A global Navier-Stokes-type equation of bubbles

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Nr. 90/1989

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1. Introduction

In the last years one has observed a growing interest of scientists in chemistry and fluid mechanics in studying the fluid motion in a surface or interface (cf.[E],[Sc]).

The mathematical complexity of such a study is due on one hand to the global, non-Euclidian geometrical character of the flow, on the other hand to the non-linear physical phenomena, like capillarity, which govern the motion and have to be taken into account.

It is the aim of the present paper to give a short presentation of a global model for the fluid flow of the bubble.

2. The model

Let us consider a bubble moving in the atmosphere in $\mathbb{R}^3$.

By a bubble we mean a skin of a deformable medium enclosing an open 3-dimensional region in $\mathbb{R}^3$ filled with athmosphere, an ideal gas.

The skin shall be of even thickness e as to whether it undergoes deformations or not. We idealize this situation by assuming that the shell reduces to its middle surface, which is assumed to be a 2-dimensional smooth connected and oriented submanifold of $\mathbb{R}^3$. It is therefore given by a smooth embedding j of a 2-dimensional smooth compact connected and oriented manifold M into $\mathbb{R}^3$. The map j is called a configuration of the skin. The collection of all such configurations is called $E(M,\mathbb{R}^3)$. It is a open subset of $C^\infty(M,\mathbb{R}^3)$, the collection of all smooth $\mathbb{R}^3$-valued functions of M, endowed with the C^∞-topology (cf.[Bi,Fi],[Bi,Sn,Fi]).

For a given configuration $j \in E(M,\mathbb{R}^3)$ the atmospheric pressure, called outer pressure, is denoted by $\Pi_a(j)$, whileas the pressure of the ideal gas inside the bubble, called inner pressure will be denoted by $\Pi(j)$. For simplicity we assume that both Π_a , $\Pi : E(M,\mathbb{R}^3) \longrightarrow \mathbb{R}$ are smooth functions. Smoothness on Frèchet-manifolds is meant in the sense of [Bi,Sn,Fi]. Moreover we require the inner pressure $\Pi(j)$ to satisfy the state equation of an ideal gas

$$\Pi(j) \cdot \upsilon(j) = \mathbf{k} \cdot \mathbf{N} \cdot T(j) , \qquad \forall\ j \in E(M,\mathbb{R}^3) , \tag{2.1}$$

where **k** is the Boltzmann constant, **N** the Avogadro number of the gas inside the bubble, $T : E(M,\mathbb{R}^3) \longrightarrow \mathbb{R}$ a smooth function called the temperature and $\upsilon(j)$ the volume of the gas. A more formal definition of υ will be given later. By a parameter independent constitutive law of the skin we mean a smooth one-form

$$F : E(M,\mathbb{R}^3) \times C^\infty(M,\mathbb{R}^3) \longrightarrow \mathbb{R} . \tag{2.2}$$

Throughout these notes we assume that F admits an integral representation

$$F(j)(l) = {}_M\!\int \langle \varphi_F(j),l \rangle\ \mu(j) , \qquad \forall\ j \in E(M,\mathbb{R}^3) ,\ l \in C^\infty(M,\mathbb{R}^3) , \tag{2.3}$$

where

$$\varphi_F : E(M,\mathbb{R}^3) \longrightarrow C^\infty(M,\mathbb{R}^3) \tag{2.4}$$

is a smooth map, called the force density acting up on the bubble,

(2.5) $\langle\ ,\ \rangle$ is a fixed scalar product on the oriented $\mathbb{R}^3$,

and

(2.6) $\mu(j)$ is the Riemannian volume on M associated with the metric $m(j)$ given by

$$m(j)(X,Y) = \langle djX,djY \rangle , \tag{2.7}$$

for any two smooth vector fields X and Y on M.

By dj we mean the differential

$$dj : TM \longrightarrow \mathbb{R}^3 , \tag{2.8}$$

of $j \in E(M,\mathbb{R}^3)$ which is locally nothing else but the Fréchet derivative of the local representation of $j \in E(M,\mathbb{R}^3)$.

We say that the constitutive law F depends on a parameter varying in the parameter manifold $\mathfrak{P}$ if there is a smooth map

$$a : E(M,\mathbb{R}^3) \longrightarrow \mathfrak{P} \tag{2.9}$$

such that the constitutive law is a smooth map

$$F : E(M,\mathbb{R}^3) \times \mathfrak{P} \times C^\infty(M,\mathbb{R}^3) \longrightarrow \mathbb{R} , \tag{2.10}$$

which is required to be linear in the third argument and allows an integral representation of the form

$$F(j,a(j))(l) = {}_M\!\int \langle \varphi_F(j,a(j)),l \rangle\ \mu(j) , \tag{2.11}$$

where

$$\varphi_F : E(M,\mathbb{R}^3) \times \mathfrak{P} \longrightarrow C^\infty(M,\mathbb{R}^3) , \tag{2.12}$$

the so-called force density, is smooth. We notice that the real number $F(j,a(j))(l)$ is the work necessary to deform the material at the manifold $j(M)$ in the infinitesimal direction $l \in C^\infty(M,\mathbb{R}^3)$.

We complete the model by specifying a fixed parameter dependent constitutive law F, and in addition a density function

(2.13) $$\rho : E(M,\mathbb{R}^3) \longrightarrow C^\infty(M,\mathbb{R}) .$$

We call the real number

(2.14) $$m(j) := e \cdot {}_M\!\int \rho(j)\cdot\mu(j) ,$$

the total mass m(j) at the configuration $j \in E(M,\mathbb{R}^3)$. This density function is assumed to satisfy a continuity equation

(2.15) $$D\rho(j)(l) = -\frac{\rho(j)}{2}\ tr_{m(j)}Dm(j)(l) , \qquad \forall\ j \in E(M,\mathbb{R}^3) ,\ \forall\ l \in C^\infty(M,\mathbb{R}^3) ,$$

where $D\rho(j)(l)$ is the derivative of ρ at j in the direction of l, $tr_{m(j)}$ means the trace formed with respect to the metric m(j) and Dm(j)(l) is the derivative of m(j) at j in the direction of l. The right hand side of (2.15) steams from the derivative of the Riemannian volume reading as

(2.16) $$D\mu(j)(l) = (\tfrac{1}{2}\, tr_{m(j)}Dm(j)(l))\cdot\mu(j) , \qquad \forall\ j \in E(M,\mathbb{R}^3),\ \forall\ l \in C^\infty(M,\mathbb{R}^3).$$

If l is decomposed into its part tangential to j(M) and normal to j(M) that is into

(2.17) $$l = djX(j,l) + \Theta(j,l)\cdot N(j) ,$$

with N(j) the positively oriented unit normal vector field and $X(j,l) \in \Gamma TM$ (where ΓTM is the collection of all smooth vector fields on M), then the derivative of m(j) is given by

(2.18) $$Dm(j)(l)(X,Y) = L_{X(j,l)}(m(j))(X,Y) + 2\cdot\Theta(j,l)\cdot m(j)(W(j)X,Y) .$$

Here $L_{X(j,l)}$ denotes the Lie derivative and $W(j) : TM \longrightarrow TM$ the Weingarten map, a smooth strong bundle endomorphism of TM selfadjoint with respect to m(j) and given by

(2.19) $$W(j)X = dN(j)X .$$

As usual we set

(2.20) $$H(j) := tr\ W(j) ,$$

twice the mean curvature of the middle surface j(M).

3. Internal properties of the skin

The internal physical properties of the skin material are independent of the specific location of j(M) in $\mathbb{R}^3$. In other words the part of the constitutive law F depending on the internal physical properties only is invariant under the translation group $\mathbb{R}^3$ of $\mathbb{R}^3$. We denote it by F_{int}.

We obtain a parameter independent $\mathbb{R}^3$-invariant constitutive law by specifying a smooth one-form

(3.1) $$F_{\mathbb{R}^3} : E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \times C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \longrightarrow \mathbb{R}$$

and setting

(3.2) $$F_{int} = d^* F_{\mathbb{R}^3}$$

with

(3.3) $$d : E(M,\mathbb{R}^3) \longrightarrow E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} ,$$

the differential and d^* its pull back. Specifying $\mathfrak{P}$ to be $C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3}$ and

$$a_{\mathbb{R}^3} : E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \longrightarrow C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3}$$

to be a smooth map we obtain a parameter dependent constitutive law in the obvious way. For the sake of simplicity we write dk instead of $a_{\mathbb{R}^3}(dj)$.

To obtain now an integral representation for $F_{\mathbb{R}^3}$ we proceed as follows :

According to [Bi,1], [Bi,2] or [Bi,Soc] any smooth one-form $\alpha : TM \longrightarrow \mathbb{R}^3$ can be uniquely decomposed with respect to a differential $dj \in E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3}$ as

(3.4) $$\alpha = c(\alpha,dj)dj + dj(C(\alpha,dj) + B(\alpha,dj)) ,$$

where the coefficients of dj on the right hand side have the following meaning : The first one is a smooth map

(3.5) $$c(\alpha,dj) : M \longrightarrow so(3) ,$$

with values in the Lie-algebra so(3) of SO(3) flipping vectors tangential to j(M) into normal ones and vice versa. The other two coefficients $C(dj,\alpha)$ and $B(dj,\alpha)$ are smooth strong bundle endomorphisms of TM skew respectively selfadjoint with respect to m(j). Given any two $\mathbb{R}^3$-valued one-forms $\alpha_1, \alpha_2 : TM \longrightarrow \mathbb{R}^3$ we define the real valued dot product of α_1 and α_2 by

(3.6) $$\alpha_1 \cdot \alpha_2 := - \big(\operatorname{tr} (c(\alpha_1,dj) \cdot c(\alpha_2,dj)) + \operatorname{tr} (C(\alpha_1,dj) \cdot C(\alpha_2,dj)) \big) + \operatorname{tr} (B(\alpha_1,dj) \cdot B(\alpha_2,dj)) ,$$

where the dot on the right hand side of the equation means pointwise composition. For the sequel we specify one parameter dependent constitutive law

(3.7) $$F_{\mathbb{R}^3} : E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \times C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \times C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \longrightarrow \mathbb{R} ,$$

by

(3.8) $$F_{\mathbb{R}^3}(dj,dk)(dl) = {}_M\!\int \alpha(dj,dk)\cdot dl \; \mu(j) ,$$

with

(3.9) $$\alpha : E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \times C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \longrightarrow A^1(M,\mathbb{R}^3) ,$$

a smooth map called stress form. Its range is the space of all smooth $\mathbb{R}^3$-valued one-forms endowed with the C^∞-topology.

The smooth two-tensor T(dj,dk) defined by

(3.10) $$T(dj,dk)(X,Y) := \langle \alpha(dj,dk)X, djY \rangle ,$$

for all $X,Y \in \Gamma TM$ is the so-called stress tensor.

Now let us show that F admits an integral representation (cf. [Bi,2] or [Bi,Soc]).

Theorem 1 :

For each $dj \in E(M,\mathbb{R}^3)|_{\mathbb{R}^3}$ and each $dk \in C^\infty(M,\mathbb{R}^3)|_{\mathbb{R}^3}$ the density of the internal forces $\varphi_{F_{int}}$ is given by

(3.11) $$\varphi_{F_{int}}(dj,dk) = - dj\,(\mathrm{div}_j\,(B(\alpha(dj,dk),dj) + C(\alpha(dj,dk),dj)) - 2 \cdot W(j)\,U(\alpha(dj,dk))) + (\,\mathrm{tr}\, B(\alpha(dj,dk),dj) \cdot W(j)) \cdot N(j)\ .$$

Here $U(\alpha(dj,dk))$ is defined by

(3.12) $$dj\,U(\alpha(dj,dk)) = c(\alpha(dj,dk),dj) \cdot N(j)$$

and $\mathrm{div}_j A$ for some smooth strong bundle endomorphism A of TM by

(3.13) $$\mathrm{div}_j A = \sum_{s=1}^{2} \nabla(j)_{e_s}(A)\, e_s\ ,$$

where e_1, e_2 is a moving frame on M orthonormal with respect to m(j) and $\nabla(j)$ denotes the Levi-Civita connection of m(j).

As shown in [Bi,2] or in [Bi,Soc] we have the following :

Theorem 2 :

Let $\alpha \in A^1(M,\mathbb{R}^3)$ and $j \in E(M,\mathbb{R}^3)$. Then there is a unique $h \in C^\infty(M,\mathbb{R}^3)$, up to a constant in $\mathbb{R}^3$, such that

(3.14) $$\alpha = dh + \beta\ ,$$

satisfying

(3.15) $$\int_M \alpha \cdot dl\ \mu(j) = \int_M dh \cdot dl\ \mu(j)\ .$$

dh is called the integrable part of α.

We sketch the proof :

Let a_1, a_2, a_3 be an orthonormal basis of $\mathbb{R}^3$ (with respect to $\langle\ ,\ \rangle$).

Then we set

(3.16) $$\alpha(X) = \sum_{r=1}^{3} m(j)(Y^r,X)\, a_r\ , \qquad \forall\ X \in TM\ .$$

Decomposing $Y^r = \mathrm{grad}\,\rho_r + Y_r^0$ with $\mathrm{div}_j Y_r^0 := \mathrm{tr}\,\nabla(j) Y_r^0 = 0$ for r=1,2,3 we set :

(3.17) $$h := \sum_{r=1}^{3} \tau^r \cdot a_r$$

and

(3.18) $$\beta\, X := \sum_{r=1}^{3} m(j)(Y_r^0,X) a_r\ , \qquad \forall\ X \in \Gamma TM\ .$$

This yields the first part of the theorem.

Now we represent dh and $d\beta$ with respect to dj yielding

(3.19) $$dh = c(dh,dj)\cdot dj + dj\,\big(C(dh,dj) + B(dh,dj)\big)$$

and

(3.20) $$\beta = c(\beta,dj)\cdot dj + dj\,\big(C(\beta,dj) + B(\beta,dj)\big)\ .$$

If $E_r \in \Gamma TM$ is defined by $dj\,E_r = a_r$ for r=1,2,3 then for each r=1,2,3

(3.21) $$dj\ \mathrm{grad}\ \tau_r = -\,c(dh,dj)\,a_r - dj\big(C(dh,dj) - B(dh,dj)\big)\,E_r$$

and

(3.22) $$dj\ Y^0_r = -\,c(\beta,dj)\,a_r - dj\big(C(\beta,dj) - B(\beta,dj)\big)\,E_r\ .$$

(3.23) $$\sum_{r=1}^{3} m(j)(\mathrm{grad}\ \tau_r,\ Y^0_r) = dh\cdot\beta\ .$$

Since $_M\!\int m(j)(\mathrm{grad}\ \tau_r,\ Y^0_r)\ \mu(j) = 0$ the second part of the theorem is established.

From now on we therefore assume that

(3.24) $$\alpha(dj,dk) = dh(dj,dk)$$

for a smooth map

(3.25) $$h : E(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \times C^\infty(M,\mathbb{R}^3)\big|_{\mathbb{R}^3} \longrightarrow \mathbb{R}\ .$$

The total force density φ_F is the sum of $\varphi_{F_{int}}$ and $\varphi_{F_{ext}}$, i.e.

(3.26) $$\varphi_F(j,k) = \varphi_{F_{int}}(dj,dk) + \big(\Pi_a(j) - \Pi(j)\big)\cdot N(j)\ , \qquad \forall\ j \in E(M,\mathbb{R}^3),\ \forall\ k \in C^\infty(M,\mathbb{R}^3).$$

Let now

(3.27) $$V(j) = {}_M\!\int \mu(j)\ ,\quad \forall\ j \in E(M,\mathbb{R}^3)$$

be the area of j(M). Then

(3.28) $$DV(j)(l) = {}_M\!\int H(j)\cdot\langle N(j),l\rangle\ \mu(j)\ , \qquad \forall\ j \in E(M,\mathbb{R}^3)\ ,\ \forall\ l \in C^\infty(M,\mathbb{R}^3)\ .$$

This remark allows us to understand what part of $\mathrm{tr}\ B(dj,dk)\cdot W(j)$ yields the volume work. We thus decompose $\mathrm{tr}\ B(dj,dk)\cdot W(j)$ along H(j) and normal to it in the Hilbert space of all $\mu(j)$–L_2–functions on M. Hence we have (cf.[Bi,3])

(3.29) $$\mathrm{tr}\ B(dj,dk)\cdot W(j) = q(dj,dk)\cdot H(j) + \big(\mathrm{tr}\ B(dj,dk)\cdot W(j)\big)^{\perp}\ .$$

Since the skin is bounded by two surfaces we set

(3.30) $$q(dj,dk) = 2\cdot\pi(dj,dk)\ .$$

Thus we get for the total force density at $(j,k) \in E(M,\mathbb{R}^3) \times C^\infty(M,\mathbb{R}^3)$ by

(3.31) $$\begin{aligned}\varphi_F(j,k) = &-dj\,\big(\mathrm{div}_j\,\big(B(\alpha(dj,dk),dj) + C(\alpha(dj,dk),dj)\big) - 2\cdot W(j)\ U(\alpha(dj,dk),dj)\big)\\ &+ \big(2\pi(j,k)\cdot H(j) + (\Pi(j) - \Pi_a(j))\big)\cdot N(j)\\ &+ \big(\mathrm{tr}\ B(\alpha(dj,dk),dj)\cdot W(j)\big)^{\perp}\cdot N(j)\ .\end{aligned}$$

If $\varphi_F(j,k) = 0$ for some $(j,k) \in E(M,\mathbb{R}^3) \times C^\infty(M,\mathbb{R}^3)$, an equilibrium condition, and if the material satisfies $B(\alpha(dj,dk),dj) = \pi(j,k)\cdot H(j)\cdot id$ then

$$2\cdot\pi(j,k)\cdot H(j) = \Pi_a(j) - \Pi(j) \,, \tag{3.32}$$

a relation playing the role of Laplacian law if we interpret $\pi(j,k)$ as structural capillarity determined by one skin boundary surface. Since $\Pi_a(j)$, $\Pi(j)$ and $\pi(j,k)$ are all reals, $H(j)$ has to be a real too. Thus $j(M)$ is a minimal surface in $\mathbb{R}^3$.

Let us turn back to the volume $\mathfrak{v} : E(M,\mathbb{R}^3) \longrightarrow \mathbb{R}$, of the ideal gas given by

$$\mathfrak{v}(j) = {}_M\!\int \langle j,N(j)\rangle \, \mu(j) \,, \qquad \forall j \in E(M,\mathbb{R}^3). \tag{3.33}$$

Since for all $j \in E(M,\mathbb{R}^3)$ and all $h \in C^\infty(M,\mathbb{R}^3)$

$$D\mathfrak{v}(j)(h) = {}_M\!\int \langle h,N(j)\rangle \, \mu(j) \,, \tag{3.34}$$

we see that

$$(\Pi_a(j) - \Pi(j)) \cdot D\mathfrak{v}(j)(h) + 2\,\pi(j)\, DV(j)(h) \tag{3.35}$$

is the work needed to deform the volume $\mathfrak{v}(j)$ and the areas $2\,V(j)$ of the two surfaces bounding the skin.

4. The structural viscosity and the equation of motion

As previously shown it suffices to specify a smooth map

$$h : E(M,\mathbb{R}^3) \times C^\infty(M,\mathbb{R}^3) \longrightarrow C^\infty(M,\mathbb{R}^3) \,, \tag{4.1}$$

in order to obtain a constitutive law $F_{\mathbb{R}^n}$, where, as mentioned above, h is only determined up to a constant in $\mathbb{R}^3$. Moreover it is possible to exhibit a structural viscosity of the deformable medium under consideration in the following way :

First write for any $u \in C^\infty(M,\mathbb{R}^3)$

$$u = djX(u,j) + \Theta(u,j)\cdot N(j) \,, \qquad \text{where } X(u,j) \in \Gamma TM \,, \tag{4.2}$$

and decompose $X(u,j)$ into

$$X(u,j) = \mathrm{grad}_j \Psi(u,j) + X^0(u,j) \tag{4.3}$$

with $\mathrm{div}_j X^0(u,j) = 0$. As shown in [Bi,3] or [Bi,Soc] the component $X^0(u,j)$ is uniquely determined by du. Thus for any $j \in E(M,\mathbb{R}^3)$ and any $k \in E(M,\mathbb{R}^3)$ the vector field $X^0(h(j,k),j)$ is uniquely determined by $dh(j,k)$ and dk respectively.

We then may set

$$X^0(h(j,k),j) = \nu(h(j,k),j)\cdot X^0(k,j) + \hat{X}(h(j,k),j) \,, \tag{4.4}$$

where $\hat{X}(h(j,k),j)$ is pointwise perpendicular to $X^0(k,j)$.

The function $\nu(h(j,k),j)$ is called the coefficient of structural viscosity (cf.[Bi,3]). Choosing a stress form of a particular kind the equation of the motion of the material will satisfy a Navier–Stokes type of equation as we will see bellow. To get this equation we use a simplified version of (4.5) and set

$$(4.5)\qquad h(j,k) := \nu\cdot dj\ X^0(k,j) + \Theta(h(j,k),j)\cdot N(j)\ ,$$

for all $j \in E(M,\mathbb{R}^3)$ and all $k \in C^\infty(M,\mathbb{R}^3)$ and a constant ν. Clearly for any $j \in E(M,\mathbb{R}^3)$, any $k \in C^\infty(M,\mathbb{R}^3)$ and any $X \in \Gamma TM$

$$(4.6)\qquad \begin{aligned} dh(j,k)X = {} & m(j)(\operatorname{grad}_j\Theta(h(j,k),j) - \nu\cdot W(j)\ X^0(k,j)\ ,\ X)\cdot N(j) \\ & + dj\ \nu\cdot\nabla(j)\ X^0(k,j) + dj\ \Theta(h(j,k),j)\cdot W(j)\ X\ . \end{aligned}$$

Thus we read off

$$(4.7)\qquad c(dj,dh(j,k))\ dj\ X = m(j)(\operatorname{grad}_j\Theta(h(j,k),j) - W(j)X^0(k,j)\ ,\ X)\cdot N(j)\ ,$$

$$(4.8)\qquad C(dh(j,k),dj) = \tfrac{\nu}{2}\cdot(\nabla(j)\ X^0(k,j) - \tilde{\nabla}(j)\ X^0(k,j))\ ,$$

$$(4.9)\qquad B(dh(j,k),dj) = \tfrac{\nu}{2}\cdot(\nabla(j)\ X^0(k,j) + \tilde{\nabla}(j)\ X^0(k,j) + \Theta(h(j,k),j)\cdot W(j)\ ,$$

with $\tilde{\nabla}(j)\ X$ the adjoint of $\nabla(j)\ X$ for any $X \in \Gamma TM$. Let us remark that

$$(4.10)\qquad \mu(j)(X,Y) = m(j)(\mathfrak{V}(j)X,Y)\ ,\qquad \forall\ X,Y \in \Gamma TM$$

for $\mathfrak{V}(j)$ a skew adjoint strong bundle isomorphism of TM. Hence

$$(4.11)\qquad C(dh(j,k),dj) = \zeta(dh(j,k),dj)\cdot\mathfrak{V}(j)\ ,$$

with $\zeta(dh(j,k),dj) \in C^\infty(M,\mathbb{R})$ satisfying

$$(4.12)\qquad \Delta(j)\ \xi(dh(j,k),dj) = 0\ ,\qquad \forall\ j \in E(M,\mathbb{R}^3)\ ,\ \forall\ k \in C^\infty(M,\mathbb{R}^3)\ ,$$

with $\Delta(j)$ the Laplacian determined by $m(j)$ as $-\operatorname{div}_j\operatorname{grad}_j$.

We call ζ the vorticity and ξ its corresponding stream function.

If we write

$$(4.13)\qquad \tfrac{1}{2}(\nabla(j)X^0(k,j)^0 - \tilde{\nabla}(j)X^0(k,j) = \zeta(k,j)\cdot\mathfrak{V}(j)\ ,$$

then

$$(4.14)\qquad C(dh(j,k),dj) = \nu\cdot\zeta(k,j)\cdot\mathfrak{V}(j)\ ,$$

which implies

$$(4.15)\qquad \operatorname{div}_j C(dh(j,k),dj) = \nu\cdot\mathfrak{V}(j)(\operatorname{grad}_j\zeta(k,j))\ ,$$

Since $\operatorname{tr}\ \nu\cdot\nabla(j)X^0(k,j) = 0$ we find

$$(4.16)\qquad \operatorname{tr}\ B(dh(j,k),dj) = \Theta(h(j,k),j)\cdot H(j)\ .$$

Let us therefore write

$$(4.17)\qquad \begin{aligned} B(dh(j,k),dj) = {} & \tfrac{1}{2}\Theta(h(j,k),j)\cdot H(j)\cdot id \\ & + \tfrac{\nu}{2}\cdot(\nabla(j)X(k,j) - \tilde{\nabla}(j)X(k,j))) \\ & + \Theta(h(j,k),j)\cdot(W(j) - \tfrac{1}{2}\ H(j)\cdot id)\ . \end{aligned}$$

Denoting $\frac{1}{2}\Theta(h(j,k),j)\cdot H(j)$ by $\tau(j,k)$ and $B(dh(j,k)dj) - \tau(j,k)\cdot id$ by $\hat{B}(dh(j,k),dj)$

we get

$$(4.18)\qquad B(dh(j,k)dj) = \tau(j,k)\cdot \mathrm{id} + \hat{B}(dh(j,k),dj)\ .$$

If we set moreover

$$(4.19)\qquad \mathrm{tr}\, B(dh(j,k),dj)\cdot W(j) = \big(\tau(j,k) + 2\pi(j)\big)\cdot H(j) + b(dh(j,k),dj)\ ,$$

where

$$(4.20)\qquad b(dh(j,k),dj) = \mathrm{tr}\, \hat{B}(dh(j,k),dj)\cdot W(j)^{\perp}\ ,$$

then we find for our force density at any $j \in E(M,\mathbb{R}^3)$ and any $h \in C^{\infty}(M,\mathbb{R}^3)$

$$(4.21)\qquad \begin{aligned} \varphi_F(j,k) = &- dj\,(\mathrm{grad}_j\tau(j,k) + \nu\cdot\Delta(j)\, X^0(j,k) \\ &- \nu\cdot\mathfrak{B}(j)\,(\mathrm{grad}_j\, \zeta(k,j)) \\ &- 2\, W(j)(\nu W(j)X^0(j,k) - \mathrm{grad}_j\Theta(h(j,k),j))) \\ &+ ((\tau(j,k) + 2\pi(j))\cdot H(j) + \Pi_a(j) - \Pi(j))\cdot N(j) \\ &+ b(dh(j,k),dj)\cdot N(j)\ . \end{aligned}$$

This formula motivates us to choose h(j,k) (just for the sake of simplicity) in such a way that

$$(4.22)\qquad b(dh(j,k),dj) = 0\ ,$$

which yields an equation for $\Theta(h(j,k),j)$. In this case the total force density to use for our equations of motion is for any $j \in E(M,\mathbb{R}^3)$ and any parameter $k \in C^{\infty}(M,\mathbb{R}^3)$

$$(4.23)\qquad \begin{aligned} \varphi_F(j,k) = &- dj(\mathrm{grad}_j\tau(j,k) + \nu\cdot\Delta(j)X^0(j,k) \\ &- \nu\cdot\mathfrak{B}(j)(\mathrm{grad}_j\zeta(k,j)) - 2\, W(j)(\nu W(j)X^0(j,k) - \mathrm{grad}_j\Theta(h(j,k),j)) \\ &+ ((\tau(j,k) + 2\pi(j))\cdot H(j) + \Pi_a(j) - \Pi(j))\cdot N(j)\ . \end{aligned}$$

Here the Laplacian $\Delta(j)$ determined by m(j) applied to any $X \in \Gamma TM$ is given by $-\mathrm{tr}(\nabla X)^2$. The equation of motion for a smooth curve

$$\sigma : (-\lambda,\lambda) \longrightarrow E(M,\mathbb{R}^3)$$

is determined via the d'Alembert principle (cf.[He]) and reads as

$$(4.24)\qquad \begin{aligned} \rho(\sigma(t))\cdot\ddot{\sigma}(t) &= \varphi_{F_{int}}(\sigma(t),\dot{\sigma}(t)) + (\Pi_a(\sigma(t)) - \Pi(\sigma(t)))\cdot N(\sigma(t)) \\ &= \varphi_F(\sigma(t),\dot{\sigma}(t))\ . \end{aligned}$$

Choosing for any t $a_{\mathbb{R}^3}$ such that $a_{\mathbb{R}^3}(\sigma(t)) = d\sigma(t)Z(t)$ and splitting $\dot{\sigma}(t)$ into

$$(4.25)\qquad \dot{\sigma}(t) = d\sigma(t)Z(t) + \epsilon(\sigma(t),\dot{\sigma}(t))\cdot N(\sigma(t))\ ,$$

for a well-determined $Z(t) \in \Gamma TM$ we assume that $Z(t) = Z^0(t)$ and obtain

$$(4.26)\qquad X(h(\sigma(t),\dot{\sigma}(t),\sigma(t)) = \nu\cdot Z(t)\ ,$$

Equation (4.24) yields thus as an equation of motion along the surface $\sigma(t)(M)$ for all

$t \in (-\lambda,\lambda)$

$$(4.27)\qquad \begin{aligned} &\rho(\sigma(t))(\nabla(\sigma(t))_{Z(t)}Z(t) + \dot{Z}(t) + 2\cdot\epsilon(\sigma(t),\dot{\sigma}(t))\cdot W(\sigma(t))\, Z(t) \\ &\quad - \epsilon(\sigma(t),\dot{\sigma}(t))\cdot \mathrm{grad}_{\sigma(t)}\epsilon(\sigma(t),\dot{\sigma}(t)) \\ &= - (\mathrm{grad}_{\sigma(t)}\tau(\sigma(t),\dot{\sigma}(t)) + \nu\cdot\Delta(\sigma(t))\, Z(t) \\ &\quad - \nu\cdot\mathfrak{D}(\sigma(t))\cdot\mathrm{grad}_{\sigma(t)}\zeta(\sigma(t),\dot{\sigma}(t)) \\ &\quad - 2\cdot W(\sigma(t))(\nu\cdot W(\sigma(t))\, Z(t) - \mathrm{grad}_{\sigma(t)}\Theta(h(\sigma(t),\dot{\sigma}(t)),\sigma(t))) \ , \end{aligned}$$

an equation which reduces on all flat parts of $\sigma(t)(M)$ to a Navier–Stokes–type of equation if ϵ is equal to zero. The motion normal to $\sigma(t)(M)$ is governed by

$$(4.28)\qquad \begin{aligned} \rho(\sigma(t))\, \dot{\epsilon}(\sigma(t),\dot{\sigma}(t)) &= m(\sigma(t))(W(\sigma(t))\, Z(t),\, Z(t)) \\ &\quad - d\epsilon(\sigma(t),\dot{\sigma}(t))(Z(t)) + \big(\tau(\sigma(t),\dot{\sigma}(t)) + 2\pi(\sigma(t)) \\ &\quad + \Pi_a(\sigma(t)) - \Pi(\sigma(t))\big)\cdot N(\sigma(t)) \ , \end{aligned}$$

an equation also implied by (4.24).

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Remark :

This paper is in final form and no similar paper has been submitted elsewhere.