On Generalized Bernstein Polynomials in CAGD

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Abstract: A central topic in CAGD is the representation of curves and surfaces by polynomial interpolation operators, in particular by Bernstein polynomials.

In this paper we present two different types of generalized Bernstein polynomials. The first one goes back to an idea of D.D.Stancu [5, 6, 7]; here a class of polynomials $q_{n\nu}$ is defined, which depend continuously on an additional parameter a. For $a = 0$, the $q_{n\nu}$ coincide with the ordinary Bernstein polynomials, whereas for $a = -1/n$ they are identical with the Lagrange polynomials.

A second type of generalization is essentially due to G.G.Lorentz [3]; here, the Bernstein polynomials are defined with respect to a generalized polynomial space, consisting of functions of the form $\sum_{\nu=0}^{n} a_\nu x^{\alpha_\nu}$ for some $0 = \alpha_0 \leq \cdots \leq \alpha_n$.

Very important for application in CAGD is the fact that the "nice" properties of the Bernstein polynomials, such as positivity, partition of unity and the recursive computability carry over to these generalizations.

Keywords: Bernstein Polynomials, Stancu Operators, Curve Fitting.

1. Introduction

Very important tools in CAGD are so-called linear interpolation operators for the representation of curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ and also – if one uses for example tensor product methods – of surfaces in $\mathbb{R}^3$.

Here, following DeVore [2, p.26], an operator L is called linear interpolation operator, if it is of the form

$$L(f,x) = \sum_{\nu=0}^{n} f(x_\nu) h_{n\nu}(x) ,$$

where f denotes a certain prescribed function, and the $x_0, \ldots, x_n$ as well as the variable x are real numbers, usually restricted to the interval [0,1]. It is *not* meant that L really *interpolates* the function f in the points x_ν ; the term interpolation only indicates that the shape of $L(f, \ )$ depends only on a finite number of values $f(x_\nu)$ – in contrast, for example – to integral operators.

For application in CAGD, one replaces the $f(x_\nu)$ by the *control points* b_ν, where $\mathsf{b}_\nu \in \mathbb{R}^2$ or $\mathbb{R}^3$, $\nu = 0, \ldots, n$, and the resulting

$$L(x) = \sum_{\nu=0}^{n} \mathsf{b}_\nu h_{n\nu}(x)$$

represents a curve which depends only on the control points b_ν (cf., also for the following, the excellent survey paper of Böhm, Farin and Kahmann [1]).

The two most important choices for the basis functions $h_{n\nu}$ are the Lagrange polynomials

$$l_{n\nu}(x) = \prod_{\substack{\lambda=0 \\ \lambda\neq\nu}}^{n} \frac{x - x_\lambda}{x_\nu - x_\lambda}$$

and the Bernstein (basis) polynomials

$$p_{n\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

for $x \in [0,1]$. In particular the latter ones are of great interest; the theoretical background is given by the well-known theorem of Bernstein and Korovkin, which says that for $f \in C[0,1]$ the polynomial operators

$$B_n(f,x) = \sum_{\nu=0}^{n} f(\frac{n}{\nu}) p_{n\nu}(x)$$

converge to f, as n tends to infinity.

The crucial point for the applicability of the corresponding operators

$$B_n(x) = \sum_{\nu=0}^{n} \mathsf{b}_\nu p_{n\nu}(x) \tag{1.1}$$

in CAGD are the following four properties:

$$\left.\begin{array}{ll} (1) & p_{n\nu}(x) \geq 0 \text{ for } x \in [0,1], \\ (2) & \sum_{\nu=0}^{n} p_{n\nu}(x) \equiv 1, \\ (3) & B_n(0) = \mathsf{b}_0 \text{ and } B_n(1) = \mathsf{b}_n, \\ (4) & \text{For } n \in I\!N \text{ and } 1 \leq \nu \leq n-1: \\ & p_{n\nu}(x) = (1-x)\, p_{n-1,\nu}(x) + x\, p_{n-1,\nu-1}(x), \end{array}\right\} \tag{1.2}$$

where in particular (4) is important, because it leads to the development of de Casteljau type algorithms; (3) guarantees that two operators of the type (1.1) can be continuously connected.

In this paper we present two different generalizations of the Bernstein polynomials, which are both suited to improve the flexibility of curve representation methods in CAGD.

The first one goes back to an idea of D.D.Stancu [5] and will be presented in the next section; a class of polynomials $q_{n\nu}$ is defined, which depend on *just one* additional real parameter, say a, and cover, for example, the Lagrange as well as the Bernstein polynomials as special cases. To our believe, this concept will be of great interest for designers of CAGD systems.

In section 3, we present another type of generalization, which is due to G.G.Lorentz [3]; here the basic idea is to use Bernstein polynomials, which are defined with respect to the space $\{x^{\alpha_0}, \ldots, x^{\alpha_n}\}$, $0 = \alpha_0 < \ldots < \alpha_n \in I\!R$, instead of the space Π_n.

It is important that both types of generalizations preserve the properties (1.2) of the Bernstein operators.

2. The Stancu Polynomials and Operators

As it is known the Bernstein operators (1.1) are quite suitable for the representation of curves, due to their nice properties (1.2); furthermore, they do not tend to an oscillating curve, as n increases – in contrast, for example, to Lagrange polynomials. On the other hand, one sometimes would like to have (polynomial) operators which react a little bit more sensitive on alterations of the control points b_ν ; and, even more, that the degree of this influence can be controlled by a user-defined parameter.

Such a possibility is given by using the following operators, which were introduced by D.D.Stancu in [5] and further investigated by the same author in [6, 7] and G.Mühlbach [4]:

Definition 2.1: Let x and a be arbitrary real numbers and define for $\nu \in I\!N_0$:

$$\varphi_0(x,a) := 1\,, \quad \varphi_\nu(x,a) := \prod_{\lambda=0}^{\nu-1}(x+\lambda a), \quad \text{and}$$

$$q_{n\nu}(x,a) := \binom{n}{\nu}\frac{\varphi_\nu(x,a)\cdot\varphi_{n-\nu}(1-x,a)}{\varphi_n(1,a)}\,, \quad n \in I\!N\,.$$

Then for $\nu = 0,1,\ldots,n$ the $q_{n\nu}$ are polynomials (in x) of degree n, which depend on the parameter a (assuming $\varphi_n(1,a) \neq 0$); they will be denoted as *Stancu polynomials.*

The corresponding operators

$$Q_n(x,a) \;:=\; \sum_{\nu=0}^{n} \mathrm{b}_\nu q_{n\nu}(x,a) \tag{2.1}$$

will be called *Stancu operators.*

Their main properties, which should be compared to (1.2), are summarized in terms of the following theorem (cf. [4]):

Theorem 2.2:

(1) For $x \in X$, $q_{n\nu}(x,a) \geq 0$, where

$$\begin{cases} X = [0,1], & \text{if } a \geq 0, \\ X = [a(1-n), 1-a(1-n)], & \text{if } a < 0. \end{cases}$$

(2) For fixed a,

$$\sum_{\nu=0}^{n} q_{n\nu}(x,a) \equiv 1.$$

(3) $Q_n(0,a) = \mathrm{b}_0$ and $Q_n(1,a) = \mathrm{b}_n$.

(4) For $n \in I\!N$ and $1 \le \nu \le n-1$:

$$q_{n\nu}(x,a) = \frac{1}{1+(n-1)a} \cdot \big((1-x+(n-\nu-1)\,a)\; q_{n-1,\nu}(x,a) + (x+(\nu-1)\,a)\; q_{n-1,\nu-1}(x,a) \big).$$

Proof: The properties (1), (2) and (4) can already be found in G.Mühlbach's paper [4]; the proof of (3), however, can be done by straightforward calculations, using the fact that

$$\binom{n-1}{\nu} + \binom{n-1}{\nu-1} = \binom{n}{\nu}. \quad \Delta$$

Now is is interesting to analyse the dependence of $Q_n(x,a)$ on the parameter a: Stancu already pointed out that $q_{n\nu}(x,a)$ coincides with the Bernstein polynomial $p_{n\nu}(x)$, if $a = 0$, and with the Lagrange polynomial $l_{n\nu}(x)$, if $a = -1/n$, where in the latter case the x_ν must be chosen equal to ν/n [6].

Now we have a closer look onto the behaviour of $Q_n(x,a)$ for several values of a; first, if $a < -1/n$, we have a highly oscillating curve, which has – to our believe – no practical applications.

Much more interesting is the range $-1/n \le a \le 0$; here we get operators, which lie "between" the Lagrange and the Bernstein operators (and depend continuously on the parameter a). More precisely, the Q_n behave as follows: starting, for $a = 0$, with the well-known Berstein operators, one gets, as a tends to $-1/n$, operators, which become more and more interpolatory with respect to the control points b_ν, until one ends up at the Lagrange operators, which do in fact interpolate the b_ν. Figure 1 sketches this behaviour for $n = 3$ and $a = 0, -1/3n, -2/3n, -1/n$.

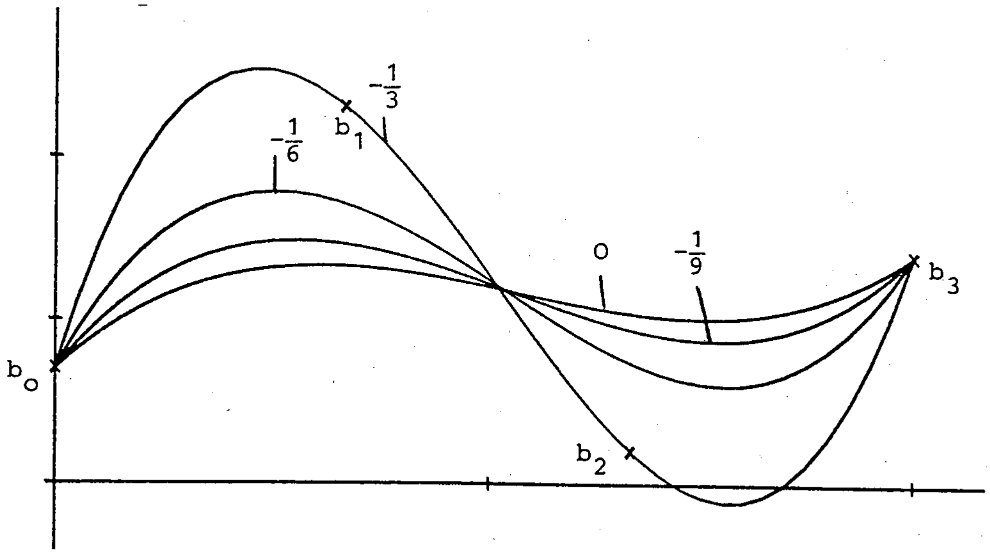

Figure 1: $Q_3(x,a)$ for various values of a

Finally we show that for $a \to +\infty$, the resulting curves tend to a straigth line:

Lemma 2.3: For fixed $x \in [0,1]$, let $Q_n(x,a)$ denote the Stancu operator (2.1); then

$$\lim_{a\to+\infty} Q_n(x,a) = (1-x)\,\mathrm{b}_0 + x\,\mathrm{b}_n \, .$$

Proof: First we note that

$$q_{n0}(x,a) = \prod_{\lambda=0}^{n-1} \frac{(1-x+\lambda a)}{(1+\lambda a)}$$

$$= \frac{(1-x)(n-1)!\,a^{n-1} + O(a^{n-2})}{(n-1)!\,a^{n-1} + O(a^{n-2})} \quad \text{for } a \to \infty \, .$$

Therefore

$$\lim_{a\to+\infty} q_{n0}(x,a) = 1-x \, ,$$

and, as it can be shown in a completely analogous manner,

$$\lim_{a\to+\infty} q_{nn}(x,a) = x \, .$$

So we are left to prove that

$$\lim_{a\to+\infty} q_{n\nu}(x,a) = 0 \quad \text{for } 1 < \nu < n \, .$$

But this follows at once, because

$$q_{n\nu}(x,a) = \binom{n}{\nu} \frac{((\nu-1)!\,a^{\nu-1}+O(a^{\nu-2}))\cdot((n-\nu-1)!\,a^{n-\nu-1}+O(a^{n-\nu-2}))}{(n-1)!\,a^{n-1}+O(a^{n-2})}$$

$$= \frac{O(a^{n-2})}{(n-1)!\,a^{n-1}+O(a^{n-2})} \to 0 \qquad \text{for } a \to +\infty\,. \quad \Delta$$

So, roughly spoken, the shape of the curves defined by the Stancu operators (2.1) varies from a highly oscillating polynomial curve to a straight line, as a moves from $-\infty$ to $+\infty$; the most interesting range for CAGD applications is given by $-1/n \le a \le 0$.

For example, one could leave the definition of the parameter a free to the user of the CAGD system; if he wants to construct a curve (or surface), which depends stronger on the control points, he would have to choose a value near $-1/n$. If, on the other hand, a curve is wanted which is not so sensitive against moves the control points, a value of a near 0 would be suited.
From this point of view, one could denote a as the "sensitivity parameter" of the system.

3. The Lorentz Polynomials and Operators

In the following, we sketch a completely different type of generalized Bernstein polynomials, which is due to G.G.Lorentz. We restrict ourselves to a very short presentation of the mathematical background and refer the interested reader to Lorentz' book [3].

Let $\alpha_0 = 0$ and $\alpha_\nu \ge 0$ be arbitrary and $x \in [0,1]$. Define, for $n \in I\!N$,

$$\tilde{p}_{nn}(x) := 1\,, \qquad \text{and for } \nu = 0,\ldots,n-1:$$
$$\tilde{p}_{n\nu}(x) := (-1)^{n-\nu}\,\alpha_{\nu+1}\cdots\alpha_n \cdot \frac{1}{2\pi i}\int_C \frac{x^z\,dz}{(z-\alpha_\nu)\cdots(z-\alpha_n)}\,, \tag{3.1}$$

where C is a simply closed curve in the complex plane such that all the points $\alpha_1,\ldots,\alpha_n$ lie in the interior of C.

If the α_ν are mutually different, then

$$\tilde{p}_{n\nu}(x) = (-1)^{n-\nu}\alpha_{\nu+1}\cdots\alpha_n \sum_{\mu=\nu}^{n} x^{\alpha_\mu} \Big/ \prod_{\substack{i=\nu \\ i\neq\mu}}^{n} (\alpha_\mu - \alpha_i) \,. \tag{3.2}$$

In particular, if $\alpha_\mu = \mu$ for $\mu = 0,\ldots,n$, $\tilde{p}_{n\nu}$ coincides with the ordinary Bernstein basis polynomials $p_{n\nu}$. In this sense the $\tilde{p}_{n\nu}$ can be denoted as generalized Bernstein polynomials.

For completeness we treat also the case that some of the α_μ coincide; then $\tilde{p}_{n\nu}$ is a linear combination of functions of the form

$$x^{\alpha_\mu}\log^k(x)\,,\ k = 0,\ldots,k_\mu - 1\,,$$

where k_μ denotes the multiplicity of α_μ.

It can be shown (cf. [3, sect. 2.7]) that the properties (1.2) carry over also to this type of generalization. For numbers (1) through (3) this is true for all $0 = \alpha_0 \le \cdots \le \alpha_n$. For example, using the identity

$$\frac{1}{z} = \frac{1}{z-\alpha_n} - \frac{\alpha_n}{(z-\alpha_{n-1})(z-\alpha_n)} + \cdots + \frac{\alpha_1\cdots\alpha_n}{(z-\alpha_0)\cdots(z-\alpha_n)}$$

(note that $\alpha_0 = 0$) and the definition (3.1), one proves that

$$\sum_{\nu=0}^{n} \tilde{p}_{n\nu}(x) = \frac{1}{2\pi i}\int_C \frac{x^z}{z}\,dz = 1\,.$$

The validity of recursion formulas like (1.2), (4) cannot be proved in general, but for some special choices of the exponents α_μ. For example, in the important case that

$$\alpha_\mu = \rho\cdot\mu$$

for some $\rho \in I\!R$, $\rho > 0$, one easily shows – e.g. using (3.2) – that

$$\tilde{p}_{n\nu}(x) = (1 - x^\rho)\,\tilde{p}_{n-1,\nu}(x) + x^\rho\,\tilde{p}_{n-1,\nu-1}(x)$$

for $n \in I\!N$ and $1 \le \nu \le n-1$.

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