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One forms on $E(M,\mathbb{R}^n)$ with integral representation

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ONE FORMS ON $E(M,\mathbb{R}^n)$ WITH INTEGRAL REPRESENTATION.

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Let M be a smooth, oriented and compact manifold of dimension $n-1$. By $E(M,\mathbb{R}^n)$ we denote the collection of all smooth $\mathbb{R}^n$-valued embeddings of M. This set is open in $C^\infty(M,\mathbb{R}^n)$, the $\mathbb{R}$-vector space of all smooth $\mathbb{R}^n$-valued maps of M, (the operations are defined pointwise), which carries the C^∞-topology. Since $C^\infty(M,\mathbb{R}^n)$ is a Fréchet space, $E(M,\mathbb{R}^n)$ is a smooth Fréchet manifold. Here smoothness is always ment in the sense of [Gu].

A smooth map $F: E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$ is a one form provided the partial map $F(j)$ sending each $k \in C^\infty(M,\mathbb{R}^n)$ into $F(j,k)$ is linear for each fixed $j \in E(M,\mathbb{R}^n)$. Call the collection of all smooth one forms on $E(M,\mathbb{R}^n)$ by $A^1(E(M,\mathbb{R}^n),\mathbb{R})$. To handle such forms we define what is ment here by an integral representation. Specifying terminology, let $\mathbb{R}^n$ be oriented and $\langle\ ;\ \rangle$ be a fixed scalar product. Each $j \in E(M,\mathbb{R}^n)$ defines hence a Riemannian metric $m(j)$ given by $m(j)(X,Y) = \langle djX;djY\rangle$ for all X,Y in the collection ΓTM of all smooth vector fields on M. dj is locally identical with the Fréchet differential of j. Finally $N(j)$ denotes the oriented unit normal field along j.

F admits an integral representation provided there is a smooth map $\varphi_F: E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)$ such that for each $j \in E(M,\mathbb{R}^n)$ and any $k \in C^\infty(M,\mathbb{R}^n)$

$$F(j,k) = \int \langle\varphi_F(j);k\rangle\ \mu(j)$$

holds. The map $\langle\varphi_F(j);k\rangle$ assigns to any $p \in M$ the value $\langle\varphi_F(j)(p);k(p)\rangle$ and $\mu(j)$ is the Riemannian volume element determined by $m(j)$. If φ_F exists, then it is uniquely determined by F.

Let F admit an integral representation. Then each $j \in E(M,\mathbb{R}^n)$ splits into $\varphi_F(j) = djY(j) + \omega_F(j)\cdot N(j)$ with $Y(j) \in \Gamma TM$ and $\omega_F(j) \in C^\infty(M,\mathbb{R}^n)$. Splitting furthermore $Y(j)$ according to Hodge's decomposition uniquely into a gradient with respect to

$m(j)$ and a divergence free vector field we have

$$\varphi_F(j) = djY^\circ(j) + dj\ \mathrm{grad}_j \tau_F(j) + \omega_F(j)\cdot N(j)$$

with $\mathrm{div}_j Y^\circ(j) = 0$ and $\tau_F(j) \in C^\infty(M,\mathbb{R}^n)$ for each $j \in E(M,\mathbb{R}^n)$.

$\mathbb{R}^n$-invariant one forms on $E(M,\mathbb{R}^n)$.

The Abelian group $\mathbb{R}^n$ operates on $\mathbb{R}^n$ as the group of translations. This operation lifts to an operation Φ of $\mathbb{R}^n$ on $C^\infty(M,\mathbb{R}^n)$ by letting $\Phi(h,z) = h + z$ for each $h \in C^\infty(M,\mathbb{R}^n)$ and each $z \in \mathbb{R}^n$ identified with the constant map assuming z as its only value. The differential

$$d: C^\infty(M,\mathbb{R}^n) \longrightarrow \{dh/\ h \in C^\infty(M,\mathbb{R}^n)\}$$

yields hence a bijection of $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ on to its range. The action Φ restricts to $E(M,\mathbb{R}^n)$ since $j + z \in E(M,\mathbb{R}^n)$ for each $j \in E(M,\mathbb{R}^n)$ and each $z \in \mathbb{R}^n$. Hence both $\{dj/\ j \in E(M,\mathbb{R}^n)\}$ and $\{dh/\ h \in C^\infty(M,\mathbb{R}^n)\}$ are Fréchet manifolds when endowed with the respective quotient topology.

$F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ is called $\mathbb{R}^n$-invariant provided that there is a smooth one form

$$F_{\mathbb{R}^n}: C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

such that $F = d^* F_{\mathbb{R}^n}$. To define the notion of an integral representation for $F_{\mathbb{R}^n}$ we fist introduce the two tensor $T(\alpha,j)$ for each $\alpha \in A^1(M,\mathbb{R}^n)$ by setting

$$T(\alpha,j)(X,Y) = \langle\alpha(X);djY\rangle \quad \forall\ X,Y \in \Gamma TM.$$

This two tensor is represented by a smooth strong bundle map $P(\alpha,j)$ of TM as $T(\alpha,j)(X,Y) = m(j)(P(\alpha,j)X,Y)$. Decomposing $P(\alpha,j)$ into its skew- and self-adjoint part $C(\alpha,j)$ and $B(\alpha,j)$ respectively yields for each $X \in \Gamma TM$

$$\alpha(X) = c(\alpha,j)djX + dj\cdot C(\alpha,j)X + dj\cdot B(\alpha,j)X$$

for a well determined map $c(\alpha,j) \in C^\infty(M,so(n))$ (with $so(n)$ the Lie algebra of $SO(n)$) which maps $\mathbb{R}\cdot N(j)$ to $TjTM$ and vice versa. Writing the analogous decomposition of $dh \in C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ we define $\alpha\cdot dh$ by

$$\alpha\cdot dh = \mathrm{tr}\,c(\alpha,j)\circ c(dh,j) + \mathrm{tr}\,C(\alpha,j)\circ C(dh,j) + \mathrm{tr}\,B(\alpha,j)\circ B(dh,j).$$

$F_{\mathbb{R}^n}$ is said to admit an integral representation provided that there is a smooth map $\alpha: E(M,\mathbb{R}^n)/\mathbb{R}^n \longrightarrow A^1(M,\mathbb{R}^n)$, ($A^1(M,\mathbb{R}^n)$ carrying the C^∞-topology) such that

$$F_{\mathbb{R}^n}(dj,dh) = \int \alpha(dj)\cdot dh\ \mu(j).$$

α is not uniquely determined by $F_{\mathbb{R}^n}$!

The relation between the two integral representations.

Let $F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ be $\mathbb{R}^n$-invariant i.e. $F = d^*F_{\mathbb{R}^n}$. Assume that $F_{\mathbb{R}^n}$ admits an integral representation by α.
We will solve

$$\int \langle\varphi(j);h\rangle\mu(j) = \int \alpha(dj)\cdot dh\ \mu(j) \text{ given } \forall\ h \in C^\infty(M,\mathbb{R}^n)$$

for the smooth map $\varphi: E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)$. To this end we write $h \in C^\infty(M,\mathbb{R}^n)$ as $h = djX_h + \theta_h\cdot N(j)$ for $X_h \in \Gamma TM$ and $\theta_h \in C^\infty(M,\mathbb{R})$. Moreover we express the Lie derivative $L_X(m(j))$ via $m(j)$ by a smooth strong bundle map $\mathbb{L}_X$ of TM to read as $L_X(m(j))(Y,Z) = m(j)(\mathbb{L}_X Y,Z)$ for any choice of all $X,Y,Z \in \Gamma TM$. Then if $\nabla(j)$ denotes the Levi-Civita connection of $m(j)$ and $\overset{*}{\nabla}(j)X$ the adjoint of $\nabla(j)X$ formed with respect to $m(j)$ we immediately deduce
$B(dh,j) = \frac{1}{2}\cdot\mathbb{L}_{X_h} + \theta_h\cdot W(j)$, $C(dh,j) = \frac{1}{2}\cdot(\nabla(j)X_h - \overset{*}{\nabla}(j)X_h)$, and
$c(dh,j)djX = -\ m(j)(W(j)X_h,X)\cdot N(j) + d\theta_h(X)\cdot N(j)$.
$W(j)$ being the Weingarten map of j. If we define the covariant divergence $\operatorname{div}_j P$ of a smooth strong bundle endomorphism P of TM via a mooving (with respect to $m(j)$) orthonormal frame $e_1,\dots,e_{n-1}$ as

$$\operatorname{div}_j P := \sum_{r=1}^{n-1} \nabla(j)_{e_r}(P)e_r$$

then we easily obtain for $\theta_h = 1$

$$\operatorname{tr} B(\alpha,j)\circ B(dh,j) = \operatorname{div}_j B(\alpha,j)X_h - m(j)(\operatorname{div}_j B(\alpha,j),X_h) + \langle \operatorname{tr} B(\alpha,j)\circ W(j)\cdot N(j);N(j)\rangle$$

$$\operatorname{tr} C(\alpha,j)\circ C(dh,j) = \operatorname{div}_j C(\alpha,j)X_h - m(j)(\operatorname{div}_j C(\alpha,j),X_h)$$

and

$$\operatorname{tr} c(\alpha,j)\circ c(dh,j) = -2m(j)(W(j)U(\alpha,j),X_h)$$

with $c(\alpha,j)N(j) = djU(\alpha,j)$. Hence if we define

$$\varphi(j) := -\ dj\ \operatorname{div}_j(B(\alpha,j) + C(\alpha,j)) - 2dj\ W(j)U(\alpha,j) + (\operatorname{tr} B(\alpha,j)\circ W(j))\cdot N(j),\ \forall\ j\in E(M,\mathbb{R}^n)$$

we immediately deduce from the above equations the following:

Theorem 1 Let $F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ be $\mathbb{R}^n$-invariant such that $F = d^* F_{\mathbb{R}^n}$, where $F_{\mathbb{R}^n}$ admits an integral representation by α. Then F admits an integral rerpresentation by φ given for each $j \in E(M,\mathbb{R}^n)$ by

$$\tau_F(j) = - \operatorname{tr} B(\alpha,j)$$

$$(Y_F)^{\circ}(j) = - \operatorname{div}_j(B(\alpha,j) - (\operatorname{tr}B(\alpha,j)\cdot\frac{\mathrm{id}}{\dim M} + C(\alpha,j)) - 2\, W(j)U(\alpha,j)$$

and

$$\omega_F(j) = \operatorname{tr} B(\alpha,j)\circ W(j).$$

The splitting of $\gamma \in A^1(M,\mathbb{R}^n)$

Let $\gamma \in A^1(M,\mathbb{R}^n)$ and $\gamma = dk + \gamma'$ for some $dk \in C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ and some $\gamma' \in A^1(M,\mathbb{R}^n)$. The differential dk is called an integrabel part of γ. Given next $j \in E(M,\mathbb{R}^n)$, and any orthonormal basis in $\mathbb{R}^n$, writing γ as a linear combination of forms in $A^1(M,\mathbb{R})$, and expressing them by vector fields via $m(j)$, then Hodge's decomposition of these fields yield a unique splitting $\gamma = dk + \gamma'$ where γ' has only zero as an integrabel part. dk is called the maximal integrabel part of γ. Moreover $\int\gamma'\cdot dh\ \mu(j) = 0$ for all $h \in E(M,\mathbb{R}^n)$. This implies:

Corollary 2 Let $F \in A^1(E(M,\mathbb{R}^n)/_{\mathbb{R}^n},\mathbb{R})$ with $F = d^* F_{\mathbb{R}^n}$ where $F_{\mathbb{R}^n}$ is represented by α, which we write as

$$\alpha(dj) = dk(j) + \alpha'(j) \quad \forall\, j \in E(M,\mathbb{R}^n),$$

where $\alpha'(j)$ has only zero as an integrabel part, and set moreover

$$k(j) = djX_k(j) + \Theta_k(j)\cdot N(j)$$

then dk is uniquely determined by $F_{\mathbb{R}^n}$, and the following equations hold:

$$\operatorname{div}_j(B(k(j),j) + C(k(j),j) = \Delta(j)X_k(j)$$

with $\Delta(j)$ as the Laplace Beltrami operator of j.

$$\operatorname{div}_j(B(\alpha'(j),j) + C(\alpha'(j),j)) = 0$$

as well as $\int \operatorname{tr} W(j)\circ B_{\alpha'}(j)\ \mu(j) = 0$. Moreover

$$Y_F(j) = -\Delta(j)X_k(j) + W(j)\,\mathrm{grad}_j\Theta_k(j) - \Theta_k(j)\cdot\mathrm{grad}_jH(j) - 2\cdot W(j)^2\,X_k(j)$$

and

$$\omega_F(j) = \mathrm{div}_jW(j)X_k - dH(j)(X_k) + \Theta_k(j)\mathrm{tr}W(j)^2.$$

Stationary points

Let $F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ be represented by $\varphi_F \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$. F is said to be stationary at j if $F(j)(h) = 0$ for all $h \in C^\infty(M,\mathbb{R}^n)$. Hence F is stationary at j iff $\tau_F(j) = 0$, $(Y_F)^\circ(j) = 0$ and $\omega_F(j) = 0$. If moreover $F = d^*F_{\mathbb{R}^n}$ where $F_{\mathbb{R}^n}$ is represented by α with dk as maximal integrabel part, then using the terminology of corollary 2 F is stationary at j iff the following equations hold

$$\mathrm{tr}\,B(\alpha,j) = 0 = \mathrm{div}_jX_k + \Theta_k\cdot H(j)$$

$$\Delta(j)X_k(j) = W(j)\,\mathrm{grad}_j\,\Theta_k(j) - \Theta_k(j)\cdot\mathrm{grad}_j\,H(j) - 2\,W(j)^2X_k(j)$$

$$dH(j)(X_k) = \mathrm{div}_jW(j)X_k + \Theta_k(j)\cdot\mathrm{tr}W(j)^2.$$

Symmetries of F and $F_{\mathbb{R}^n}$

Let $F \in A^1(E(M,\mathbb{R}^n),\mathbb{R})$ be represented by $\varphi \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n)$. Call $a \in \mathrm{Diff}(M)$ to be a symmetry of F if

$$F(j\circ a)(h\circ a) = F(j)(h) \quad \forall\ j \in E(M,\mathbb{R}^n) \text{ and } \forall\ k \in C^\infty(M,\mathbb{R}^n)$$

which holds provided iff $\varphi(j\circ a) = \varphi(j)\circ a$ for all $j \in E(M,\mathbb{R}^n)$. The collection Diff_FM of all symmetries of F is a closed subgroup of DiffM. Its formal Lie algebra $\partial\mathrm{iff}_FM$ consists of all $X \in \Gamma TM$ for which

$$DF(j)(djX)(h) = -F(j)(djX) \quad \forall\ j \in E(M,\mathbb{R}^n) \text{ and } \forall\ h \in C^\infty(M,\mathbb{R}^n)$$

which is equivalent to say that

$$D\varphi(j)(djX) = d\varphi(j)(X) \quad \forall\ j \in E(M,\mathbb{R}^n).$$

In fact $DF(j)(djX)$ is represented by $d\varphi(j)(X)$.

In case $F = d^*F_{\mathbb{R}^n}$ and $F_{\mathbb{R}^n}$ admits an integral representation, then $\mathrm{Diff}_FM = \mathrm{DiffM}$.

Examples 1) Consider $V: E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$ given by $V(j) = \int\mu(j)$ for each $j \in E(M,\mathbb{R}^n)$. Hence

$$DV(j)(h) = \int\langle H(j)\cdot N(j),h\rangle\mu(j)$$

$$\forall\ j \in E(M,\mathbb{R}^n) \text{ and } \forall\ h \in C^\infty(M,\mathbb{R}^n).$$

Since $N(j\circ a) = N(j)\circ a$ und $H(j\circ a) = H(j)\circ a$ for any choice of $j \in E(M,\mathbb{R}^n)$ and $a \in \mathrm{Diff}M$ we have

$$D(H(j)\cdot N(j))(djX) = dH(j)(X)\cdot N(j) + H(j)\cdot djW(j)X$$

showing hence that DV is DiffM-invariant. This however is obvious from the fact that V factors over the Fréchet manifold $E(M,\mathbb{R}^n)/\mathrm{Diff}M$. The one form DV of $E(M,\mathbb{R}^n)$ is $\mathbb{R}^n$ invariant and $(DV)_{\mathbb{R}^n}$ is represented by dj. Hence $T(dj,dj) = m(j)$. $DV(j)$ is stationary at j iff $H(j) = 0$.

2) Next consider $F_{\mathbb{R}^n}$ given by $F_{\mathbb{R}^n}(dj)(dh) = \int djW(j)\cdot dh\ \mu(j)$ for all $j \in E(M,\mathbb{R}^n)$ and all $h \in C^\infty(M,\mathbb{R}^n)$. Hence

$$\varphi(j) = -\mathrm{div}_j W(j) + (\mathrm{tr}\ W(j)^2)\cdot N(j)$$
$$= -dj\ \mathrm{grad}_j H(j) - (\lambda(j)\cdot N(j) - H(j)^2)\cdot N(j)$$

where $\lambda(j)$ is the scalar curvature of $m(j)$. Then

$$F(j)(h) = -\int\langle dj\ \mathrm{grad}_j H(j) - (H(j)^2 - \lambda(j))\cdot N(j),h\rangle\ \mu(j).$$

The tensor $T(djW,dj)$ is identical with the second fundamental form of j. Clearly F is stationary at j iff $H(j)^2 = \mathrm{const} = \lambda(j)$.

References:

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