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Given a smooth function f on an n-dimensional real affine space E we can associate (under certain conditions) another function $\mathcal{R}$, which is defined on the set of hyperplanes

(1) $$(\xi;x) = \sum_{j=1}^{n} \xi_j x^j = p, \ \xi \in S^{n-1}, \ p \in \mathbb{R}$$

by

(2) $$\mathcal{R}f(\xi,p) = \int_{(\xi;x)=p} f(x)\omega,$$

where ω is given by

(3) $$\omega = (-1)^{j-1} \frac{dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n}{\xi_j}$$

(cf. [GGV]). Using the δ-function, we can write (2) in the convenient form

(4) $$\mathcal{R}f(\xi,p) = \int_{\mathbb{R}^n} f(x)\ \delta(p-(x;\xi))dx.$$

The transformation $\mathcal{R}$ is called the Radon transform. We wish to obtain a formula expressing f(x) in terms of its integrals over hyperplanes, or in other words to invert equation (4). This formula depends on whether the space has even or odd dimension. A few successful approaches had been made to obtain a single formula for f in terms of $\mathcal{R}f$, and one way to obtain such a formula is presented in [GSt]. In our notes we show the splitting of this (single) formula into the wellknown formulas for even and odd dimensional spaces, by simply using elementary properties of the Fourier transform together with special homogeneous generalized densities.

In order for the integral (2) to converge for all values of ξ and p we need to require that f(x) be absolutely summable over the entire space, i.e.

$$(5) \qquad \int_{\mathbb{R}^n} |f(x)|\,dx < \infty.$$

For our considerations however we place stronger requirements on f(x). We shall assume, that f(x) is infinitely differentiable rapidly decreasing as are all of its derivatives. Then the Radon transform of f is an infinitely differentiable function of ξ and p, (cf. [GGV]).

Let

$$(6) \qquad \mathcal{F}f(\xi) = (2\pi)^{-n/2} \int f(x)e^{-i(\xi;x)}\,dx$$

denote the Fourier transform of f. There is a simple relation between the Radon transform of f, and its Fourier transform. The Fourier transform can be written directly in terms of integrals over hyperplanes. Specifically, in order to calculate (6) we firstly integrate over the $(\xi;x) = p$ hyperplane and then integrate the expression so obtained over p for fixed ξ. This yields

$$(7) \qquad \mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathcal{R}f(\xi,p)e^{-ip}\,dp.$$

If we replace ξ by $\alpha\xi$ with $\alpha \neq 0$ and then change variables in the integrand, writing αp for p we obtain, using the homogeneity of $\mathcal{R}f(\xi,p)$, (we have $\mathcal{R}f(\alpha\xi,\alpha p) = |\alpha|^{-1}\mathcal{R}f(\xi,p)$, $\forall\ \alpha \in \mathbb{R}\backslash\{0\}$)

$$(8) \qquad \mathcal{F}f(\alpha\xi) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathcal{R}f(\xi,p)\ e^{-i\alpha p}\,dp$$

$$= (2\pi)^{-(n-1)/2}\ \mathcal{F}(\mathcal{R}f)(\alpha).$$

This shows that the Fourier transform in n dimensions reduces to the Radon transform followed by a one-dimensional Fourier transform. Although the analogon of the Radon transform exists in many homogeneous spaces, we should keep in mind that, the second of these transforms is peculiar only to Euclidean space. (cf. [GGV]). Applying the inverse Fourier transform in (8) we obtain

$$(9)\qquad \mathcal{R}f(\xi,p) = (2\pi)^{-\frac{1}{2}(n+2)} \int_{-\infty}^{\infty} \mathcal{F}f(\alpha\xi)\, e^{i\alpha p}\, dp.$$

In order to invert

$$(10)\qquad \mathcal{R}f(\xi,p) = \int f(x)\, \delta(p-(x;\xi))dx,$$

we use the following relation given by the Fourier inversion formula

$$(11)\qquad f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\eta}\, \mathcal{F}f(\eta)\, d\eta.$$

Cutting the integration over $\mathbb{R}^n$ into two pieces by firstly integrating over $t \in \mathbb{R}_+$, and then over S^{n-1}, we obtain

$$(12)\qquad f(x) = (2\pi)^{-n/2} \int_{S^{n-1}} \int_0^{\infty} e^{it\Omega\cdot x}\, \mathcal{F}f(t\Omega)\, t^{n-1}\, dt\, d\Omega,$$

where $d\Omega$ is the volume density on S^{n-1} induced by the standard density on $\mathbb{R}^n$. If we define

$$(13)\qquad t_+^{n-1} := \begin{cases} t^{n-1} & \text{für } t > 0 \\ 0 & \text{für } t < 0 \end{cases}$$

and use furthermore formula (8) we can write (12) as

$$(14)\qquad f(x) = (2\pi)^{-n+\frac{1}{2}} \int_{S^{n-1}} \int_{-\infty}^{\infty} e^{it\Omega\cdot x}\, \mathcal{F}(\mathcal{R}f)(t)\, t_+^{n-1}\, dt\, d\Omega.$$

This is the inversion formula we need in the sequel for our computations. For the sake of completeness however, we present above all the relation with the Radon inversion formula in [GSt]. For this reason we introduce the operator I^r on distributions on $\mathbb{R}$ defined by

$$(15)\qquad \mathcal{F}(I^r\nu)(t) = t_+^r\mathcal{F}(\nu)(t)$$

or equivalently

$$(16)\qquad (I^r\nu)(c) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{itc}\, t_+^r\, \mathcal{F}(\nu)(t)\, dt.$$

Using the last expression (16), and formula (14) we arrive at

$$(17)\qquad f(x) = (2\pi)^{-(n-1)} \int_{S^{n-1}} I^{n-1}[\mathcal{R}f](\Omega\cdot x)d\Omega$$

yielding the formula in [GSt].

Let us now go back to equation (14). Tempered distributions such as t_+^{n-1} are examined in detail in [GSh1]. In particular the following relation holds:

(18) $t_+^{n-1} = ie^{i(n-1)\pi/2}\,\Gamma(n)\cdot(2\pi)^{-1/2}\,\mathcal{F}((p+i0)^{-n})(t).$

For the inner integral in (14) we obtain

(19) $$\int e^{it\Omega\cdot x}\,\mathcal{F}(\mathcal{R}f)(t)\;t_+^{n-1}\,dt = (2\pi)^{-1/2}\,\mathcal{F}^{-1}(\mathcal{F}(\mathcal{R}f)(t)\cdot t_+^{n-1})(\Omega\cdot x).$$

In (19) we have to interpret t_+^{n-1} as a tempered distribution. Let us set

(20) $$c := i\cdot(2\pi)^{-1/2}\,e^{i(n-1)\pi/2}\,\Gamma(n) = i\cdot(2\pi)^{-1/2}\,(n-1)!\cdot e^{i(n-1)\pi/2},$$

where Γ denotes the gamma function. Replacing t_+^{n-1} by $c\cdot\mathcal{F}((x+i0)^{-n})(t)$ we obtain

(21) $$\mathcal{F}^{-1}(\mathcal{F}(\mathcal{R}f)(t)\cdot t_+^{n-1})(\Omega\cdot x) = c\cdot\mathcal{F}^{-1}(\mathcal{F}(\mathcal{R}f)(t)\cdot\mathcal{F}((x+i0)^{-n})(t))(\Omega\cdot x).$$

Using $\mathcal{F}f(\xi)\cdot\mathcal{F}g(\xi) = (2\pi)^{-1/2}\,\mathcal{F}(f*g)(\xi)$ we can write

(22) $$\begin{aligned}\mathcal{F}^{-1}(\mathcal{F}(\mathcal{R}f)(t)\cdot t_+^{n-1})(\Omega\cdot x) &= c\cdot(2\pi)^{-1/2}\,\mathcal{F}^{-1}(\mathcal{F}(\mathcal{R}f(\Omega,p)*(p+i0)^{-n})(t))(\Omega\cdot x)\\ &= c\cdot(2\pi)^{1/2}\,\mathcal{R}f(\Omega,\Omega\cdot x)*(\Omega\cdot x+i0)^{-n}\\ &= c\cdot(2\pi)^{-1/2}\int_{-\infty}^{\infty}\mathcal{R}f(\Omega,p)\cdot(\Omega\cdot x-p+i0)^{-n}\,dp,\end{aligned}$$

and formula (19) becomes

(23) $$\begin{aligned}\int_{-\infty}^{\infty} e^{it\Omega\cdot x}\,\mathcal{F}(\mathcal{R}f)(t)\cdot t_+^{n-1}\,dt &= (2\pi)^{1/2}\cdot c\cdot(2\pi)^{-1/2}\int_{-\infty}^{\infty}\mathcal{R}f(\Omega,p)\cdot(\Omega\cdot x-p+i0)^{-n}\,dp\\ &= i\cdot(2\pi)^{-1/2}(n-1)!\cdot e^{i(n-1)\pi/2}\int_{-\infty}^{\infty}\mathcal{R}f(\Omega,p)\cdot(\Omega\cdot x-p+i0)^{-n}\,dp.\end{aligned}$$

The fundamental relation, (cf. [GSh1], p.94)

(24) $$(x+i0)^{-n} = x^{-n} - \frac{(-1)^{n-1}i\pi}{(n-1)!}\cdot\delta^{(n-1)}(x)$$

yields

(25) $$(\Omega\cdot x-p+i0)^{-n} = (\Omega\cdot x-p)^{-n} - \frac{(-1)^{n-1}i\pi}{(n-1)!}\cdot\delta^{(n-1)}(\Omega\cdot x-p).$$

Inserting (25) into the last equation of (23) we obtain

(26) $$\int_{-\infty}^{\infty} e^{it\Omega\cdot x}\,\mathcal{F}(\mathcal{R}f)(t)\cdot t_+^{n-1}\,dt =$$

$$= i(2\pi)^{-\frac{1}{2}} \cdot e^{\frac{1}{2}i(n-1)\pi}(n-1)! \cdot \int_{-\infty}^{\infty} \mathcal{R}f(\Omega,p) \cdot (\Omega \cdot x-p)^{-n}\, dp \,+$$

$$+ \tfrac{1}{2}(2\pi)^{+\frac{1}{2}}\, e^{\frac{1}{2}i(n-1)\pi} \int_{-\infty}^{\infty} \mathcal{R}f(\Omega,p) \cdot \delta^{(n-1)}(\Omega \cdot x-p)\, dp$$

$$= i(2\pi)^{-\frac{1}{2}} \cdot e^{\frac{1}{2}i(n-1)\pi}(n-1)! \cdot \int_{-\infty}^{\infty} \mathcal{R}f(\Omega,p) \cdot (\Omega \cdot x-p)^{-n}\, dp \,+$$

$$+ \tfrac{1}{2}(2\pi)^{+\frac{1}{2}}\, e^{\frac{1}{2}i(n-1)\pi}\, \mathcal{R}_p^{(n-1)}f(\Omega,\Omega \cdot x).$$

($\mathcal{R}_p^{(n-1)}(\Omega,p)$ denotes (n-1) times differentiation with respect to p). For $\alpha \in \mathbb{R}\backslash\{0\}$ we have (cf. [GGV])

$$(27) \qquad \mathcal{R}_p^{(n-1)}f(\alpha\Omega,\alpha p) = \alpha^{-n} \cdot \mathrm{sgn}(\alpha) \cdot \mathcal{R}_p^{(n-1)}f(\Omega,p)$$

and in particular

$$(28) \qquad \mathcal{R}f(-\Omega,-\alpha) = \mathcal{R}f(\Omega,\alpha),$$

i.e. $\mathcal{R}f$ is an even function.

Let n be odd, then $(\Omega \cdot x-p)^{-n}$ is an odd function and this is also true for the product $\mathcal{R}f(\Omega,p) \cdot (\Omega \cdot x-p)^{-n}$. Thus the integral after the last equality sign in (26) vanishes. Therefore we obtain with (14) for n odd the formula

$$(29) \qquad f(x) = \frac{(-1)^{\frac{1}{2}(n-1)}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \mathcal{R}_p^{(n-1)}f(\Omega,\Omega \cdot x)\, d\Omega.$$

If n is even, then the very last expression in Formula (26) is odd in ξ and therefore the integral over this expression vanishes identically for fixed x, yielding

$$(30) \quad f(x) = (-1)^{\frac{1}{2}n}(n-1)!(2\pi)^{-n} \int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}f(\Omega,p)(\Omega x-p)^{-n} dp\, d\Omega,$$

where the integral over p is understood in terms of its regularization. (cf. [GSh1], p.335).

The formulas obtained by our elementary computations coincide with the results presented in [GGV], thus showing a direct interpretation of the unification formula in [GSt].

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