\infty)$. The validity of (16) follows immediately.

2. Polynomial approximation. The solution of the integral equation for $r \equiv 0$,

$$z(x) = 1 - \int_x^\infty G(x,t)(\rho(\rho+1) + p(t))z(t)\frac{dt}{t^2} \qquad (17)$$

in $\tilde{C}[T,\infty)$, where T satisfies (15), will be denoted by $z_0(x)$. If p possesses an asymptotic expansion of the type

$$p(x) = \sum_{\nu=0}^{N} \frac{p_\nu}{x^\nu} + O\left(\frac{1}{x^{N+1}}\right) \qquad \text{for} \quad x \to \infty , \qquad (18)$$

$N = 0,1,\ldots$, then z_0 possesses an asymptotic expansion of the same type:

$$z_0(x) = \sum_{\nu=0}^{N} \frac{c_\nu}{x^\nu} + O\left(\frac{1}{x^{N+1}}\right) \qquad \text{for} \quad x \to \infty, \qquad (19)$$

$N = 0,1,\dots$, with $c_0 = 1$. This can be derived from (17), using the iteration procedure for contractions (cp. [3]). It therefore seems natural to approximate $z_0(x)$ by polynomials of the variable $1/x$.
Let us restrict now to the case

$$p(x) \equiv p_0 \ . \qquad (20)$$

The approximations and estimations, which follow, will be slightly modified if only

$$p(x) = p_0 + O\left(\frac{1}{x}\right) \qquad \text{for} \quad x \to \infty$$

holds.
In the sequel we use the abbreviations

$$g_\nu := (\rho+\nu)(\rho+\nu+1) + p_0 \quad , \quad \nu \in \mathbb{N}_0 \ . \qquad (21)$$

We assume

$$g_\nu \neq 0 \qquad \text{for all} \quad \nu \in \mathbb{N}_0 \ . \qquad (22)$$

This restriction is not a serious one, because one may conclude easily from the linear system (31) below, that $z_0(x)$ coincides with a polynomial of the variable $1/x$ of degree $\tilde{\nu}$, if (and only if) $\tilde{\nu}$ is the smallest integer from $\mathbb{N}_0$ such that $g_{\tilde{\nu}} = 0$.
We now consider the Tchebycheff polynomials $T_n(u)$ of the first kind, defined recursively by

$$T_0(u) \equiv 1 \quad , \quad T_1(u) = u \quad ,$$
$$T_{n+1}(u) = 2uT_n(u) - T_{n-1}(u) \qquad \text{for} \quad n \in \mathbb{N} \quad , \tag{23}$$

or, explicitly by the representation (cp. [8]):

$$T_n(u) = \frac{1}{2} \sum_{\nu=0}^{[n/2]} (-1)^\nu \frac{n}{n-\nu} \binom{n-\nu}{\nu} (2u)^{n-2\nu} \quad , n \in \mathbb{N} . \tag{24}$$

From (23) one gets the relation

$$T_n(2w - 1) = T_{2n}(\sqrt{w}) \tag{25}$$

for real $w \in [0,1]$ and all $n \in \mathbb{N}_0$. One deduces from (24) and (25) the

<u>Lemma 2</u>: *Let* $x \geq 1$. *Then*

$$T_{n+1}(1 - \frac{2}{x}) = \sum_{\mu=0}^{n+1} \frac{d_\mu}{x^\mu} \quad , \quad n \in \mathbb{N}_0 \quad ,$$

where

$$d_\mu = (-1)^\mu \frac{n+1}{n+1+\mu} \binom{n+1+\mu}{n+1-\mu} 4^\mu \quad , \mu=0,1,\ldots,n+1. \tag{26}$$

<u>Remark</u>: It is well known that for $T > 0$ we have

$$\| T_{n+1}(1 - \frac{2T}{x}) \|_T = 1 \quad , \tag{27}$$

and these functions share the alternant property

$$\left.\begin{aligned} &T_{n+1}(1) = 1 \quad , \\ &T_{n+1}\left(1 - \frac{2T}{x_\lambda}\right) = (-1)^\lambda \quad , \\ &\text{with} \\ &x_\lambda = \frac{2T}{1 - \cos\left(\frac{\lambda\pi}{n+1}\right)} \quad ; \quad \lambda = 1, 2, \ldots, n+1 \end{aligned}\right\} \tag{28}$$

<u>Theorem 2</u>: *Let* T *satisfy condition* (15) *and, according to* (20), $p(x) \equiv p_0$. *Then there exists uniquely a number* $A_n \neq 0$ *such that the unique solution of the integral equation* (10) *with*

$$r(x) = A_n T_{n+1}\left(1 - \frac{2T}{x}\right)$$

is a polynomial of degree $n+1$ *of the variable* $1/x$ *of the form*

$$z(x) = 1 + \sum_{\nu=1}^{n+1} \frac{a_\nu}{x^\nu} \quad , \tag{29}$$

provided

$$\delta := d_0 + \sum_{\nu=1}^{n+1} d_\nu \frac{(-2\tau T)^\nu \nu!}{g_1 g_2 \cdots g_\nu} \neq 0 \quad . \tag{30}$$

<u>Remark</u>: The inequality (30) can always be satisfied by choosing T sufficiently large.

Proof of theorem 2: Inserting $z(x)$ from (29) into the differential equation (8) we get the following system of linear equations for the coefficients:

$$
\begin{array}{lr}
2\tau a_1 & = A_n d_0 - g_0 \\
g_1 a_1 + 4\tau a_2 & = A_n d_1 T \\
- - - - - - - - - - - - - - - - & \\
\qquad g_\nu a_\nu + 2(\nu+1)\tau a_{\nu+1} & = A_n d_\nu T^\nu \\
- - - - - - - - - - - - - - - - & \\
g_{n+1} a_{n+1} & = A_n d_{n+1} T^{n+1}
\end{array} \qquad (31)
$$

Multiplying the μ^{th} equation of (31) by the factor

$$\frac{(-2\tau)^{\mu-1}(\mu-1)!}{g_1 g_2 \cdots g_{\mu-1}} \quad ; \quad \mu=2,3,\ldots,n+2 \quad ;$$

and adding then all equations leads to

$$A_n \delta = g_0 \quad .$$

Therefore the question, if $\delta \neq 0$ decides the solvability (and uniqueness) of the system (31).

The coefficients can now be given explicitly: We get

$$a_\nu = \frac{g_0}{\delta \cdot \nu!} \sum_{\mu=\nu}^{n+1} d_\mu \frac{(-2\tau)^{\mu-\nu} T^\mu \mu!}{g_\nu g_{\nu+1} \cdots g_\mu} \qquad (32)$$

for $\nu=1,2,\ldots,n+1$, and

$$A_n = \frac{g_0}{\delta} \quad . \qquad (33)$$

Theorem 3: *Let* T *satisfy* (15) *and let* δ *from* (30) *be* $\neq 0$. *Then for the polynomial solution* $z(x)$ *from* (29) *we get the following error estimation in approximating* z_0 , *the solution of the integral equation* (17):

$$\| z_0 - z \|_T \le |A_n| \frac{\kappa(T)}{T(1-\omega)} \quad . \tag{34}$$

Proof: The assertion is a direct application of (16), using the fact stated in (27).

Theorem 4: *The function*

$$p_n(x) = z(x) - \frac{A_n}{g_{n+1}} T_{n+1}\left(1 - \frac{2T}{x}\right) \tag{35}$$

belongs to the space Π_n *of polynomials of degree at most* n *in the variable* $1/x$. *We denote by*

$$\operatorname{dist}(z_0,\Pi_n) = \inf_{p\in \Pi_n} \| z_0 - p \|_T \tag{36}$$

the minimal deviation in approximating the function z_0 *by polynomials from the space* Π_n *with respect to the interval* $[T,\infty)$. *Then the following estimations from above and from below are valid:*

$$\operatorname{dist}(z_0,\Pi_n) \le \| z_0 - p_n \|_T \le |A_n| \left(\frac{1}{|g_{n+1}|} + \frac{\kappa(T)}{T(1-\omega)} \right), \tag{37}$$

and

$$\mathrm{dist}(z_0,\Pi_n) \geq |A_n|\left(\frac{1}{|g_{n+1}|} - \frac{\kappa(T)}{T(1-\omega)}\right) \quad . \tag{38}$$

Proof: Using (31) the coefficient of $x^{-(n+1)}$ in $z(x)$ can be expressed by

$$a_{n+1} = \frac{A_n}{g_{n+1}} d_{n+1}T^{n+1} \quad .$$

The coefficient of $x^{-(n+1)}$ in

$$T_{n+1}(1 - \frac{2T}{x})$$

is by definition equal to

$$d_{n+1}T^{n+1} \quad .$$

Therefore $p_n \in \Pi_n$.- The upper bound (37) follows now easily from (34) and (35). To prove the validity of (38) we may restrict to the case

$$\frac{\kappa(T)}{T(1-\omega)} < \frac{1}{|g_{n+1}|} \quad . \tag{39}$$

Using (28) the de la Vallée Poussin theorem (cp. [6], p. 82) leads to (38).

Remark: Choosing T large enough we are always able to satisfy (39). One then knows, how close in terms of the given norm the constructed polynomial in (35) is to the best approximating polynomial. Even more, one can make the upper and lower bound for the minimal deviation as close as one likes. We call p_n therefore a near-best approximation to z_0 on $[T,\infty)$.

It should be remarked that this method to get good uniform approximation of polynomial type from the differential equation has in its basic idea some connection with the socalled Lanczos Tau method. One should compare here the paper [7] by E.L. Ortiz.

3. Rational approximation. Let us use the transformation

$$x = \frac{2T}{1-u} \quad . \tag{40}$$

Then the function $z(x)$ from (29) coincides with a continuous function $w(u)$ on $[-1,+1]$. It can be uniquely represented by the sum of Tchebycheff polynomials

$$w(u) = 2\sum_{\nu=0}^{n+1} b_\nu T_\nu(u) \tag{41}$$

with real coefficients. Obviously

$$b_{n+1} = \frac{A_n}{2g_{n+1}} \tag{42}$$

and

$$2\sum_{\nu=0}^{n+1} b_\nu = 1 \quad .$$

In the following we assume

$$b_{n+1} \neq 0 \quad . \tag{43}$$

We describe now a method to construct explicitly the best approximation of $w(u)$ by real rational functions of maximal degree n in the numerator as well as in the denominator. This method can be found in [2] and can be considered as a special case of the constructions given in [4] and in [9]. The assumptions in theorem 5 are satisfied in most cases.

Theorem 5 (cp. [2], [4], [9]): *Let us denote by* H *the Hankel matrix*

$$H = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_{n+1} \\ b_2 & b_3 & & \cdot^{\cdot^{\cdot}} & 0 \\ b_3 & & \cdot^{\cdot^{\cdot}} & & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \vdots \\ b_{n+1} & 0 & \cdots & \cdots & 0 \end{pmatrix} .$$

We assume that H *possesses only one eigenvalue* $\lambda = \lambda_n$ *of smallest absolute value, and that its multiplicity is* 1. *Let* $h \in \mathbb{R}^{n+1}$ *be an eigenvector to the eigenvalue* λ_n *with components*

$$h = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{pmatrix} .$$

Then

$$h_0 \neq 0 \quad , \quad h_n \neq 0 \quad ,$$

and the polynomial

$$h(v) = \sum_{\nu=0}^{n} h_\nu v^\nu$$

has all its zeros in the unit disk. We denote these zeros by

$$\alpha_1, \alpha_2, \ldots, \alpha_n \quad .$$

Let

$$R(u) = \frac{\tilde{p}(u)}{\tilde{q}(u)}$$

be the best real rational approximation of the function $w(u)$ *with respect to the set*

$$V_{n,n} = \left\{ \frac{p}{q} \;\middle|\; p \in \Pi_n \, , q \in \Pi_n \, , q(u) > 0 \text{ in } [-1,+1] \right\}$$

on the interval $[-1,+1]$.

Then

$$w(u) - R(u) = \frac{\lambda_n}{2} \left\{ v^{n+1} k(v) + v^{-n-1} k\left(\frac{1}{v}\right) \right\} \quad , \tag{44}$$

where

$$k(v) = \prod_{\mu=1}^{n} \frac{v-\alpha_\mu}{1-\bar{\alpha}_\mu v} \tag{45}$$

and

$$u = \frac{1}{2}(v+\frac{1}{v}) \quad \textit{for} \quad |v| = 1 \; .$$

For the proof of theorem 5 we refer to the original papers.

Remark: Using the well known formula

$$T_\nu(u) = \frac{1}{2}(v^\nu + v^{-\nu}) \quad , \quad \nu \in \mathbb{N}_0 \; ,$$

one gets from (44) at once the denominator $\tilde{q}(u)$ of $R(u)$. By multiplying $w(u)$ by $\tilde{q}(u)$ and using (44) again, one gets the numerator $\tilde{p}(u)$ of $R(u)$.

4. Examples. For real s and $x > 0$ we consider the incomplete Gamme function

$$\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1} dt \; . \tag{46}$$

Let

$$y(x) = e^{\frac{x}{2}} x^{\frac{1-s}{2}} \Gamma(s,x) \; .$$

Then obviously

$$\frac{d}{dx}\left(e^{x}x^{1-s}\,\frac{d}{dx}\left(e^{-\frac{x}{2}}\,x^{\frac{s-1}{2}}\,y\right)\right) = 0 \quad ,$$

a relation which is equivalent to the special Whittaker equation (cp. (3) and [1])

$$y'' + \left(-\frac{1}{4} + \frac{s-1}{2x} + \frac{1-s^2}{4x^2}\right)y = 0 \quad .$$

According to (4), (5) and (20) we have

$$\tau = \frac{1}{2}\ ,\ \rho = \frac{1-s}{2},\ p_0 = \frac{1-s^2}{4}\ . \tag{47}$$

The function

$$z_0(x) = e^{x}x^{1-s}\Gamma(s,x) \tag{48}$$

satisfies the differential equation (cp.(7))

$$z'' - \left(1 + \frac{1-s}{x}\right)z' + \frac{1-s}{x^2}\,z = 0 \quad . \tag{49}$$

Using integration by parts in (46) we get easily the asymptotic expansion

$$\Gamma(s,x) \sim e^{-x}x^{s-1}\left(1 + \frac{s-1}{x} + \frac{(s-1)(s-2)}{x^2} + \ldots\right) \tag{50}$$

for $x \to \infty$. Therefore the function z_0 from (48) has the property

$$\lim_{x\to\infty} z_0(x) = 1 \quad .$$

For fixed x it is not possible to compute the value of the function $\Gamma(s,x)$ from the asymptotic expansion (50) to arbitrary accuracy. The reason is that the series in (50) does not converge, unless s is a natural number.

We restrict now to the case

$$0 \leq s < 1 \quad . \tag{51}$$

Because of the functional equation

$$\Gamma(s,x) = e^{-x}x^{s-1} + (s-1)\Gamma(s-1,x)$$

the restriction (51) is not a serious one.

Furthermore we have (cp. (13)):

$$\kappa(T) = 1$$

and (cp. (15))

$$\omega = \frac{1-s}{T} \quad .$$

Therefore we have to assume that

$$T > 1-s \tag{52}$$

holds. The expressions g_ν (cp. (21)) have the form

$$g_\nu = (\nu+1)(\nu+1-s) \quad , \quad \nu \in \mathbb{N}_0 \quad .$$

They are all different from zero.

Now we choose $n \in \mathbb{N}$. Then (26) and (30) yield, that

$$\delta = \delta_n = 1 + \sum_{\nu=1}^{n+1} \frac{(n+1)(n+\nu)!\,4^\nu T^\nu}{(\nu+1)(2\nu)!(n+1-\nu)!(2-s)\dots(\nu+1-s)} \tag{53}$$

is always positiv. So (cp. (29)) there exists uniquely a function

$$z(x) = z_n(x) = 1 + \sum_{\nu=1}^{n+1} \frac{a_\nu}{x^\nu}$$

with coefficients given in (32), which are all different from zero and alternating in sign, such that

$$\| z_0 - z_n \|_T \leq \frac{1-s}{\delta_n (T-1+s)} \tag{54}$$

holds.

It is interesting how the coefficients a_ν are for fixed n and for $T \to \infty$. A little computation shows

$$\lim_{T\to\infty} a_\nu = (s-1)(s-2)\ldots(s-\nu)$$

for $\nu=1,2,\ldots,n+1$, i.e. our polynomial approximation in the variable $1/x$ tends to the partial sum of the asymptotic expansion (50), if $T \to \infty$.

Let us remark, that the approximation of z_0 furnished by z_n holds uniformly in x for all $x \geq T$ and uniformly in s for all $s \in [0,1]$, since

$$\Gamma(1,x) = e^{-x} ,$$

hence

$$z_0(x) \equiv z_n(x) \equiv 1 \quad , \quad s = 1 \quad .$$

We now turn to the Gaussian probability function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \ .$$

We have

$$\Phi(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$= 1 - \frac{1}{2\sqrt{\pi}} \Gamma(\tfrac{1}{2}, \tfrac{x^2}{2})$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{x^2}{2}}}{x} \cdot z_0(\tfrac{x^2}{2}) \quad ,$$

where z_0 belongs to the value $s = \frac{1}{2}$.

For $n = 3$ and $T = 5$ we get the approximation

$$z_3(x) = \frac{1}{487738}\left(487738 - \frac{243680}{x} + \frac{350400}{x^2} - \frac{624000}{x^3} + \frac{672000}{x^4}\right)$$

for the function $z_0(x)$, which in terms of Tchebycheff polynomials

$$T_\nu(u) \quad , \quad u = 1 - \frac{10}{x} \quad ,$$

can be written

$$z_3(x) = \frac{1}{2438690}\left\{2336800\, T_0(u) + 96148\, T_1(u) + 5256\, T_2(u) + \right.$$

$$\left. + 444\, T_3(u) + 42\, T_4(u)\right\} \ . \quad (55)$$

We have (cp. (54)) with

$$\varepsilon_1(x) = z_0(x) - z_3(x) \tag{56}$$

the error estimation

$$|\varepsilon_1(x)| \leq \frac{42}{487738} \approx 8.62 \cdot 10^{-5} \quad , \quad x \geq 5 \; .$$

This approximation leads for the error function

$$\varepsilon_2(x) = \Phi(x) - \left\{1 - \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}\, z_3\left(\frac{x^2}{2}\right)\right\} \tag{57}$$

to the estimation

$$|\varepsilon_2(x)| \leq 3.44 \cdot 10^{-5} \cdot \frac{e^{-\frac{x^2}{2}}}{x} \quad .$$

The coefficients in the representation (55) decrease quite fast, so that it is not necessary to choose T large enough in order to get lower bounds for the minimal deviation. Obviously the function $z_0(x)$ can be approximated very well by polynomials in the variable $1/x$. It is not known which analytic properties of z_0 are possibly responsible for this fact. - We abandon to derive likewise good rational approximations to z_0 . To make the approximation problem more transparent let us close with considering the incomplete Gamma function again, assuming

$$0 < s < 1 \quad .$$

Then

$$\Gamma(s,x) = \Gamma(s) - \int_0^x e^{-t} t^{s-1} dt$$

$$= \Gamma(s) - x^s f(s,x) \qquad (58)$$

with

$$f(s,x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu x^\nu}{\nu!(\nu+s)} \quad .$$

The function f is with respect to x the restriction of an entire function in the complex plane. Therefore it can be approximated in every finite interval by polynomials in x very well. Using the comparison theorem 61 from [6] we get with a little computation the following asymptotic result for the minimal deviation $E_n(f,T)$ in approximating f by polynomials of degree n at most in the interval $[0,T]$:

$$E_n(f;T) = \frac{T^{n+1} e^{-\frac{T}{2}}}{2^{2n+1}(n+1)!(s+n+1)} \left(1 + O\left(\frac{1}{n+1}\right)\right) \qquad (59)$$

for $n \to \infty$. A sequence of near-best approximating polynomials can be constructed quite easily too. Replacing $f(s,x)$ by those polynomials in (58) yields good approximations for $\Gamma(s,x)$ in $[0,T]$.

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