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Introduction

In [Bi,1] we deduced the force density of an internal constitutive law for a deformable medium, moving in $\mathbb{R}^n$. The geometric type of the body was given by a smooth orientable compact manifold of dim $n-1$. In the normal component of that force density a pressure is hidden, which describes the energy needed to change the instantaneous volume of the deformable medium.

In these notes this pressure will be exhibited.

In addition to that a natural coefficient of viscosity, called the structural coefficient of viscosity will be exhibited in case that the internal constitutive law depends on an external parameter, namely a tangent vector of the phase space.

We conclude by investigating the interplay between a densit map and scalar curvature, mostly in case of dim $M = 2$. Finally we show that the qualitative behavior of the volume active pressure for two-dimensional surfaces depends heavily on the Euler characteristic of the body.

We thank D. Socolescu for suggesting the term "structural viscosity" for the type of viscosity we observed.

1) Differential geometric background

Let M be a connected, compact, smooth and oerientable manifold.

The collection $C^\infty(M,\mathbb{R}^n)$ *of all smooth* $\mathbb{R}^n$*-valued maps is an* $\mathbb{R}$*-vector space* under the pointwise defined operation and moreover is a complete metrizable and locally convex vector space if endowed with the C^∞-topology. Hence $C^\infty(M,\mathbb{R}^n)$ is referred to as Frèchet space.

The subset of $C^\infty(M,\mathbb{R}^n)$ *consisting of all embeddings of* M *into* $\mathbb{R}^n$ is denoted by $E(M,\mathbb{R}^n)$.

Since $E(M,\mathbb{R}^n)$ *is open in* $C^\infty(M,\mathbb{R}^n)$ *it is a Frèchet manifold.*

The tangent manifold of $E(M,\mathbb{R}^n)$ is

$$TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)$$

To introduce a metric on $E(M,\mathbb{R}^n)$ *we fix a scalar product* $\langle,\rangle$ *on* $\mathbb{R}^n$*. Then a* metric $\bar{G}$ on $E(M,\mathbb{R}^n)$ is given by

$$\bar{G}(j)(h,k) = \int_M \langle h,k\rangle \mu(j).$$

Here $\mu(j)$ is the Riemannian volume associated to the Riemannian metric $m(j)$, defined by

$$m(j)(v_q,w_q) := \langle dj\ v_q, dj\ w_q\rangle \quad \forall\ v_q,w_q \in T_qM \quad \forall\ q \in M.$$

By dj we mean the one $\mathbb{R}^n$-valued form on M which locally is determined by the Frèchet derivative of j.

Given any $h \in C^\infty(M,\mathbb{R}^n)$ and an embedding $j \in E(M,\mathbb{R}^n)$ the map h splits into

1.1) $$h = dj\, X_h(j) + h^{\perp}$$

with $X_h(j)$ in ΓTM, the collection of all smooth vector fields on M, and where $h^{\perp}$ is pointwise orthogonal to $djX_h(j)$.

Using Hodges decomposition we furthermore get

1.2) $$X_h(j) = \mathrm{grad}_j\, \tau_h\, (j) + X_h^0(j)$$

Here $\tau_h(j) \in C^{\infty}(M,\mathbb{R})$, grad_j denotes the gradient with respect to m(j) and $X_h^0(j)$ is the divergence free part of $X_h(j)$. By the divergence of any $Y \in \Gamma TM$ we mean

$$\mathrm{div}_j\, Y : = \mathrm{tr}\, \nabla(j)\, Y$$

where $\nabla(j)$ is the Levi-Civita connection of m(j).

Let $n = 1 + \dim M$. We fix an orientation on each of $\mathbb{R}^n$ and M. The positively oriented unite normal vector field associated with any embedding $j \in E(M,\mathbb{R}^n)$ is called N(j). Hence (1.1) turns into

1.3) $$h = djX_h(j) + \Theta_h(j)\cdot N(j)$$

with $\Theta_h(j) \in C^{\infty}(M,\mathbb{R})$.

We conclude this section by the description of any smooth $\mathbb{R}^n$-valued one form α on M relative to a given embedding $j \in E(M,\mathbb{R}^n)$.

Let $A^1(Q,\mathbb{R})$ denote the smooth $\mathbb{R}^n$-valued one form defined on the manifold Q. This manifold may be finite or infinite dimensional.

Given α, a map $h \in C^{\infty}(M,\mathbb{R}^n)$ is called the integrable part of α, if

$$\alpha = dh + \beta(j)$$

where $\beta(j)$ has only constants as integrable parts.

Proposition 2 :

Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$, then

$$\alpha = dh(i) + \beta(i)$$

for some $h(i) \in C^\infty(M,\mathbb{R}^n)$, determined up to a constant, where $\beta(j)$ has only constants as integrable parts. If for some $h^1(i) \in C^\infty(M,\mathbb{R}^n)$ with

$h^1(i) = h(i)+z$ and $z \in \mathbb{R}^n$ then

$$\alpha = dh^1(i) + \beta(i)$$

and moreover

$$X^0_h(j) = X^0_{h^1}(j) .$$

Here $X^0_h(j)$ and $X^0_{h^1}(j)$ *denote the divergence free part of* $X_h(j)$ *and* $X_{h^1}(j)$ *respectively. Hence* $X^0_h(j)$ *depends only on* dh.

2) The volume active pressure

Given a smooth constitution law on $E(M,\mathbb{R}^n)$, that is a smooth $\mathbb{R}$-valued one form

$$F: E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

the value $F(j)(k)$ for any $j \in E(M,\mathbb{R}^n)$ and any $h \in C^\infty(M,\mathbb{R})$ can be regarded as the energy necessary to deforme $j(M)$ infinitesimally in the direction of h. What we are concerned with in this paragraph is to find to what extend the change of the volume of $j(M)$ contributes to $F(p)(h)$. We do this by introducing the volume active pressure π_F associated with F.

To execute this program let us assume that

2.1) $$F(j)(h) = \int_M \langle \varphi_F(j),h\rangle \mu(j) \,, \qquad \forall\, h \in C^\infty(M,\mathbb{R}^n)$$

for some smooth map

2.2) $$\varphi_F : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)$$

called the force density.

Clearly if F is interpreted as an energy functional, it splits from a physical point of view, into the energy determined by internal and external force densities.

To specify F further as to express the deformation energy determined by internal force densities let us identify $C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ with $\{dh \mid h \in C^\infty(M,\mathbb{R}^n)\}$ via the differential

$$C^\infty(M,\mathbb{R}^n) \xrightarrow{\ d\ } \{dh \mid h \in C^\infty(M,\mathbb{R}^n)\}$$

Similarily $E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$ denotes $\{dj \mid j \in E(M,\mathbb{R}^n)\}$. Both $C^\infty(M,\mathbb{R}^n)$ and in turn $E(M,\mathbb{R}^n)$ are assumed to carry the C^∞-topology. Now we require the existence of a smooth one form

$$F_{\mathbb{R}^n} : E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

(both factors of the domain carry the C^∞*-topology) such that*

2.3) $$F=d^* F_{\mathbb{R}^n}.$$

Next we demand that $F_{\mathbb{R}^n}$ admits an integral representation of the form

2.4) $$F_{\mathbb{R}^n}(dj)dh = \int \alpha(dj)\cdot dh \; \mu(j)$$

where the $\cdot$ on the right hand side means the following:

Each $\gamma \in A^1(M,\mathbb{R}^n)$ *is uniquely represented with respect to any given* $j \in E(M,\mathbb{R}^n)$ as

2.5) $$\gamma = c_\gamma(j)\cdot dj + dj\cdot C_\gamma(j) + dj\cdot B_\gamma(j)$$

as shown in [Bi,1]. Here $C_\gamma(j)$ and $B_\gamma(j)$ are smooth strong bundle endomorphisms of TM skew respectively selfadjoint with respect to $m(j)$ and $c_j(j) \in C^\infty(M,so(n))$. Then the integrand in (2.4) is defined as :

2.6) $$\alpha(dj)\cdot dh := -\,tr\big(c_{\alpha(dj)}\cdot c_{dh} + C_{\alpha(dj)}\cdot C_{dh}\big) + tr\, B_{\alpha(dj)}\cdot B_{dh}\,.$$

Since according to proposition 1

$$\alpha(dj) = dl(dj) + \beta(dj) \qquad \forall\, dj \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$$

with $l(dj)$ as the integrable part of $\alpha(dj)$, the force density φ_F is determined by

$$\varphi_F(dj) = -\, dj\, div_j\big(C_{dl(dj)}(j) + B_{dl(dj)}(j)\big) \\ \qquad - 2dj\; U_{dl(dj)}(j) - \big(tr\, B_{dl(dj)}(j)\cdot W(dj)\big)\cdot N(j).$$

Here $dj\; U_{dl(dj)}(j)$ is by definition $c_{dl(dj)}(j)\cdot N(j)$.

The form α in (2.4) is called the stress form associated with F.

The volume active pressure of F is now determined as follows:

The map

$$\mathrm{Vol}: E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

sending each $j \in (M,\mathbb{R}^n)$ into $\int \mu(j)$ is obviously smooth. An easy exercise shows that the gradient $\mathrm{Grad}(\mathrm{Vol}(j))$ with respect to $\bar{G}$ of $D\,\mathrm{Vol}(j)$ at j is

$$\mathrm{Grad}\ \mathrm{Vol}(j) = H(j)\cdot N(j) \quad \forall\ j \in E(M,\mathbb{R}^n)$$

since

$$DV(j)(h) = \int \Theta_h \cdot H(j)\mu(j)\ .$$

To find the part of $F(j)(h)$ which is due to the infinitesimal expansion of the volume in the direction of h we split

$$\mathrm{tr}\ B_{dl(dj)}(j)\cdot W(j)\ ,$$

an element of $L^2(M,\mathbb{R})$, the Hilbert space of all $\mathbb{R}$-valued L^2-integrable functions of M, into a component along $H(j)$ (also contained in that space) and into a component normal to $H(j)$. This splitting then reads as

2.9) $$\mathrm{tr}\ B_{dl(dj)}(j)\cdot W(j) = \pi_F(dj)\cdot H(j) + g(j) \quad \forall\ j \in E(M,\mathbb{R}^n)\ .$$

Here $\pi_F(dj) \in \mathbb{R}$ and $g(j) \in C^\infty(M,\mathbb{R})$. Clearly

2.10) $$g(j) = \mathrm{tr}(B_{dl(dj)}(j)\cdot W(j) - \pi_F(dj)\cdot \mathrm{id}_{TM}\ .$$

We call $\pi_F(dj)$, which varies linearily in $dl(dj)$, the volume active pressure of F at j.

In an analoguous way we proceed in the case of such constitutive laws F that admit an integral representation by φ_F, but are not of the form $F = d^* F_{\mathbb{R}^n}$ for any $F_{\mathbb{R}^n} \in A^1(E(M,\mathbb{R}^n)/\mathbb{R}^n,\mathbb{R})$. Since

$$\varphi_F(j) = dj\ X_F(j) + \Theta_F(j)\cdot N(j)$$

we treat $\Theta_F(j)$ just as in the same way as $\mathrm{tr}B_{dl(j)}\cdot W(j)$ and obtain the *volume active pressure* $\pi_F(dj)$ *for any* $j \in E(M,\mathbb{R}^n)$.

The reason that we call $\pi_F(dj)$ the volume active pressure is the following theorem, now easily proved by the reader.

<u>Theorem 2</u> Let $F \in A(E(M,\mathbb{R}^n),\mathbb{R})$ admit an integral representation by $\varphi_F \in C^\infty(M,\mathbb{R}^n)$ as

$$F(j)(h) = \int_M \langle \varphi_F(j),h\rangle \, \mu(j) \ , \ \forall \ h \in C^\infty(M,\mathbb{R}^n) \ .$$

Then if $\varphi_F(j)$ split into

$$\varphi_F(j)=dj \ X_F(j) + \Theta_F(j)\cdot N(j) \qquad \forall \ j \in (M,\mathbb{R}^n)$$

with $\Theta_F(j) \in C^\infty(M,\mathbb{R})$ then there exists a unique number $\pi_F(dj) \in \mathbb{R}^n$ and $\hat{\varphi}_F(j) \in C^\infty(M,\mathbb{R}^n)$, such that

2.11) $$F(j)(h) = \int \langle\hat{\varphi}_F(j),h\rangle\mu(j) + \pi_F(dj)\cdot D \ \mathrm{Vol}(j)(h) \ .$$

Here $\hat{\varphi}_F(j)$ is given by

2.12) $$\hat{\varphi}_F(j) := \varphi_F(j) - \pi(dj)\cdot N(j)$$

and satisfies

2.13) $$\int \langle\hat{\varphi}_F(j),H(j)\cdot N(j)\rangle\mu(j) = 0 \ .$$

In case $F = d^* F_{\mathbb{R}^n}$ and $F_{\mathbb{R}^n}$ admits at (dj) an integral representation by the stress form $l(dj)$, the

2.14) $$\mathrm{tr} \ B_{dl(dj)}\cdot W(j) = \pi_F(dj)\cdot H(j) + g(j) \quad \forall \ j \in E(M,\mathbb{R}^n)$$

where $g(j)$ is L^2-orthogonal to $H(j)$. Moreover $\pi_F(dj)$ depends linearily on $\varphi_F(j)$ and hence linearily on $dl(dj)$.

If $B^0_{dl(dj)}$ and $\tau(dj,dl)$ denote the trace free part of $B_{dl(dj)}$ and the trace of $B_{dl(dj)}$ then the following holds for all $j \in E(M,\mathbb{R}^n)$

2.15) $$(\tau(dj,dl) - \pi_F(dj))\cdot H(j) = \varphi(j) - \mathrm{tr} \ B^0_{dl(dj)}(j)\cdot W(j)$$

Corollary :

If $F(j)(h) = 0$ for some $j \in E(M,\mathbb{R}^n)$ and all $h \in C^\infty(M,\mathbb{R}^N)$ then

2.16) $$\hat{\varphi}_F(j) = 0 \text{ and } \pi_F(dj) = 0 .$$

Proof: We have

$$\int \langle \hat{\varphi}_F(j),h\rangle \mu(j) = \pi_F(dj) \int \langle H(j)\cdot N(j),h\rangle \mu(j)$$

and hence

$$\int \langle \hat{\varphi}_F(j) - \pi_F(j)H(j),h\rangle \mu(j) = o \quad \forall\, h \in C^\infty(M,\mathbb{R}^n)$$

showing

$$\varphi_F(j) = \pi_F(dj)\cdot H(j)$$

However

$$\int \langle \hat{\varphi}_F(j),H(j)\cdot N(j)\rangle \mu(j) = o$$

proving our assertion.

The following is evident:

Corollary 4 If $F \in A^1(E,M,\mathbb{R}^n)$ splits into

2.17) $$F = F_1+F_2$$

for some $F_1,F_2 \in A^1(E,(M,\mathbb{R}^n),\mathbb{R})$ admitting both integral representations by φ_{F_1} and φ_{F_2}, respectively then for each $dj \in E(M,\mathbb{R}^n)/_{\mathbb{R}^n}$

2.18) $$\pi_F(dj) = \pi_{F_1}(dj) + \pi_{F_2}(dj) \text{ and } \hat{\varphi}_{F_1}(j) = \hat{\varphi}_{F_2}(j) .$$

Hence if $F(j) = o$ for some $j \in E(M,\mathbb{R}^n)$

2.19) $$\pi_{F_1}(dj) = - \pi_{F_2}(dj) \text{ and } \hat{\varphi}_F(j) = - \hat{\varphi}_{F_2}(j) .$$

3) The concept of structural viscosity

In this section we assume that F depends on a parameter. Moreover let $F_{\mathbb{R}^n}$ be a smooth map

$$F_{\mathbb{R}^n} : C^{\infty}(M,\mathbb{R}^n)/_{\mathbb{R}^n} \longrightarrow A^1(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}\mathbb{R})$$

of which we require

1.) $\qquad F = d^*F_{\mathbb{R}^n}$

and that $F_{\mathbb{R}^n}$ satisfies

2.) $$F_{\mathbb{R}^n}(dk,dj)(dk) = \int dl(dk,dj)\cdot dk\ \mu(j) \quad \forall\ dl,dj \in C^{\infty}(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} .$$

As mentioned before $l(dk,dj)$ decomposes into

$$l(dk,dj) = dj\ \mathrm{grad}_j\ X_l^0\ (dk,dj) + \Theta_l(dk,dj)\cdot N(p)$$

with $X_l^0(j) \in \Gamma TM$ being divergence free. *In fact* $X_l^0(j)$ *only depends* on $dl(dk,dj)$, as expressed in proposition 1. We split k into

$$k = dj\ \mathrm{grad}_{\psi_k} + dj\ X_k^0(j) + \Theta_k(j)\cdot N(j).$$

The splitting allows us to decompose $X_l^0(dk.dj)$ *into a component along* $X_l^0(j)$ *and another one pointwise perpendicular to* $X_k^0\,(j)$ *as follows :*

$$X_l^0(dk,dj) = \nu(dk,dj)\ X_k^0(j) + X_l(dk,dj)$$

with $\nu(dj,dk) \in C^{\infty}(M,\mathbb{R}^n)$. The coefficient $\nu(dk,dj)$ is called the coefficient of structural viscosity.

A routine calculation shows that $B_{dl(dk,dj)}(j)$, the symmetric coefficient in the representation of $dl(dk,dj)$ represented by dj can be written as

$$B_{dl(dk,dj)}(j) = \frac{1}{2}\left(L_{\nu(dk,dj)X_k^0}(j) + L_{\hat{X}_k(dk,dj)}\right) + \frac{1}{2} L_{\operatorname{grad}_j \psi_l(dk,dj)} + \Theta_l(dk,dj)\cdot W(j)$$

Hence $\pi_F(dk,dj)$ is given by

$$\pi_F(dk,dj) = \operatorname{div} \nu(dk,dj)X_k^0 + \operatorname{div} \hat{X}_k(dk,dj) - \Delta(j)\, \psi_l(dh,dj) - \Theta_l(dk,dj)\operatorname{tr} W(j) - \operatorname{tr} \hat{B}_{dl(dk,dj)}(j)$$

where $\hat{B}_{dl(dk,dj)}(j) := B_{dl(dk,dj)}(j) - \pi_F(dk,dj)\cdot id_{TM}$. The $\mathbb{R}^n$-valued one-form to which $\hat{B}_l(dk,dj)$ belongs is evidently

$$d\hat{l}(dk,dj) = dl(dk,dj) - \pi_F(dk,dj)\cdot dj$$

for any $j \in E(M,\mathbb{R}^n)$ and any $k \in C^\infty(M,\mathbb{R})$.

4) Density map and curvature

By a density map ρ we mean a smooth map

$$\rho : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n) ,$$

for which the real $m(j)$, called the mass at j, being defined as

$$m(j) := \int \rho(j)\, \mu(j) \quad , \qquad \forall\ j \in E(M,\mathbb{R}^n)$$

is positive, and for which the following continuity equation

4.1) $$D\rho(j)(h) = -\tfrac{1}{2}\, tr_{m(j)} Dm(j)(h)$$

is satisfied for any $j \in E(M,\mathbb{R}^n)$ and any $h \in C^\infty(M,\mathbb{R}^n)$.

By $tr_{m(j)}$ we mean the trace taken with respect to the Riemannian metric $m(j)$. Clearly

$$Dm(j)(h)(X,Y) = \langle dhX,djY\rangle + \langle djX,dhY\rangle$$

for any $j \in E(M,\mathbb{R}^n)$ and any $h \in C^\infty(M,\mathbb{R}^n)$.

To show the existance of such a density map ρ , let us consider $i \in E(M,\mathbb{R}^n)$ and its connected component $O_i \subset E(M,\mathbb{R}^n)$. Then by [Bi,2] any $dj \in E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}$ with $j \in O_i$ satisfies

4.2) $$dj = g(j)\cdot dj\cdot f(j)$$

with $g(j) \in C^\infty(M,so(n))$ and $f(j) : TM \longrightarrow TM$ being a smooth strong bundle isomorphism, positiv definit with respect to $m(j)$. By $so(n)$ we denote the Lie-algebra of $SO(n)$.

Then

4.3) $$\mu(j) = \det f(j)\cdot\mu(i)$$

and hence

4.4) $$\rho(j) := \rho(i)\cdot\det f^{-1}(j)$$

is a density map provided that

$$\int \rho(i)\ \mu(i)$$

is a positive real.

(Clearly the assumption of codim $M = 1$ did not play any role in constructing $\rho(i)$.)

A straight forward computation shows the following set of reformulations of the continouity equation

4.5) $$D\rho(j)(h) = -\frac{\rho(j)}{2}\, tr_{m(j)} Dm(j)(h) = -\rho(j)\ tr\ f^{-1}(j)\cdot Df(j)(h)$$

$$= -\rho(j)\cdot tr\ B_{dh}(j) = -\rho(j)\cdot tr\ f^{-1}(j)\ B_{dh}(i) \quad ,$$

for any j *in* O_i, *the connected component of* $i \in E(M,\mathbb{R}^n)$ *and any* $h \in C^{\infty}(M,\mathbb{R}^n)$.

Hence if $h = N(i)$ then

4.6) $$D\rho(j)(N(j)) = -\rho(j)\cdot H(j) \ ,$$

with $H(j) := tr\ W(j)$ and $W(j)$ the Weingarten map associated with $N(j)$.

The link to the scalar curvature is made by computing the second derivative of $\rho(j)$. To exhibit this, let us first find the derivatives of $W(j)$ and of its trace, the mean curvature $H(j)$.

Since for any $j \in E(M,\mathbb{R}^n)$

$$dj\, W(j)\, X = dN(j)\, X \qquad \forall\, X \in \Gamma TM$$

we obtain by differentiation

$$dh\, W(j)\, X + dj\, DW(j)(h)\, X = d\, DN(j)(h)\, X .$$

The equations

$$\langle N(j), djX\rangle = 0 \quad \text{and} \quad \langle N(j), N(j)\rangle = 1$$

yield by differentiation

4.7) $$\langle DN(j)(h), djX\rangle = -\langle N(j), dhX\rangle = \langle c_{dh}(j)N(j), djX\rangle .$$

If we replace $c_{dh}N(j)$ by

4.8) $$c_{dh}(j)N(j) = dj\, U_{dh}(j)$$

for some $U_{dh}(j) \in \Gamma TM$, we find

4.9) $$DN(j)(h) = dj\, U_{dh}(j)$$

and consequently

4.10) $$DW(j)(h) = \nabla(j)\, U_{dh}(j) - \big(C_{dh}(j) + B_{dh}(j)\big) \cdot W(j) .$$

To represent U_{dh} in terms of h, let us write h as

$$h = djX_h(j) + \Theta_h(j)\cdot N(j)$$

for some $X_h \in \Gamma TM$ and some $\Theta_h \in C^\infty(M, \mathbb{R}^n)$.

Then

4.11)
$$dh = c_{dh}(j)\cdot dj + dj\cdot(C_{dh}(j) + B_{dh}(j))$$
$$= S(j)(X_h,\dots) + d\Theta_h(j)(\dots)\cdot N(j) + dj\nabla(j)X_h(j) + \Theta_h(j)\cdot djW(j).$$

Therefore $c_{dh}N(j)$ is given by

$$\langle c_{dh}(j)N(j), djY\rangle = m(j)(W(j)X_h(j),Y) - m(j)(\mathrm{grad}_j\Theta_h(j),Y)$$

for all $Y \in \Gamma TM$ saying that

4.12)
$$U_{dh}(j) = W(j)X_h(j) - \mathrm{grad}\ \Theta_h(j)\ .$$

Thus $DW(j)(h)$ rewrites by (4.11) and (4.12) as

$$DW(j)(h) = \nabla(j)\big(W(j)X_h(j) - \mathrm{grad}\ \Theta_h(j)\big) - \big(\nabla(j)X_h(j) + \Theta_h(j)\cdot W(j)\big)\cdot W(j)\ .$$

Hence

4.13)
$$DW(j)(h) = \nabla(j)W(j)X_h(j) + W(j)\cdot\nabla(j)X_h(j) - \nabla(j)X_h(j)\cdot W(j)$$
$$- \nabla(j)\cdot\mathrm{grad}\ \Theta_h(j) - \Theta_h(j)\cdot W(j)^2\ .$$

Let us introduce L_{X_h} , which we define by

4.14)
$$M(j)\big(L_{X_h(j)}X,Y\big) = L_{X_h(j)}(m(j))(X,Y)\ , \qquad \forall\ X,Y \in \Gamma TM\ ,$$

with $L_{X_h(j)}$ the Lie-derivative with respect to X_h.
Since $DW(j)(h)$, $\nabla(j)W(j)X_h(j)$, $\nabla(j)\cdot\mathrm{grad}\ \Theta_h(j)$ and $\Theta_h(j)\cdot W(j)^2$ are selfadjoint with respect to $m(j)$ and since

$$W(j)\cdot C_{dh}(j) - C_{dh}\cdot W(j)$$

is selfadjoint as well as we deduce from (4.13) for any $j \in E(M,\mathbb{R}^n)$ the equation

4.15)
$$W(j)\cdot L_X = L_X\cdot W(j) \qquad \forall\ X \in \Gamma TM\ .$$

As a consequence of (4.10) we get :

$$4.16)\quad DH(j)(h) = \operatorname{div} U_{dh}(j) - \operatorname{tr} B_{dh}(j)\cdot W(j)$$
$$= \Delta(j)\Theta_h(j) + \operatorname{div}_j W(j)X_h(j) - \tfrac{1}{2}\operatorname{tr} L_{X_h(j)}\cdot W(j) - \Theta_h(j)\cdot \operatorname{tr} W(j)^2 ,$$

where $\Delta(j)$ denotes the Laplace–Beltrami operator of $m(j)$.

Since for any $X \in \Gamma TM$

$$4.17)\qquad \tfrac{1}{2}\operatorname{tr} L_X\cdot W(j) = \operatorname{tr} \nabla(j)X\cdot W(j) = \operatorname{div}_j W(j)X - dH(j)(X) ,$$

(4.15) turns into

$$4.18)\qquad DH(j)(h) = dH(j)(X_h(j)) + \Delta(j)\Theta_h(j) - \Theta_h(j)\cdot \operatorname{tr} W(j)^2 .$$

Let us observe that $DH(j)(h) = 0$ and $dH(j)(X_h) = 0$ can not hold simultaniously for all $h \in C^\infty(M,\mathbb{R}^n)$, fore otherwise

$$0 = \int DH(j)(N(j))\mu(j) = \int \operatorname{tr} W(j)^2\mu(j) .$$

However the right-hand term is unequal to zero since in case of $W(j) = 0$, M has to be a piece of a plane in $\mathbb{R}^n$ contradicting the compactness of M. Hence $H : E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$ does not have critical values.

Clearly if $h = N(j)$ then

$$4.19)\qquad DW(j)(N(j)) = - W(j)^2$$

and hence

$$4.20)\qquad DH(j)(N(j)) = - \operatorname{tr} W(j)^2 = 2\,\lambda(j) - H(j)^2 ,$$

where $\lambda(j)$ is the scalar curvature (that is the metric trace of the Ricci tensor of the metric $m(j)$) .

Combinating (4.20) together with the requirement that $\dim M = 2$ and with the theorem of Gauß–Bonnet (cf.[G.H.V]) , saying that

4.21) $$\frac{1}{2\pi}\int \lambda(j) = X(M) \ ,$$

where $X(M)$ the Euler characteristic of M, yields immediately :

Theorem 4.1 :

In case of $\dim M = 2$ *the following equation holds for any* $j \in E(M,\mathbb{R}^n)$

4.22) $$X(M) = \frac{1}{\pi}\cdot\int \left(H(j)^2 - DH(j)(N(j))\right)\ \mu(j)$$

and

4.23) $$\int H(j)\ \mu(j) = \int DH(j)(N(j))\ \mu(j) \quad \textit{iff} \quad X(M) = 0.$$

Theorem 2 and equation (4.15) yield

Theorem 4.2 :

Given any smooth stress form $dl : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n)$, then the following *equation holds for any* $h \in C^\infty(M,\mathbb{R}^n)$ decomposed as

$$h = djX_h(j) + \Theta_h(j)\cdot N(j)$$

4.24) $$\begin{aligned}\pi_F(dj)\cdot DV(j)(h) &= \int \Theta_h(j)\cdot \operatorname{tr} B_{dl}(j)\cdot W(j)\ \mu(j)\\ &= -\int \Theta_h(j)\cdot \left(DH(j)(l) - \operatorname{div} U_{dl}(j)\right)\ \mu(j)\end{aligned}$$

and in particular

4.25) $$\pi_F(dj)\cdot DV(j)(N(j)) = -\int DH(j)(l)\ \mu(j)\ .$$

Here F is the force density determined by the stress form dl.

Now let us turn back to the second derivative of ρ.

We have by

4.27) $$\begin{aligned}D^2\rho(j)(N(j),h) &= -\left(D\rho(j)(h)\cdot H(j) + \rho(j)\cdot DH(j)\right) - D\rho(j)(DN(j)(h))\\ &= \rho(j)\left(H(j)\cdot \operatorname{tr} B_{dh}(j) + \operatorname{tr} B_{dh}(j)\cdot W(j) - \operatorname{div} U_{dh}(j)\right)\\ &\quad - \rho(j) \operatorname{div} U_{dh}(j)\end{aligned}$$

and thus

4.28) $$D^2\rho(j)(N(j),h) = \rho(j)(H(j)\cdot \operatorname{tr} B_{dh}(j) + \operatorname{tr} B_{dh}(j)\cdot W(j)) .$$

Therefore we find

4.29) $$\begin{aligned} D^2\rho(j)(N(j),N(j)) &= \rho(j)\cdot(H(j)^2 + \operatorname{tr} W(j)^2) \\ &= 2\ \rho(j)(H(j)^2 - \lambda(j)) \quad . \end{aligned}$$

In case $\rho(j)(p) > 0$ for all $p \in M$, then

4.30) $$D \log \rho(j)(N(j)) = - H(j)$$

and hence

4.31) $$\begin{aligned} D^2 \log \rho(j)(N(j),h) &= DH(j)(h) - D \log \rho(j)(DN(j)(h)) \\ &= \operatorname{tr} B_{dh}(j)\cdot W(j) \quad . \end{aligned}$$

This yields :

Theorem 4.3 :
In case of dim $M = 2$, then

4.32) $$\int D^2 \log \rho(j)(N(j),N(j)) = \int H(j)^2\ \mu(j) - \pi\cdot X(M) \quad ,$$

with $X(M)$ the Euler characteristic of M.

Equation (4.31) together with (2.9) yields

Theorem 4.4 :
Given any stress form $dl \in E(M,\mathbb{R}^n)$ and any *density function* with $\rho(i)(p) > 0$ for all $p \in M$, then for any j in the connected component O_i of i and any $h \in C^\infty(M,\mathbb{R}^n)$ following equation holds

4.33) $$\Pi_F(dj)\cdot DV(j)(h) = \int \Theta_h(j)\cdot D^2 \log \rho(j)(N(j),h)\ \mu(j)$$

with

$$h = djX_h + \Theta_h \cdot N(j) \; . \tag{4.34}$$

Moreover

$$\int D^2 \log \rho(j)(N(j),h) = \int \rho(j) \cdot \tau(j,dl) \cdot H(j) \, \mu(j) - \pi_F(dj) \cdot DV(j)(N(j)) + \int k\hat{B}^0(j,dl) \cdot W(j) \cdot \mu(j) \tag{4.35}$$

holds if

$$\tau(j,dl) := \operatorname{tr} \hat{B}_{dl}(j,dl) \tag{4.36}$$

$$\text{and} \quad \hat{B}(j,dl) := B(j,dl) + \pi(dj) \cdot H(j) \cdot id - \operatorname{tr} B(j,dl) \cdot id \; .$$

Here F is the force density determined by the stress form dl.

Next let us turn to dim $M = 2$ and to equation (4.21).

By differentiation of (4.21) we get for any $j \in E(M,\mathbb{R}^n)$ and any $h \in C^\infty(M,\mathbb{R}^n)$:

$$\int \left(D\lambda(j)(h) + \frac{\lambda(j)}{2} \operatorname{tr} Dm(j)(h) \right) \mu(j) = 0 \quad , \tag{4.37}$$

a continously equation in integral form.

Let us hence compute $D\lambda(j)(h)$.

By the Cayley–Hamilton theorem we have due to dim $M = 2$

$$W(j)^2 - H(j) \cdot W(j) + \lambda(j) = 0 \tag{4.38}$$

and hence

$$\lambda(j) = \frac{1}{2} \left(H(j)^2 - \operatorname{tr} W(j)^2 \right) . \tag{4.39}$$

Therefore we find

4.40) $$D\lambda(j)(h) = H(j)\cdot DH(j)(h) - \operatorname{tr} DW(j)(h)\cdot W(j)$$

or

4.41) $$\begin{aligned} D\lambda(j)(h) &= H(j)\big(\operatorname{div} U_{dh}(j) - \operatorname{tr} B_{dh}(j)\cdot W(j)\big) \\ &\quad - \operatorname{tr} \nabla(j)\, U_{dh}(j)\cdot W(j) + \operatorname{tr} B_{dh}(j)\cdot W(j)^2 \\ &= H(j) \operatorname{div} U_{dh}(j) - \operatorname{tr} \nabla(j)\, U_{dh}(j)\cdot W(j) - 2\,\lambda(j)\cdot \operatorname{tr} B_{dh}(j) \end{aligned}$$

or reordered

4.42) $$D\lambda(j)(h) = -\lambda(j)\cdot \operatorname{tr} B_{dh}(j) + H(j)\cdot \operatorname{div} U_{dh}(j) - \operatorname{tr} \nabla(j)\, U_{dh}(j)\cdot W(j) .$$

Since

4.43) $$\operatorname{tr} W(j)\, \nabla(j)\, X = \operatorname{div}_j W(j)\, X - dH(j)(X) ,$$

for any $X \in \Gamma TM$, (4.42) yields

4.44) $$D\lambda(j)(h) = -\big(\lambda(j) \operatorname{tr} B_{dh}(j) + \operatorname{div}_j((W(j) - H(j)\cdot id)U_{dh}(j))\big)$$

and in particular

4.45) $$D\lambda(j)(N(j)) = -\lambda(j)\cdot H(j) .$$

By (4.12) we get a more refined version of 4.43) in ternms of $X_h(j)$ and $\Theta_h(j)$ as follows

4.46) $$\begin{aligned} \operatorname{div}_j\big((W(j) - H(j)\cdot id)U_{dh}(j)\big) &= \operatorname{div}_j\big(W(j)^2 X_h - H(j)W(j)X_h\big) \\ &\quad - \operatorname{div}_j H(j)\cdot W(j)X_h \\ &\quad - \operatorname{div}_j\big(W(j)\cdot \operatorname{grad}_j \Theta_h - H(j)\cdot \operatorname{grad}_j \Theta_h\big) . \end{aligned}$$

Since moreover the Ricci-tensor Ric(j) of m(j) is

4.47) $$Ric(j)(X,Y) = -\, m(j)\big((W(j)^2 - H(j)W(j))X,Y\big)$$

as easily shown by using the Gauss-equations and

$$4.48) \qquad \operatorname{div}_j W(j)Y = m(j)\,(\operatorname{div}_j W(j),Y) + \operatorname{tr} \nabla(j)Y\cdot W(j)$$
$$= dH(j)(Y) + \operatorname{tr} \nabla(j)Y\cdot W(j)$$

and

$$4.49) \qquad \operatorname{div}_j(W(j)^2 - H(j)W(j)) = -\operatorname{grad} j\lambda(j)\ ,$$

we find

$$4.50) \qquad \operatorname{div}_j(W(j)^2X_h - H(j)W(j)X_h) = \frac{1}{2}\,d\lambda(j)(X_h) - dH(j)(W(j)X_h)$$
$$+\ dH(j)(\operatorname{grad}\Theta_h) - \operatorname{div}_j W(j)\operatorname{grad}_j\Theta_h\ .$$

Therefore $D\lambda(j)(h)$ reads as

$$4.51) \qquad D\lambda(j)(h) = -\ \lambda(j)\cdot\operatorname{tr} B_{dh}(j) + \frac{1}{2}\,d\lambda(j)(X_h)$$
$$+\ dH(j)(\operatorname{grad}\Theta_j - W(j)X_h) - \operatorname{div}_j W(j)\operatorname{grad}_j\Theta_h\ .$$

Let us pause here to remark that there is no continuity equation of λ along the manifold $j(M)$ for any $j \in E(M,\mathbb{R}^n)$, however as (4.45) shows, λ satisfies such an equation along the $N(j)$ of $j(M)$ for every $j \in E(M,\mathbb{R}^n)$.

Now let us compute $D^2\lambda(j)(N(j),j)$ for any $h \in C^\infty(M,\mathbb{R}^n)$. We have

$$D^2\lambda(j)(N(j),h) = -\ D\lambda(j)(h)\cdot H(j) - \lambda(j)\cdot DH(j)(h) - D\lambda(j)(DN(j)(h))\ ,$$

and then

$$4.52) \qquad D^2\lambda(j)(N(j),h) = H(j)\cdot\lambda(j)\cdot\operatorname{tr} B_{dh}(j)$$
$$+\ H(j)\cdot\operatorname{div}_j\big((W(j) - H(j)\cdot id)U_{dh}(j)\big)$$
$$+\ \lambda(j)\cdot\operatorname{tr} B_{dh}(j)\cdot W(j)\ \ .$$

Thus we conclude by (4.27) :

Lemma :

Let dim $M = 2$ and ρ be a density map.

If $i \in E(M,\mathbb{R}^n)$ satisfies

(i) $\lambda(i)(p) > 0 \qquad \forall\, p \in M$

and

(ii) $\rho(i)(p) > 0 \qquad \forall\, p \in M$,

then there is an open neighbourhood V of i in $E(M,\mathbb{R}^n)$ such that

4.53) $$D^2 \log \lambda(j)(N(j),h) = D^2 \log \rho(j)(N(j),h) \qquad \forall\, j \in V .$$

This Lemma yields immediately.

Theorem 4.6 :

Let dim $M = 2$, ρ a density map, $dl : E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n)$ a smooth stress-form determining a constitutive law F. Then if

(i) $\lambda(i)(p) > 0 \qquad \forall\, p \in M$

and

(ii) $\rho(i)(p) > 0 \qquad \forall\, p \in M$,

then there is an open neighbourhood V on which the following equations hold for any $h \in C^\infty(M,\mathbb{R}^n)$:

4.54) $$D^2 \log \lambda(j)(N(j),l) = \operatorname{tr} B_{dl}(j) W(j) \quad ,$$

4.55) $$\int \Theta_h(j) \cdot D^2 \log \lambda(j)(N(j),l)\, \mu(j) = \int \Theta_h(j) \cdot \operatorname{tr} \hat{B}(j.dl) - \pi_F(j) \cdot DV(j)(h) \quad ,$$

where

4.56) $$\tau(dj,dl) := \operatorname{tr} B_{dl}(j)$$
$$\text{and} \quad \hat{B}(j,dl) = B(j,dl) + \pi_F(dj)\, DV(j)(h) .$$

Equation (4.54) shows how the force density in normal direction (determined by the stress form) affects the geometry of $j(M)$ and vica veria.

Let us turn back to (4.49) in case the embedding $i \in E(M,\mathbb{R}^n)$ satisfies the following :

$$\lambda(i)(p) > 0 \qquad \forall\, p \in M$$

and the density map fulfills accordingly

$$\rho(i)(p) > 0 \qquad \forall\, p \in M\,.$$

Then if $n : (-a,a) \longrightarrow E(M,\mathbb{R}^n)$ for a positive real $a \in \mathbb{R}$ is a smooth map for which

$$n(0) = i$$

and

$$\frac{d}{dt}\, n(t) = N(n(t)) \qquad \forall\, t \in (-a,a)\ ,$$

we immediately deduce

4.57) $$\log \lambda(n(t)) = \log \rho(n(t)) + q(n(t)) \quad \forall\, t \in (-a,a)\,.$$

Hence for any t in the normal evolution of the scalar curvature λ is determined as follows :

Proposition 4.7 :

Suppose $\dim M = 2$. Let moreover $\rho \in C^\infty(E(M,\mathbb{R}^n),C^\infty(M,\mathbb{R}^n))$ a density map and $i \in E(M,\mathbb{R}^n)$ be fixed, such that i and ρ satisfy the following couple of equations

4.58) (i) $\lambda(i)(p) > 0 \qquad \forall\, p \in M$ and

4.59) (ii) $\rho(i)(p) > 0 \qquad \forall\, p \in M.$

Then if $n : (-a,a) \longrightarrow E(M,\mathbb{R}^n)$ for some small enough positive real $a \in \mathbb{R}$ is a smooth curve for which

$$n(0) = i$$

and

$$\frac{d}{dt}\, n(t) = N(n(t)) \qquad \forall\, t \in (-a,a)$$

then the scalar curvature λ is given related to ρ by

4.60) $$\lambda(n(t)) = \frac{\lambda(i)}{\rho(i)} \cdot \rho(n(t)) = \lambda(i)\cdot \det f(n(t))^{-1}\ ,$$

where f(n(t)) is determined by the equation

4.61) $$m(n(t))(X,Y) = m(i)(f(n(t))^2X,Y) \qquad \forall\ X,Y \in \Gamma TM,\ t \in (-a,a).$$

Let us turn our attention to the volume function, which occurs quite often in our formulas.

We will describe the value of the volume function via the volume of a given embedding $i \in E(M,\mathbb{R}^n)$ and a unite normal vectorfield $N(i)$ in case of $\dim M = 2$ and a special sort of embedding j, namely $j = i + \tau\cdot N(j)$ for a small positive real $\tau \in \mathbb{R}$.

In doing so we will observe the influence of $H(j)$ and $\lambda(j)$ on the normal evolution of the volume function.

By Taylor's formula we obtain via (4.20), we get for any small real $\tau > 0$, for which $i + \tau\cdot N(i) \in E(M,\mathbb{R}^n)$ the formula

4.62) $$V(i + \tau\cdot N(i)) = V(i) + \tau\cdot\int H(i)\mu(i) - \tau^2\cdot\int\lambda(i)\mu(i)$$

and hence

4.63) $$V(i + \tau\cdot N(i)) = V(i) - \pi\cdot X(M)\cdot\tau^2 + \tau\cdot\int H(i)\mu(i)\ .$$

Therefore if ρ is a density map the evolution of the volume along $N(j)$ in terms of matter density is given by the equation

$$V(i + \tau\cdot N(i)) = V(i) - \pi\cdot X(M)\cdot\tau^2 - \tau\cdot\int D\log\rho(i)(N(i))\cdot\mu(i)\ ,$$

provided that $\rho(i)(p) > 0 \ \ \forall\ p \in M$, and that $\dim M = 2$.

Similarily we obtain

4.64) $$\rho(i + \tau\cdot N(i)) = \rho(i)(1 + \tau\cdot H(i) + \frac{\tau^2}{2}(H(i)^2 - \operatorname{tr} W(i)^2))\ .$$

Finally we return to the volume active pressure π_F associated with a constitutive law determined via (2.9) by a given smooth stress form

$$dl : E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n)\ ,$$

for any oriented, smooth M with $\dim M = n-1$.

The purpose of the last part of this section is to study the influence of the topology of the body M to π_F.

According to (2.14) $\pi_F(dj)$ splits as follows

$$\pi_F(dj)\cdot H(j) + g(dj) = \mathrm{tr}\, B_{dl(dj)}(j)\cdot W(j) \ . \tag{4.65}$$

To use the form of the term on the right hand side of (4.65), we observe that if $L(TM,TM)$ denotes the collection of all smooth strong bundle maps of TM, then

$$(A,B) := \int \mathrm{tr}\, A\cdot B\, \mu(dj) \qquad \forall\ A,B \in L(TM,TM)$$

is a scalar product.

Hence we can split $H(j)\cdot B_{dl(dj)}(j)$ with respect to (,) into a component $a(dj,dl(dj))\cdot W(j)$ along $W(j)$ and a component $\overline{W}(j)$ normal to $W(j)$.

Thus we have

$$\pi_F(dj)\cdot \int H(j)^2\mu(j) = a(dj,dl(dj))\cdot \int \mathrm{tr}\, W(j)^2\mu(j) \ .$$

By the Cayley–Hamilton theorem applied on $W(j)$, we therefore find

$$\pi_F(dj) = a(dj,dl(dj)) \cdot \left(1 - \frac{\pi\cdot X(M)}{\int H(j)^2\mu(j)}\right) \ . \tag{4.66}$$

Another splitting of $B_{dl(dj)}(j)$ is

$$B_{dl(dj)}(j) = \tau(dj,dl(dj))\cdot id_{TM} + B^0_{dl(dj)}(j) \ , \tag{4.67}$$

and then

$$\begin{aligned} a(dj,dl(dj))\cdot \int \mathrm{tr}\, W(j)^2\mu(j) &= \\ \int \tau(dj,dl(dj))\cdot H(j)^2\cdot\mu(j) &+ a^0(dj,dl(dj))\cdot\int \mathrm{tr}\, W(j)^2 \ , \end{aligned} \tag{4.68}$$

where $a^0(dj,dl(dj))\cdot W(j)$ is the component of $H(j)\cdot B^0_{dl(dj)}(j)$ along $W(j)$.

Decomposing $\tau(dj,dl(dj))\cdot H(j)$ according to the L^2-scalar-product into a component

$$\gamma(dj,dl(dj))\cdot H(j)$$

along H(j) then (4.68) turns into

$$\gamma(dj,dl(dj))\cdot\int H(j)^2 = (a-a^0)\ (dj,dl(dj))\cdot\left(\int H(j)^2\mu(j) - \pi\cdot X(M)\right)$$

or

4.69) $$\gamma(dj,dl(dj)) = (a-a^0)\ (dj,dl(dj))\cdot\left(1 - \frac{\pi\cdot X(M)}{\int H(j)^2\mu(j)}\right) .$$

Thus we get

$$\begin{aligned}\pi_F(dj)\cdot\int H(j)^2 M(j) &= \gamma(dj,dl(dj))\cdot\int H(j)^2\mu(j) \\ &\quad + a^0(dj,dl(dj))\cdot\int \mathrm{tr}\ W(j)^2\end{aligned}$$

or

$$\pi_F(dj) = \gamma(dj,dl(dj)) + a^0(dj,dl(dj))\cdot\left(1 - \frac{\pi\cdot X(M)}{\int H(j)^2\mu(j)}\right) .$$

With the above defined terminology we have

Theorem 4.8 :

Let $\dim M = 2$ *and* $dl : E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n)$ *a smooth stress form determining a constitutive law* $F : TE(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$.

Then

4.70) $$\pi_F(dj) = a(dj,dl(dj))\cdot\left(1 - \frac{\pi\cdot X(M)}{\int H(j)^2\mu(j)}\right)$$

and

4.71) $$\pi_F(dj) = \gamma(dj,dl(dj)) + a^0(dj,dl(dj))\cdot\left(1 - \frac{\pi\cdot X(M)}{\int H(j)^2\mu(j)}\right) .$$

In particular, if $X(M) = 0$ then

4.72) $$\pi_F(dj) = a(dj,dl(dj)) = \gamma(dj,dl(dj)) + a^0(dj,dl(dj)) .$$

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