An Algorithm for Minimizing a Convex-Concave Function over a Convex Set

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Abstract – A branch-and-bound method is proposed for minimizing a convex-concave function over a convex set. The minimization of a dc-function is a special case, where the subproblems connected with the bounding operation can be solved effectively.

1. Introduction. In what follows we propose a branch-and-bound method for minimizing a convex-concave function over a convex set. A similar scheme for minimizing an indefinite quadratic function over a convex set has been described in our earlier paper [3]. Here, due to the more general form of the objective function, the branching operation must be different from the one used in [3], whereas the bounding operation is essentially the same and is based on a suitable relaxation of the constraint set. An important special case is the minimization of a dc-function (i.e., a function which is representable as the difference of two convex functions – see [1], [4]). In this case the subproblems occuring in the bounding operation can be solved effectively.

2. Problem Statement. Let $S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed convex set. Let the continuous function $f(\cdot,\cdot) : S \to \mathbb{R}$ be convex in the first argument and concave in the second argument. We consider the problem

$$\text{(P)} \qquad \min \{f(x,y) | x \in \mathbb{R}^n, y \in \mathbb{R}^m, (x,y) \in S\}.$$

We suppose that problem (P) admits a solution, and we denote by f^* the optimal value of (P). We suppose furthermore that we can fix two compact convex polyhedra $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ such that $X \times Y$ contains a solution of (P). Given a compact polyhedral subset $B \subset Y$ we shall have occasion to consider the problem

$$\text{R}(B) \qquad \min \{f(x,y) | x \in X, y \in B, u \in B, (x,u) \in S\}.$$

By $\beta(B)$ we denote the optimal value of $\text{R}(B)$ (we set $\beta(B) := \infty$, if $\text{R}(B)$ has no feasible points). If (x^B, y^B, u^B) is a solution of $\text{R}(B)$, then clearly

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$$\beta(B) = f(x^B, y^B) \le \min\ \{f(x,u) | x \in X, u \in B, (x,u) \in S\} \le f(x^B, u^B)$$

and $f^* \le f(x^B, u^B)$. If $X \times B$ contains a solution of (P), then $\beta(B) = f(x^B, y^B) \le f^*$. The solution of $R(B)$ will be discussed below in connection with the dc-problem.

3. Description of the Algorithm. The algorithm can now be described as follows (comments are inserted in brackets).

Initialization. Set $\Gamma_0 := \{Y\}$, $\alpha_{-1} := \infty$. Solve $R(Y)$.

Iteration k. At the beginning of iteration k $(k = 0, 1, \ldots)$ we have a collection Γ_k of compact polyhedral subsets $B \subset Y$ such that $X \times \cup\{B | B \in \Gamma_k\}$ contains a solution of (P). For each $B \in \Gamma_k$ we have determined $\beta(B)$ and, if $\beta(B) < \infty$, a solution (x^B, y^B, u^B) of $R(B)$. Furthermore we are given $\alpha_{k-1} \ge f^*$.

Let $\alpha_k := \min\ \{\alpha_{k-1}, \min\ \{f(x^B, u^B) | B \in \Gamma_k, \beta(B) < \infty\}\}$ $[\Rightarrow f^* \le \alpha_k]$.

Select $B_k \in \Gamma_k$ such that $\beta(B_k) = \min\ \{\beta(B) | B \in \Gamma_k\}$.

Let (x^k, y^k, u^k) be a solution of $R(B_k)$ $[\Rightarrow f(x^k, y^k) \le f^* \le f(x^k, u^k)]$.

If $f(x^k, y^k) \ge f(x^k, u^k)$, then terminate: (x^k, u^k) solves (P).

If $f(x^k, y^k) < f(x^k, u^k)$, then let $l_k(y) := \langle u^k - y^k, y\rangle$ and $c_k := (l_k(y^k) + l_k(u^k))/2$, and set

$$B_k^- := \{y \in B_k | l_k(y) \le c_k\},\ B_k^+ := \{y \in B_k | l_k(y) \ge c_k\}$$

$[\Rightarrow y^k \in B_k^- \ne \emptyset,\ u^k \in B_k^+ \ne \emptyset]$.

Solve $R(B_k^-)$, $R(B_k^+)$.

Let $\Delta_k := \{B \in \Gamma_k | \beta(B) \le \alpha_k\}$ $[\Rightarrow B_k \in \Delta_k]$.

Let $\Gamma_{k+1} := \Delta_k \setminus \{B_k\} \cup \{B_k^-, B_k^+\}$.

Go to iteration $k+1$.

This completes the description of iteration k.

4. Convergence of the Algorithm. If the algorithm terminates at iteration k, then $f(x^k, y^k) = f^* = f(x^k, u^k)$, and $(x^k, u^k) \in S$ is clearly a solution of (P). Otherwise we have again that $X \times \cup\{B | B \in \Gamma_{k+1}\}$ contains a solution of (P). Moreover we have $\beta(B_k) \le \beta(B_{k+1})$, hence $f(x^k, y^k) \le f(x^{k+1}, y^{k+1}) \le f^*$. If the algorithm does not terminate, then the sequence $\{(x^k, u^k)\}$ has a cluster point.

Theorem. *If the algorithm does not terminate, then every cluster point of* $\{(x^k, u^k)\}$ *is a solution of* (P). *Moreover* $f(x^k, y^k) \nearrow f^*$.

Proof: Let $(\overline{x}, \overline{u})$ be a cluster point of $\{(x^k, u^k)\}$. By extracting a subsequence, if necessary, we may assume that $x^k \to \overline{x}$, $u^k \to \overline{u}$, $y^k \to \overline{y}$, and furthermore that either $B_{k+1} \subset B_k^-$ for all k or $B_{k+1} \subset B_k^+$ for all k. If $B_{k+1} \subset B_k^-$ for all k, then in particular $u^{k+1} \in B_k^-$, hence $l_k(u^{k+1}) \leq c_k$. This gives

$$\|u^k - y^k\|^2 = l_k(u^k) - l_k(y^k) = 2(l_k(u^k) - c_k) \leq 2(l_k(u^k) - l_k(u^{k+1}))$$
$$\leq 2\|u^k - y^k\| \cdot \|u^k - u^{k+1}\|,$$

hence

$$\|u^k - y^k\| \leq 2\|u^k - u^{k+1}\| \to 0.$$

If $B_{k+1} \subset B_k^+$ for all k, then we use $y^{k+1} \in B_k^+$ to obtain in a similar way

$$\|u^k - y^k\| \leq 2\|y^{k+1} - y^k\| \to 0.$$

Hence in both cases we obtain $\overline{u} = \overline{y}$. Therefore $f(x^k, y^k) \nearrow f(\overline{x}, \overline{u})$, and from $f(x^k, y^k) \leq f^* \leq f(x^k, u^k)$ follows $f(\overline{x}, \overline{u}) = f^*$, i.e., $(\overline{x}, \overline{u}) \in S$ is a solution of (P).

q.e.d.

5. DC-Problems. The above algorithm can be applied to the so-called dc-problem

(DC) $$\min \{g(x) - h(x) | x \in G\},$$

where $G \subset I\!R^m$ is a closed convex set, and $g, h : G \to I\!R$ are continuous convex functions (supposed to be known explicitly). This problem has earned considerable interest recently, see [1], [4]. We bring problem (DC) into the form (P) by choosing

$$f(x, y) := g(x) - h(y) : G \times G \to I\!R,$$
$$S := \{(x, y) \in G \times G | x = y\} \subset I\!R^m \times I\!R^m.$$

We need a compact convex polyhedron $Y \subset I\!R^m$ such that Y contains a solution of (DC). Then, if $B \subset Y$ is a compact polyhedral subset , the problem R(B) with the above choices of f and S and with $X := Y$ takes the form

R(B) $$\min \{g(x) - h(y) | x \in G \cap B, y \in B, u = x\}.$$

Clearly we may drop the variable u from R(B) and substitute in the description of the algorithm x^B for u^B and x^k for u^k. Every cluster point of the sequence $\{x^k\}$ generated by the algorithm solves (DC). The bounding problem R(B) becomes manageable in this case. Namely, if v^i $(i = 1, \ldots, i_B)$ are the vertices of B, then due to the concavity of $-h(\cdot)$

one has $\min_{y \in B} -h(y) = \min_i -h(v^i)$, and therefore R($B$) with the variable u suppressed becomes

$$\mathrm{R}(B) \qquad \min \{g(x) | x \in G \cap B\} + \min_i -h(v^i).$$

Hence solution of R(B) requires solving a standard convex programming problem and searching the vertices of B. The latter problem can be solved with reasonable effort, due to the fact that B is generated from some predecessor B' by adding an affine inequality, see [2]. The starting polyhedron Y should be chosen as a simplex or as a rectangle, so that the vertices of Y are easily at hand.

REFERENCES

[1] Hiriart-Urruty J.-B.: Generalized differentiability, duality and optimization for problems dealing with differences of convex functions. Lecture Notes in Economics and Mathematical Systems **256**, 37-69 (1986).

[2] Horst R., de Vries J., Thoai N.V.: On finding new vertices and redundant constraints in cutting plane algorithms for global optimization. Operations Research Letters **7**, 85-90 (1988).

[3] Muu L.D., Oettli W.: An algorithm for indefinite quadratic programming with convex constraints. Submitted.

[4] Tuy H.: Global minimization of a difference of two convex functions. Mathematical Programming Study **30**, 150-182 (1987).