ON A GLOBAL DIFFERENTIAL GEOMETRIC APPROACH TO THE RATIONAL MECHANICS OF DEFORMABLE MEDIA

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0. Introduction

In the past the rational mechanics of deformable media was largely concerned with materials governed by linear constitutive equations. In recent years, the theory has expanded considerably towards covering materials for which the constitutive equations are inherently nonlinear, and/or whose mechanical properties resemble in some respects those of a fluid and in others those of a solid (cf[Tr,No],[Le,Fi]).

In the present article we formulate a satisfactory global mathematical theory of moving deformable media, which includes all these aspects.

As we shall see, in our theory the stress tensor is neither necessarily local nor symmetric. In fact it does not even determine the equations of motion. It is a more general object, namely, the stress form, which governs the motion. Typical for our considerations is the study of the motion of a soap bubble, i.e. of a closed, deformable, two-dimensional material surface in $\mathbb{R}^3$. It is intuitively clear that this complex motion can be described as the superposition of two different ones. These are on one hand the "elastic" *deformation of the soap bubble in "radial" direction, and the* "instantaneous", "viscous" fluid flow of the same soap bubble along its surface, that is "transversally" to its "elastic" deformation on the other.
For our general case let us assume that at any instant the deformable medium in $\mathbb{R}^n$ forms a manifold and that the diffeomorphism type of this manifold does not change. Hence these manifolds are all diffeomorphic to a fixed one, which we denote by M.
As we shall show, this fascinating representative problem of mechanics of continua as well as the general problem of motion of a deformable medium leads to a dynamical system on a suitably chosen infinite-dimensional manifold. In order to explain the main ideas of our global approach we introduce at first the differential geometric framework.
The manifold M is supposed to be smooth, compact, oriented and of dimension less or equal to $n-1$. The ambient euclidean space $\mathbb{R}^n$ is assumed to be equipped with a fixed scalar product $\langle\ ,\ \rangle$.
Hence an instantaneous configuration of the medium is given by a smooth embedding of M into $\mathbb{R}^n$. Therefore the configuration space is $E(M,\mathbb{R}^n)$, the space of all smooth embeddings of M into $\mathbb{R}^n$. As shown in [Bi,Fi], $E(M,\mathbb{R}^n)$ can be given a smooth principal bundle structure. More precisely let Diff M be the group of smooth diffeomorphisms of M, and define the action Φ of Diff M on $E(M,\mathbb{R}^n)$ as follows

$$(0.1) \qquad \Phi(j,g) = j \circ g, \qquad \forall\ j \in E(M,\mathbb{R}^n),\ g \in \text{Diff } M.$$

Let us denote the quotient of $E(M,\mathbb{R}^n)$ by this action by $U(M,\mathbb{R}^n)$, and identify it with the set of all smooth submanifolds of M in $\mathbb{R}^n$ diffeomorphic with M. Further denote by Π the projection of $E(M,\mathbb{R}^n)$ onto $U(M,\mathbb{R}^n)$. Endowed with the C^∞-topology, $E(M,\mathbb{R}^n)$, $U(M,\mathbb{R}^n)$ and Diff M become Fréchet manifolds. The quadruple $(E(M,\mathbb{R}^n),\Pi,U(M,\mathbb{R}^n),\text{Diff } M)$ is then a principal bundle with Diff M as its structure group. Hence the fibres of this principal bundle have the form

$$(0.2) \qquad j \circ \mathrm{Diff}\, M \quad , \; j \in E(M,\mathbb{R}^n).$$

In the particular case of the soap bubble we now visualize the two motions described above as follows :
The "instantaneous" fluid flow along its surface is described by a curve in one of the fibres of the above principal bundle, while the "radial" deformation is given by a curve which is transverse to the fibres of $E(M,\mathbb{R}^n)$.

Each configuration $j \in E(M,\mathbb{R}^n)$ yields a Riemannian metric $m(j)$, assuming on any pair of tangent vectors $v,w \in TM$ the value

$$(0.3) \qquad m(j)(v,w) := \langle dj\, v, dj\, w\rangle ,$$

where the scalar product is to be taken pointwise.
The "instantaneous" metrical properties of the moving body are described in this metric. Suppose now that the deformable medium is moving. We furnish the description of its motion by assuming that we know the work done by the forces acting upon M. It is in this work that all the constitutive information on the medium is coded. We therefore call it the constitutive law. The fluid component of the medium is expressed through the dependence of the work on an extra parameter. Accordingly the constitutive law is then given by

$$(0.4) \qquad F : C^\infty(M,\mathbb{R}^n) \times E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R} ,$$

where F is linear in the third argument, the first factor in the cartesian product is the space of extra parameters and furthermore the trivial tangent bundle $TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)$ is the phase space of motions in $E(M,\mathbb{R}^n)$.
We concentrate on those constitutive laws which admit an integral representation. More precisely, we assume that F is given by

$$(0.5) \qquad F(k)(j,h) = {}_M\!\int \langle \varphi_F(j,k),h\rangle \mu(j), \qquad \forall\; j \in E(M,\mathbb{R}^n),\; h,k \in C^\infty(M,\mathbb{R}^n),$$

with $\varphi_F : TE(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)$ being a smooth map called the force density.
The equation of motion on $E(M,\mathbb{R}^n)$ described by a smooth curve

$$(0.6) \qquad \sigma : (-\lambda,\lambda) \longrightarrow E(M,\mathbb{R}^n), \qquad \lambda > 0,$$

is given then by

$$(0.7) \qquad \begin{aligned} F(\dot\sigma(t))(\sigma(t),h) &= {}_M\!\int \langle \varphi_f(\sigma(t),\dot\sigma(t)),h\rangle \mu(\sigma(t)) \\ &= {}_M\!\int \rho(\sigma(t))\langle \ddot\sigma(t),h\rangle \mu(\sigma(t)) \qquad \forall\; h \in C^\infty(M,\mathbb{R}^n). \end{aligned}$$

We note that in (0.7) we have assumed for simplicity that the constitutive law F depends on the "velocity" $\dot{\sigma}(t)$, i.e. $k=\dot{\sigma}(t)$. Interpreting h as a virtual displacement, (0.7) is just d'Alembert's principle of virtual work, which was formulated for the mechanics of continua by [He]. But (0.7) implies easily

$$(0.8) \qquad \rho(\sigma(t))\ddot{\sigma}(t) = \varphi_F(\sigma(t),\dot{\sigma}(t)), \qquad \forall\ t\in (-\lambda,\lambda).$$

To obtain a more refined form, let us denote by "T" and "⊥" respectively the tangential and the normal component with respect to $\sigma(t)(M)$. The equation of motion (0.8) splits into the coupled system

$$(0.9) \qquad \begin{cases} \nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + W(\sigma(t),\dot{\sigma}(t)^{\perp})Z(t) + [(\dot{\sigma}(t)^{\perp})^{\cdot}]^{T} \\ \quad = \rho^{-1}(\sigma(t))\ Y(\sigma(t),\dot{\sigma}(t)), \\ \\ [(\dot{\sigma}(t))^{\perp})^{\cdot}]^{\perp} = \\ \quad \rho^{-1}(\sigma(t))\varphi_F(\sigma(t),\dot{\sigma}(t))^{\perp} - [d\dot{\sigma}(t)^{\perp}Z(t)]^{\perp} - S(\sigma(t))(Z(t),Z(t)). \end{cases}$$

Here $\nabla(\sigma(t))$ denotes the Levi-Civita connection of $m(\sigma(t))$, the metric given by $\sigma(t)$, $S(\sigma(t))$ is the second fundamental tensor, $Z(t)$ and $Y(\sigma(t),\dot{\sigma}(t))$ belong to ΓTM. Furthermore $W(j,N)$ is the unique bundle map of TM associated with a smooth map $N : M \longrightarrow \mathbb{R}^n$ satisfying

$$(0.10) \qquad \langle dj\ X(p),N(p)\rangle = 0, \qquad \forall\ X\in \Gamma TM,\ p\in M$$

and which is determined by

$$(0.11) \qquad dj\ W(j,N)Y = (dN,Y)^{T}.$$

Among the force densities acting on M we distinguish between internal forces and external ones. Of a special interest is the study of the motion of the deformable medium M subjected to an internal force density. Clearly, internal physical properties of the moving medium are described by constitutive laws invariant under the translation group $\mathbb{R}^n$. Evidently, the $\mathbb{R}^n$-invariant configurations are differentials of embeddings. We hence identify

$$(0.12) \qquad E(M,\mathbb{R}^n)/_{\mathbb{R}^n} \text{ with } \{dj \mid j\in E(M,\mathbb{R}^n)\}$$

and more generally

$$(0.13) \qquad C^{\infty}(M,\mathbb{R}^n)/_{\mathbb{R}^n} \text{ and } \{dh \mid h\in C^{\infty}(M,\mathbb{R}^n)\}.$$

The phase space for the $\mathbb{R}^n$-invariant motion is hence

$$(0.14) \qquad T(E(M,\mathbb{R}^n)/_{\mathbb{R}^n}) = \{dj \mid j\in E(M,\mathbb{R}^n)\} \times \{dh \mid h\in C^{\infty}(M,\mathbb{R}^n)\}.$$

We require that the internal constitutive law F admits the representation

$$(0.15) \qquad F = F_{\mathbb{R}^n} \circ Td ,$$

where

$$(0.16) \qquad F_{\mathbb{R}^n} : C^\infty(M,\mathbb{R}^n)|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)|_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

is a parameter depending one form (the parameter varies in the front factor in (0.16)) and Td is the tangent map of the differential

$$(0.17) \qquad d : E(M,\mathbb{R}^n) \longrightarrow E(M,\mathbb{R}^n)/_{\mathbb{R}^n} .$$

To get a detailed description of the motion of the deformable medium, we assume now that $F_{\mathbb{R}^n}$ itself has an integral representation

$$(0.18) \qquad F_{\mathbb{R}^n}(dk)(dj,dl) = {}_M\!\int \alpha(dj,dk)dl \ \mu(j), \qquad \forall \ j \in E(M,\mathbb{R}^n),\ k,l \in C^\infty(M,\mathbb{R}^n),$$

where α is an $\mathbb{R}^n$-valued one-form, the so-called stress form, depending itself on an extra parameter, i.e.

$$(0.19) \qquad \alpha : E(M,\mathbb{R}^n)|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n).$$

The stress form α decomposes naturally at each

$$(dk,dj) \in TE(M,\mathbb{R}^n)|_{\mathbb{R}^n} = C^\infty(M,\mathbb{R}^n)|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)|_{\mathbb{R}^n}$$

into

$$(0.20) \qquad \alpha(dj,dk) = c_\alpha(dj,dk)\ dj + dj\ C_\alpha(dj,dk) + dj\ B_\alpha(dj,dk) ,$$

with $C_\alpha : TM \longrightarrow TM$ and $B_\alpha : TM \longrightarrow TM$ being smooth, strong bundle endomorphisms, which are respectively skew- and selfadjoint with respect to m(i) and $c_\alpha \in C^\infty(M,so(n))$. Here so(n) denotes the Lie algebra of the group of all proper rotations SO(n). In case that M is of codimension 1, the equations of motion (0.9) read as

$$(0.21) \qquad \begin{cases} \nabla(\sigma(t))_{Z(t)}Z(t)+\dot Z(t)+2\cdot\epsilon(\sigma(t),\dot\sigma(t))W(\sigma(t)Z(t)-\mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)) \\ \qquad = -\rho^{-1}(\sigma(t))\mathrm{div}_{\sigma(t)}T_\alpha(d\sigma(t),d\dot\sigma(t))-2\ W(\sigma(t))U_\alpha(d\sigma(t),d\dot\sigma(t)), \\ \\ \dot\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\ \mathrm{tr}\ (B_\alpha(d\sigma(t),d\dot\sigma(t))\ W(\sigma(t)) \\ \qquad - d\epsilon(\sigma(t),\dot\sigma(t))\ Z(t) + \mathfrak{h}(\sigma(t))(Z(t),Z(t)). \end{cases}$$

Here $\epsilon(\sigma(t),\dot\sigma(t)) \in C^\infty(M,\mathbb{R})$, $U_\alpha(d\sigma(t)) \in \Gamma TM$, $W(\sigma(t))$ is the Weingarten map, $\mathrm{div}_{\sigma(t)}$ is the divergence taken with respect to $m(\sigma(t))$, $\mathfrak{h}(\sigma(t))$ is the second fundamental form, tr denotes the trace and $T_\alpha(dj,dk)$ is the so-called stress tensor, defined as

$$(0.22) \qquad T_\alpha(dj,dk)(X,Y) = m(j)((B_\alpha+C_\alpha)(dj,dk)X,Y) , \quad \forall\ X,Y \in \Gamma TM.$$

Each $\alpha \in A^1(M,\mathbb{R}^n)$, and hence the parameter depending stress form splits relative to an embedding $i \in E(M,\mathbb{R}^n)$ into

$$\alpha = dh + \beta \,, \tag{0.23}$$

where $h \in C^\infty(M,\mathbb{R}^n)$, the so called integrable part of α, is uniquely determined up to a constant. Moreover h splits into parts tangential and normal to $j(M)$, i.e.

$$h = di\, X_h + h^\perp \tag{0.24}$$

(with $h^\perp = \Theta_h \cdot N(i)$, $\Theta_h \in C^\infty(M,\mathbb{R}^n)$, in case of $\dim M = n-1$) for a well determined vector field $X_h \in \Gamma TM$. Using the Hodge decomposition

$$\begin{cases} X_h = X_h^0 + \mathrm{grad}_i \Psi_h \,, \\ \mathrm{div}_i X_h^0 = 0 \,, \end{cases} \tag{0.25}$$

we thus obtain immediately

$$\alpha(X) = di \nabla(i)_X X_h + di W_h(i) X + S(i)(X_h, X) + (dh^\perp(X))^\perp + \beta(X), \quad \forall\, X \in \Gamma TM. \tag{0.26}$$

This allows us to read off the coefficients in (0.22) as

$$\begin{cases} c_\alpha di = (dh^\perp)^\perp + S(i)(X_h, \cdot) + c_\beta di \,, \\ C_\alpha = \frac{1}{2}\,[\nabla(i) X_h^0 - \tilde{\nabla}(i) X_h^0] + C_\beta \,, \\ B_\alpha = \frac{1}{2}\, L_{X_h^0} + \mathrm{grad}_i \Psi_h + W_h(i) + B_\beta \,. \end{cases} \tag{0.27}$$

$W_h(i)$ denotes here the strong smooth bundle map of TM given by

$$di W_h(i)\, X := (dh^\perp(X))^T \,, \qquad \forall\, X \in \Gamma TM \,. \tag{0.28}$$

$\tilde{\nabla}(i) X_h$ is the adjoint of $\nabla(i) X_h$ with respect to $m(i)$ formed fibrewise, so that each $v_p \in T_p M$ is sent into $\tilde{\nabla}(i) X_h(v_p)$, $\forall\, p \in M$. Moreover

$$L_{X_h} : TM \longrightarrow TM \tag{0.29}$$

is the strong smooth bundle endomorphism of TM defined by the Lie derivative $L_{X_h}(m(i))$ via the equation

$$m(i)(L_{X_h} X, Y) := L_{X_h}(m(i))(X,Y) \,, \qquad \forall\, X_h, X, Y \in \Gamma TM. \tag{0.30}$$

Using now the definition of the Laplace-Beltrami operator $\Delta(i)$

$$\mathrm{div}_i(\nabla(i) X_h) = \Delta(i) X_h = -\mathrm{tr}\, \nabla^2(i) X_h \,, \tag{0.31}$$

and introducing $R(i)X_h$ via

(0.32) $$m(i)(R(i)X_h,Y) = \mathrm{Ric}(m(i))(X_h,Y)\ , \qquad \forall\ Y \in \Gamma TM\ ,$$

where Ric(m(i)) denotes the Ricci tensor of m(i), we obtain in the case of codimension 1 the formulas

(0.33) $$\begin{cases} \mathrm{div}_i\ B_{dh} = \frac{1}{2}\,\Delta(i)X_h + \frac{1}{2}\,R(i)X_h \\ \qquad\qquad + W(i)\ \mathrm{grad}_i\Theta_h + \Theta_h\ \mathrm{grad}_i H(i)\ , \\ \mathrm{div}_i\ C_{dh} = \frac{1}{2}\,\Delta(i)X_h - \frac{1}{2}\,R(i)X_h - \frac{1}{2}\,\mathrm{grad}_i\mathrm{div}_i X_h\ , \\ \mathrm{tr}\ B_{dh} = -\ \Delta(i)\ \Psi_h + \mathrm{tr}(\Theta_h W(i))\ . \end{cases}$$

a fixed embedding i.
Here the unnormalized mean curvature $H(i)$ is defined to be tr $W(i)$. Next we introduce the notion of structural viscosity. To this end we consider on the one hand the decompositions (0.23), (0.24) and (0.25) for the stress form $\alpha(dk,dj)$, which now depends on an additional parameter dk with $k \in C^\infty(M,\mathbb{R}^n)$. On the other hand, we use the decomposition for k, i.e.

(0.34) $$k = djX_k + k^\perp,$$

(0.35) $$\begin{cases} X_k = X_k^0 + \mathrm{grad}_j\Psi_k\ , \\ \mathrm{div}_j X_k^0 = 0\ . \end{cases}$$

Even though dk is determined only up to a constant, X_k^0 depends uniquely on dk. This allows us to relate X_k^0 and X_h^0 uniquely to each other by

(0.36) $$X_h^0(dj,dk) = \nu(dj,dk)X_k^0 + \hat{X}_h(dj,dk)\ ,$$

where $\nu(dj,dk) \in C^\infty(M,\mathbb{R})$ and $\hat{X}(dj,dk) \in \Gamma TM$ is pointwise orthogonal to X_k^0. We call the function $\nu(dj,dk)$, the coefficient of structural viscosity.
Accordingly we call these deformable media, whose constitutive laws depend *only on* $k^\perp$, *frictionless deformable media, while the deformable media,* whose constitutive laws depend on the whole of k, will be called frictional ones.
Furnished with the structure developed so far, we deduce next the equations of motion of a deformable medium M subjected to a general constitutive law

$$F : E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

To do this, we assume that F splits into

(0.37) $$F = F_{ext} + F_{int}$$

and that F_{int} is of the form

(0.38) $$F_{int} = d^*F_{\mathbb{R}^n}\ .$$

Furthermore, we require that F_{ext} and $F_{\mathbb{R}^n}$ both admit integral representation and denote the resulting force densities by φ_{ext} and φ_{int} respectively. Using Hodge's decomposition, we obtain for all $j \in E(M,\mathbb{R}^n)$ and all $k \in C^\infty(M,\mathbb{R}^n)$

$$(0.39)\qquad \begin{cases} \varphi_{int}(j,k) = dj\, \mathrm{grad}_j \tau_{int}(j,k) + dj\, Y^0_{int}(j,k) + \varphi^\perp_{int}(j,k) , \\ \varphi_{ext}(j,k) = dj\, \mathrm{grad}_j \tau_{ext}(j,k) + dj\, Y^0_{ext}(j,k) + \varphi^\perp_{ext}(j,k) , \end{cases}$$

and in turn

$$(0.40)\qquad \begin{aligned} \varphi(j,k) &= dj\, (\mathrm{grad}_j \tau_{int}(j,k) + \mathrm{grad}_j \tau_{ext}(j,k)) \\ &\quad + dj\, (Y^0_{int}(j,k) + Y^0_{ext}(j,k)) + \varphi^\perp_{int}(j,k) + \varphi^\perp_{ext}(j,k) \\ &:= dj\, \mathrm{grad}_j\, \tau(j,k) + dj\, Y^0(j,k) + \varphi^\perp(j,k) , \\ &\qquad \forall\ j \in E(M,\mathbb{R}^n),\ \ k \in C^\infty(M,\mathbb{R}^n). \end{aligned}$$

Hence the equations of motion in case of dim $M = n-1$ are

$$(0.41)\qquad \begin{cases} \nabla(\sigma(t))_{Z(t)} Z(t) + \dot{Z}(t) + 2\cdot\epsilon(\sigma(t),\dot\sigma(t))\ W(\sigma(t))Z(t) \\ -\ \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)) \\ = \rho^{-1}(\sigma(t))(\mathrm{grad}_{\sigma(t)}\tau(\sigma(t),\dot\sigma(t)) - \Delta(\sigma(t))[\nu(d\sigma(t),d\dot\sigma(t))Z^0(t) \\ +\ \hat{Z}_h(d\sigma(t),d\dot\sigma(t)) + \mathrm{grad}_{\sigma(t)}\psi(\sigma(t),\dot\sigma(t))] \\ -\ W(\sigma(t))[\mathrm{grad}_{\sigma(t)}\Theta_h(d\sigma(t),d\dot\sigma(t)) + 2(W(\sigma(t))X_h - \mathrm{grad}\ \Theta_h)] \\ -\ \Theta_h(d\sigma(t),d\dot\sigma(t))\mathrm{grad}_{\sigma(t)}H(\sigma(t)]) , \\ \\ \dot\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))[-\tau_{int}(d\sigma(t),d\dot\sigma(t))\ H(\sigma(t)) \\ -\ dH(\sigma(t))[\nu(d\sigma(t),d\dot\sigma(t))\ Z^0(t) + \hat{Z}_h(d\sigma(t),\dot\sigma(t))] \\ +\ \mathrm{div}_{\sigma(t)}\nu(d\sigma(t),d\dot\sigma(t))W(\sigma(t))Z^0(t) \\ +\ \mathrm{div}_{\sigma(t)}W(\sigma(t))\hat{X}_h(d\sigma(t),d\dot\sigma(t)) \\ -\ \Theta_h(d\sigma(t),d\dot\sigma(t))\ \mathrm{tr}\ W(\sigma(t))^2] + \mathfrak{h}(\sigma(t))\ (Z(t),Z(t)) \\ -\ d\epsilon(\sigma(t),\dot\sigma(t))\ Z(t) + \kappa_{ext}(\sigma(t),\dot\sigma(t))] \end{cases}$$

where $\varphi_{ext}{}^\perp(\sigma(t),\dot\sigma(t)) = \kappa_{ext}(\sigma(t),\dot\sigma(t))\ N(\sigma(t))$.
In case the motion follows a fixed surface $i(M)\subset\mathbb{R}^n$ given by a fixed embedding $i \in E(M,\mathbb{R}^n)$, the equation (0.41) reduces to

$$(0.42)\qquad \begin{cases} \nabla(i)_{X(t)}X(t) + \dot{X}(t) = \rho^{-1}(X(t))[-\mathrm{grad}_i\tau_{int}(X(t),\dot X(t)) \\ \qquad -\ \Delta(i)[\nu(X(t),\dot X(t))\ X^0(t) + \hat{X}_h(X(t),\dot X(t)) \\ \qquad +\ \mathrm{grad}_i\psi(X(t),\dot X(t))] - W(i)[\mathrm{grad}_i\ \Theta_h(X(t),\dot X(t)) \\ \qquad +\ 2\ (W(\sigma(t))X_h - \mathrm{grad}\ \Theta_h)] - \Theta_h(X(t),\dot X(t))\ \mathrm{grad}_i H(i) , \\ \\ 0 = \rho^{-1}(X(t))\ (-\tau_{int}(X(t),\dot X(t))\cdot H(i) - dH(i)[\nu(X(t),\dot X(t))X^0(t) \\ \qquad +\ \mathrm{div}_i\nu(X(t),\dot X(t))W(i)X^0(t) + \mathrm{div}_i W(i)\hat{X}_h(X(t),\dot X(t)) \\ \qquad +\ \hat{X}_h(X(t),\dot X(t))] - \Theta_h(X(t),\dot X(t))\ \mathrm{tr}\ W(i)^2) \\ \qquad +\ \mathfrak{h}(i)(X(t),X(t)) + \kappa_{ext}(X(t),\dot X(t)) , \end{cases}$$

where $X(t)$ is the push-forward of $Z(t)$ by $g(t) \in \mathrm{Diff}\, M$, i.e.

$$X(t) = Tg(t)\, Z(t)\, g(t)^{-1}, \qquad \forall\, t \in (-\lambda,\lambda)\,. \tag{0.43}$$

At the end of the paper we remark how to introduce a volume active pressure $\pi(dj,dk)$, which allows us to decompose $F(dj,dk)$ into

$$F(dj,dk) = \bar{F}(dj,dk) - \pi(dj,dk)\cdot DV(j)\,, \tag{0.44}$$

where $V(j)$ denotes the volume of $j(M)$.
$\pi(dj,dk)\cdot DV(j)$ is the work used against the infinitesimal volume change by $DV(j)$. Let us point out that $\pi(dj,dk)$ is not identical with $\tau_{int}(dj,dk)$, the former is a real, the latter a smooth function.

We have omitted to discuss the influence of thermodynamics to the deformations of the medium. We will do these studies in a forthcoming paper.

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1. The space of configurations as a principal bundle

As already mentioned a configuration of the moving deformable medium M is described by a smooth embedding of M into an euclidean ambient space $\mathbb{R}^n$. In the present paper we assume that the dimension $\dim M$ of the manifold M satisfies the inequality

$$(1.1) \qquad \dim M \leq n-1.$$

Let us recall that a smooth embedding $j : M \longrightarrow \mathbb{R}^n$ is a smooth map satisfying the following conditions (i) the tangent map

$$(1.2) \qquad Tj(p) : T_pM \longrightarrow \{j(p)\} \times \mathbb{R}^n$$

of j at $p \in M$ is injective for any $p \in M$,

(ii)

$$(1.3) \qquad j : M \longrightarrow j(M) \subset \mathbb{R}^n$$

is a homeomorphism.

We point out at this occasion that the tangent map

$$(1.4) \qquad Tj : TM \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

splits naturally into

$$(1.5) \qquad Tj = (j, dj),$$

where

$$(1.6) \qquad dj = pr_2 \circ Tj$$

and

$$(1.7) \qquad pr_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is the projection onto the second factor along the first one, i.e.

$$(1.8) \qquad pr_2(a,b) = b \qquad \text{for all pairs } (a,b) \in \mathbb{R}^n \times \mathbb{R}^n .$$

Thus dj represents locally nothing else but the Frèchet differential of the local representative of j.

We denote by $E(M,\mathbb{R}^n)$ the set of all smooth embeddings of M into $\mathbb{R}^n$. Hence $E(M,\mathbb{R}^n)$ is the set of all configurations of our moving deformable medium. If we equip $E(M,\mathbb{R}^n)$ with the Whitney C^∞– topology (cf. [Gui,Go]), then it becomes an infinite dimensional Frèchet manifold.

The reason is the following one :

Consider the set $C^\infty(M,\mathbb{R}^n)$ of all smooth maps of M into $\mathbb{R}^n$, which is endowed

with Whitney's C^∞- topology and note that together with the pointwise defined operations of addition and multiplication with scalars $C^\infty(M,\mathbb{R}^n)$ becomes a complete metrizable, locally convex space, a so-called Fréchet space.
Since as shown in [Hi] $E(M,\mathbb{R}^n)$ is open in $C^\infty(M,\mathbb{R}^n)$, it hence carries the structure of a Fréchet manifold (cf. [Bi,Fi]).
Using now the differential calculus in locally convex spaces constructed either in [Ba], [Gu], [Mi] or [Fr,Kr], it is evident that the tangent space $T_j\, E(M,\mathbb{R}^n)$ at $j \in E(M,\mathbb{R}^n)$ is nothing else but $C^\infty(M,\mathbb{R}^n)$.
Therefore the tangent bundle $TE(M,\mathbb{R}^n)$ is trivial, i.e.

(1.9) $$TE(M,\mathbb{R}^n) = E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n).$$

We note that $TE(M,\mathbb{R}^n)$ is the phase space of motions in $E(M,\mathbb{R}^n)$.
Next we introduce the principal bundle structure of $E(M,\mathbb{R}^n)$, which is crucial for our formalism to describe the motion of the deformable medium M.
Following [Bi,Fi] we first describe the group action. Let Diff M be the group of all smooth diffeomorphisms of M equipped with the C^∞- topology. Diff M is a Fréchet manifold, in which the operations are smooth in either one of the above mentioned notions of differentiability.
Consequently we call Diff M a differentiable group. The tangent space at the identity in Diff M is naturally identified with ΓTM, the set of smooth vector fields on M.
The operation Φ of Diff M on $E(M,\mathbb{R}^n)$ given by

(1.10) $$\Phi(j,g) = j \circ g \qquad \forall\ j \in E(M,\mathbb{R}^n),\ g \in \text{Diff } M,$$

is smooth.
Consequently $E(M,\mathbb{R}^n)$ can be represented as

(1.11) $$E(M,\mathbb{R}^n) = \bigcup_{i \in E(M,\mathbb{R}^n)} i \circ \text{Diff } M \qquad ,$$

i.e. as the collection of smooth fibers. The quotient $U(M,\mathbb{R}^n)$ of $E(M,\mathbb{R}^n)$ by Diff M

(1.12) $$U(M,\mathbb{R}^n) = E(M,\mathbb{R}^n)/_{\text{Diff } M}$$

inherits a smooth Fréchet manifold and it is naturally identified with the collection of all smooth submanifols of M in $\mathbb{R}^n$ which are diffeomorphic to M.
Let us denote the quotient map, i.e. the projection of $E(M,\mathbb{R}^n)$ onto $U(M,\mathbb{R}^n)$, by Π. The quadruple ($E(M,\mathbb{R}^n)$, Π, $U(M,\mathbb{R}^n)$, Diff M) is then a principal bundle in the sense of [Gr,H,V].
Each configuration $j \in E(M,\mathbb{R}^n)$ induces a Riemannian metric $m(j)$ on M defined

by

$$(1.13) \qquad m(j)(X,Y) := \langle djX, djY \rangle \qquad \forall\, X,Y \in \Gamma TM.$$

Denoting now by $\mathfrak{M}(M)$ the set of all smooth Riemannian metrics on M equipped with the C^∞- topology, we obtain then a natural map

$$(1.14) \qquad m : E(M,\mathbb{R}^n) \longrightarrow \mathfrak{M}(M).$$

Noting that $\mathfrak{M}(M)$ is an open cone in $S^2(M)$, the Frèchet space of smooth symmetric two tensors on M, we deduce that $\mathfrak{M}(M)$ is also a Frèchet manifold and that the map m is smooth (cf. [Sch]).
Moreover the tangent bundle $T\mathfrak{M}(M)$ of $\mathfrak{M}(M)$ is trivial, i.e.

$$(1.15) \qquad T\mathfrak{M}(M) = \mathfrak{M}(M) \times S^2(M).$$

For later use we calculate at this point the derivative $Dm(j)(k)$ of m at $j \in E(M,\mathbb{R}^n)$ in the direction of $h \in C^\infty(M,\mathbb{R}^n)$. Due to the $\mathbb{R}$ – bilinearity of m in the variable $j \in E(M,\mathbb{R}^n)$ we have

$$(1.16) \qquad Dm(j)(h)(X,Y) = \langle djX, dhY \rangle + \langle dhX, djY \rangle, \qquad \forall\, X,Y \in \Gamma TM.$$

Splitting h into

$$h = dj\, X_h + h^\perp$$

with $X_h \in \Gamma TM$ and $h^\perp$ being pointwise normal to $j(M)$ in $\mathbb{R}^n$, equation (1.16) turns into

$$(1.17) \qquad Dm(j)(h)(X,Y) = \langle djX,d(djX_h)(Y)\rangle + \langle d(djX_h)X,djY\rangle + \langle djX,dh^\perp Y\rangle + \langle dh^\perp X,djY\rangle .$$

Since $d(djX_h)Y = dj\nabla(j)_Y X_h + S(i)(X_h,Y)$, where $\nabla(i)$ is the Levi-Civita connection of $m(j)$, (1.17) turns into

$$(1.18) \qquad \begin{aligned} Dm(j)(h)(X,Y) &= m(j)(X,\nabla(j)_Y X_h) + m(j)(\nabla(j)_X X_h,Y) \\ &\quad + \langle djX,dh^\perp Y\rangle + \langle dh^\perp X,djY\rangle \\ &= L_{X_h}(m(j))(X,Y) + \langle djX,dh^\perp Y\rangle + \langle dh^\perp X,djX\rangle , \end{aligned}$$

with $L_{X_h}(m(j))$ being the Lie derivative of $m(j)$.
In case $h^\perp = \Theta_h \cdot N(j)$ then

$$(1.19) \qquad Dm(j)(h)(X,Y) = L_{X_h}(m(j))(X,Y) + 2\cdot\Theta_h\cdot\mathfrak{h}(j)(X,Y) ,$$

with $\mathfrak{h}(j)$ the second fundamental form of j defined by

$$\mathfrak{h}(j)(X,Y) := m(j)(W(j)X,Y) = \langle dN(j)X,djY\rangle .$$

Here $W(j)$ denotes the Weingarten map of j given by

$$djW(j)X = dN(j)X\,, \qquad \forall\, X \in \Gamma TM\,. \tag{1.20}$$

If $H(j)$ denotes $\operatorname{tr} W(j)$, then

$$\operatorname{tr} Dm(j)(h)(X,Y) = 2\cdot\operatorname{div}_j X_h + \Theta_h\cdot H(j)\,. \tag{1.21}$$

By $\operatorname{div}_j X_h$ we denote the divergence of X_h formed with respect to $m(j)$. This means,

$$\operatorname{div}_j X_h = \operatorname{tr} \nabla(j)X_h\,.$$

The function $\frac{H(j)}{\dim M}$ is called the mean curvature of j, while $H(j)$ denotes the unnormalized mean curvature.

2. The metric $\mathcal{G}_E$ on the configuration space $E(M,\mathbb{R}^n)$

In order to define a metric on $E(M,\mathbb{R}^n)$, which is adapted to the mass distribution of our moving deformable medium M, we first introduce a density map

$$(2.1) \qquad \rho : E(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}),$$

which is supposed to be smooth in either sense of the above mentioned notions of differentiability. In addition we require that ρ fullfills a continuity equation, namely

$$(2.2) \qquad D\rho(j)(h) = -\tfrac{1}{2}\,\rho(j)\,\mathrm{tr}_j\, Dm(j)(h), \quad \forall\ j\in E(M,\mathbb{R}^n),\ h\in C^\infty(M,\mathbb{R}^n),$$

where tr_j denotes the trace taken with respect to $m(j)$. Using the fact that the derivative at j in the direction of any $h\in C^\infty(M,\mathbb{R}^n)$ of the Riemannian volume form $\mu(j)$ has the form

$$(2.3) \qquad D\mu(j)(h) = \tfrac{1}{2}\,\mu(j)\,\mathrm{tr}_j\, Dm(j)(h)$$

as shown in [Bi,1], it follows that (2.2) is indeed the continuity equation. Consequently the total mass $m(j)$ attached to any $j\in E(M,\mathbb{R}^n)$ via the formula

$$(2.4) \qquad m(j) = {}_M\!\int \rho(j)\,\mu(j)$$

is constant in j.
The existence of such a function ρ can be established as follows :
Let $i\in E(M,\mathbb{R}^n)$ be any embedding and denote by O_i its connected component in $E(M,\mathbb{R}^n)$. Then for any $j\in O_i$ the differential dj is related to di in the following way

$$(2.5) \qquad dj = g \circ di \circ f,$$

where $g\in C^\infty(M,SO(n))$ and f is a strong bundle isomorphism of TM which is fibrewise positive with respect to $m(i)$. One easily verifies that the Riemannian volume forms $\mu(j)$ and $\mu(i)$ of j and i respectively are related by

$$(2.6) \qquad \mu(j) = \det f\cdot\mu(i)\ .$$

We then set

$$(2.7) \qquad \rho(j) := \rho(i)\cdot\det f^{-1}$$

with $\rho(i)$ chosen such that

$$(2.8) \qquad \rho(i)(p) > 0\ , \qquad \forall\ p\in M,$$

and note that this map satisfies the continuity equation (2.2).
Next we introduce the metric $\mathcal{G}_E$ on $E(M,\mathbb{R}^n)$ by the formula

$$(2.9) \qquad \mathcal{G}_E(j)(h,k) := {}_M\!\int \rho(j)\,\langle h,k\rangle\,\mu(j)\,, \quad \forall\, h,k \in C^\infty(M,\mathbb{R}^n).$$

Due to (2.2) $\mathcal{G}_E$ is constant in j. Therefore the geodesics of $\mathcal{G}_E$ are straight line segments as shown in [Bi,1].

3. The constitutive law and the general equations of motion

By a constitutive law we understand a smooth parameter depending one form, the so-called work

(3.1) $$F : C^\infty(M,\mathbb{R}^n) \times E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

The first factor $C^\infty(M,\mathbb{R}^n)$ in the above cartesian product is the parameter space. We often will regard F as a map

$$F : C^\infty(M,\mathbb{R}^n) \longrightarrow A^1(E(M,\mathbb{R}^n),\mathbb{R}).$$

The domain of F is the parameter space, the range the collection of all smooth one forms on $E(M,\mathbb{R}^n)$ with values in $\mathbb{R}$. To handle this abstract notion (3.1) we require an integral representation for F given by

(3.2) $$F(k)(j,h) = {}_M\!\int \langle \varphi_F(j,k),h\rangle\ \mu(j)\ , \quad \forall\ j\in E(M,\mathbb{R}^n),\ h,k\in C^\infty(M,\mathbb{R}^n),$$

where $\varphi_F : TE(M,\mathbb{R}^n) \longrightarrow C^\infty(M,\mathbb{R}^n)$, the so-called force density, is assumed to be a smooth map. We point out that $F(k)(j,h)$ varies linearly only in h and that it furthermore depends on the maps j and k globally.
The equation of motion on $E(M,\mathbb{R}^n)$ described by a smooth curve

(3.3) $$\sigma : (-\lambda,\lambda) \longrightarrow E(M,\mathbb{R}^n)\ ,$$

for some positive real λ, is given by

(3.4) $$\begin{aligned} F(\dot\sigma(t))(\sigma(t),h) &= \mathcal{B}(j)(\ddot\sigma(t),h) = {}_M\!\int \langle \varphi_F(\sigma(t),\dot\sigma(t)),h\rangle\ \mu(\sigma(t)) \\ &= {}_M\!\int \rho(\sigma(t))\ \langle \ddot\sigma(t),h\rangle\ \mu(\sigma(t)) \\ &\qquad\qquad\qquad \forall\ h\in C^\infty(M,\mathbb{R}^n), \end{aligned}$$

where for the sake of simplicity we have taken $k=\dot\sigma(t)$.
It is obvious that this equation implies

(3.5) $$\rho(\sigma(t))\ \ddot\sigma(t) = \varphi_F\ (\sigma(t),\dot\sigma(t)), \quad \forall\ t\in(-\lambda,\lambda).$$

We note that (3.5) is a second order differential equation on $E(M,\mathbb{R}^n)$ and not on parts of $\mathbb{R}^n$.
We rewrite it now according to the principal bundle structure of $E(M,\mathbb{R}^n)$, by proceeding as follows :
At first we note that $\dot\sigma(t)$ admits in $\mathbb{R}^n$ the pointwise splitting

(3.6) $$\dot\sigma(t) = d\sigma(t)\ Z(t) + \dot\sigma(t)^\perp\ ,$$

where $Z(t)\in \Gamma TM$ is uniquely determined and $\dot\sigma(t)^\perp$ is, according to the definition, pointwise perpendicular to $\sigma(t)(M)$ for each $t\in(-\lambda,\lambda)$.
Consequently $\ddot\sigma(t)$ is given by

$$(3.7)\qquad \ddot{\sigma}(t) = d\dot{\sigma}(t)\, Z(t) + d\sigma(t)\, \dot{Z}(t) + (\dot{\sigma}(t)^{\perp})^{\cdot}\ ,\quad t\in(-\lambda,\lambda),$$

where

$$(3.8)\qquad \begin{aligned} d\dot{\sigma}(t)\, Z(t) &= d(d\sigma(t)\, Z(t))\, Z(t) + d\dot{\sigma}(t)^{\perp}\, Z(t) \\ &= d\sigma(t)\, \nabla(\sigma(t))_{Z(t)}\, Z(t) + S(\sigma(t))\, (Z(t),Z(t)) \\ &\quad + d\sigma(t)\, W(\sigma(t),\dot{\sigma}(t)^{\perp})\, Z(t) + (d\dot{\sigma}(t)^{\perp}\, Z(t))^{\perp}. \end{aligned}$$

Here $\nabla(\sigma(t))$ means the Levi – Civita connection of $m(\sigma(t))$, and $W(\sigma(t),\dot{\sigma}(t))$, the Weingarten map of $\sigma(t)$ is defined as follows :
Let $\quad N : M \longrightarrow \mathbb{R}^n \quad$ be any vector field along $j\in E(M,\mathbb{R}^n)$, such that

$$(3.9)\qquad \langle dj\, X(p), N(p)\rangle = 0\ ,\qquad \forall\, X\in \Gamma TM,\ p\in M.$$

Then the Weingarten map $W(j,N)$ of j given by N is the uniquely determined bundle map of TM for which

$$(3.10)\qquad dj\ W(j,N)\ Y = (dN\ Y)^{T}\ .$$

In the particular case where M is oriented and of dimension equal to $n-1$ and N coincides with the unit normal vector field $N(j)$ of $j(M)$ in $\mathbb{R}^n$, then $W(j,N(j))$ is nothing else but the Weingarten map, denoted in this particular case just by $W(j)$.
Let us turn back to $\ddot{\sigma}(t)$. Obviously (compare cf. [Bi,6])

$$(3.11)\qquad \begin{aligned} \ddot{\sigma}(t) &= d\sigma(t)\, \nabla(\sigma(t))_{Z(t)}Z(t) + d\sigma(t)\, W(\sigma(t),\dot{\sigma}(t)^{\perp})\, Z(t) \\ &\quad + ((\dot{\sigma}(t)^{\perp})^{\cdot})^{T} + d\sigma(t)\, \dot{Z}(t) + S(\sigma(t))\, (Z(t),Z(t)) \\ &\quad + (d\dot{\sigma}(t)^{\perp}\, Z(t))^{\perp} + ((\dot{\sigma}(t)^{\perp})^{\cdot})^{\perp}\ . \end{aligned}$$

For unifying terminology we set

$$(3.12)\qquad ((\dot{\sigma}(t)^{\perp})^{\cdot})^{T} := d\sigma(t)\ U(\sigma(t),\dot{\sigma}(t))$$

for each $t\in(-\lambda,\lambda)$ and a well defined vector field $U(\sigma(t),\dot{\sigma}(t))\in\Gamma TM$.
If we split now $\varphi_F(\sigma(t),\dot{\sigma}(t))$ into a tangential and normal part respectively then for all $t\in(-\lambda,\lambda)$

$$(3.13)\qquad \begin{aligned} \varphi_F(\sigma(t),\dot{\sigma}(t)) &= \varphi_F(\sigma(t),\dot{\sigma}(t))^{T} + \varphi_F(\sigma(t),\dot{\sigma}(t))^{\perp} \\ &= d\sigma(t)\ Y_F(\sigma(t),\dot{\sigma}(t)) + \varphi_F(\sigma(t),\dot{\sigma}(t))^{\perp} \end{aligned}$$

holds for a uniquely determined $Y_F(\sigma(t),\dot{\sigma}(t))\in\Gamma TM$.
The equation of motion splits thus into the coupled system

$$(3.14)\qquad \left\{ \begin{aligned} &\nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + W(\sigma(t),\dot{\sigma}(t)^{\perp})\, Z(t) + U(\sigma(t),\dot{\sigma}(t)) \\ &\qquad = \rho^{-1}(\sigma(t))\ Y_F(\sigma(t),\dot{\sigma}(t)), \\ &((\dot{\sigma}(t)^{\perp})^{\cdot})^{\perp} = \rho^{-1}\,(\sigma(t))\ \varphi_F(\sigma(t),\dot{\sigma}(t))^{\perp} \\ &\qquad - (d\dot{\sigma}(t)^{\perp}\ Z(t))^{\perp} - S(\sigma(t))\ (Z(t),Z(t))\ . \end{aligned} \right.$$

The first equation multiplied on both sides by $d\sigma(t)$ yields an equation of vectors which are tangential to $\sigma(t) \circ \mathrm{Diff}\, M$, while the second one is an equation of vectors in $C^\infty(M,\mathbb{R}^n)$, which are normal to $\sigma(t) \circ \mathrm{Diff}\, M$.

Hence the above coupled system (3.14) is a splitting of the equation of motion according to the principal bundle structure of $E(M,\mathbb{R}^n)$ as mentioned above.
In the particular case $\dim M = n-1$ we obtain

$$(3.15) \qquad \dot\sigma(t)^\perp = \epsilon(\sigma(t),\dot\sigma(t))\; N(\sigma(t))$$

$$(3.16) \qquad \varphi_F(\sigma(t),\dot\sigma(t))^\perp = \kappa_F(\sigma(t),\dot\sigma(t))\; N(\sigma(t))$$

for well determined $\epsilon(\sigma(t),\dot\sigma(t)),\ \kappa_F(\sigma(t),\dot\sigma(t)) \in C^\infty(M,\mathbb{R})$.
Hence

$$(3.17) \qquad (\,\dot\sigma(t)^\perp)^{\cdot} = \dot\epsilon(\sigma(t),\dot\sigma(t))\; N(\sigma(t)) + \epsilon(\sigma(t),\dot\sigma(t))\; N(\sigma(t))^{\cdot}.$$

It remains now to calculate $N(\sigma(t))^{\cdot}$. To this end we prove at first that

$$(3.18) \qquad DN(j)(\tau\cdot N(j)) = -\, dj\ \mathrm{grad}_j\tau\,, \qquad \forall\ \tau\in C^\infty(M,\mathbb{R})\,.$$

Indeed let

$$j(t) = j + t\cdot\tau\cdot N(j)\,, \qquad \forall\ t\in\mathbb{R}.$$

Then we get

$$(3.19) \qquad \begin{aligned} \tfrac{d}{dt}\ \langle dj(t),N(j(t))\rangle\Big|_{t=0} &= 0 \\ &= \langle\tau\cdot dj W(j),N(j)\rangle + d\tau + \langle dj,DN(j)(\tau\cdot N(j))\rangle. \end{aligned}$$

Using that $\langle N(j),N(j)\rangle = 1$, (3.18) then follows.
Next consider a smooth curve $\gamma(t)\in \mathrm{Diff}\, M$, $t\in(-\lambda,\lambda)$, $\lambda>0$, and note that

$$(3.20) \qquad N(j\circ\gamma(t)) = N(j)\circ\gamma(t)\,.$$

Differentiating (3.20) we get

$$(3.21) \qquad DN(j)(dj\ X) = dN(j)\ X = dj\ W(j)\ X\,,$$

where $X = \dot\gamma(0)$. From (3.6), (3.15), (3.18) and (3.21) it follows that

$$(3.22) \qquad \begin{aligned} N(\sigma(t))^{\cdot} &= DN(\sigma(t))\; N(d\sigma(t)\ Z(t) + \epsilon(\sigma(t),\dot\sigma(t))\; N(\sigma(t)) \\ &= d\sigma(t)\ \big(W(\sigma(t))\ Z(t) - \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t))\big)\,, \end{aligned}$$

and hence

$$(3.23_1) \qquad U(\sigma(t),\dot\sigma(t)) = \epsilon(\sigma(t),\dot\sigma(t))\; W(\sigma(t))Z(t) - \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t))$$

$$(3.23_2) \qquad ((\dot\sigma(t)^\perp)^{\cdot})^\perp = \dot\epsilon(\sigma(t),\dot\sigma(t))\; N(\sigma(t)).$$

Moreover

$$(3.24)\qquad d(\dot\sigma(t)^{\perp})\, Z(t) = d\epsilon(\sigma(t),\dot\sigma(t))\, Z(t)\, N(\sigma(t)) + \epsilon(\sigma(t),\dot\sigma(t))\, d\sigma(t)\, W(\sigma(t))\, Z(t) - d\sigma(t)\, \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)).$$

Thus (3.14) rewrites as

$$(3.25)\qquad \begin{cases} \nabla(\sigma(t))_{Z(t)}\, Z(t) + \dot Z(t) + 2\,\epsilon(\sigma(t),\dot\sigma(t))\, W(\sigma(t)\, Z(t)) \\ \quad - \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\, Y(\sigma(t),\dot\sigma(t)), \\ \\ \dot\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\, \kappa_F(\sigma(t),\dot\sigma(t)) + \mathfrak{h}(\sigma(t))\, (Z(t),Z(t)) \\ \quad - d\epsilon(\sigma(t),\dot\sigma(t))\, Z(t)\,. \end{cases}$$

We refer to (3.25) as the general equations of motion of a deformable medium.

Let us now split $Y_F(\sigma(t),\dot\sigma(t))$ with respect to $m(\sigma(t))$ according to the Hodge decomposition into

$$(3.26)\qquad Y_F(\sigma(t),\dot\sigma(t)) = Y^o{}_F(\sigma(t),\dot\sigma(t)) + \mathrm{grad}_{\sigma(t)}\, \tau_F(\sigma(t),\dot\sigma(t)),$$

where $\tau(\sigma(t),\dot\sigma(t)) \in C^\infty(M,\mathbb{R}^n)$ and

$$(3.27)\qquad \mathrm{div}_{\sigma(t)}\, Y^o{}_F(\sigma(t),\dot\sigma(t)) = 0\,,$$

i.e. the divergence of $Y^o{}_F(\sigma(t),\dot\sigma(t))$ taken with respect to $m(\sigma(t))$ vanishes, $\tau_F(\sigma(t),\dot\sigma(t)) \in C^\infty(M,\mathbb{R})$, and $\mathrm{grad}_{\sigma(t)}$ means the gradient taken with respect to $m(\sigma(t))$.

Using (3.26) we rewrite the first equation (3.25) as

$$(3.25')\qquad \begin{aligned} &\nabla(\sigma(t))_{Z(t)}\, Z(t) + \dot Z(t) + 2\,\epsilon(\sigma(t),\dot\sigma(t))\, W(\sigma(t)\, Z(t) \\ &- \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t\ ,\dot\sigma(t)) \\ &= \rho^{-1}(\sigma(t))\, [\, \mathrm{grad}_{\sigma(t)}\tau_F(\sigma(t),\dot\sigma(t)) + Y^o{}_F(\sigma(t),\dot\sigma(t))]. \end{aligned}$$

The above Hodge decomposition of $Y_F(\sigma(t),\dot\sigma(t))$ yields a decomposition of φ_F into

$$(3.28)\qquad \varphi_F(\sigma(t),\dot\sigma(t)) = \varphi^0_F(\sigma(t),\dot\sigma(t)) + \varphi'_F(\sigma(t),\dot\sigma(t))\,,$$

where

$$(3.29)\qquad \begin{cases} \varphi^0_F(\sigma(t),\dot\sigma(t)) = d\sigma(t)\, \mathrm{grad}_{\sigma(t)}\tau_F(\sigma(t),\dot\sigma(t)), \\ \\ \varphi'_F(\sigma(t),\dot\sigma(t)) = d\sigma(t)\, Y^o{}_F(\sigma(t),\dot\sigma(t)) + \varphi_F(\sigma(t),\dot\sigma(t))^{\perp}. \end{cases}$$

We note that the tangential part of the force density φ'_F is divergence free. Its corresponding work, i.e. the one form F', will be called the reduced constitutive law.

4. The motion along a fixed surface $i(M) \subset \mathbb{R}^n$

Let us consider again the coupled system (3.14) *describing the motion of the deformable medium* M. Noting that the embedding $\sigma(t)$ varies with t and that the submanifolds $\sigma(t_1)(M)$ and $\sigma(t_2)(M)$ of $\mathbb{R}^n$ differ generically from each other for different $t_1, t_2 \in (-\lambda,\lambda)$, we obtain that the first equation (3.14) describes the instantaneous *motion of the deformable medium along the* submanifold M.

In this section we assume that the submanifolds $\sigma(t)(M)$ of $\mathbb{R}^n$ are identical for all $t \in (-\lambda,\lambda)$.
As a visualising example we image a fluid moving on a sphere of fixed radius .
More generally let $i \in E(M,\mathbb{R}^n)$ be fixed.
Thus $i(M)$ is a submanifold of $\mathbb{R}^n$ on which a deformable medium moves according to a constitutive law to be specified below. A configuration of this motion is an embedding j of M onto i(M) and is hence of the form

$$(4.1) \qquad j = i \circ g,$$

for some $g \in \text{Diff } M$.
Consequently the configuration space is $i \circ \text{Diff } M$. It remains now to specify the constitutive law on $T(i \circ \text{Diff } M)$, the phase space of the motions on i(M). To this end let us first study the nature of a tangent vector h to $i \circ \text{Diff } M$ at $i \circ g$, i.e. $h \in T_{i\circ g}\, i \circ \text{Diff } M$. If we denote by R_g the right translation by g, i.e.

$$(4.2) \qquad \begin{cases} R_g : \text{Diff } M \longrightarrow \text{Diff } M, \\ g' \mapsto g \circ g' , \end{cases}$$

then the tangent map $TR_g(id)$ sends any tangent vector $X \in \Gamma TM$ at Id Diff M into a tangent vector in $T_g\text{Diff } M$. Moreover

$$(4.3) \qquad TR_g : \Gamma TM \longrightarrow T_g \text{Diff } M$$

is obviously surjective. Regarding $j \circ g \circ \text{Diff } M$ as a submanifold of $E(M,\mathbb{R}^n)$, any $h \in T_{j\circ g}\, E(M,\mathbb{R}^n)$ tangential to $j \circ g \circ \text{Diff } M$ is thus of the form

$$(4.4) \qquad h = d(i \circ g)\, X_h ,$$

for a uniquely defined vector field $X_h \in \Gamma TM$.
Thus we have a natural bijection

$$(4.5) \qquad d(i \circ g) : \Gamma TM \longrightarrow T_{i\circ g}\, i \circ \text{Diff } M \subset T_{i\circ g}\, E(M,\mathbb{R}^n) ,$$

sending each $X \in \Gamma TM$ into $d(i \circ g)\, X \in T_{i\circ g}\, i \circ \mathrm{Diff}\, M$.
By (4.5) we see that $T\,(\, i \circ \mathrm{Diff}\, M\,)$ is trivialized via right translations as

$$(4.6) \qquad T(\, i \circ \mathrm{Diff}\, M\,) = i \circ \mathrm{Diff}\, M \;\dot{\times}\; \Gamma TM\,.$$

Clearly

$$(4.7) \qquad d(i \circ g)\, X = di \circ T_g X \quad .$$

We now introduce the constitutive law via the smooth map

$$(4.8) \qquad F : di\, \Gamma TM \times i \circ \mathrm{Diff}\, M \times C^{\infty}(M,\mathbb{R}^n) \longrightarrow \mathbb{R}\,,$$

which is linear in $h \in C^{\infty}(M,\mathbb{R}^n)$. We note that in analogy to (3.1) we put the parameter space $di\, \Gamma TM$ as the first factor of the domain of definition. The justification for choosing the third factor as being $C^{\infty}(M,\mathbb{R}^n)$ instead or ΓTM will be given later.
As in the preceding section we require an integral representation for the constitutive law reading as

$$(4.9) \qquad F(d(i \circ g)X)(i\circ g,h) = \int \langle \varphi_F(i \circ g, d(i \circ g)X, h\rangle\, \mu(i \circ g)$$
$$\forall\, h \in C^{\infty}(M,\mathbb{R}^n),\ X \in \Gamma TM,\ g \in \mathrm{Diff}\, M,$$

where the force density

$$(4.10) \qquad \varphi_F : i \circ \mathrm{Diff}\, M \;\times\; \Gamma TM \longrightarrow C^{\infty}(M,\mathbb{R}^n)$$

is a smooth map.
The equation of motion on $i \circ \mathrm{Diff}\, M$ described by a smooth curve

$$(4.11) \qquad \sigma : (-\lambda,\lambda) \longrightarrow i \circ \mathrm{Diff}\, M\,, \qquad \lambda > 0,$$

subjected to the above constitutive law is hence

$$(4.12) \qquad \rho(\sigma(t))\ddot{\sigma}(t) = \varphi_F(\sigma(t),\dot{\sigma}(t)),$$

or, equivalently,

$$(4.13) \qquad \begin{cases} \rho(\sigma(t))(\ddot{\sigma}(t))^{T} = (\varphi_F(\sigma(t),\dot{\sigma}(t)))^{T}\,, & \\ \rho(\sigma(t))(\ddot{\sigma}(t))^{\perp} = (\varphi_F(\sigma(t),\dot{\sigma}(t)))^{\perp}\,, & t \in (-\lambda,\lambda). \end{cases}$$

We obviously have

$$(4.14) \qquad (\varphi_F(\sigma(t),\dot{\sigma}(t)))^{T} = d\sigma(t)\; Y_F(\sigma(t),\dot{\sigma}(t))\,,$$

for a well defined vector field $Y_F(\sigma(t),\dot{\sigma}(t)) \in \Gamma TM$. Since a solution $\sigma(t)$ of (4.13) has the form

(4.15) $\sigma(t) = i \circ g(t)$,

for a smooth map

$$g : (-\lambda,\lambda) \longrightarrow \mathrm{Diff}\, M ,$$

it follows that

(4.16) $$\begin{aligned}\dot\sigma(t) &= d(\, i \circ g(t))\, Z(t)\\ &= di \circ T_g(t)\, Z(t) ,\end{aligned}$$

where $Z(t) \in \Gamma TM$ is well determined, and consequently

(4.17) $$\begin{aligned}\ddot\sigma(t) &= (d(\, i \circ g(t))\, Z(t))^{\cdot}\\ &= d(\, i \circ \dot g(t))\, Z(t) + d(\, i \circ g(t))\, \dot Z(t)\\ &= d(\, i \circ g(t))\, \nabla(\, i \circ g(t))_{Z(t)} Z(t) +\\ &\quad d(\, i \circ g(t))\, \dot Z(t) + S(\, i \circ g(t))\, (Z(t),Z(t)) .\end{aligned}$$

Using (4.17), from (4.13) we obtain, $\forall\, t \in (-\lambda,\lambda)$,

(4.18) $$\begin{cases} \rho(\, i \circ g(t))(\nabla(\, i \circ g(t))_{Z(t)} Z(t) + \dot Z(t)) = Y_F(g(t),Z(t)), \\ \rho(\, i \circ g(t))\, S(\, i \circ g(t))\, (Z(t),Z(t)) = \varphi_F(g(t),Z(t))^{\perp} , \end{cases}$$

where we have used the notation

(4.19) $$f\,(g(t),Z(t)) := f\,(\, i \circ g(t),\, d(\, i \circ g(t))\, Z(t)) .$$

By comparing (4.18) with (3.14) we observe that the last equations are obtained from the first ones by setting $\dot\sigma(t)^{\perp} = 0$, in accordance with the fact that $i(M)$ is a fixed surface.
We note that we can remove the instantaneous connection $\nabla(i \circ g(t))$ and the instantaneous second fundamental tensor $S(i \circ g(t))$ in (4.18) by using the push-forward of $Z(t)$ by $g(t) \in \mathrm{Diff}\, M$, that is we introduce $X(t)$ by

(4.20) $$X(t) := Tg(t)Z(t)g(t)^{-1} , \qquad t \in (-\lambda,\lambda) .$$

Using (4.20) we obtain on one hand

(4.21) $$\begin{cases} \dot\sigma(t) = di\, X(t)g(t) = d(\, i \circ g(t))\, Z(t) , \\ \ddot\sigma(t) = di\, \dot X(t)g(t) + d(\, di\, X(t))\, \dot g(t) . \end{cases}$$

On the other hand the equation

(4.22) $$\dot\sigma(t) = di\, \dot g(t) = di\, X(t)g(t)$$

yields

(4.23) $$\dot g(t) = X(t)g(t) .$$

(4.21) and (4.23) imply

$$\ddot{\sigma}(t)g(t)^{-1} = \mathrm{di}\ \dot{X}(t) + \mathrm{di}\ \nabla(i)_{X(t)}X(t) + S(i)(X(t),X(t))\ . \tag{4.24}$$

Setting now

$$\begin{cases} Y_F(g(t),X(t)) := Y_F(g(t),Z(t))\ g(t)\ , \\ \varphi_F(g(t),X(t))^{\perp} := (\varphi_F(g(t),Z(t)))^{\perp}\ g(t)\ , \qquad t\in(-\lambda,\lambda) \end{cases} \tag{4.25}$$

and observing that the map

$$\rho : E(M,\mathbb{R}^n) \longrightarrow C^{\infty}(M,\mathbb{R})$$

does depend by construction on dj rather than on j itself the system (4.18) turns into

$$\begin{cases} \rho(d\sigma(t))(\nabla(i)_{X(t)}X(t) + \dot{X}(t)) = Y_F(g(t),X(t))\ , \\ \rho(d\sigma(t))S(i)(X(t),X(t)) = \varphi_F(g(t),X(t))^{\perp}\ . \end{cases} \tag{4.26}$$

Let us note that in the case when we would require that $F(d(i \circ g))$ would act on ΓTM rather than on $C^{\infty}(M,\mathbb{R}^n)$, we would obtain only the equations (4.13), by missing the observation that the normal forces are up to the density ρ of a geometric nature.

The next step is to decompose $Y_F(g(t),X(t))$ with respect to $m(i \circ g(t))$ according to Hodge uniquely into

$$\begin{cases} Y_F(g(t),X(t)) = \mathrm{grad}_i\tau_F(g(t),X(t)) + Y^0{}_F(g(t),X(t))\ , \\ \mathrm{div}_i\ Y^0{}_F(g(t),X(t)) = 0, \end{cases} \tag{4.27}$$

where $\tau_F(g(t),X(t)) \in C^{\infty}(M,\mathbb{R})$.
Thus the first equation (4.26) becomes

$$\begin{aligned} \rho(X(t))(\nabla(i)_{X(t)}X(t) &+ \dot{X}(t)) \\ &= \mathrm{grad}_i\tau_F(g(t),X(t)) + Y^0(g(t),X(t))\ . \end{aligned} \tag{4.26'}$$

If we require

$$\sigma((-\lambda,\lambda)) \subset i \circ \mathrm{Diff}_{\mu(i)}M\ , \tag{4.28}$$

where $\mathrm{Diff}_{\mu(i)}M$ is the subgroup of all elements in Diff M which leave $\mu(i)$ invariant, then $X(t)$ has to be divergence free for all $t\in(-\lambda,\lambda)$. This is due to the fact that

$$(4.29) \qquad T_{id}\, \mathrm{Diff}\, M = \{\, X \in \Gamma TM \mid \mathrm{div}_i\, X = 0 \,\} .$$

In this case $\sigma(t)$ has to satisfy the system of equations

$$(4.30) \qquad \begin{cases} \rho\, (\nabla(i)_{X(t)} X(t) + \dot{X}(t)) \\ \quad = \mathrm{grad}_i\, \tau_F(g(t), X(t)) + Y^0{}_F(g(t), X(t)), \\ \\ \rho\, S(i)\, (X(t), X(t)) = \varphi_F(\sigma(t), X(t))^{\perp} , \\ \mathrm{div}_i\, X(t) = 0 , \end{cases}$$

with $\rho : i \circ \mathrm{Diff}\, M \longrightarrow \mathbb{R}$ being a constant function.

5. The $\mathbb{R}^n$ - invariance of internal constitutive laws

Let us assume now that the motion of the deformable medium M is subjected to an internal constitution law F, which admits an integral representation. The fact that the corresponding force density is an internal one requires it to be independent of the region in $\mathbb{R}^n$ in which the deformable medium moves. Hence an internal force density has to be invariant under the translation or, more precisely, under the action of the translation group $\mathbb{R}^n$ of $\mathbb{R}^n$ (cf.[Bi,4]).

Let us describe next this action of $\mathbb{R}^n$ on $TE(M,\mathbb{R}^n)$.

At first we recall that the translation group $\mathbb{R}^n$ of the vector space $\mathbb{R}^n$ is the underlying abelian group of the $\mathbb{R}$-vector space $\mathbb{R}^n$.

The action

(5.1) $$r : C^\infty(M,\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow C^\infty(M,\mathbb{R}^n)$$

on $C^\infty(M,\mathbb{R}^n)$ is given by

(5.2) $$r\,(h,z) = h + z\;, \qquad \forall\, h\in C^\infty(M,\mathbb{R}^n),\ z\in \mathbb{R}^n\;,$$

where by $h + z$ we mean the map defined via

(5.3) $$(h + z)(p) = h(p) + z\;, \quad \forall\, p\in M\;.$$

Hence $z\in \mathbb{R}^n$ is naturally identified with the constant map in $C^\infty(M,\mathbb{R}^n)$ assuming z as its value.

Clearly in the particular case where $h = j \in E(M,\mathbb{R}^n)$

(5.4) $$r\,(j,z) = j + z\;, \qquad \forall\, z\in E(M,\mathbb{R}^n),$$

belongs to $E(M,\mathbb{R}^n)$. Hence r reduces to

(5.5) $$r : E(M,\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow E(M,\mathbb{R}^n).$$

The tangent map Tr of r is given by

(5.6) $$Tr\,(h,z)(k,u) = (\,h+z,\ k+u)\;,$$
$$\forall\, k\in T_hC^\infty(M,\mathbb{R}^n),\ h\in C^\infty(M,\mathbb{R}^n),\ z,u\in \mathbb{R}^n.$$

Hence r induces an action on $TC^\infty(M,\mathbb{R}^n)$ defined by

(5.7) $$Tr : TC^\infty(M,\mathbb{R}^n) \times T\mathbb{R}^n \longrightarrow TC^\infty(M,\mathbb{R}^n),$$
$$((h,k),(z,u)) \mapsto (\,h+z,\ k+u)\;,$$

where $k\in T_hC^\infty(M,\mathbb{R}^n) = C^\infty(M,\mathbb{R}^n)$ and $u\in T_z\mathbb{R}^n = \mathbb{R}^n$, and respectively on $TE(M,\mathbb{R}^n)$ given by

(5.8) $$Tr : TE(M,\mathbb{R}^n) \times T\mathbb{R}^n \longrightarrow TE(M,\mathbb{R}^n)\;,$$
$$((j,k),(z,u)) \mapsto (\,j+z,\ k+u)\;,$$

where $k \in T_j E(M,\mathbb{R}^n) = C^\infty(M,\mathbb{R}^n)$ and $u \in T_z \mathbb{R}^n = \mathbb{R}^n$.

Given now a parameter depending smooth constitutive law

$$(5.9) \qquad F : C^\infty(M,\mathbb{R}^n) \longrightarrow A^1(E(M,\mathbb{R}^n),\mathbb{R})$$

and continuing to write

$$(5.10) \qquad F(j,k) \text{ instead of } F(k,j)\ ,\quad j \in E(M,\mathbb{R}^n)\ ,\ k \in C^\infty(M,\mathbb{R}^n)$$

we form next

$$(5.11) \qquad F \circ Tr((j,k),(z,u)) : C^\infty(M,\mathbb{R}^n) = T_{j+z}E(M,\mathbb{R}^n) \longrightarrow \mathbb{R}\ .$$

The requirement

$$(5.12) \qquad F\ (j+z,\ k+u) = F\ (j,k) \qquad \forall\ j \in E(M,\mathbb{R}^n),\ k \in C^\infty(M,\mathbb{R}^n),\ z,u \in \mathbb{R}^n,$$

does then yield the type of constitutive law we want to work with.
In order to construct the desired type of $\mathbb{R}^n$-invariant constitutive laws, we consider the quotients of the actions r and Tr. To this end we note that the map

$$(5.13) \qquad d : C^\infty(M,\mathbb{R}^n) \longrightarrow \{\ dh\ \mid\ h \in C^\infty(M,\mathbb{R}^n)\}$$

has the property that

$$(5.14) \qquad d^{-1}(dh) = \{\ h+z\ \mid\ z \in \mathbb{R}\ \}.$$

Hence if we quotient out the action of $\mathbb{R}^n$ on $C^\infty(M,\mathbb{R}^n)$ we obtain a bijection again called d

$$(5.15) \qquad d : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow \{\ dh\ \mid\ h \in C^\infty(M,\mathbb{R}^n)\}.$$

We equip $\{\ dh\ \mid\ h \in C^\infty(M,\mathbb{R}^n)\ \}$ with the uniquely determined topology making d to a homeomorphism to $C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}$ carrying the quotient topology. Note that both topological spaces are Frechet manifolds.
Next we identify them via d. Hence we have identified also the two Fréchet manifolds $E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}$ and $\{\ dj\ \mid\ j \in E(M,\mathbb{R}^n)\ \}$ yielding

$$(5.16) \qquad TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} = \{\ dj\ \mid\ j \in E(M,\mathbb{R}^n)\ \} \times \{\ dh\ \mid\ h \in C^\infty(M,\mathbb{R}^n)\ \}.$$

Therefore we obtain the following

Lemma 5.1 :
Given a smooth map

$$(5.17) \qquad F_{\mathbb{R}^n} : C^{\infty}(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^{\infty}(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow \mathbb{R} ,$$

linear in the third argument, then the resulting (parameter depending) one form F given by

$$(5.18) \qquad F = F_{\mathbb{R}^n} \circ (d,Td)$$

is a (parameter depending) $\mathbb{R}^n$-invariant one form on $E(M,\mathbb{R}^n)$.
Here Td is the tangent map of

$$(5.19) \qquad d : E(M,\mathbb{R}^n) \longrightarrow E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} .$$

Remark 5.1 :
In the following we write

$$(5.20) \qquad d^* F_{\mathbb{R}^n} \quad \text{instead of} \quad F_{\mathbb{R}^n} \circ (d,Td) .$$

Remark 5.2 :
The above lemma allows us to study the constitutive laws of the type $d^* F_{\mathbb{R}^n}$ rather than constitutive laws invariant under Tr.

6 . On the characterization of $\mathbb{R}^n$-valued one-forms relative to embeddings

Let throughout this section $i \in E(M,\mathbb{R}^n)$ be a fixed smooth embedding and $\in A^1(M,\mathbb{R}^n)$ be a fixed smooth $\mathbb{R}^n$-valued one-form. We follow [Bi,2].
As the first observation we formulate the following

Proposition 6.1 :
Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given.
Then the following decomposition holds

$$(6.1) \qquad \alpha = dh + \beta ,$$

where $h \in C^\infty(M,\mathbb{R}^n)$, the so-called integrable part of α, is uniquely determined up to a constant. Moreover this decomposition is maximal in the sense that the integrable part of β is a constant.

Proof :
Let $e_1,\dots,e_n$ be an orthonormal basis of $\mathbb{R}^n$. Then we get

$$(6.2) \qquad \alpha(X) = \sum_{s=1}^{n} \alpha^s(X)\, e_s , \qquad \forall\, X \in \Gamma TM ,$$

for an uniquely determined family $\alpha^1,\dots,\alpha^n$ of smooth $\mathbb{R}$- valued one-forms on M, i.e. $\alpha^s \in A^1(M,\mathbb{R})$, $s= 1,\dots,n$. Clearly

$$(6.3) \qquad \alpha^s(X) = \langle \alpha(X), e_s \rangle , \qquad \forall\, X \in \Gamma TM,\ s= 1,\dots,n.$$

In addition α^s , $s= 1,\dots,n$, can be represented as

$$(6.4) \qquad \alpha^s(X) = m(i)\,(Y_s, X) , \qquad \forall\, X \in \Gamma TM,$$

for a well defined $Y_s \in \Gamma TM$. This vector field splits according to Hodge's decomposition uniquely into

$$(6.5) \qquad \begin{cases} Y_s = \operatorname{grad}_i \tau_s + Y_s^0 , \\ \operatorname{div}_i Y_s^0 = 0 , \end{cases}$$

where $\tau_s \in C^\infty(M,\mathbb{R}^n)$ and $Y_s^0 \in \Gamma TM$.
Hence

$$(6.6) \qquad \alpha^s(X) = d\tau_s(X) + m(i)(Y_s^0,X) , \qquad \forall\, X \in \Gamma TM .$$

Next we define the integrable part h of α by

$$(6.7) \qquad h := \sum_{s=1}^{n} \tau_s\, e_s ,$$

and the non-integrable part β by

$$(6.8) \qquad \beta(X) := \sum_{s=1} m(i)\, (Y_s^0, X)\, e_s\,, \qquad \forall\, X \in \Gamma TM\,.$$

Inserting (6.6) into (6.2) and using (6.7) and (6.8) yields the decomposition (6.1). It remains only to show that (6.1) does not depend on the choice of the basis of $\mathbb{R}^n$. To this end let $\bar{e}_1,\ldots,\bar{e}_n \in \mathbb{R}^n$ be another orthonormal basis of $\mathbb{R}^n$ and define $\bar{\alpha}$, $\bar{\tau}$, $\bar{Y}^0$, $\bar{h}$ and $\bar{\beta}$ accordingly.
Then

$$
\begin{aligned}
(6.9) \qquad \bar{\alpha}(X) &= \langle \alpha(X), \bar{e} \rangle \\
&= \langle dh(X), \bar{e} \rangle + \langle \beta(X), \bar{e} \rangle \\
&= \langle \sum_{s=1}^{n} d\tau_s(X)\, e_s, \bar{e} \rangle + \langle \sum_{s=1}^{n} m(i)\, (Y_s^0, X)\, e_s, \bar{e} \rangle \\
&= m(i)\, \Big(\sum_{s=1}^{n} \operatorname{grad}_i \tau_s \langle e_s, \bar{e} \rangle\,,\, X \Big) \\
&\quad + m(i)\, \Big(\sum_{s=1}^{n} Y_s^0 \langle e_s, \bar{e} \rangle\,,\, X \Big) \\
&= \langle d\bar{h}(X)\,,\, \bar{e} \rangle + \langle \bar{\beta}(X)\,,\, \bar{e} \rangle \\
&= m(i)\, (\operatorname{grad}_i \bar{\tau}\,, X) + m(i)\, (\bar{Y}^0, X).
\end{aligned}
$$

Since on one hand

$$(6.10) \qquad \sum_{s=1}^{n} \operatorname{grad}_i \tau_s \langle e_s, \bar{e} \rangle = \operatorname{grad}_i \Big(\sum_{s=1}^{n} \tau_s \langle e_s, \bar{e} \rangle \Big)\,,$$

on the other hand

$$(6.11) \qquad \operatorname{div}_i \Big(\sum_{s=1}^{n} Y_s^0 \langle e_s, \bar{e} \rangle \Big) = \sum_{s=1}^{n} (\operatorname{div}_i Y_s^0)\, \langle e_s, \bar{e} \rangle = 0\,,$$

we conclude due to the uniqueness of Hodge's decomposition the following relations

$$(6.12) \qquad \left\{ \begin{aligned} &\sum_{s=1}^{n} \operatorname{grad}_i \big(\tau_s \langle e_s, \bar{e} \rangle \big) = \operatorname{grad}_i \bar{\tau}\,, \\ &\sum_{s=1}^{n} Y_s^0 \langle e_s, \bar{e} \rangle = \bar{Y}^0\,. \end{aligned} \right.$$

Consequently the uniqueness of the decomposition (6.1) follows, namely

$$(6.13) \qquad \left\{ \begin{aligned} dh(X) &= \sum_{=1}^{n} \langle dh(X), \bar{e} \rangle\, \bar{e} = \sum_{=1}^{n} \langle d\bar{h}(X), \bar{e} \rangle\, \bar{e} = d\bar{h}(X)\,, \\ \beta(X) &= \sum_{=1}^{n} \langle \beta(X), \bar{e} \rangle\, \bar{e} = \sum_{=1}^{n} \langle \beta(X), \bar{e} \rangle\, \bar{e} = \bar{\beta}(X)\,, \\ &\qquad\qquad \forall\, X \in \Gamma TM\,. \end{aligned} \right.$$

Let us detail now the decomposition (6.1).
For this purpose we note first that h can be given the form

$$(6.14) \qquad h = di\, X_h + h^{\perp} ,$$

where $X_h \in \Gamma TM$ is well defined and $h^{\perp}$ denotes the pointwise formed component of h normal to i(M).
Using the fact that X_h splits into

$$(6.15) \qquad \begin{cases} X_h = X_h^0 + \operatorname{grad}_i \psi_h , \\ \operatorname{div}_i X_h^0 = 0 , \end{cases}$$

where $\psi_h \in C^{\infty}(M,\mathbb{R})$, $X_h^0 \in \Gamma TM$, we deduce from (6.14) that

$$(6.16) \qquad \begin{aligned} dh(X) = {} & di\, \nabla(i)_X X_h^0 + di\, (\nabla(i)_X \operatorname{grad}_i \psi_h + W_h(i)\, X) \\ & + S(i)\, (X_h^0, X) + (dh^{\perp}(X))^{\perp} . \end{aligned}$$

For the sake of readability we remind that $W_h(i)$ defined via

$$(6.17) \qquad di\, W_h(i)\, X := (dh^{\perp}(X))^T ,$$

is a smooth strong bundle endomorphism of TM, which is selfadjoint with respect to m(i).
Let us show *next that the divergence-free part* X_h^0 *of* X_h *is uniquely* determined by h. To this end we use the fact that according to the above proposition the integrable part h of α is uniquely determined up to a constant, i.e.

$$(6.18) \qquad h' = h + z ,$$

for some $z \in \mathbb{R}^n$. Regarding z as a constant map in $C^{\infty}(M,\mathbb{R}^n)$ we write it in the form

$$(6.19) \qquad z = di\, X_z + z^{\perp} .$$

But the vector field Z on $\mathbb{R}^n$ assigning to any $z \in \mathbb{R}^n$ the vector $Z(z) = Z \in \mathbb{R}^n$ is the gradient of some map $\varphi \in C^{\infty}(M,\mathbb{R}^n)$ and hence

$$(6.20) \qquad X_z = \operatorname{grad}_i (\varphi \circ i) .$$

Therefore

$$(6.21) \qquad \begin{aligned} h' & = di\, X_h^0 + di\, \operatorname{grad}_i (\psi_h + \varphi \circ i) + h^{\perp} + z^{\perp} \\ & = di\, X_{h'}^0 + di\, \operatorname{grad}_i \psi_{h'} + h'^{\perp} \end{aligned}$$

or, equivalently

$$\text{(6.22)} \qquad di\,(X^0_{h'} - X^0_h) + di\,\mathrm{grad}_i(\,\psi_{h'} - \psi_h - \varphi \circ i\,) = h^\perp + z^\perp - h'^\perp = 0\,.$$

But (6.22) implies

$$\text{(6.23)} \qquad X^0_{h'} - X^0_h + \mathrm{grad}_i\,(\,\psi_{h'} - \psi_h - \varphi \circ i\,) = 0\,.$$

Using once more the uniqueness of Hodge's decomposition we conclude then (cf.[Bi,2])

Proposition 6.2 :

Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given, and denote by $h \in C^\infty(M,\mathbb{R}^n)$ the integral part of α, which is uniquely detemined up to a constant

$$\alpha = dh + \beta\,.$$

Splitting h into

$$h = di\,X_h + h^\perp\,,$$

where $X_h \in \Gamma TM$ is well defined, then the divergence-free part X^0_h of X_h is uniquely determined.

Next we characterize $\alpha \in A^1(M,\mathbb{R}^n)$ relative to $i \in E(M,\mathbb{R}^n)$ *from* a quite different point of view. To this end let us introduce the following two tensor T_α on M

$$\text{(6.24)} \qquad T_\alpha(X,Y) := \langle\, \alpha(X),\, di\,Y\,\rangle\,, \qquad \forall\, X,Y \in \Gamma TM\,.$$

Clearly T_α is smooth. Next we denote by

$$\text{(6.25)} \qquad P : TM \longrightarrow TM$$

the unique smooth strong bundle endomorphism for which

$$\text{(6.26)} \qquad T_\alpha(X,Y) = m(i)\,(PX,Y)\,,$$

and by $\tilde{P}$ the fibre-wise formed adjoint of P with respect to m(i). The symmetric and the antisymmetric part of T_α, the tensors T^s_α and T^a_α respectively have the form

$$\text{(6.27)} \qquad T^s_\alpha\,(X,Y) = m(i)\,(\tfrac{1}{2}(P + \tilde{P})\,X,Y)\,,$$

$$\text{(6.28)} \qquad T^a_\alpha\,(X,Y) = m(i)\,(\tfrac{1}{2}(P - \tilde{P})\,X,Y)\,.$$

Setting now

$$\text{(6.29)} \qquad \begin{cases} B_\alpha := \frac{1}{2}\,(\,P + \tilde{P}\,)P, \\[1ex] C_\alpha := \frac{1}{2}\,(\,P - \tilde{P}\,)P, \end{cases}$$

we obtain that

$$(6.30) \qquad \alpha(X) = \alpha'(X) + di\, C_\alpha X + di\, B_\alpha X , \qquad \forall X \in \Gamma TM.$$

Clearly $\alpha'(X)(p)$ is a vector in the normal space of TiT_pM , $\forall\, p\in M$.
Hence there is a unique smooth map

$$(6.31) \qquad c_\alpha \in C^\infty(M,so(n)) ,$$

where $so(n)$ denotes the Lie algebra of the group of all proper rotations of $SO(n)$, such that

$$(6.32) \qquad \alpha'(X) = c_\alpha diX ,$$

with

$$(6.33) \qquad c_\alpha(N(j)) \perp Ker\, c_\alpha , \qquad \forall X \in \Gamma TM.$$

We may now state the following

<u>**Proposition 6.3 :**</u>

Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given. Then there exist two uniquely determined smooth, strong bundle endomorphisms

$$C_\alpha : TM \longrightarrow TM$$

and

$$B_\alpha : TM \longrightarrow TM ,$$

which are skew- and respectively selfadjoint with respect to $m(i)$, and a uniquely determined map $c_\alpha \in C^\infty(M,so(n))$, such that the following relation holds

$$(6.34) \qquad \alpha(X) = c_\alpha\, di\, X + di\, C_\alpha X + di\, B_\alpha X , \qquad \forall X \in \Gamma TM.$$

<u>Remark 6.1 :</u>

Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$, then

$$(6.35) \qquad \delta T^a_\alpha = 0 \qquad \text{iff} \quad \delta\alpha = 0 .$$

Indeed, let us consider the one form $\langle i,\alpha\rangle \in A^1(M,\mathbb{R}^n)$, which assigns to any $X \in \Gamma TM$ the real function $\langle i,\alpha(X)\rangle$.
Since

$$(6.36) \qquad \delta\langle i,\alpha\rangle = T^a_\alpha \qquad \text{iff} \quad \delta\alpha = 0,$$

(6.35) then follows immediately.

Next we link the two characterizations of $\mathbb{R}^n$-valued one-forms relative to embeddings, as expressed by the two propositions above. To this end let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given.
Using (6.1) and (6.14) $\alpha(X)$ turns into

$$
\begin{aligned}
\alpha(X) &= di\, \nabla(i)_X X_h + di\, W_h(i)\, X + S(i)(X_h,X) \\
&\quad + (dh^{\perp}(X))^{\perp} + \beta(X)\,.
\end{aligned} \tag{6.37}
$$

Inserting (6.37) in (6.24) we get

$$
\begin{aligned}
T_\alpha(X,Y) &= \langle\, \alpha(X), di\, Y \,\rangle \\
&= \langle\, di\, \nabla(i)_X X_h + W_h(i)\, X,\ di\, Y \,\rangle + T_\beta(X,Y) \\
&= m(i)\ (\nabla(i)_X X_h, Y) + m(i)\ (W_h(i)\ X,Y) + T_\beta(X,Y)\ , \\
&\qquad\qquad \forall\ X,Y \in \Gamma TM.
\end{aligned} \tag{6.38}
$$

Therefore

$$
\begin{aligned}
T^s_\alpha(X,Y) &= \tfrac{1}{2}\, [\ m(i)\ (\nabla(i)_X X_h, Y) + m(i)\ (\nabla(i)_Y X_h, X)\ ] \\
&\quad + m(i)\ (W_h(i)\ X,Y) + T^s_\beta(X,Y) \\
&= \tfrac{1}{2}\, L_{X_h}\ (m(i))\ (X,Y) + m(i)\ (W_h(i)\ X,Y) + T^s_\beta(X,Y)\ ,
\end{aligned} \tag{6.39}
$$

$$
T^a_\alpha(X,Y) = \tfrac{1}{2}\, [\ m(i)\ (\nabla(i)_X X_h, Y) - m(i)\ (\nabla(i)_Y X_h, X)] + T^a_\beta(X,Y), \tag{6.40}
$$

Rewriting the Lie derivative $L_{Z_h}(m(i))$ of $m(i)$ in the direction of Z_h with the help of the Theorem of Fischer and Riesz as

$$
L_{Z_h}\ (m(i)\ (X,Y) = m(i)\ (\mathbf{L}_{Z_h}\ X,Y) \qquad \forall\ Z_h \in \Gamma TM\ , \tag{6.41}
$$

by a uniquely determined strong smooth bundle endomorphism

$$
\mathbf{L}_{Z_h}\ : TM \longrightarrow TM\ , \tag{6.42}
$$

from (6.34) and (6.37) we infer the following formulas for c_α, C_α and B_α :

$$
c_\alpha\ di\ X = (dh^{\perp}(X))^{\perp} + S(i)\ (X_h,X) + c_\beta\ di\ X\ , \tag{6.43}
$$

$$
C_\alpha\ X = \tfrac{1}{2}\, [\ \nabla(i)\ X_h - \tilde{\nabla}(i)\ X_h\ ]\ X + C_\beta\ X\ , \tag{6.44}
$$

$$
\begin{aligned}
B_\alpha\ X &= \tfrac{1}{2}\, [\ \nabla(i)\ X_h + \tilde{\nabla}(i)\ X_h\ ]\ X + W_h(i)\ X + B_\beta\ X \\
&= (\ \tfrac{1}{2}\, \mathbf{L}_{X_h} + W_h(i) + B_\beta\ )\ X\ .
\end{aligned} \tag{6.45}
$$

Instead of $\nabla(i)_v X_h$ we often write $\nabla(i)X_h(v)$ for any $v \in T_pM$. Similarily we use $\tilde{\nabla}(i)X_h(v)$ instead of $\tilde{\nabla}(i)_v X_h$.
Using next the Hodge decomposition of X_h, i.e. (6.15) and taking into account that

$$
m(i)\ ((\nabla(i)\ \mathrm{grad}_i\ \psi_h - \tilde{\nabla}(i)\ \mathrm{grad}_i\ \psi_h)\ X,Y) = 0\ , \tag{6.46}
$$

we obtain finally the following

Proposition 6.4 :

Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given. Then *the following relations hold*

$$\alpha = dh + \beta ,$$
$$\alpha(X) = c_\alpha \, di \, X + di \, C_\alpha X + di \, B_\alpha X , \qquad \forall \, X \in \Gamma TM,$$

where the integrable part $h \in C^\infty(M,\mathbb{R}^n)$ is uniquely determined up to a constant , $c_\alpha \in C^\infty(M,so(n))$ is a uniquely determined, $C_\alpha : TM \longrightarrow TM$ is a uniquely determined smooth, strong and skew-adjoint bundle endomorphism, $B_\alpha : TM \rightarrow TM$ is a uniquely determined smooth, strong and selfadjoint bundle endomorphism.

Writing

$$h = di \, X_h + h^\perp ,$$

where $X_h \in \Gamma TM$, and using Hodge's decomposition

$$(6.47) \qquad \begin{cases} X_h = X_h^0 + \mathrm{grad}_i \, \psi_h , \\ \mathrm{div}_i X_h^0 = 0 , \end{cases}$$

we obtain finally

$$(6.48) \qquad c_\alpha \, di = (dh^\perp)^\perp + S(i)\,(X_h, \,.\,) + c_\beta \, di ,$$

$$(6.49) \qquad C_\alpha = \tfrac{1}{2} [\, \nabla(i) \, X_h^0 - \tilde{\nabla}(i) \, X_h^0 \,] + C_\beta ,$$

$$(6.50) \qquad B_\alpha = \tfrac{1}{2} \, L_{X_h^0 + \mathrm{grad}_i \psi_h} + W_h(i) + B_\beta .$$

Hence

$$(6.51) \qquad \begin{aligned} \mathrm{tr}\, B_\alpha &= \mathrm{div}_i \, X_h + \mathrm{tr}\, W_h(i) + \mathrm{tr}\, B_\beta \\ &= - \Delta(i) \, \psi_h + \mathrm{tr}\, W_h(i) + \mathrm{tr}\, B_\beta , \end{aligned}$$

where $\Delta(i)$ is the Laplace-Beltrami operator of $m(i)$.
Let us calculate now the covariant divergence of B_α and C_α . To this end we *recall at first the* covariant *divergence* $\mathrm{div}_i \, A$ *of a smooth strong bundle* endomorphism

$$(6.52) \qquad A : TM \longrightarrow TM .$$

Let $e_1,\ldots,e_m$ be a moving orthonormal frame of TM, and set

$$(6.53) \qquad \mathrm{div}_i \, A = \sum_{r=1}^{n} \nabla(i)_{e_r} (A) \, e_r .$$

At first we compute $\mathrm{div}_i \, \nabla(i) \, X_h$. Using the equation

(6.54) $$m(i)\,(\nabla(i)_{e_r}(\nabla(i)\,X_h)\,e_r,Y)$$
$$= m(i)\,(\nabla(i)_{e_r}(\nabla(i)_{e_r} X_h) - \nabla(i)_{\nabla(i)_{e_r}e_r} X_h,Y),$$
$$\forall\ Y \in \Gamma TM,$$

we get

(6.55) $$\mathrm{div}_i(\nabla(i)\,X_h) = \Delta(i)\,X_h,$$

where $\Delta(i)X_h$ is the Laplace–Beltrami operator of m(i) applied to X_h which by definition is $-\,\mathrm{tr}\,\nabla(i)^2\,X_h$.
In order to compute $\mathrm{div}_i\tilde{\nabla}(i)\,X_h$ we consider the equations

(6.56) $$m(i)\,(\nabla(i)_{e_r}(\tilde{\nabla}(i)\,X_h)(e_r),Y)$$
$$= e_r\,(m(i)\,(\tilde{\nabla}(i)\,X_h(e_r),Y) - m(i)\,(\tilde{\nabla}(i)\,X_h\,(\nabla(i)_{e_r}e_r),Y)$$
$$- m(i)\,(\tilde{\nabla}(i)\,X_h(e_r),\nabla(i)_{e_r}Y)$$
$$= m(i)\,(e_r,\nabla(i)_{e_r}\nabla(i)_Y X_h) - m(i)\,(e_r,\nabla(i)_{\nabla(i)_{e_r}Y}\,X_h)$$
$$= m(i)\,(e_r,\nabla(i)_{e_r}\,(\nabla(i)\,X_h)\,Y)\ ,$$

(6.57) $$m(i)\,(\nabla(i)_Y(\tilde{\nabla}(i)\,X_h)(e_r),e_r)$$
$$= m(i)\,(e_r,\nabla(i)_Y\nabla(i)_{e_r}X_h) - m(i)\,(\ e_r,\nabla(i)_{\nabla(i)_Y e_r}X_h)$$
$$= m(i)\,(e_r,\ \nabla(i)_Y(\nabla(i)\,X_h)(e_r))$$

and find

(6.58) $$\sum_{r=1}^{n}[\ m(i)\,(\nabla(i)_{e_r}(\tilde{\nabla}(i)X_h)(e_r),Y) - m(i)\,(\nabla(i)_Y(\tilde{\nabla}(i)X_h)(e_r),e_r)]$$
$$= \mathrm{Ric}\,(m(i))(Y,X_h)\ ,$$

where $\mathrm{Ric}(m(i))$ denotes the Ricci tensor of $m(i)$. Hence

(6.59) $$m(i)\,(\mathrm{div}_i(\tilde{\nabla}(i)X_h),Y) = \mathrm{tr}\ \nabla(i)_Y(\nabla(i)X_h) + \mathrm{Ric}(m(i))(X,X_h).$$

But (6.59) yields

(6.60) $$\mathrm{div}_i(\tilde{\nabla}(i)X_h) = \mathrm{grad}_i\mathrm{div}_i\ X_h + R(i)\ X_h\ ,$$

where $R(i)X_h$ is defined via

(6.61) $$m(i)\,(R(i)X_h,Y) = \mathrm{Ric}(m(i))\,(X_h,Y)\ , \qquad \forall\ Y \in \Gamma TM.$$

From (6.49),(6.50),(6.55) and (6.60) we deduce

(6.62) $$\mathrm{div}_i L_{X_h} = \ \Delta(i)\ X_h + R(i)\ X_h + \mathrm{grad}_i\mathrm{div}_i X_h\ ,$$

(6.63) $$2\ \mathrm{div}_i\ C_h = \Delta(i)\ X_h - R(i)\ X_h - \mathrm{grad}_i\mathrm{div}_i\ X_h$$

and consequently

$$\text{(6.64)} \qquad \operatorname{div}_i \left(\tfrac{1}{2} L_{X_h} + C_h\right) = \Delta(i)\, X_h \,,$$

$$\text{(6.65)} \qquad \operatorname{div}_i \left(\tfrac{1}{2} L_{X_h} - C_h\right) = R(i)\, X_h + \operatorname{grad}_i \operatorname{div}_i X_h \,.$$

Let us restrict our attention to the case where M has codimension 1. Since M is oriented we have an oriented unit normal vector field $N(i)$ along i. Hence $h \in C^\infty(M,\mathbb{R}^n)$ splits uniquely into

$$\text{(6.66)} \qquad h = di\, X_h + \Theta_h\, N(i) \,,$$

where $X_h \in \Gamma TM$, $\Theta_h \in C^\infty(M,\mathbb{R})$.
Thus

$$\text{(6.67)} \qquad W_h(i) = W(i) \qquad \text{if} \quad \Theta_h = 1 \,.$$

Defining the mean curvature $H(i)$ of i by

$$\text{(6.68)} \qquad \operatorname{tr} W(i) = H(i)$$

we immediately find

$$\text{(6.69)} \qquad \begin{aligned} m(i)\,(\operatorname{div}_i(\Theta_h\, W(i))\,,\, Y\,) &= \sum_{r=1}^{n-1} m(i)\,(\nabla(i)_{e_r}(\Theta_h\, W(i))\, e_r, Y) \\ &= m(i)\,(\operatorname{grad}_i \Theta_h, W(i)\, Y) + m(i)\,(\Theta_h \operatorname{div}_i W(i), Y) \end{aligned}$$

and hence

$$\text{(6.70)} \qquad \operatorname{div}_i(\Theta_h\, W(i)) = W(i) \operatorname{grad}_i \Theta_h + \Theta_h \operatorname{div}_i W(i) \,.$$

On the other hand by Codazzi's equation (cf. [Kl])

$$\text{(6.71)} \qquad \begin{aligned} \sum_{r=1}^{m} m(i)\,(\nabla(i)_{e_r}(W(i))\, e_r, Y) &= \sum_{r=1}^{n-1} m(i)\,(\nabla(i)_Y(W(i))\, e_r, e_r) \\ &= m(i)\,(\operatorname{grad}_i H(i), Y) \end{aligned}$$

and consequently

$$\text{(6.72)} \qquad \operatorname{div}_i(\Theta_h\, W(i)) = W(i) \operatorname{grad}_i \Theta_h + \Theta_h \operatorname{grad}_i H(i) \,.$$

From (6.49), (6.60), (6.67), (6.68) and (6.72) we infer then

$$\text{(6.73)} \qquad \operatorname{div}_i (B_h + C_h) = \Delta(i)\, X_h + W(i) \operatorname{grad}_i \Theta_h + \Theta_h \operatorname{grad}_i H(i)$$

$$\text{(6.74)} \qquad \begin{aligned} \operatorname{div}_i (B_h - C_h) = {} & R(i)\, X_h + \operatorname{grad}_i \operatorname{div}_i X_h \\ & + W(i) \operatorname{grad}_i \Theta_h + \Theta_h \operatorname{grad}_i H(i) \,, \end{aligned}$$

$$\text{(6.75)} \qquad \operatorname{div}_i B_h = \tfrac{1}{2}\,\Delta(i)\, X_h + \tfrac{1}{2}\, R(i)\, X_h + W(i)\, \operatorname{grad}_i \Theta_h + \Theta_h \operatorname{grad}_i H(i)\ .$$

We conclude this section by proving the following

Lemma 6.5 :

Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $i \in E(M,\mathbb{R}^n)$ be given.
If α has no integrable part, i.e. $\alpha = \beta$, then

$$\text{(6.76)} \qquad \operatorname{div}_i\,(C_\beta + B_\beta) = 0\ .$$

Proof :

Denoting by $\bar{e}_1,\dots,\bar{e}_n$ an orthonormal basis of $\mathbb{R}^n$ we have, in accordance with (6.8) ,

$$\text{(6.77)} \qquad \begin{aligned} T_\beta(X,Y) &= \langle\ \beta(X), di\ Y\rangle \\ &= m(i)\ ((B_\beta + C_\beta)\ X,Y) \\ &= \sum_{s=1}^{n} m(i)\ (Y_s^0,X)\ \langle\bar{e}_s,di\ Y\rangle \\ &= m(i)\ (\sum_{s=1}^{n} \langle\bar{e}_s,di\ Y\rangle\ Y_s^0,X)\ , \qquad \forall\ X,Y \in \Gamma TM. \end{aligned}$$

If now $e_1,\dots,e_m$ is a moving orthonormal frame in TM, then we get

$$\text{(6.78)} \qquad \begin{aligned} m(i)\ (\operatorname{div}_i(B_\beta + C_\beta),Y) &= \sum_{r=1}^{m} m(i)\ (\nabla(i)_{e_r}(B_\beta + C_\beta)\ e_r,Y) \\ &= \sum_{r=1}^{m}\sum_{s=1}^{n} m(i)\ (\nabla(i)_{e_r} \langle\bar{e}_s,di\ Y\rangle\ Y_s^0,\ e_r) \end{aligned}$$

By interchanging the summation the assertion then follows.

7. The equations of motion of a deformable medium subjected to an internal constitutive law

Let M be a moving deformable medium of codimension 1, i.e. $\dim M = n-1$, and assume that its motion is due only to an internal constitutive law F. As we have already seen F is $\mathbb{R}^n$-invariant and admits the representation

$$F = d^* F_{\mathbb{R}^n}$$

where

$$F_{\mathbb{R}^n} : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$$

is a smooth map. Next we assume that $F_{\mathbb{R}^n}$ has an integral representation

$$F_{\mathbb{R}^n}(dk)(dj,dl) := \int_M \alpha(dj,dk)\cdot dl\ \mu(j)\ , \quad \forall\ j \in E(M,\mathbb{R}^n),\ k,l \in C^\infty(M,\mathbb{R}^n), \tag{7.1}$$

where α is an $\mathbb{R}^n$-valued one form, the so-called stress-form, i.e.

$$\alpha : TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n). \tag{7.2}$$

We note that the integrand

$$\alpha(dj,dk)\cdot dl \tag{7.3}$$

in (7.1) is defined in the following way :
Let first represent $\alpha(dj,dk)$ and dl according to Proposition 6.3 as

$$\alpha(dj,dk) = c_\alpha(dj,dk)\cdot dl + dj\cdot C_\alpha(dj,dk) + dj\cdot B_\alpha(dj,dk)$$

and

$$dl = c_l\cdot dj + dj\cdot C_l + dj\cdot B_l$$

respectively. Then we set

$$\alpha(dj,dk)\cdot dl := \operatorname{tr} B_\alpha(dj,dk)\cdot B_{dl} + \operatorname{tr} C_\alpha(dj,dk)\cdot C_{dl} + \operatorname{tr} c_\alpha(dj,dk)\cdot c_{dl}\ . \tag{7.4}$$

Using now (5.20) and (7.1), it is easy to see that the internal constitutive law F admits an integral representation by a force density φ_F, which depends on (the coefficients of) the stress form α. Indeed, to this end we have to solve the equation

$$\int_M \langle \varphi_F(dj,dk), l\rangle \mu(j) = \int_M \alpha(dj,dk)\cdot dl\ \mu(j)\ , \quad \forall\ j \in E(M,\mathbb{R}^n),\ k,l \in C^\infty(M,\mathbb{R}^n). \tag{7.5}$$

Before doing so, however, we point out that α in (7.5) is not uniquely determined. In fact we have (cf.[Bi,4]).

Theorem 7.1 :
Given $\alpha \in A^1(M,\mathbb{R}^n)$ and $j \in E(M,\mathbb{R}^n)$ with dh as the integrable part of α then

$$(7.6) \qquad \int \alpha\cdot dk\ \mu(j) = \int dh\cdot dk\ \mu(j)\ ,$$

expressing the fact that the non-integrable part β of $\alpha = dh + \beta$ is orthogonal to $C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}$, regarded as a subspace of $A^1(M,\mathbb{R}^n)$.

Proof :
By Proposition 6.1 we have, $\forall\ X \in \Gamma TM$,

$$dhX = \sum_{s=1}^{n} m(j)\ (V_s,X)e_s,$$

$$\beta(X) = \sum_{s=1}^{n} m(j)(Y_s^0,X)e_s\ ,$$

where $V_s = \mathrm{grad}_j \tau_s$, $\mathrm{div}_j Y_s^0 = 0$, $s = 1,\ldots,n$. Then from (6.24), (6.26) and (6.29) it follows

$$(7.7) \qquad (B_{dh} + C_{dh})X = \sum_{s=1}^{n} m(j)\ (V_s,X)Y_s$$

and

$$(7.8) \qquad (B_\beta + C_\beta)X = \sum_{s=1}^{n} m(j)\ (Y_s^0,X)Y_s\ .$$

where $Y_s = V_s + Y_s^0$, $s = 1,\ldots,n$, satisfies the equation

$$m(j)\ (Y_s,X) = \langle\alpha(X),e_s\rangle\ , \qquad \forall\ \ X \in \Gamma TM.$$

Hence

$$(7.9) \qquad (B_{dh} + C_{dh}) \circ (B_\beta + C_\beta)X = \sum_{s=1}^{n} \sum_{s'=1}^{n} m(j)\ (Y_s^0,X)\ m(j)\ (V_{s'},Y_s)\ Y_{s'}\ .$$

Therefore if $\bar{e}_1,\ldots,\bar{e}_{n-1}$ is an orthonormal moving frame on M

$$(7.10) \qquad \begin{aligned} \mathrm{tr}(B_{dh} + C_{dh}) \circ (B_\beta + C_\beta) &= \mathrm{tr}(B_{dh} \circ B_\beta + C_{dh} \circ C_\beta) \\ &= \sum_{r=1}^{n-1} \sum_{s'=1}^{n} \sum_{s=1}^{n} m(j)\ (Y_s^0,\bar{e}_r)m(j)\ (V_{s'},Y_s)m(j)\ (Y_{s'},\bar{e}_r) \\ &= \sum_{s=1}^{n} m(j)\ (Y_s^0,V_s)\ . \end{aligned}$$

Next we form

$$(7.11) \qquad \begin{cases} c_{dh}\ djX = \sum_{s=1}^{n} m(j)\ (V_s,X)\cdot\langle e_s,N(j)\rangle\cdot N(j) \\ c_{dh}\ N(j) = \sum^{n} \langle e_s,N(j)\rangle\cdot dj\ V_s\ . \end{cases}$$

Similar equations hold for c_β with V_s replaced by Y_s^0 for each s. Hence

$$(7.12)\qquad \begin{cases} c_{dh} \circ c_\beta \, dj\, X = \sum_{s'=1}^{n} \sum_{s=1}^{n} m(j)\ (Y_s^0,X)\cdot\langle e_s,N(j)\rangle\cdot\langle e_{s'},N(j)\rangle\cdot V_s\ , \\ c_{dh} \circ c_\beta\ N(j) = \sum_{s'=1}^{n} \sum_{s=1}^{n} m(j)\ (V_s,Y_{s'})\cdot\langle e_s,N(j)\rangle\cdot\langle e_{s'},N(j)\rangle\cdot N(j)\ . \end{cases}$$

Therefore

$$(7.13)\qquad \operatorname{tr} c_{dh} \circ c_\beta = 2\cdot \sum_{s=1}^{n} m(j)\ (Y_s^0,V_s)$$

and hence

$$(7.14)\qquad \operatorname{tr}(B_{dh} \circ B_\beta + C_{dh} \circ C_\beta + c_{dh} \circ c_\beta) = 3\cdot \sum_{s=1}^{n} m(j)\ (Y_s^0,V_s)\ .$$

But (7.14) implies

$$(7.15)\qquad {}_M\!\int dh\cdot\beta\ \mu(j) = 3\cdot \sum_{s=1}^{n} {}_M\!\int m(j)\ (Y_s^0,V_s)\ \mu(j) = 0$$

since Y_s^0 and V_s are L_2-orthogonal, $s = 1,\dots,n$.

Let us now turn back to (7.5).
At first we study the equation

$$(7.16)\qquad \langle \varphi_F,l\rangle = \operatorname{tr}\,(\,B_\alpha \circ B_{dl} + C_\alpha \circ C_{dl} + c_\alpha \circ c_{dl}\,)\ ,$$

where for the sake of simplicity we omited the arguments of α and φ_F. On one hand we note that $l \in C^\infty(M,\mathbb{R}^n)$ can be represented as

$$(7.17)\qquad l = dj\ X_l + \Theta_l\ N(j)$$

and that it will be sufficient to consider only those l for which $\Theta_l = 1$, i.e.

$$(7.18)\qquad l = dj\ X_l + N(j)\ .$$

On the other hand according to Proposition 6.4 we have for a fixed orthonormal moving frame $e_1,\dots,e_{n-1} \in \Gamma TM$

$$\begin{aligned} (7.19)\qquad \operatorname{tr} B_\alpha \circ B_{dl} &= \sum_{s=1}^{n-1} \tfrac{1}{2}\, m(j)\ ((B_\alpha \circ (\nabla(j))\ X_l + \tilde{\nabla}(j)X_l)\ e_s,\ e_s) \\ &\quad + \sum_{s=1}^{n-1} m(j)\ (B_\alpha \circ W(j)e_s,\ e_s) \\ &= \sum_{s=1}^{n-1} m(j)\ (B_\alpha \circ \nabla(j)_{e_s}X_l,\ e_s) - \sum_{s=1}^{n-1} \langle N(j),\ S(j)(e_s,B_\alpha e_s)\rangle \\ &= \operatorname{div}_j B_\alpha X_l - m(j)(\operatorname{div}_j B_\alpha, X_l) \\ &\quad + \langle \operatorname{tr}\,(B_\alpha \circ W(j))\cdot N(j),\ N(j)\rangle\ . \end{aligned}$$

We calculate next in the same way $\operatorname{tr} C_\alpha \circ C_{dl}$ and obtain, using the fact that

$$\operatorname{tr} W(j) \circ C_\alpha = 0, \tag{7.20}$$

$$\operatorname{tr} C_\alpha \circ C_{dl} = \operatorname{div}_j C_\alpha X_l - m(j)(\operatorname{div}_j C_\alpha, X_l) . \tag{7.21}$$

It remains now to calculate $\operatorname{tr} c_\alpha \circ c_{dl}$. Recalling to this purpose that

$$c_{dl}\, dj\, Y = S(j)\,(X_l, Y)\,, \qquad \forall\, Y \in \Gamma TM, \tag{7.22}$$

and writing $c_\alpha N(j)$ with the help of a field, say U_α, in ΓTM as

$$c_\alpha N(j) = dj\, U_\alpha \tag{7.23}$$

we then get

$$\begin{aligned}
\operatorname{tr} c_\alpha \circ c_{dl} &= \sum_{s=1}^{n-1} \langle c_\alpha \circ c_{dl} dj e_s, dj e_s\rangle + \langle c_{dl} \circ c_\alpha N(j), N(j)\rangle \\
&= \sum_{s=1}^{n-1} \langle c_\alpha S(j)(X_l, e_s), dj e_s\rangle + \langle c_{dl} dj\, U_\alpha, N(j)\rangle \\
&= - \sum_{s=1}^{n-1} m(j)(W(j)X_l, e_s)\, \langle c_\alpha N(j), dj e_s\rangle \\
&\quad - m(j)(W(j)U_\alpha, X_l) \\
&= - \sum_{s=1}^{n-1} m(j)(W(j)X_l, e_s)\cdot m(j)(U_\alpha, e_s) - m(j)(W(j)U_\alpha, X_l) \\
&= - 2\, m(j)\,(W(j)U_\alpha, X_l) .
\end{aligned}$$

The equation for φ_F becomes then

$$\begin{aligned}
{}_M\!\int \langle \varphi_f, l\rangle \mu(j) = {}_M\!\int [&\operatorname{div}_j((B_\alpha + C_\alpha)X_l) \\
&- \langle dj(\operatorname{div}_j(B_\alpha + C_\alpha) + 2W(j)U_\alpha) \\
&+ \operatorname{tr}(B_\alpha \circ W(j))N(j), l\rangle]\ \mu(j).
\end{aligned} \tag{7.24}$$

Using Gauss' theorem this equation yields

Proposition 7.2 :

Let $F_{\mathbb{R}^n}$ admits an integral representation given by

$$F_{\mathbb{R}^n}(dk)(dj,dl) = {}_M\!\int \alpha(dj,dk)\cdot dl\ \mu(j)\ , \qquad \forall\ j \in E(M,\mathbb{R}^n),\ k,l \in C^\infty(M,\mathbb{R}^n),$$

where the stress form

$$\alpha : TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n)$$

splits into

$$\alpha = c_\alpha \cdot dj + dj\cdot C_\alpha + dj\cdot B_\alpha .$$

Then F admits an integral representation with a force density φ_F given at (dj,dk) by

$$(7.25)\qquad \varphi(dj,dk) = -\, dj\,(\mathrm{div}\,(B_\alpha(dj,dk) + C_\alpha(dj,dk) + 2\,W(j)U_\alpha(dj,dk)) + \mathrm{tr}\,(B_\alpha(dj,dk)\circ W(j))\cdot N(j)\ .$$

Since α in (7.1) can be replaced by its integrable part as expressed in (7.6) a redundancy occurs in (7.23). Let us therefore rewrite this equation by replacing $\alpha(dj,dk)$ by its integrable part $dh(dj,dk)$. To this end we first rewrite $U_{dh}(dj,dk)$ in terms of $h(dj,dk)$ as done by [Bi,5] and restated in the following :

Lemma 7.3 :
Let $h = dj\,X_h + \Theta_h\cdot N(j)$ *for any* $h \in C^\infty(M,\mathbb{R}^n)$ *then* U_{dh} , as defined in (7.23) takes the form

$$(7.26)\qquad U_{dh} = W(j)X_h - \mathrm{grad}_j\Theta_h\ .$$

Proof :
Equation (7.23) reads in the case under consideration as

$$c_{dh}\,N(j) = dj\,U_{dh}\ .$$

By (6.44) we have moreover

$$\begin{aligned} c_{dh}\,dj\,X &= S(j)(X_h,X) + d\Theta_h(X)\cdot N(j) \\ &= -\,m(j)(W(j)X_h - \mathrm{grad}_j\Theta_h,X)\cdot N(j)\ . \end{aligned}$$

Therefore

$$(7.27)\qquad \begin{aligned} c_{dh}^{\,2}\,N(j) &= c_{dh}\cdot dj\,U_{dh} \\ &= -\,m(j)(W(j)X_h - \mathrm{grad}_j\Theta_h,U_{dh})\,N(j) \end{aligned}$$

implying (7.26).
In view of (7.6) and (7.26) we obtain

Corollary 7.8 :
Since for any $(dj,dk) \in TE(M,\mathbb{R}^n)|_{\mathbb{R}^n}$ the stress-form α splits uniquely into

$$\alpha(dj,dk) = dh(dj,dk) + \beta(dj,dk)$$

with $h(dj,dk)$ the integrable and $\beta(dj,dk)$ the non-integrable part respectively, equation (7.25) turns into

$$(7.28)\qquad \begin{aligned} \varphi(dj,dk) = -\,\mathrm{div}\,B_{dh}(dj,dk) &+ C_{dh}(dj,dk) \\ &+ 2\,W(j)^2X_h - 2\,W(j)\,\mathrm{grad}_j\Theta_h \end{aligned}$$

where

$$h = dj\,X_h + \Theta_h\,N(j)$$

with $X_h \in \Gamma TM$ and $\Theta_h \in C^\infty(M,\mathbb{R})$.

Using now the eqnations of motion (3.25) of the deformable medium M we immediately obtain the following

Main theorem :

Let M be a moving deformable medium of codimension 1 subjected to an internal constitutive law

$$F : C^\infty(M,\mathbb{R}^n) \times E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R} ,$$

with

$$F = d^* F_{\mathbb{R}^n} ,$$

and

$$F_{\mathbb{R}^n} : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow \mathbb{R} ,$$

a smooth map admitting an integral representation given by the so-called stress form

$$\alpha : TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n) .$$

This stress form α decomposes according to (6.34) at each

$$(dj,dk) \in TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} = E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}$$

into

$$\alpha(dj,dk) = c_\alpha(dj,dk)\cdot dj + dj\cdot C_\alpha(dj,dk) + dj\cdot B_\alpha(dj,dk) .$$

The equation of motion on $E(M,\mathbb{R}^n)$ described by a smooth curve

$$\sigma : (-\lambda,\lambda) \longrightarrow E(M,\mathbb{R}^n), \qquad \lambda > 0 ,$$

is then given by

(7.29)
$$\int_M \rho(\sigma(t))\, \langle\ddot\sigma(t),l\rangle\, \mu(\sigma(t)) = \int_M \alpha(d\sigma(t),d\dot\sigma(t))\cdot dl\ \mu(\sigma(t)) = \int_M \langle\varphi_F(d\sigma(t),d\dot\sigma(t)),l\rangle\, \mu(\sigma(t)),$$

where the force density φ_F satisfies (7.25). Using the fact that (7.29) implies

(7.30)
$$\rho(\sigma(t))\ddot\sigma(t) = \varphi_F(d\sigma(t),d\dot\sigma(t)) , \qquad \forall\ t \in (-\lambda,\lambda)$$

and writing $\dot\sigma(t)$ in $\mathbb{R}^n$ as

$$\dot\sigma(t) = d\sigma(t)Z(t) + \epsilon(\sigma(t),\dot\sigma(t))N(\sigma(t)) , \qquad \forall\ t \in (-\lambda,\lambda) ,$$

where $Z(t) \in \Gamma TM$ and $\epsilon(\sigma(t),\dot\sigma(t)) \in C^\infty(M,\mathbb{R})$, the equation of motion (7.30) splits into the coupled system

$$
(7.31)\quad \begin{cases}
\nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + 2\,\epsilon(\sigma(t),\dot\sigma(t))W(\sigma(t))Z(t) \\
-\mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)) \\
= \rho^{-1}(\sigma(t))\,[-\,\mathrm{div}_{\sigma(t)}T_\alpha(d\sigma(t),d\dot\sigma(t)) - 2\,W(\sigma(t))U_\alpha)], \\
\\
\dot\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\ \mathrm{tr}\ B_\alpha(d\sigma(t),d\dot\sigma(t))\ W(\sigma(t)) \\
\qquad - d\epsilon(\sigma(t),\dot\sigma(t))Z(t) + \mathfrak{h}(\sigma(t))(Z(t),Z(t)),
\end{cases}
$$

where the smooth two tensor $T_\alpha(dj,dk)$, the so-called stress tensor, is defined by

$$
(7.32)\qquad T_\alpha(dj,dk)(X,Y) = m(j)(B_\alpha+C_\alpha)(dj,dk)X,Y), \qquad \forall\ X,Y \in \Gamma TM
$$

and $\mathfrak{h}(\sigma(t))$ denotes the second fundamental form of $\sigma(t)$.

Remark :
(7.31) corresponds to Cauchy's law in the mechanics of continua.

Using Corollary 7.8 equations (7.31) turn into

Corollary :
Using the splitting of the stress-form into an integrable dh and a non-integrable part β respectively, (7.31) reads

$$
(7.33)\quad \begin{cases}
\nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + 2\,\epsilon(\sigma(t),\dot\sigma(t))W(\sigma(t))Z(t) \\
-\mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\,[-\,\mathrm{div}_{\sigma(t)}T_{dh}(d\sigma(t),d\dot\sigma(t)) \\
\qquad - 2\,W(\sigma(t))(W(\sigma(t))X_h - \mathrm{grad}_{\sigma(t)}\Theta_h)], \\
\dot\epsilon(\sigma(t),\dot\sigma(t)) = \rho^{-1}(\sigma(t))\ \mathrm{tr}\ B_{dh}(d\sigma(t),d\dot\sigma(t))\ W(\sigma(t)) \\
\qquad - d\epsilon(\sigma(t),\dot\sigma(t))Z(t) + \mathfrak{h}(\sigma(t))(Z(t),Z(t)).
\end{cases}
$$

Example 7.9 :

Let α be given by

$$
(7.34)\qquad \alpha(dj,dk) = -\,\tau(dj,dk)\cdot dj\,, \qquad \forall\ j\in E(M,\mathbb{R}^n),\ k\in C^\infty(M,\mathbb{R}^n),
$$

where

$$
(7.35)\qquad \tau : TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow C^\infty(M,\mathbb{R}^n)
$$

is a smooth map. Then we have

$$
(7.36)\quad \begin{cases}
B_\alpha(dj,dk) = \tau(dj,dk)\cdot Id_{TM}\,, \\
C_\alpha(dj,dk) = 0\,, \\
c_\alpha(dj,dk) = 0\,.
\end{cases} \qquad \forall\ (dj,dk) \in TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n}\,,
$$

Therefore the stress tensor T_α and the force density φ_F become

$$T_\alpha(dj,dk)(X,Y) = \tau(dj,dk)\cdot m(j)(X,Y)\ , \qquad \forall\ X,Y \in \Gamma TM\ , \tag{7.37}$$

and

$$\varphi_F(dj,dk) = dj\ \mathrm{grad}\ \tau(dj,dk) + \tau(dj,dk)\cdot H(j)\ N(j)\ . \tag{7.38}$$

Then the Main theorem reduces to the following

Proposition 7.10 :

Let the hypotheses of the Main theorem hold and assume that α is given by (7.34). Then the equations of motion of the deformable medium M are given by

$$\begin{cases} \nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + 2\cdot\epsilon(\sigma(t),\dot{\sigma}(t))\ W(\sigma(t))Z(t) \\ -\mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot{\sigma}(t)) = \rho^{-1}(\sigma(t))\ \mathrm{grad}_{\sigma(t)}\tau(d\sigma(t),d\dot{\sigma}(t))\ , \\ \\ \dot{\epsilon}(\sigma(t),\dot{\sigma}(t)) = \rho^{-1}(\sigma(t))\ \tau(d\sigma(t),d\dot{\sigma}(t))\cdot H(\sigma(t)) \\ \qquad\qquad - d\epsilon(\sigma(t),\dot{\sigma}(t))Z(t) + \mathfrak{h}(\sigma(t))\ (Z(t),Z(t))\ . \end{cases} \tag{7.39}$$

We call (7.39) the equations of motion of a perfect deformable medium. The corresponding constitutive law will be refered too as the perfect constitutive law.
The above corollary relates with the motion induced by a reduced constitutive law introduced earlier as shown by the following

Corollary 7.11 :
Let F be a smooth $\mathbb{R}^n$-invariant constitutive law splitting into the sum

$$F = F^0 + F'\ , \tag{7.40}$$

where F^0 and F' are smooth $\mathbb{R}^n$-invariant constitutive laws admitting both stress forms α^0 and α' respectively. Assume moreover that α^0 and α' have the decompositions

$$\begin{cases} \alpha^0(dj,dk) = \tau(dj,dk)\cdot dj\ , \\ \alpha'(dj,dk) = c_\alpha'(dj,dk)\cdot dj + dj\cdot C_\alpha'(dj,dk) + dj\cdot B_\alpha'(dj,dk)\ , \\ \qquad\qquad \forall\ (dj,dk) \in TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \end{cases} \tag{7.41}$$

with

$$\mathrm{tr}\ B_\alpha'(dj,dk) = 0\ . \tag{7.42}$$

Then the motion

$$\sigma : (-\lambda,\lambda) \longrightarrow E(M,\mathbb{R}^n)$$

induced by F satisfies the equations

$$(7.43)\quad \begin{cases} \nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) \\ \quad + 2\,\epsilon(\sigma(t),\dot{\sigma}(t))W(\sigma(t))Z(t) - \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot{\sigma}(t)) \\ = \rho^{-1}(\sigma(t))\,[\mathrm{grad}_{\sigma(t)}\tau(d\sigma(t),d\dot{\sigma}(t)) - \mathrm{div}_{\sigma(t)}T'_{\alpha}(d\sigma(t),d\dot{\sigma}(t)) \\ \qquad - 2\,W(\sigma(t))\,U_{\alpha})] \\ \dot{\epsilon}(\sigma(t),\dot{\sigma}(t)) = \rho^{-1}(\sigma(t))\,[\mathrm{tr}\big(B_{\alpha}'(d\sigma(t),d\dot{\sigma}(t))\cdot W(\sigma(t))\big) \\ \qquad\qquad - \tau(d\sigma(t),d\dot{\sigma}(t))\,H(\sigma(t))] \\ \qquad\qquad - d\epsilon(\sigma(t),\dot{\sigma}(t))\,Z(t) + h(\sigma(t))\,(Z(t),Z(t))\,. \end{cases}$$

8. The structural viscosity

As known the notion of viscosity was first introduced by Newton as "the resistance which arises from the lack of slipperiness of the parts of the liquid". He made the assumption that the viscosity "is proportional to the velocity with which the parts of the liquid are separated from one another". As a measure of the viscous resistance one has introduced the coefficient of viscosity ν.

In this section we introduce the notion of structural viscosity, as done in [Bi,5], i.e. the notion of viscosity within our apparatus.

Let again M be a moving deformable medium of codimension 1 and assume that its motion is due only to an internal constitutive law F. As shown F is $\mathbb{R}^n$-invariant and admits the representation

$$F = d^* F_{\mathbb{R}^n}$$

with

$$F_{\mathbb{R}^n} : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow \mathbb{R}$$

a smooth map. In addition we assume that $F_{\mathbb{R}^n}$ admits a stress form

$$\alpha : TE(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow A^1(M,\mathbb{R}^n) .$$

According to Proposition 6.1 the $\mathbb{R}^n$-valued one-form α admits the decomposition

$$\alpha(dj,dk) = dh(dj,dk) + \beta(dj,dk),$$

where the integrable part h $\in C^\infty(M,\mathbb{R}^n)$ *is uniquely determined up to a* constant. Next we give h and k the equivalent forms

$$h(dj,dk) = dj\ X_h(dj,dk) + h^\perp(dj,dk)$$

and respectively

$$k = dj\ X_k + k^\perp .$$

Moreover we set, using the fact that M has codimension 1,

$$h^\perp(dj,dk) = \Theta_h(dj,dk)\ N(j) . \tag{8.1}$$

We split now according to Hodge's theorem X_h and X_k into

$$X_h = X_h^0 + \mathrm{grad}_j \psi_h ,$$
$$\mathrm{div}_j X_h^0 = 0 ,$$

and

$$X_k = X_k^0 + \mathrm{grad}_j \psi_k ,$$
$$\mathrm{div}_j X_k^0 = 0$$

respectively. Let us remind of proposition 6.2, expressing in particular that X^0_h and X^0_k are uniquely determined by α and dk respectively.
We relate next X^0_h and X^0_k uniquely to each other by setting

$$X^0_h(dj,dk) = \nu(dj,dk)X^0_k + \hat{X}_h(dj,dk) , \tag{8.2}$$

where $\nu(dj,dk) \in C^\infty(M,\mathbb{R})$ and $\hat{X}_h(dj,dk) \in \Gamma TM$ is pointwise orthogonal to X^0_k, and call the function $\nu(dj,dk)$ coefficient of structural viscosity.
Accordingly we call the deformable media, whose constitutive laws depend only on $k^\perp$, frictionless deformable media, while the deformable media whose constitutive laws depend on the whole of k, will be called frictional ones. Taking now into account (8.1) and (8.2), Proposition 6.4 becomes

Proposition 8.1 :

Let $\alpha \in A^1(M,\mathbb{R}^n)$ and $j \in E(M,\mathbb{R}^n)$ be given and assume that M has codimension 1. Then the following relations hold :

$$\left\{ \begin{array}{l} \alpha = dh + \beta , \\ \alpha(X) = c_\alpha \, dj \, X + dj \, C_\alpha \, X + dj \, B_\alpha \, X , \qquad \forall\, X \in \Gamma TM , \\ h(dj,dk) = dj \, X^0_h(dj,dk) + dj \, \mathrm{grad}_j \Psi_h + \Theta_h(dj,dk) \, N(j) , \\ \mathrm{div}_j \, X^0_h = 0 , \\ X_k = X^0_k + \mathrm{grad}_j \Psi_k , \\ \mathrm{div}_j \, X^0_h = 0 , \\ X^0_h = \nu(dj,dk)X^0_k + \hat{X}_k(dj,dk) , \end{array} \right.$$

$$\begin{aligned} c_\alpha(dj,dk)\cdot dj = {} & (dh^\perp(dj,dk))^\perp + \nu(dj,dk)S(j)(X^0_k, \,.\,) \\ & + S(j)(\hat{X}, \,.\,) + S(j)(\mathrm{grad}_j \Psi_h, \,.\,) + c_\beta(dj,dk)\cdot dj , \end{aligned} \tag{8.3}$$

$$\begin{aligned} C_\alpha(dj,dk) = {} & \tfrac{1}{2} [\nabla(j) \, \nu(dj,dk)X^0_k - \tilde{\nabla}(j) \, \nu(dj,dk)X^0_k] \\ & + \tfrac{1}{2} [\nabla(j)\hat{X}_h(dj,dk) - \tilde{\nabla}(j)\hat{X}_h(dj,dk)] + C_\beta(dj,dk) , \end{aligned} \tag{8.4}$$

$$\begin{aligned} B_\alpha(dj,dk) = {} & \tfrac{1}{2} L_{\nu(dj,dk)X^0_k} + \tfrac{1}{2} L_{\hat{X}_h(dj,dk)} \\ & + \tfrac{1}{2} L_{\mathrm{grad}_j \Psi_h(dj,dk)} + \Theta_h(dj,dk) \, W(j) + B_\beta(dj,dk) , \\ & \forall\, (dj,dk) \in TE(M,\mathbb{R}^n)|_{\mathbb{R}^n} . \end{aligned} \tag{8.5}$$

If now $\nu(dj,dk)$ is a constant map in $C^\infty(M,\mathbb{R})$, then we get

$$\begin{aligned} C_\alpha(dj,dk) = {} & \tfrac{1}{2} \nu(dj,dk) \, (\nabla(j) \, X^0_k - \tilde{\nabla}(j) \, X^0_k) \\ & + \tfrac{1}{2} [\nabla(j)\hat{X}_h(dj,dk) - \tilde{\nabla}(j)\hat{X}_h(dj,dk)] + C_\beta(dj,dk) , \end{aligned} \tag{8.6}$$

$$\begin{aligned} B_\alpha(dj,dk) = {} & \tfrac{1}{2} \nu(dj,dk) \, L_{X^0_k} + \tfrac{1}{2} L_{\hat{X}_h(dj,dk)} \\ & + \tfrac{1}{2} L_{\mathrm{grad}_j \Psi_h(dj,dk)} + \Theta_h(dj,dk) \, W(j) + B_\beta(dj,dk) , \\ & \forall\, (dj,dk) \in TE(M,\mathbb{R}^n)|_{\mathbb{R}^n} . \end{aligned} \tag{8.7}$$

Using now the definition (7.32) of the stress tensor we get

$$
\begin{aligned}
(8.8) \qquad T_{\alpha}(dj,dk)(X,Y) = m(j)((\tfrac{1}{2}\,\nu(dj,dk)\,L_{X_k^0} + \tfrac{1}{2}\,\nu(dj,dk) \\
(\nabla(j)X_k^0 - \tilde{\nabla}(j)X_k^0) + \tfrac{1}{2}\,L_{\hat{X}_h(dj,dk)} \\
+ \tfrac{1}{2}\,[\nabla(j)\hat{X}_h(dj,dk) - \tilde{\nabla}(j)\,\hat{X}_h(dj,dk)] \\
- \tfrac{1}{2}\,L_{\mathrm{grad}_j \psi_h(dj,dk)} + \Theta_h(dj,dk)\cdot W(j) \\
+ B_{\beta}(dj,dk) + C_{\beta}(dj,dk))\,X\,,\,Y)\,.
\end{aligned}
$$

It is this equation which motivates us to call the function $\nu(dj,dk)$ coefficient of structural viscosity.

9. The equations of motion of a deformable medium subjected to a general constitutive law

Let us suppose that the motion of the deformable medium M is governed by the smooth constitutive law

$$F : C^\infty(M,\mathbb{R}^n) \times E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R} \ ,$$

and assume that F splits into

$$(9.1) \qquad F = F_{ext} + F_{int} \ ,$$

where the internal constitutive law F_{int} and the external one F_{ext} are also smooth. As shown F_{int} is $\mathbb{R}^n$-invariant and admits the representation

$$F_{int} = d^* F_{\mathbb{R}^n}$$

with

$$F_{\mathbb{R}^n} : C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times E(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \times C^\infty(M,\mathbb{R}^n)\big|_{\mathbb{R}^n} \longrightarrow \mathbb{R} \ .$$

Finally we assume that both F_{ext} and $F_{\mathbb{R}^n}$ admit integral representations and denote by φ_{ext} and φ_{int} the corresponding force densities. According to (3.13) and (3.26) we have the splittings

$$(9.2) \qquad \varphi_{F_{int}}(j,k) = dj\ \mathrm{grad}_j \tau_{int}(j,k) + dj\ Y^0_{int}(j,k) + \varphi_{F_{int}}(j,k)^\perp \ ,$$

$$(9.3) \qquad \varphi_{F_{ext}}(j,k) = dj\ \mathrm{grad}_j \tau_{ext}(j,k) + dj\ Y^0_{ext}(j,k) + \varphi_{F_{ext}}(j,k)^\perp \ ,$$

$$\forall\ j \in E(M,\mathbb{R}^n),\ k \in C^\infty(M,\mathbb{R}^n) \ ,$$

and hence

$$(9.4) \qquad \varphi_F(j,k) = dj\ \mathrm{grad}_j \tau(j,k) + dj\ Y^0(j,k) + \varphi_F(j,k)^\perp \ ,$$

where

$$(9.5) \qquad \begin{aligned} &\tau_{int}(j,k) + \tau_{ext}(j,k) = \tau(j,k) \ , \\ &Y^0_{int}(j,k) + Y^0_{ext}(j,k) = Y^0(j,k) \ , \\ &\varphi_{F_{int}}(j,k)^\perp + \varphi_{F_{ext}}(j,k)^\perp = \varphi_F(j,k)^\perp \ , \\ &\qquad \forall\ j \in E(M,\mathbb{R}^n),\ k \in C^\infty(M,\mathbb{R}^n). \end{aligned}$$

Since $\varphi_{F_{int}}$ is $\mathbb{R}^n$- invariant, its tangential and normal parts $\mathrm{grad}\ \tau_{int} + Y^0_{int}$ and $\varphi_{F_{int}}^\perp$ respectively are also $\mathbb{R}^n$-invariant. Using now the $\mathbb{R}^n$-invariance of the Laplace-Beltrami operator Δ, we infer that τ and consequently $\mathrm{grad}\ \tau$ are $\mathbb{R}^n$-invariant. Accordingly we split F_{int} into

$$(9.6) \qquad F_{int} = F^0_{int} + F'_{int} \ ,$$

where F^0_{int} has the force density $\varphi^0_{F_{int}}$ defined by

$$\varphi^0_{F_{int}}(j,k) = dj\ \mathrm{grad}_j \tau_{int}(dj,dk) \tag{9.7}$$

and F'_{int} admits the stress form

$$\left\{ \begin{array}{l} \alpha : TE(M,\mathbb{R}^n) \longrightarrow A^1(M,\mathbb{R}^n) \\ \\ \alpha'(dj,dk) = c_\alpha'(dj,dk)\cdot dj + dj\ C_\alpha'(dj,dk) + dj\ B_\alpha'(dj,dk)\ . \end{array} \right. \tag{9.8}$$

Hence

$$\begin{aligned} \alpha(dj,dk) = {} & c_\alpha'(dj,dk)\cdot dj + dj\ C_\alpha'(dj,dk) + \\ & dj\ (-\tau_{int}(dj,dk)\ Id_{TM} + B_\alpha'(dj,dk))\ . \end{aligned} \tag{9.9}$$

Using (7.40) we then find

$$\begin{aligned} \varphi_{F_{int}}(j,k) &= \varphi^0_{F_{int}}(j,k) + \varphi'_{F_{int}}(j,k) \\ &= dj\ \mathrm{grad}_j\tau_{int}(dj,dk) + dj\ Y^0_{int}(dj,dk) + \varphi_{F_{int}}(dj,dk)^\perp \\ &= dj\ \mathrm{grad}_j\tau_{int}(dj,dk) - dj(\mathrm{div}_j B_\alpha'(dj,dk) + \mathrm{div}_j C_\alpha'(dj,dk) \\ &\quad + 2\ W(j)\ (W(j)X_h - \mathrm{grad}\ \Theta_h)) + [\mathrm{tr}\ (B_\alpha'(dj,dk)\cdot W(j)) - \\ &\quad \tau_{int}(dj,dk)\cdot H(j)]\ N(j)\ , \end{aligned} \tag{9.10}$$

$$\begin{aligned} T_\alpha(j,k)(X,Y) = {} & -\tau_{int}(dj,dk)\ m(j)(X,Y) \\ & + m(j)((B_\alpha' + C_\alpha')(dj,dk)X,Y)\ ,\ \forall\ X,Y \in \Gamma TM\ . \end{aligned} \tag{9.11}$$

On the other hand α' splits according to Proposition 6.4 at (dj,dk) into

$$\alpha'(dj,dk) = dh\ (dj,dk) + \beta(dj,dk)\ ,$$

where $h(dj,dk) \in C^\infty(M,\mathbb{R}^n)$. Writing

$$h(dj,dk) = dj\ X_h(dj,dk) + \Theta_h(dj,dk)\ N(j)\ ,$$

$$k(dj,dk) = dj\ X_k(dj,dk) + \Theta_k(dj,dk)\ N(j)\ ,$$

with $\Theta_h, \Theta_k \in C^\infty(M,\mathbb{R})$, we obtain by (9.10) and Proposition 6.4

$$\begin{aligned} \varphi_{F_{int}}(j,k) = {} & dj\ \mathrm{grad}_j\tau_{int}(dj,dk) - dj(\Delta(j)\ X_h \\ & + W(j)\mathrm{grad}_j\Theta_h(dj,dk) + \Theta_h(dj,dk)\mathrm{grad}_j H(j) \\ & + 2\ W(j)(W(j)X_h - \mathrm{grad}\ \Theta_h)) \\ & + (\mathrm{tr}\ ([\tfrac{1}{2} L_{X_{h(dj,dk)}} + \Theta_h(dj,dk)\ W(j)] \circ W(j)) \\ & - \tau_{int}(dj,dk)\ H(j))\ N(j)\ . \end{aligned} \tag{9.12}$$

Decomposing now $X_h(dj,dk)$ and X_k by the Hodge theorem into

$$(9.13)\qquad \begin{cases} \mathcal{X}_h(dj,dk) = \mathcal{X}_h^0(dj,dk) + \text{grad}_j\ \Psi_h(dj,dk)\ , \\ \\ \text{div}_j\ \mathcal{X}_h^0 = 0\ , \end{cases}$$

$$(9.14)\qquad \begin{cases} \mathcal{X}_k(dj,dk) = \mathcal{X}_k^0(dj,dk) + \text{grad}_j\ \Psi_k(dj,dk)\ , \\ \\ \text{div}_j\ \mathcal{X}_k^0 = 0\ , \end{cases}$$

introducing the structural viscosity $\nu(dj,dk) \in C^\infty(M,\mathbb{R})$ via

$$(9.15)\qquad \mathcal{X}_h^0(dj,dk) = \nu(dj,dk)\ \mathcal{X}_k^0 + \hat{\mathcal{X}}_h(dj,dk)$$

and noting that

$$(9.16)\qquad \tfrac{1}{2}\text{tr}\ (L_Z\cdot W(j)) = \text{div}_j W(j)Z - dH(j)Z$$

from (9.2), (9.3) and (9.11) we obtain

$$(9.17)\qquad \begin{aligned} \varphi_F(j,k) = {} & \varphi_{F_{ext}}(j,k) + dj\ (\text{grad}_j\tau(j,k) - \Delta(j)\ [\ \nu(dj,dk)\mathcal{X}_k^0 \\ & + \hat{\mathcal{X}}_k(dj,dk) + \text{grad}_j\Psi_h(dj,dk)] - W(j\ )\ [\text{grad}_j\Theta_h(dj,dk) \\ & + 2\ (W(j)\mathcal{X}_h - \text{grad}\ \Theta_h)] - \Theta_h(dj,dk)\ \text{grad}_j H(j)] \\ & - (\tau_{int}(dj,dk)\ H(j) + dH(j)\ [\nu(dj,dk)\ \mathcal{X}_k^0 + \hat{\mathcal{X}}_h(dj,dk)] \\ & - \text{div}_j\ \nu(dj,dk)W(j)\mathcal{X}_k^0 - \text{div}_j W(j)\ \hat{\mathcal{X}}_h(dj,dk) \\ & - \Theta_h(dj,dk)\ \text{tr}\ W(j)^2)\ N(j), \end{aligned}$$

$$(9.18)\qquad \begin{aligned} T_\alpha(dj,dk)(\mathcal{X},\mathcal{Y}) = {} & -\tau_{int}(dj,dk)\ m(j)\ (\mathcal{X},\mathcal{Y}) \\ & + \tfrac{1}{2}\ (L_{\nu(dj,dk)\ \mathcal{X}_k^0} + L_{\hat{\mathcal{X}}_k(dj,dk)} + L_{\text{grad}_j\Psi_h(dj,dk)}) \\ & (m(j))(\mathcal{X},\mathcal{Y}) + m(j)(W_h(j)\mathcal{X},\mathcal{Y}) \\ & + m(j)(C_\alpha{}'(dj,dk)\ \mathcal{X},\mathcal{Y})\ . \end{aligned}$$

Consequently we may state the following theorem, based on (9.17), (9.18) and (3.25)

Theorem 9.1 :

Let $F : C^\infty(M,\mathbb{R}^n) \times E(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \longrightarrow \mathbb{R}$ be a smooth constitutive law admitting the splitting (9.1), i.e.

$$F = F_{int} + F_{ext}\ ,$$

where both the internal and the external constitutive laws, F_{int} and F_{ext} respectively, admit integral representations with the respective force densities $\varphi_{F_{int}}$ and $\varphi_{F_{ext}}$.

Then the general equations of motion of a deformable medium are given by

$$
(9.19)\quad \begin{cases}
\nabla(\sigma(t))_{Z(t)}Z(t)+\dot{Z}(t)+2\,\epsilon(\sigma(t),\dot{\sigma}(t))W(\sigma(t))Z(t)-\mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot{\sigma}(t)) \\
\quad = \rho^{-1}(\sigma(t))(\mathrm{grad}_{\sigma(t)}\tau_{int}(\sigma(t),\dot{\sigma}(t)) - \Delta(\sigma(t))\,[\nu(d\sigma(t),d\dot{\sigma}(t))Z^0(t) \\
\quad + \hat{X}_h(d\sigma(t),d\dot{\sigma}(t)) + \mathrm{grad}_{\sigma(t)}\Psi_h(\sigma(t),\dot{\sigma}(t))] \\
\quad - W(\sigma(t))[\mathrm{grad}_{\sigma(t)}\Theta_h(d\sigma(t),d\dot{\sigma}(t)) + 2\,(W(\sigma(t))X_h(\sigma(t),\dot{\sigma}(t)) \\
\quad - \mathrm{grad}\,\Theta_h)(\sigma(t),\dot{\sigma}(t))] - \Theta_h(d\sigma(t),d\dot{\sigma}(t))\mathrm{grad}_{\sigma(t)}H(\sigma(t)) \\
\quad + \rho^{-1}(\sigma(t))Y_{ext}(\sigma(t),\dot{\sigma}(t))\,, \\
\\
\dot{\epsilon}(\sigma(t),\dot{\sigma}(t)) = \rho^{-1}(\sigma(t))[-\tau_{int}(d\sigma(t),d\dot{\sigma}(t))\,H(\sigma(t)) \\
\quad - dH(\sigma(t))[\nu(d\sigma(t),d\dot{\sigma}(t))\,Z^0(t) + \hat{X}_h(d\sigma(t),\dot{\sigma}(t))] \\
\quad + \mathrm{div}_{\sigma(t)}\nu(d\sigma(t),d\dot{\sigma}(t))W(\sigma(t))Z^0(t) \\
\quad + \mathrm{div}_{\sigma(t)}W(\sigma(t))\hat{X}_h(d\sigma(t),d\dot{\sigma}(t)) \\
\quad - \Theta_h(d\sigma(t),d\dot{\sigma}(t))\,\mathrm{tr}\,W(\sigma(t))^2] + \mathfrak{h}(\sigma(t))\,(Z(t),Z(t)) \\
\quad - d\epsilon(\sigma(t),\dot{\sigma}(t))\,Z(t) + \kappa_{ext}(\sigma(t),\dot{\sigma}(t))]\,,
\end{cases}
$$

where $\mathcal{F}^{\perp}_{ext}(\sigma(t),\dot{\sigma}(t)) = \kappa_{ext}(\sigma(t),\dot{\sigma}(t))\,N(\sigma(t))$.
The motion of a deformable medium along a fixed surface $i(M)\subset \mathbb{R}^n$ is given by

$$
(9.20)\quad \begin{cases}
\nabla(i)_{X(t)}X(t) + \dot{X}(t) = \rho^{-1}(X(t))[-\mathrm{grad}_i\tau_{int}(X(t),\dot{X}(t)) \\
\qquad - \Delta(i)[\nu(X(t),\dot{X}(t))\;X^0(t) + \hat{X}_h(X(t),\dot{X}(t)) \\
\qquad + 2\,(W(i)X_h - \mathrm{grad}\,\Theta_h)] - \Theta_h(X(t),\dot{X}(t))\;\mathrm{grad}_i H(i) \\
\qquad + \rho^{-1}(X(t))\;Y_{ext}(X(t),\dot{X}(t)) \\
\\
0 = \rho^{-1}(X(t))\;(-\tau_{int}(X(t),\dot{X}(t))\cdot H(i) \\
\qquad - dH(i)[\nu(X(t),\dot{X}(t))X^0(t) + \hat{X}_h(X(t),\dot{X}(t))] \\
\qquad + \mathrm{div}_i\nu(X(t),\dot{X}(t))W(i)X^0(t) + \mathrm{div}_i W(i)\hat{X}_h(X(t),\dot{X}(t)) \\
\qquad - \Theta_h(X(t),\dot{X}(t))\mathrm{tr}W(i)^2) + \mathfrak{h}(i)(X(t),X(t)) \\
\qquad + \kappa_{ext}(X(t),\dot{X}(t)),
\end{cases}
$$

where $X(t)$ is the push-forward (4.20) of $Z(t)$ by $g(t)\in \mathrm{Diff}\,M$.
As an example let us consider the stress form

$$
(9.21)\qquad \alpha(dj,dk) = -\,\tau_{int}(dj,dk)\cdot dj + \nu\; dj\cdot L_{X_k}
$$

with a constant $\nu\in\mathbb{R}$. Then the motion along $i(M)\subset\mathbb{R}^n$ is governed by

$$
(9.22)\qquad \rho(X(t))(\nabla(i)_{X(t)}X(t) + \dot{X}(t)) \\
= -\,\mathrm{grad}_i\tau_{int}(g(t),X(t)) - \nu\cdot\Delta(i)\;X(t) - \nu\cdot\mathrm{Ric}(i)\;X(t)
$$

in case $\mathrm{div}_i X(t) = 0$. Thus the equation (9.22), a Navier-Stokes type of equation, is an approximation to (9.20).

Remark:
As done in [Bi,5] a type of pressure $\pi(dj,dh)$ can be introduced by forming the L^2-component of $\mathrm{tr}\;B_\alpha(dj,dk)\circ W(j)$ (in 7.25) along $H(j)$ yielding the decomposition

$$(9.23) \qquad B_\alpha(dj,dk) = -\pi(dj,dk)\cdot H(j) + \bar{B}_\alpha(dj,dk)\ .$$

This allows us to split $F(dj,dk)$ into

$$(9.24) \qquad F(dj,dk) = \bar{F}(dj,dk) - \pi(dj,dk)\cdot DV(j)$$

where $V(j)$ is the volume of $j(M)$. Both of these equations hold for all dj, dk and dh. Motivated by the last equation we call $\pi(dj,dk)$ the volume active pressure. $\pi(dj,dk)\cdot DV(j)$ is the work needed to change the volume by $DV(j)$. Clearly this type of pressure also exists in the realm of section 3 and is clearly not identical (in general) with $\tau_{int}(dj,dk)$.

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