BASES FOR BIVARIATE SPLINE SPACES
- A CONSTRUCTIVE APPROACH

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Abstract: We give a basis for bivariate spline spaces on crosscut partitions which can be computed without solving systems of linear equations. In particular, we develop a recursion formula for these basis functions that cannot be written as polynomials or truncated power functions.

0. Introduction

One of the main problems in the theory of spline spaces is to compute their dimension and to construct basis functions. In the case of one variable, these problems have well-known solutions: if knots

$$a = y_0 < y_1 < \ldots < y_{k+1} = b$$

and multiplicities $M = (m_1, \ldots, m_k)$ are given, then the spline space is defined by

$$\begin{aligned} S_q(y_1, \ldots, y_k, M) := \{ s \in C[a,b] \mid s_i := s\,|_{[y_i, y_{i+1}]} \in \Pi_q \quad \text{and} \\ s_{i-1}^{(\nu)}(y_i) = s_i^{(\nu)}(y_i) \quad \text{for } \nu = 0, \ldots, q - m_i \\ \text{and } i = 1, \ldots, k \}. \end{aligned}$$

A basis can be constructed by using polynomials and truncated power functions, but a local basis consisting of so-called B-splines is of more interest.
Since we are interested in spline spaces satisfying certain boundary conditions, we study the following situation: let $\widetilde{M} = (m_0, \ldots, m_{k+1})$ be given and set

$$\begin{aligned} \widetilde{S}_q(y_0, \ldots, y_{k+1}, \widetilde{M}) := \{ s \in C(I\!R) \mid s_{|[a,b]} \in S_q(y_1, \ldots, y_k, M), \\ s(x) = 0 \quad \text{for } x \le y_0 \quad \text{and } x \ge y_{k+1}, \\ s^{(\nu)}(y_0) = 0 \quad \text{for } \nu = 0, \ldots, q - m_0, \\ s^{(\nu)}(y_{k+1}) = 0 \quad \text{for } \nu = 0, \ldots, q - m_{k+1} \}. \end{aligned}$$

Then a basis of $\widetilde{S}_q$ is built up by the B-splines

$$B_i^q(x) := (-1)^{q+1}[x_i, \ldots, x_{i+q+1}](x - t)_+^q,$$

where

$$(x_0 \le x_1 \le \cdots \le x_m) := (y_0, \ldots, y_0, y_1, \ldots, y_1, \ldots, y_{k+1}, \ldots, y_{k+1}),$$

$m = m_0 + \cdots + m_{k+1}$ and $[x_i, \ldots, x_{i+q+1}](x-t)_+^q$ is the divided difference of the function $f(t) = (x - t)_+^q$ over the points $x_i, \ldots, x_{i+q+1}$. It is well-known that support $(B_i^q) = [x_i, x_{i+q+1}]$ and B_i^q satisfies a recursion relation (cp. [6]).

The aim of this paper is to develop a kind of B-spline basis for certain spaces of bivariate splines. Let $\Omega \subseteq I\!R^2$ be a simply connected domain and Δ be a crosscut partition of Ω (i.e. Δ is built up by straight lines $\Gamma_1, \ldots, \Gamma_N$, so-called crosscuts which cross the whole domain Ω). A basis of such a space is given by C.K.Chui and R.H. Wang in [2] consisting of polynomials, truncated power functions with respect to the lines $\Gamma_1, \ldots, \Gamma_N$ and some spline functions which cannot be given explicitly but have to be computed by solving systems of linear equations.

In this paper we will show how these functions can be computed without solving linear equations and will give a recursion formula. In fact, we deal with the following more general problem: let $(\overline{x}, \overline{y}) \in \mathbb{R}^2$ and lines $L_1, \ldots, L_n$ be given in the following way:

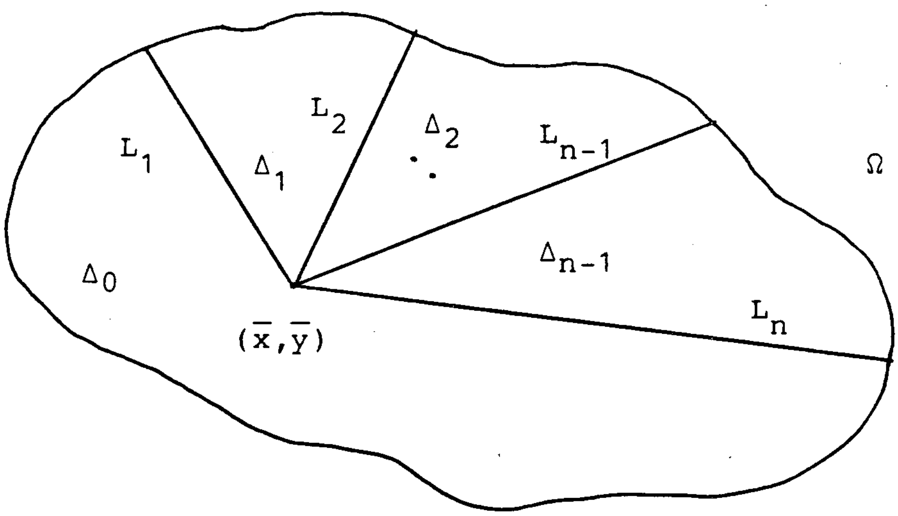

Figure 1: Partition of Ω

For $r_1, \ldots, r_n \in \mathbb{N}_0$ we define a bivariate spline space

$$S := S_q(L_1, \ldots, L_n; r_1, \ldots, r_n)$$

by

$$S := \{ s \in C(\Omega) \mid s_0 = s \mid \Delta_0 = 0, s_i = s \mid \Delta_i \in \Pi_q^2 \quad \text{for } i = 1, \ldots, n-1,$$
$$\frac{\delta^\nu}{\delta x^a \delta y^b} s_{i-1}(x,y) = \frac{\delta^\nu}{\delta x^a \delta y^b} s_i(x,y) \quad \text{for } \nu = 0, \ldots, r_i,\ (x,y) \in L_i \},$$

where $s_{n+1} = s_0 = 0$ and

$$\Pi_q^2 := \{p : \Omega \longrightarrow I\!R \mid p(x,y) = \sum_{i+j\leq q} a_{i,j} x^i y^j\}$$

denotes the space of bivariate polynomials of degree not larger than q. For $s \in S$, we say that L_i is an edge of s of multiplicity $q - r_i$ and write

$$l^1, \ldots, l^m = L_1, \ldots, L_1, \ldots\ldots, L_n, \ldots, L_n,$$

where $m = nq - \sum_{i=1}^{n} r_i$.
In Section 1 we construct an explicit basis of S and apply our results to crosscut-partitions. In Section 2 we prove a recursion formula for the basis splines.

1. Construction of a basis

For given $\alpha \in I\!R$ we define a class of lines l_λ by: $(x,y) \in l_\lambda$ if and only if

$$y - \overline{y} + \alpha \cdot (x - \overline{x}) = \lambda.$$

If $\Delta_0, \ldots, \Delta_{n-1}$ denote the open parts of Ω as above, we say that l_λ is admissible if $l_\lambda \cap \Delta_i \neq \emptyset$ for $i = 0, \ldots, n-1$. The following geometrical lemma about admissible lines will be helpful.

1.1. Lemma Let $\alpha \in I\!R$ and $\lambda_0 \neq 0$ be given such that l_{λ_0} is admissible. Then l_λ is admissible if $\lambda \cdot \lambda_0 > 0$. Moreover, for each $(x,y) \in \Delta_1 \cup \ldots \cup \Delta_{n-1}$ there is λ with $\lambda \cdot \lambda_0 > 0$ and $(x,y) \in l_\lambda$.

Since Lemma 1.1 is geometrically evident, we omit the proof.

From now on, without loss of generality we assume that $(\overline{x}, \overline{y}) = (0,0)$. Then the grid line L_i can be described by $a_i x + b_i y = 0$ and $l_\lambda \cap L_i = (c_i\lambda, d_i\lambda)$, where $c_i = -\frac{b_i}{a_i - \alpha b_i}$, $d_i = \frac{a_i}{a_i - \alpha b_i}$ (note that $a_i - \alpha b_i \neq 0$, because l_λ is admissible).
By construction, we have $c_1\lambda < \cdots < c_n\lambda$ or $c_n\lambda < \cdots < c_1\lambda$. Setting $x_i^\lambda = c_{i+1}\lambda$, $m_i = q - r_{i+1}$ in the first case, and $x_i^\lambda = c_{n-i}\lambda$, $m_i = q - r_{n-i}$ in the second case, we can construct bivariate splines by considering a class of univariate splines.

1.2. Lemma Let $\alpha \in I\!R, \lambda_0 \neq 0$ be given such that λ_0 is admissible. Moreover, let $s : \Omega \longrightarrow I\!R$ satisfy:

(i) for each λ with $\lambda \cdot \lambda_0 > 0$ we have

$$s\,|_{l_\lambda} \in \widetilde{S}_q(x_0^\lambda, \ldots, x_{n-1}^\lambda, \widetilde{M}),$$

where $\widetilde{M} = (m_0, \ldots, m_{n-1})$;

(ii) for $\nu = 1, \ldots, n-1$ and $\lambda \cdot \lambda_0 > 0$, $s\,|_{l_\lambda \cap \Delta_\nu}$ is a polynomial in x and λ of total degree not larger than q, i.e. there are coefficients a_{ij}^ν not depending on λ such that

$$s\,|_{l_\lambda}(x) = \sum_{i+j \leq q} a_{ij}^\nu x^i \lambda^j$$

if $x \in \Delta_\nu$, $\nu = 1, \ldots, n-1$;

(iii) $s\,|_{\Delta_0} = 0$.

Then $s \in S_q(L_1, \ldots, L_n; r_1, \ldots, r_n)$.

Proof: We prove the claim in the case $c_1\lambda < \ldots < c_n\lambda$. The second case can be treated analogously.

For fixed λ with $\lambda \cdot \lambda_0 > 0$, by Lemma 1.1. l_λ is admissible and we can define $s_\lambda := s\,|_{l_\lambda} : I\!R \longrightarrow I\!R$. Then $s_\lambda(x) = 0$ for $x \notin [x_0^\lambda, x_{n-1}^\lambda]$ and s_λ is a univariate spline of degree q and smoothness $r_1, \ldots, r_n$. Thus there are univariate polynomials $p_{\nu,\lambda} \in \Pi_{q-r_\nu-1}(x)$ satisfying

$$s_\lambda(x) = \sum_{\nu=1}^{n-1} (x - x_{\nu-1}^\lambda)_+^{r_\nu+1} \cdot p_{\nu,\lambda}(x),$$

$$\text{and } s_\lambda(x) = (x_{n-1}^\lambda - x)_+^{r_n+1} \cdot p_{n,\lambda}(x) \quad \text{if } x \in [c_{n-1}\lambda, c_n\lambda],$$

where a_+^m denotes the well-known truncated power. Moreover, for $1 \leq \mu \leq n-1$, $(x,y) \in \Delta_\mu$ and $y + \alpha \cdot x = \lambda$ by condition (ii) we have

$$\begin{aligned} s(x,y) = s_\lambda(x) &= \sum_{i+j \leq q} a_{ij}^{(\mu)} x^i \lambda^j \\ &= \sum_{i+j \leq q} a_{ij}^{(\mu)} x^i (y + \alpha \cdot x)^j \\ &= \sum_{i+j \leq q} a_{ij}^{(\mu)} \cdot (x^i \cdot \sum_{t=0}^{j} \binom{j}{t} \alpha^{j-t} x^{j-t} y^t), \end{aligned}$$

and since the coefficients $a_{ij}^{(\mu)}$ do not depend on λ, we have $s|_{\Delta_\mu} \in \Pi_q^2$.
Using condition (ii), for $(x,y) \in \Delta_\mu$ one obtains

$$s(x,y) = s_\lambda(x) = \sum_{\nu=1}^{\mu}(x - x_{\nu-1}^{\lambda})^{r_\nu+1} \cdot p_{\nu,\lambda}(x)$$
$$= \sum_{\nu=1}^{\mu}(x + \frac{b_\nu y + b_\nu \alpha x}{a_\nu - b_\nu \alpha})^{r_\nu+1} \cdot p_{\nu,\lambda}(x)$$
$$= \sum_{\nu=1}^{\mu}(\frac{1}{a_\nu - b_\nu \alpha})^{r_\nu+1} \cdot (a_\nu x + b_\nu y)^{r_\nu+1} \cdot p_{\nu,\lambda}(x),$$

and we have to show that $p_{\nu,\lambda}$ can be written as a polynomial in x and y.
For $\mu = 1$ and $x \in [x_0^\lambda, x_1^\lambda]$, because of condition (ii) $p(x,\lambda) := s_\lambda(x)$ is a polynomial in x and λ which is divided by the factor $(x - c_1\lambda)^{r_1+1}$. Hence, there exists a polymomial $\tilde{p} \in \Pi_{q-r_1-1}(x,\lambda)$ satisfying:

$$(x - c_1\lambda)^{r_1+1} \cdot p_{1,\lambda}(x) = s_\lambda(x)$$
$$= p(x,\lambda)$$
$$= (x - c_1\lambda)^{r_1+1} \cdot \tilde{p}(x,\lambda),$$

and this implies that there are coefficients $b_{ij}^{(1)}$, not depending on λ, such that

$$p_{1,\lambda}(x) = \tilde{p}(x,\lambda) = \sum_{i+j \le q-r_1-1} b_{ij}^{(1)} x^i \lambda^j$$
$$= \sum_{i+j \le q-r_1-1} b_{ij}^{(1)} x^i \cdot (y + \alpha \cdot x)^j.$$

Therefore, $s|_{\Delta_1}$ contains the factor $(a_1 x + b_1 y)^{r_1+1}$, and this implies that all derivatives up to the order r_1 exist for each $(x,y) \in L_1$.
For $1 < \mu \le n-1, (x,y) \in \Delta_\mu$ and $y + \alpha \cdot x = \lambda$ there exist polynomials $\bar{p} \in \Pi_q^2(x,\lambda)$, $p_{\mu,\lambda} \in \Pi_{q-r_\mu-1}^2(x,\lambda)$ such that

$$s(x,y) = s_\lambda(x) = \bar{p}(x,\lambda) + (x - c_\mu\lambda)^{r_\mu+1} \cdot p_{\mu,\lambda}(x) =: p(x,\lambda).$$

Then the polynomial $u := p - \bar{p}$ is divided by $(x - c_\mu\lambda)^{r_\mu+1}$ and thus there is $\tilde{p} \in \Pi_{q-r_\mu-1}^2(x,\lambda)$ such that

$$p_{\mu,\lambda}(x) = \tilde{p}(x,\lambda)$$
$$= \sum_{i+j \le q-r_\mu-1} b_{ij}^{(\mu)} x^i \lambda^j$$
$$= \sum_{i+j \le q-r_\mu-1} b_{ij}^{(\mu)} x^i (y + \alpha \cdot x)^j$$

with suitable coefficients $b_{ij}^{(\mu)}$.
Therefore, all derivatives of s up to the order r_μ exist for $(x,y) \in L_\mu$.
Analogously one can show the existence of the derivatives at L_n and in all we have $s \in S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$. Δ

In order to apply Lemma 1.2. to univariate B-splines, we have to introduce some further notations. We describe lines $l^1,\ldots,l^m$, $m = nq - \sum_{i=1}^{n} r_i$, by the equations $a^i x + b^i y = 0$, set $c^i = -\frac{b^i}{a^i - b^i \alpha}$,

$$(x_1^\lambda,\ldots,x_m^\lambda) = (c^1\lambda,\ldots,c^m\lambda) \quad \text{if } c^1\lambda \le \cdots \le c^m\lambda,$$

and

$$(x_1^\lambda,\ldots,x_m^\lambda) = (c^m\lambda,\ldots,c^1\lambda) \quad \text{if } c^m\lambda \le \cdots \le c^1\lambda.$$

Since we want to use the recursion formula for B-splines, at first we treat some special B-splines which can be used as starting points for the recursion.

1.3. Lemma Let $\alpha \in I\!R$, $\lambda_0 \neq 0$ be given such that l_{λ_0} is admissible, and assume there is $1 \le \mu \le n$ with $r_\mu = 0$. If for $\lambda \cdot \lambda_0 > 0$ $B_\lambda^{i,q}$ denotes the univariate B-spline of degree q with respect to the points $x_i^\lambda \le \cdots \le x_{i+q+1}^\lambda$, and if $L_\mu \cap l_\lambda \in \{x_i^\lambda,\ldots,x_{i+q+1}^\lambda\}$, then the function

$$s : \Omega \longrightarrow I\!R \quad \text{defined by}$$

$$s(x,y) = \begin{cases} \lambda^{q+1} \cdot B_\lambda^{i,q}(x) & \text{for } y + \alpha \cdot x = \lambda \\ 0 & \text{for } (x,y) \in \Delta_0 \end{cases}$$

is in $S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$.

Proof: By construction, s satisfies condition (i) and condition (iii) of Lemma 1.2. and we only have to show that condition (ii) is fulfilled. It is clear that $B_\lambda^{i,q}$ has a knot of multiplicity q and thus there is $j \in \{i, i+1, i+2\}$ such that $x_j^\lambda = \cdots x_{j+q-1}^\lambda = L_\mu \cap l_\lambda$ for each λ with $\lambda \cdot \lambda_0 > 0$. Without loss of generality, again we can assume that $c^1\lambda \le \ldots \le c^m\lambda$.
We prove by induction on q.
For $q = 1$, all considered knots are simple knots and s_λ is given by

$$s_\lambda(x) = \frac{1}{c^{i+2} - c^i} \begin{cases} \frac{x - c^i\lambda}{(c^{i+1} - c^i)\cdot\lambda^2} & , c^i\lambda \le x \le c^{i+1}\lambda \\ \frac{c^{i+2}\lambda - x}{(c^{i+2} - c^{i+1})\cdot\lambda^2} & , c^{i+1}\lambda \le x \le c^{i+2}\lambda \\ 0 & , \text{elsewhere.} \end{cases}$$

By construction, s_λ is a univariate spline on l_λ and of course $\lambda^2 \cdot s_\lambda$, too. Obviously $\lambda^2 \cdot s_\lambda$ restricted to the subintervals is a polynomial in x and λ of degree one with coefficients that do not depend on λ, and therefore, by Lemma 1.2., $s \in S$.

If our claim is true for $q-1$, we distinguish three cases.

Case 1: $j = i$. Using the well-known recursion formula for univariate B-splines (cp [6], p.120) we obtain a B-spline B_λ^1 of degree $q-1$ with respect to the knots $x_{i+1}^\lambda \leq \cdots \leq x_{i+q+1}^\lambda$ such that

$$B_\lambda^{i,q}(x) = \frac{1}{(c^{i+q+1} - c^i) \cdot \lambda}\left((x - c^i\lambda)B_\lambda^0(x) + (c^{i+q+1}\lambda - x)B_\lambda^1(x)\right),$$

where

$$B_\lambda^0(x) = \begin{cases} \frac{(c^{i+q}\lambda - x)^{q-1}}{(c^{i+q} - c^i)^q \cdot \lambda^q} & \text{if } c^i\lambda \leq x \leq c^{i+q}\lambda \\ 0 & \text{elsewhere.} \end{cases}$$

By the induction hypothesis the function $\widetilde{s} : \Omega \longrightarrow I\!R$, defined by

$$\widetilde{s}(x,y) = \begin{cases} \lambda^q \cdot B_\lambda^1(x) & \text{if } y + \alpha \cdot x = \lambda \\ 0 & \text{if } (x,y) \in \Delta_0 \end{cases}$$

satisfies conditions (i)-(iii) of Lemma 1.2. . Since also $\lambda^q \cdot B_\lambda^0 |_{\Delta_\nu}$ is a polynomial in x and λ of degree $q-1$, s satisfies condition (ii) of Lemma 1.2., too.

Case 2: $j = i+1$. Since x_i^λ and x_{i+q+1}^λ are knots of multiplicity one, whereas $x_{i+1}^\lambda = \cdots = x_{i+q}^\lambda$ has multiplicity q, $\lambda^{q+1} \cdot B_\lambda^{i,q}$ can be given explicitly by the formula

$$\lambda^{q+1} \cdot B_\lambda^{i,q}(x) = \frac{1}{(c^{i+q+1} - c^i)} \begin{cases} \frac{(x - c^i\lambda)^q}{(c^{i+1} - c^i)^q} & \text{if } c^i\lambda \leq x \leq c^{i+1}\lambda \\ \frac{(c^{i+q+1}\lambda - x)^q}{(c^{i+q+1} - c^{i+q})^q} & \text{if } c^{i+q}\lambda \leq x \leq c^{i+q+1}\lambda \\ 0 & \text{elsewhere.} \end{cases}$$

Obviously $\lambda^{q+1} \cdot B_\lambda^{i,q}$ satisfies condition (ii) of Lemma 1.2. .

Case 3: $j = i+2$. This case can be treated analogously as Case 1. Δ

With the help of Lemma 1.3. one can extend arbitrary univariate B-splines to bivariate splines in S.

1.4. Theorem Let $\alpha \in I\!R$, $\lambda_0 \neq 0$ be given such that l_{λ_0} is admissible. If $(n-1)q \geq K := 2 + \sum_{\nu=1}^{n} r_\nu$ and for $\lambda \cdot \lambda_0 > 0$ $B_\lambda^{i,q}$ denotes the univariate B-spline of degree q with respect to the points $x_i^\lambda \leq \cdots \leq x_{i+q+1}^\lambda$, then the function

$$s : \Omega \longrightarrow I\!R \quad \text{defined by}$$

$$s(x,y) = \begin{cases} \lambda^{q+1} \cdot B_\lambda^{i,q}(x) & \text{for } y + \alpha \cdot x = \lambda \\ 0 & \text{for } (x,y) \in \Delta_0 \end{cases}$$

is in $S_q(L_1, \ldots, L_n; r_1, \ldots, r_n)$.

Proof: Without loss of generality we assume that $c^1\lambda \leq \cdots \leq c^m\lambda$. Since $(n-1)q \geq K$, there is at least one $i \in \{1,\ldots,m\}$ such that a B-spline $B_\lambda^{i,q}$ exists for each λ with $\lambda \cdot \lambda_0 > 0$. We prove by induction on q .

For $q = 1$ the theorem was proved in Lemma 1.3. . If the claim is true for $q-1$, we distinguish two cases.

Case 1: There is $\mu \in \{1,\ldots,n\}$ such that $r_\mu = 0$ and $L_\mu = l^j, j \in \{i,\ldots,i+q+1\}$. Then Lemma 1.3. can be applied.

Case 2: $r_\mu \geq 1$ for each μ with $L_\mu \in \{l^i,\ldots,l^{i+q+1}\}$. Then by the univariate recursion formula there are continuous B-splines B_λ^0, B_λ^1 of degree $q-1$ such that

$$\lambda^{q+1} \cdot B_\lambda^{i,q}(x) = \frac{1}{c^{i+q+1}-c^i}\big((x-c^i\lambda)\lambda^q \cdot B_\lambda^0(x) + (c^{i+q+1}\lambda - x)\lambda^q \cdot B_\lambda^1(x)\big).$$

Since by induction hypotheses $\lambda^q \cdot B_\lambda^0$ and $\lambda^q \cdot B_\lambda^1$ satisfy conditions (i)-(iii) of Lemma 1.2., also $\lambda^{q+1} \cdot B_\lambda^{i,q}$ satisfies condition (ii). But conditions (i) and (iii) are fulfilled by construction of $B_\lambda^{i,q}$, and so the theorem is proved. Δ

Using these extended B-splines, we now build up a basis of $S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$. It is clear that for given knots $y_1 < \cdots < y_n$ and smoothness parameters $r_1,\ldots,r_n$ a B-spline supported in $[y_1,y_n]$ exists if and only if $(n-1)q \geq K$. Therefore we define

$$\overline{q} := \left[\frac{K-1}{n-1}\right] + 1 = \quad \min\ (t \in I\!N \mid (n-1)t \geq K)$$

and set $k := q - \overline{q} + 1$. For $\mu = 1,\ldots,k$ we choose parameters $\alpha_\mu \in I\!R, \lambda_\mu \neq 0$ such that the lines l_{λ_μ} , defined by $y + \alpha_\mu \cdot x = \lambda_\mu$ are admissible.

For $\mu = 1,\ldots,k$, $N_\mu := (n-1)(q-\mu+1) - K + 1$ and $i = 1,\ldots,N_\mu$ we consider the univariate B-spline $B_\lambda^{i,q-\mu+1}$ of degree $q-\mu+1$ with respect to the knots $x_i^\lambda(\mu),\ldots,x_{i+q-\mu+2}^\lambda(\mu)$, where the knots $x_j^\lambda(\mu)$ are constructed analogously to the knots x_j^λ with respect to the degree $q-\mu+1$ and multiplicities $q-\mu+1-r_\nu$ for $\lambda \cdot \lambda_0 > 0$. Then we can define

$$A_i^{q-\mu+1}, B_i^{q-\mu+1} \in S_q(L_1,\ldots,L_n;r_1,\ldots,r_n) \quad \text{by}$$

$$A_i^{q-\mu+1}(x,y) = \begin{cases} \lambda^{q-\mu+1} \cdot B_\lambda^{i,q-\mu+1}(x) & \text{if } y + \alpha_\mu \cdot x = \lambda_\mu \\ 0 & \text{if } (x,y) \in \Delta_0, \end{cases}$$

and

$$B_i^{q-\mu+1}(x,y) = A_i^{q-\mu+1}(x,y) \cdot \prod_{j=1}^{\mu-1}(y + \alpha_j x - \lambda_j),$$

where the last product is said to be one if $\mu = 1$. It turns out that the splines $B_i^{q-\mu+1}$ build up a basis of S .

1.5. Lemma The functions $B_i^{q-\mu+1}, \mu = 1,\ldots,k,\ i = 1,\ldots,N_\mu$ are linearly independent.

Proof: We assume that

$$\sum_{\mu=1}^{k}\sum_{i=1}^{N_\mu} \beta_{\mu_i} B_i^{q-\mu+1} \equiv 0.$$

We show that $\beta_{\mu_i} = 0$ for $\nu = 1,\ldots,k,\ \mu = 1,\ldots,\nu,\ i = 1,\ldots,N_\mu$.
For $\nu = 1$ and $(x,y) \in l_{\lambda_1}$ we have $y + \alpha_1 x - \lambda_1 = 0$ and therefore

$$B_i^q(x,y) = A_i^q(x, \lambda_1 - \alpha_1 x) = \lambda_1^{q+1} \cdot B_{\lambda_1}^{i,q}(x)$$

for $i = 1,\ldots,N_1$, whereas $B_i^{q-\mu+1}(x,y) = 0$ for $\mu \geq 2$. Thus

$$\lambda_1^{q+1} \cdot \sum_{i=1}^{N_1} \beta_{1,i} B_{\lambda_1}^{i,q}(x) \equiv 0$$

and since the univariate B-splines are lineraly independent, we have $\beta_{1,i} = 0$ for $i = 1,\ldots,N_1$.
Assuming now that $\beta_{\mu,i} = 0$ for $\mu = 1,\ldots,\nu-1$, we obtain

$$\sum_{\mu=\nu}^{k}\sum_{i=1}^{N_\mu} \beta_{\mu,i} \cdot A_i^{q-\mu+1}(x,y) \cdot \prod_{j=\nu}^{\mu-1}(y + \alpha_j x - \lambda_j) \equiv 0,$$

where the last product is said to be one for $\mu = \nu$. Then by restricting this equation to l_{λ_ν} we analogously obtain $\beta_{\nu,i} = 0$ for $i = 1,\ldots,N_\nu$, and thus for $\nu = k$ the claim is proved. Δ

Now we can formulate our main theorem.

1.6. Theorem Let $(\overline{x},\overline{y}) \in \Omega$ be given.

(i) If $(n-1)q < K$, then $S_q(L_1,\ldots,L_n; r_1,\ldots,r_n) = \{0\}$.

(ii) If $(n-1)q \geq K$, then let admissible lines l_{λ_μ} be given by $y - \overline{y} + \alpha_\mu(x - \overline{x}) = \lambda_\mu$. The set

$$\mathcal{B} = \{B_i^{q-\mu+1}(x - \overline{x}, y - \overline{y}) \mid \mu = 1,\ldots,k,\ i = 1,\ldots,N_\mu\}$$

is a basis of $S_q(L_1,\ldots,L_n; r_1,\ldots,r_n)$, where the functions $B_i^{q-\mu+1}$ are constructed with respect to the lines $y + \alpha_\mu x = \lambda_\mu$ and to the grid lines $L_\nu - (\overline{x},\overline{y})$.

Proof: Obviously we can assume $(\overline{x},\overline{y}) = (0,0)$.
At first, we assume that $(n-1)q < K$. Then for $s \in S$ and an arbitrary line l through $\Delta_0,\dots,\Delta_{n-1}$, $s\,|_l$ is a univariate spline the support of which is contained in $[x_1^\lambda, x_{nq-K+2}^\lambda]$. Since $nq-K+2 < q+2$ we have that $s\,|_l = 0$ and thus $s \equiv 0$.
For $(n-1)q \geq K$ we set

$$S_\mu := \operatorname{span}(A_i^{q-\mu+1} \mid i = 1,\dots,N_\mu) \subseteq S_q(L_1,\dots,L_n;r_1,\dots,r_n).$$

We prove the following claim:
For $s \in S$, there are $s_1 \in S_1,\dots,s_\mu \in S_\mu$, $s^\mu \in S_{q-\mu}(L_1,\dots,L_n;r_1,\dots,r_n)$ such that

$$s = s_1 + l_1\cdot(s_2 + l_2\cdot(\cdots(s_\mu + l_\mu\cdot s^\mu)\cdots)),$$

where $l_i(x,y) = y + \alpha_i x - \lambda_i$.
For $\mu = 1$ we choose points $u_i, i = 1,\dots,N_1$, on l_{λ_1} such that

$$t_i \in [x_i^{\lambda_1}, x_{i+q+1}^{\lambda_1}] = \text{support } B_{\lambda_1}^{i,q} \quad \text{with } u_i = (t_i, z_i).$$

By the theorem of Schoenberg and Whitney (cp. [4]) there is one and only one

$$\widetilde{s} \in \text{ span } (B_{\lambda_1}^{i,q} \mid i = 1,\dots,N_1) = \widetilde{S}_q(x_1^{\lambda_1},\dots,x_{nq-K+2}^{\lambda_1})$$

satisfying $\widetilde{s}(t_i) = \frac{s(u_i)}{\lambda_1^{q+1}}$.
If $\widetilde{s} = \sum_{i=0}^{N_1} \beta_i B_{\lambda_1}^{i,q}$, we set $s_1 = \sum_{i=0}^{N_1} \beta_i A_i^q \in S_1$ and obtain

$$\begin{aligned} s_1(u_\nu) &= \sum_{i=1}^{N_1} \beta_i A_i^q(u_\nu) \\ &= \sum_{i=1}^{N_1} \beta_i \lambda_1^{q+1}\cdot B_{\lambda_1}^{i,q}(t_\nu) \\ &= \lambda_1^{q+1}\cdot\widetilde{s}(t_\nu) = s(u_\nu). \end{aligned}$$

Since the points t_i satisfy the Schoenberg-Whitney-condition for univariate splines, l_{λ_1} is a zero-line of $s - s_1$ and therefore there is $s^1 \in S_{q-1}(L_1,\dots,L_n;r_1,\dots,r_n)$ with

$$s(x,y) - s_1(x,y) = l_1(x,y)\cdot s^1(x,y).$$

If the claim is proved for $\mu - 1$, we can treat $s^{\mu-1}$ and l_{λ_μ} analogously to s and l_{λ_1} and obtain

$$s^{\mu-1}(x,y) = s_\mu(x,y) + l_\mu(x,y)\cdot s^\mu(x,y),$$

where $s_\mu \in S_\mu$ and $s^\mu \in S_{q-\mu}(L_1, \dots, L_n; r_1, \dots, r_n)$ which proves the claim. For $\mu = k$ we get $s^k \in S_{q-k}(L_1, \dots, L_n; r_1, \dots, r_n) = \{0\}$, because $(n-1)\cdot(q-k) < K$, and this implies

$$\begin{aligned} s &= s_1 + l_1 \cdot (s_2 + l_2 \cdot (\cdots (s_{k-1} + l_{k-1} \cdot s_k) \cdots)) \\ &= s_1 + l_1 s_2 + l_1 l_2 s_3 + \cdots + l_1 l_2 \cdots l_{k-1} s_k. \end{aligned}$$

By construction of s_i we have $s_j \cdot \prod_{\nu=1}^{j-1} l_\nu \in \operatorname{span}(B_i^{q-j+1} \mid i = 1, \dots, N_j)$ and therefore $s \in \operatorname{span}(\mathcal{B})$. Since by Lemma 1.5. $\mathcal{B}$ is linearly independent, it is a basis of $S_q(L_1, \dots, L_n; r_1, \dots, r_n)$. Δ

Now the dimension of the considered spline space can be computed easily.

1.7. Corollary For $(n-1)q \geq K$ we have

$$\dim S_q(L_1, \dots, L_n; r_1, \dots, r_n) = k \cdot \Big(1 - K + (n-1)(q+1) - \frac{(k+1)(n-1)}{2}\Big),$$

where $K = 2 + \sum_{\nu=1}^{n} r_\nu$ and $k = q - \left[\frac{K-1}{n-1}\right]$.

Proof: By construction of $\mathcal{B}$ we have

$$\begin{aligned} \dim S_q(L_1, \dots, L_n; r_1, \dots, r_n) &= \sum_{i=1}^{k} ((n-1)(q-i+1) - K + 1) \\ &= k(1-K) + k(n-1)(q+1) - \frac{k(k+1)(n-1)}{2}. \end{aligned}$$

Δ

If $r_\nu = r$ for $\nu = 1, \dots, n$, this dimension formula coincides with the result given by Chui [2] and Schumaker [6]. Therefore, the basis for crosscut-partitions which was developed in [2] can be simplified by using our approach without solving systems of linear equations. Indeed, the functions $s_{i,j,t}$ being computed in [2] by solving linear equations, can be replaced by the basis functions

$$B_\nu^{q-\mu+1}(x - x_i, y - y_i), \ \mu = 1, \dots, k_i, \ \nu = 1, \dots, N_\mu,$$

where (x_i, y_i) are grid points of the crosscut partition, $k_i = q - \left[\frac{K_i - 1}{n_i - 1}\right]$, $K_i = 2 + n_i r$ and n_i is the number of crosscuts intersecting in (x_i, y_i).

2. Recursion formulas

For univariate B-splines there are well-known recursion formulas that are numerically stable. Since the basis $\mathcal{B}$ of $S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$ is built up by extending univariate B-splines it is not surprising that also in this case recursion formulas hold. To develop such formulas, we have to change our notation. Throughout this section we assume that $c^1\lambda \leq \ldots \leq c^m\lambda$ (the case $c^m\lambda \leq \ldots \leq c^1\lambda$ can be treated analogously). In contrast to Section 1 we do not fix the smoothness parameters r_i, but consider the multiplicities of the lines L_i. For this purpose we set $m_i = q - r_i$ and obtain

$$L_1 \ldots L_n = l^1 \ldots l^{m_1} l^{m_1+1} \ldots l^{M_2} \ldots l^{M_n},$$

where $M_i = m_1 + \ldots + m_i, m = M_n$ and $l^{M_{i-1}+1} = \ldots = l^{M_i} = L_i$. We say that L_i is an edge of multiplicity m_i for $s \in S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$, set $I(i) = \{i,\ldots,i+q+1\}$ and $J(i) = \{j \mid L_j \in \{l^i,\ldots,l^{i+q+1}\}\}$ and define

$$\widetilde{m}^i_j = \begin{cases} m_j & \text{if } j \notin J(i) \\ |\{\nu \in I(i) \mid l^\nu = L_j\}| & \text{if } j \in J(i), \end{cases}$$

i.e. we just count, how often L_j appears among $l^i,\ldots,l^{i+q+1}$. Then we can set

$$A^q(l^i,\ldots,l^{i+q+1}) := A^q_i \in S_q(L_1,\ldots,L_n;q-\widetilde{m}^i_1,\ldots,q-\widetilde{m}^i_n),$$

where all splines are constructed with respect to admissible lines $l_\lambda : y + \alpha x = \lambda$, α fixed.

It is clear that for given q and $r_1,\ldots,r_n$ by construction $A^q(l^i,\ldots,l^{i+q+1})$ coincides with $A^q_i \in S_q(L_1,\ldots,L_n;r_1,\ldots,r_n)$, but

$$A^{q-1}(l^i,\ldots,l^{i+q}) \in S_{q-1}(L_1,\ldots,L_n;q-1-\widetilde{m}^i_1,\ldots,q-1-\widetilde{m}^i_n)$$

and in general

$$A^{q-1}(l^i,\ldots,l^{i+q}) \notin S_{q-1}(L_1,\ldots,L_n;r_1,\ldots,r_n)$$

as the following example shows.

2.1. Example We set $n=4, q=3, r_1=r_2=r_3=r_4=1$. Then $A^{q-1}(l^1,\ldots,l^3) \in S_2(L_1,\ldots,L_4;0,0,1,1)$ is supported by the angle between L_1 and L_2, but $A^{q-1}_1 \in S_2(L_1,\ldots,L_4;1,1,1,1)$ is supported by the angle between L_1 and L_4.

Moreover, we set $d^i = \frac{1}{a^i - b^i\alpha}$ and $l^i(x,y) = a^i x + b^i y$. Then the following recursion formula holds.

2.2. Theorem If $(n-1)q \geq K$ and $A^q(l^i, \ldots, l^{i+q+1})$ has no edge of multiplicity q, then

$$A^q(l^i, \ldots, l^{i+q+1}) = \frac{1}{c^{i+q+1} - c^i} \cdot \big(d^i l^i A^{q-1}(l^i, \ldots, l^{i+q}) - d^{i+q+1} l^{i+q+1} A^{q-1}(l^{i+1}, \ldots, l^{i+q+1})\big).$$

Proof: For $(x,y) \in \Delta_0$ everything is clear. For $(x,y) \in \overline{\Delta}_1 \cup \ldots \cup \overline{\Delta}_{n-1}$ and $y + \alpha x = \lambda$ we have

$$\begin{aligned}
A^q(l^i, \ldots, l^{i+q+1})(x,y) &= \lambda^{q+1} \cdot B_\lambda^{i,q}(x) \\
&= \frac{\lambda^{q+1}}{c^{i+q+1}\lambda - c^i\lambda}\big((x - c^i\lambda)B_\lambda^{i,q-1}(x) + (c^{i+q+1}\lambda - x)B_\lambda^{i+1,q-1}(x)\big) \\
&= \frac{1}{c^{i+q+1} - c^i}\big(((1 - c^i\alpha)x - c^i y)A^{q-1}(l^i, \ldots, l^{i+q})(x,y) \\
&\qquad + ((c^{i+q+1}\alpha - 1)x + c^{i+q+1}y)A^{q-1}(l^{i+1}, \ldots, l^{i+q+1})(x,y)\big) \\
&= \frac{1}{c^{i+q+1} - c^i}\big(d^i l^i(x,y)A^{q-1}(l^i, \ldots, l^{i+q})(x,y) \\
&\qquad - d^{i+q+1}l^{i+q+1}(x,y)A^{q-1}(l^{i+1}, \ldots, l^{i+q+1})(x,y)\big).
\end{aligned}$$

Here $B_\lambda^{i,q-1}$ is the univariate B-spline with knots $x_i^\lambda, \ldots, x_{i+q}^\lambda$. Δ

To complete the description of the computation of the basis functions, we give formulas for basis splines which possess an edge of multiplicity q.

Case 1: $l^i = \ldots = l^{i+q-1}, q > 2$ if $l^{i+q} = l^{i+q+1}$.

$$A^q(l^i, \ldots, l^{i+q+1}) = \frac{1}{c^{i+q+1} - c^i} \cdot \Big(\frac{(d^i l^i)_+ \cdot (-d^{i+q}l^{i+q})_+^{q-1}}{(c^{i+q} - c^i)^q} - d^{i+q+1} \cdot l^{i+q+1} \cdot A^{q-1}(l^{i+1}, \ldots, l^{i+q+1})\Big),$$

where l^{i+1} is an edge of multiplicity $q-1$ for $A^q(l^{i+1}, \ldots, l^{i+q+1})$.

Case 2: $l^{i+1} = \ldots = l^{i+q}$.

$$A^q(l^i, \ldots, l^{i+q+1})(x,y) = \frac{1}{c^{i+q+1} - c^i} \cdot \begin{cases} \frac{(d^i l^i(x,y))^q}{(c^{i+q} - c^i)^q} & \text{if } (x,y) \in \Delta^i \\ \frac{(-d^{i+q+1}l^{i+q+1}(x,y))^q}{(c^{i+q+1} - c^{i+1})^q} & \text{if } (x,y) \in \Delta^{i+1}. \end{cases}$$

Case 3: $l^{i+2} = \ldots = l^{i+q+1}, q > 2$ if $l^i = l^{i+1}$.

$$A^q(l^i, \ldots, l^{i+q+1}) = \frac{1}{c^{i+q+1} - c^i} \cdot \Big(d^i \cdot l^i \cdot A^{q-1}(l^i, \ldots, l^{i+q}) + \frac{(-d^{i+q+1}l^{i+q+1})_+ \cdot (d^{i+1}l^{i+1})_+^{q-1}}{(c^{i+q+1} - c^{i+1})^q}\Big),$$

where l^{i+q} is an edge of multiplicity $q-1$ for $A^{q-1}(l^i,\ldots,l^{i+q})$.
Case 4: $q=2, l^i=l^{i+1}, l^{i+2}=l^{i+3}$.

$$A^2(l^i,\ldots,l^{i+3}) = -\frac{2}{(c^{i+3}-c^i)^3}\cdot d^i d^{i+3}\cdot l^i l^{i+3}.$$

This formulas can be proved by using the well-known formulas for univariate B-splines (cp.[6]).

Another example how to make use of the structure of the basis splines is the following remark which extends the univariate partition of unity.

2.3. Remark We assume that $c^1\lambda \le \ldots \le c^m\lambda$ for $\lambda\cdot\lambda_0 > 0, l^j \ne l^{j+1}, (n-1)q-K+1 \ge j \ge q+1$ and (x,y) lies between l^j and l^{j+1}. Then

$$\sum_{i=j-q}^{j}(c^{i+q+1}-c^i)\cdot A_i^q(x,y) = (y+\alpha\cdot x)^q.$$

Proof: With $y+\alpha\cdot x=\lambda$ we have

$$\begin{aligned}\sum_{i=j-q}^{j}(c^{i+q+1}-c^i)\cdot A_i^q(x,y) &= \sum_{i=j-q}^{j}\lambda^{q+1}\cdot(c^{i+q+1}-c^i)\cdot B_\lambda^{i,q}(x)\\ &= \lambda^q\cdot\sum_{i=j-q}^{j}(c^{i+q+1}\lambda-c^i\lambda)\cdot B_\lambda^{i,q}(x)\\ &= \lambda^q\sum_{i=j-q}^{j}(x_{i+q+1}^\lambda-x_i^\lambda)\cdot B_\lambda^{i,q}(x)\\ &= \lambda^q=(y+\alpha\cdot x)^q,\end{aligned}$$

because the splines $(x_{i+q+1}^\lambda - x_i^\lambda)\cdot B_\lambda^{i,q}$ are the so-called normalized B-splines which form a partition of unity (cp. [6], p.125). Δ

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