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G. Schwarz

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G. SCHWARZ

Lehrstuhl für Mathematik I
Universität Mannheim
Schloß, D - 6800 Mannheim
W. - Germany

ABSTRACT

One of the foundations of continuum mechanics is the description of forces in terms of a symmetric tensor. The fundamental observation that the existence of a symmetric stress tensor is a consequence of the material frame indifference is due to Noll [No63,Tr]. This in turn means a local symmetry of the system under the action of the Euklidean group. In this paper we will show that the assumption of locality in the axiom of frame indifference is not necessary for a wide class of modells. We will prove the existence of a symmetric stress tensor, demanding only the invariance of the system under global rigid infintesimal Eukledian group action. The localization of that global symmetry will be done by means of Hodge theory on manifolds with boundary.

1. Introduction

By means of differential geometric methods serveral progress has been made in the field of continuum mechanics within in the last two decades, cf. [AMR,Mar] and references therein. The purpose of this paper is to adopt such geometric notions in rational mechanics to the question of objectivity [No63,Tr].

To obtain a geometric appropriate description of the deformations of a piece of material, we use the embeddings of a material body $\mathcal{B}$ into the $\mathbb{R}^n$ as ambient space, cf. [HuMa], where the body is a Riemannian manifold. We denote by $E(\mathcal{B}, \mathbb{R}^n)$ the set of all embeddings $J : \mathcal{B} \to \mathbb{R}^n$, which itself carries the structure of an infinite-dimensional manifold. In the classical notation such J is also called a placement of the body and elements of the tangent space of $E(\mathcal{B}, \mathbb{R}^n)$ are refered to as virtual displacements. In this setting dynamics means to formulate of continuum mechanics in terms of curves $J(t)$ of embeddings. Since our intention is to explore the geometrical structure of a system, we restrict ourselves to statics, cf.[BSS].

Under a global model we now understand a system, where the respective non-linear equations of continuum mechanics are determined by some functional on $E(\mathcal{B}, \mathbb{R}^n)$, depending on the placement J in a possibly non-local way. Such a functional is refered to as the (non-local) constitutive function of the theory. Here we start with the principle of virtual work first introduced in continuum mechanics as d'Alambert's principle by [He] and reformulated in a geometric context by [EpSe] : There the constitutive function is the virtual work $F[J]$ which is a linear functional on the space of all virtual displacements, i.e. the tangent space of $E(\mathcal{B}, \mathbb{R}^n)$, depending on the configuration J in a non-linear and non-local way. With appropriate geometric and functional analytic specializations – cf. section 2 – the principle states that J describes an equilibrium iff the integral

$$F[J](\Lambda) = \int_{\mathcal{B}} < \Phi(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} \tag{1.1}$$

vanishs for all virtual displacements $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$. Here $\Phi(J)$ is a prescribed physical force density and $<, >_{\mathbb{R}^n}$ means the scalar product on $\mathbb{R}^n$. As a special case this principle includes the description of hyperelasticity, where the equations of continuum mechanics can be derived from a local energy fuctional or a Lagrangian [HuMa,TrTo].

Taking the principle of virtual work as such we will investigate the effect of a class of symmetries on the functional $F[J]$ and in consequence on the equations of continuum mechanics : For classical mechanics the symmetry group of special interest is the Euclidean group $E(n)$ [Thi] describing all rigid motions in the ambient space $I\!R^n$. Since the internal properties of a system are independent on the placement and on the orientation of the embedded body as a whole, they cannot be influenced by rigid motions. Hence $E(n)$ is a natural symmetry group in continuum mechanics. An explicite investigation of that symmetry is due to Noll [No63,59], who introduced it under the notion of material frame indifference or objectivety. The action of $E(n)$ on an embedding $J \in E(\mathcal{B}, I\!R^n)$ is

$$g_{(T,R)}[J] = R(J + T) \tag{1.2}$$

where $g_{(T,R)} \in E(n)$ is represented by a translation $T \in I\!R^n$ and a rotation $R \in SO(n)$. In this context Noll's axiom of objectivity reads as follows : A system is called material frame indifferent if the constitutive function is such that no work is done against any virtual displacement, which is rigid infinitesimal action of $E(n)$ restricted to an arbitrary subbody $U \subset \mathcal{B}$. This means

$$\int_U < \Phi(J), g_{(z,C)}[J] >_{I\!R^n} \mu_{\mathcal{B}} = 0 \tag{1.3}$$

for any infinitesimal action $g_{(z,C)}$ of the Euclidean groups. We note that this is a local demand. Noll's celebrate result [No63,Tr] is to prove the existence of a symmetric stress tensor, starting from that assumption.

The central result of this paper is that the locality of the $E(n)$-invariance, assumed by Noll's axiom (1.3), is redundant for the existence of a symmetric stress tensor. It suffices to start with a weaker global demand on the global functional $F[J]$, which is an integral over the whole body $\mathcal{B}$: This has to vanish if it is evaluated on all virtual displacements, which are rigid infinitesimal actions of $E(n)$, what means for the special case of the virtual work given by (1.1) that

$$F[J](g_{(z,C)}[J]) = \int_{\mathcal{B}} < \Phi(J), g_{(z,C)}[J] >_{I\!R^n} \mu_{\mathcal{B}} = 0 \tag{1.4}.$$

In continuum mechanics fundamental quantities, as the deformation gradient or the 1st Piola-Kirchhoff stress tensor, are described by two-point tensors [Er,HuMa]. Such objects may alternatively be considered as vector-valued differential forms. The motivation for using differential forms instead of the well known tensor calculus lies in the fact that there is a Hodge theory, which serves as powerfull tool for solving boundary value problems. Generalizing classical results on the Hodge theory on manifolds with boundaries [Mo58,62] in section 3 we will give a lemma concerning a Neumann and a modified Dirichlet boundary value problem for vector-valued differential forms. As a by-product we obtain a result on an interesting boundary value problem for the divergence of vector fields.

The solution of the Neumann problem will be used in section 4 to prove the tensorial character of the stress. The physical input therefore is the global demand of invariance of virtual work under infinitesimal rigid translations, which is a consequence of (1.4). By performing a Piola transformation we derive the weak form of Cauchy's equation of continuum mechanics from the principle of virtual work in the general form (1.1) and the symmetry argument. We then observe that the stress tensor is not uniquely determined from the kernel of $F[J]$, but it owns a gauge freedom.

In section 5 we will use the Dirichlet problem to show the existence of a symmetric stress tensor from the physical demand of invariance of the virtual work under rigid rotations. By Noll's theorem the existence of a symmetric stress tensor is equivalent to the local demand (1.3) of frame indifference. Hence the use of Hodge theory, required for our proof, may be considered as a localization of the global invariances.

Finally in section 6 we consider constitutive theory under the aspect of the Euclidean group acting as symmetry group in continuum mechanics. It is shown how our approach to the $E(n)$-symmetry of elasticity may be understood in the reduced phase space formalism [MaWe] of symplectic geometry.

2. The Principle of Virtual Work and Material Frame Indifference

In this paper we will describe mechanical properties of a continuous medium in terms of embeddings of a Riemannian manifold, as presented e.g. in [HuMa]. For the physical space, i.e. the ambient space of the embeddings, we take the Euclidean $\mathbb{R}^n$; a generalization to other ambient manifolds is possible, but requires more effort [BiFi]. To fix the notation we introduce the following definitions :

By a body $\mathcal{B}$ we mean a compact orinentable Riemannian C^k-manifold with boundary, where the dimension $\dim \mathcal{B} \leq n$. We denote by $G_{\mathcal{B}}$ the Riemannian metric on $\mathcal{B}$, by $\mathcal{N}$ the (outward pointing) unite normal field on the boundary $\partial\mathcal{B} \subset \mathcal{B}$ and have the Riemannian volume elements $\mu_{\mathcal{B}}$ on $\mathcal{B}$ and $\mu_{\partial} = \mathbf{i}_{\mathcal{N}}\mu_{\mathcal{B}}$ on $\partial\mathcal{B}$. Points of $\mathcal{B}$ are refered to as material points; they manifest themselves by their configurations in the ambient physical space $\mathbb{R}^n$. By a configuration (or placement) of the body $\mathcal{B}$ we then mean a C^k-embedding $J : \mathcal{B} \longrightarrow \mathbb{R}^n$ and call

$$E(\mathcal{B}, \mathbb{R}^n) := \{\ J : \mathcal{B} \to \mathbb{R}^n \mid J \text{ is a } C^k\text{-embedding}\ \} \tag{2.1}$$

the configuration space of the system. $E(\mathcal{B}, \mathbb{R}^n)$ can be given the structure of an infinite dimensional manifold with

$$TE(\mathcal{B}, \mathbb{R}^n) = \{\ \Lambda : \mathcal{B} \longrightarrow T\mathbb{R}^n \mid \Lambda \text{ is a } C^k\text{-map}, \Pi_{\mathbb{R}^n} \circ \Lambda \in E(\mathcal{B}, \mathbb{R}^n)\ \} \tag{2.2}$$

as tangent bundle, cf. [BSF,Mar]. In the language of classical mechanics [Thi] a tangent vector on the configuration space, i.e. some $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$, is called a virtual displacement. Although non-smooth configurations are important we restict our interest to C^k-embeddings with $k \geq 1$ or $k = \infty$.

Within the general framework of (infinite-dimensional) manifolds as configuration space of physical systems the principle of virtual work is naturally described by considering the generalized force as an element of the co-tangent bundle. For continuum mechanics Epstein and Segev [EpSe] gave the appropriate formulation, writing the work, done by a virtual displacement $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$ as the evaluation with some co-tangent vector $F \in T_J^* E(\mathcal{B}, \mathbb{R}^n)$. For our considerations, we restrict the dual of the infinite-dimensional space $T_J E(\mathcal{B}, \mathbb{R}^n)$ to the space of co-vectors having a special L^2-representation on the manifold with boundary $\mathcal{B}$. This means to take only such linear functionals on $T_J E(\mathcal{B}, \mathbb{R}^n)$ into account, which have an integral representation of the form

$$\begin{aligned} F[J] &: T_J E(\mathcal{B}, \mathbb{R}^n) \longrightarrow \mathbb{R} \\ F[J](\Lambda) &= \int_{\mathcal{B}} < \Phi(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi(J), \Lambda >_{\mathbb{R}^n} \mu_{\partial} \end{aligned} \tag{2.3}$$

where $<,>_{\mathbb{R}^n}$ denotes the Euclidean scalar product on $\mathbb{R}^n$. The precribed functions $\Phi(J) \in C^k(\mathcal{B}; \mathbb{R}^n)$ and $\varphi(J) \in C^k(\partial\mathcal{B}; \mathbb{R}^n)$ are to be understood as the physical force densities, effecting the material points inside the body and on the surface of $\mathcal{B}$, respectively. They may depend on the configuration J in a non-linear and non-local way. For boundaryless manifolds, the virtual work (1.1) appears as a specialization of (2.3). Without going into details about the proper treatment of forces in continuum mechanics, cf. [Tr],

we remark that our considerations hold true, independent of the interpretation of $\Phi(J)$ as a traction force, a body force or a combination of both.
Given the functional dependence of Φ and φ on the configuration J, the principle of virtual work (for a static problem) reads as :

$$J \in E(\mathcal{B}, I\!R^n) \text{ is an equilibrium configuration } \Leftrightarrow F[J](\Lambda) = 0 \text{ for all } \Lambda \in T_J E(\mathcal{B}, I\!R^n).$$

Searching for equilibrium configurations of the body by means of this principle is nothing but solving of the weak boundary value problem of non-linear elastostatic (with traction boundary conditions). It has been shown by [AnOs] that under some technical conditions the balance laws in continuum mechanics are equivalent to this principle. Also a description of dynamics can easily be included into this framework [BSS,BiSc].

For our consideration we take the principle of virtual work as such and show, how the symmetries of the ambient space $I\!R^n$ can be used to specify the form of the virtual work and consequently the equations of continuum mechanics.
In the classical treatment this question has been attacked by searching for a material frame indifferent formulation of elasticity and is answered by Noll's theorem [No63,Tr]. The symmetry group in question is the group of rigid changes of frame on $I\!R^n$ which is the semi-direct product $I\!R^n \otimes_S SO(n)$ and called the Eucildean group $E(n)$, cf. [Thi,MRW]. An element $g_{(T,R)} \in E(n)$ is uniquely represented by a translation $T \in I\!R^n$ and a rotation $R \in SO(n)$ and its action on $I\!R^n$ is given as

$$\begin{aligned} E(n) \times I\!R^n &\longrightarrow I\!R^n \\ g_{(T,R)}[v] &= R(v+T) \end{aligned} \qquad (2.4).$$

The corresponding action of the Lie algebra $e(n)$ is

$$\begin{aligned} e(n) \times I\!R^n &\longrightarrow I\!R^n \\ g_{(z,C)}[v] &= Cv + z \end{aligned} \qquad (2.5)$$

where $z \in I\!R^n$, $C \in so(n)$ and $so(n)$ denotes the Lie algebra of $SO(n)$, which is the space of all anti-symmetric $n \times n$ matrices. By pointwise action on $J(p) \in I\!R^n$ this induces naturally an action on $E(\mathcal{B}, I\!R^n)$.
Demanding the symmetry of a physical system under the group $E(n)$ is the contents of Noll's axioms of frame indifference of forces and of frame indifference of working. The first one claims

$$\Phi\Big(R(J+T)\Big) = R\Phi(J) \qquad \text{and} \qquad \widetilde{\varphi}\Big(R(J+T)\Big) = R\widetilde{\varphi}(J) \qquad \forall\,(z,R) \in E(n) \quad (2.6)$$

for all body forces $\Phi(J)$ and all surface forces $\widetilde{\varphi}(J)$. Its physical content is to consider only those forces as relevant for the theory which are due to either interior interactions between material points or tracition boundary conditions and which transform as vectors under rigid rotations $R \in SO(n)$.

The second axiom, which is fundamental for our considerations, demands that rigid Euclidean motions cause no work on any subbody $U \subset \mathcal{B}$. Originally this was posed in terms of the mechanical power of a motion, but for statics it is equivalent to the following : For any subbody $U \subset \mathcal{B}$ those virtual displacements cause no work, which are infinitesimal rigid Euclidean motions (2.5) the with domain restricted to this subbody :

$$\int\limits_{U} < \Phi(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int\limits_{\partial U} < \widetilde{\varphi}(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_{\partial} = 0 \qquad \forall\ U \subset \mathcal{B} \quad (2.7),$$

holding for any $z \in \mathbb{R}^n$ and any anti-symmetric matrix $C \in so(n)$, where $\widetilde{\varphi}(J)$ describes the force density on the surface ∂U and is a priori independent from the force acting on $\partial\mathcal{B}$. Since invariance is demanded for any $U \subset \mathcal{B}$ this axiom is a local one, which appears somewhat artificial from a global point of view. To motivate it from physical considerations requires some more arguments like the axiom of the cut principle of Euler and Cauchy [Tr] or the demand that only short distance interactions have an effect [LaLi]. Also due to the locallity of (2.7) it is not clear how results, deduced from this axiom, are influenced by prescribed boundary conditions on $\partial\mathcal{B}$.

Hence we drop the postulate of locality in the frame indifference of the working and replace (2.7) by an axiom that is more obvious from a physical point of view : We start with the demand that the prescribed force densities determines a virtual work, obeying the global invariance property

$$F[J](g_{(z,C)}[\Lambda]) = F[J](\Lambda) \qquad \forall g_{(z,C)} \in e(n) \qquad (2.8)$$

under the action of the Lie algebra of the Euclidean group. As worked out below this global property of constitutive function suffies to reproduce the classical theory.

Theorem 1

Let the prescribed force densities $\Phi(J)$ and $\varphi(J)$, acting on the body $\mathcal{B}$ and its boundary $\partial\mathcal{B}$, respectively, determine the virtual work by (2.3). If this work is $\epsilon(n)$-invariant in the sense of (2.8), i.e. if

$$\int_{\mathcal{B}} < \Phi(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \varphi(J), g_{(z,C)}[J] >_{\mathbb{R}^n} \mu_{\partial} = 0 \tag{2.9}$$

then the equilibrium configuration (respectively the motion) of the body $\mathcal{B}$ is determined by the divergence of a symmetric stress tensor.

The crucial point of this theorem is that a gobal rigid symmetry condition suffies to prove the existence of a symmetric stress tensor. Under the stronger (local) assumption (2.7) the corresponding result is known as Noll's theorem [Tr]. Similar theorems have been derived by Green, Rivelin and Naghdi [GrRi], who replaced the working axiom (2.7) by starting with an $E(n)$-invariant energy functional $\mathcal{E}[J] \in C^\infty(\mathcal{B}; \mathbb{R})$. Again this is a local invariance demand and furthermore the existence of an energy functional restricts the theory to the special case of hyperelasticity.

The proof of the theorem above is based on Hodge theory on manifolds with boundaries, which makes it possible, to obtain from the global axiom (2.8), the existence of symmetric stress tensor as a local result. In this sense the cut principle of Euler and Cauchy, which fills the gap between Noll's local axiom and the global invariance demand in the physical argumentation, may be understood as a reflection of Hodge theory. Before doing the constructions in detail we have to present some fundamental results of that theory for manifolds with boundaries.

3. Vector-valued differential forms, Hodge Theory and Boundary Value Problems

Considering $E(\mathcal{B}, \mathbb{R}^n)$ as the configuration space for elasticity, two-point tensors [Er] over the body manifold $\mathcal{B}$ are natural objects to describe the phyiscal properties of the medium. Such tensors are the canonical generalizations of vector fields and one forms over maps, respectively. Restricting the general definition [HuMa] to the case of interest we define :

A two-point tensor $\mathbf{T}$ of type $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, shortly denoted as a (r,s)-type two-point tensor, at $p \in \mathcal{B}$ over an embedding $J \in E(\mathcal{B}, \mathbb{R}^n)$ is a multilinear mapping

$$\mathbf{T} : \underbrace{(T_p\mathcal{B} \times \ldots \times T_p\mathcal{B})}_{r-times} \times \underbrace{(T^*_{J(p)}I\!R^n \times \ldots \times T^*_{J(p)}I\!R^n)}_{s-times} \longrightarrow I\!R \qquad (3.1).$$

One can think of a two-point tensor having two tensor legs, one in $\mathcal{B}$ and one in $I\!R^n$. For $s = 1$ a two-point tensor over J at $p \in \mathcal{B}$ can also be considered as a multilinear form on $\mathcal{B}$ with values in $T_{J(p)}I\!R^n$, in other words any such $\mathbf{T}$ is a $J^*(TI\!R^n)$-valued form on $\mathcal{B}$. Hence the skew-symmetric $(r, 1)$-type two-point tensors fit into the notion of vector-valued differential forms, which we define more generally for any Riemannian manifold M and any finite dimensional vector space V :

Definition and Remark [GHV]

A V-valued differential form $\omega \in \Omega^r(M; V)$ of degree r over a m-dimensional manifold M is a smooth assignment of skew-symmetric r-linear maps to the points of M, where

$$\omega_p : \underbrace{(T_pM \times \ldots \times T_pM)}_{r-times} \longrightarrow V \qquad \forall\, p \in M \qquad (3.2).$$

The algebra of all V-valued forms on M is denoted by $\Omega(M; V) = \bigoplus_{r=1}^{m} \Omega^r(M; V)$. There is a natural identification $\Omega(M; V) \cong \Omega(M; I\!R) \otimes V$, such that the algebraic and analytic structures on the algebra of usual ($I\!R$-valued) differential forms, carry over to $\Omega(M; V)$. In terms of a fixed scalar product $<,>_V$ the isomorphism can be given by means of the pairing

$$\begin{aligned} &<<,>>:\ V \otimes \Omega^r(M; V) \longrightarrow \Omega^r(M; I\!R) \\ &<< v, \omega >>(X_1, \ldots X_r) := < v, \omega(X_1, \ldots X_r) >_V \qquad \forall\, (X_1, \ldots X_r) \in \Gamma TM \end{aligned} \qquad (3.3).$$

Fundamental quantities in continuum mechanics as the deformation gradient or the 1^{st} Piola-Kirchhoff tensor, are described by $(1, 1)$-type two-point tensors on the body manifold and hence can also be considered as $I\!R^n$-valued one forms. The use of vector-valued forms instead of the well known tensor language is motivated by the fact that the Hodge theory on the algebra of differential forms is a usefull tools to solve boundary value problems on Riemannian manifolds, cf. [EbMa]. Thus the idea is to formulate boundary value problems in continuum mechanics in terms of V-valued forms $\omega \in \Omega(M; V)$ with $M = \mathcal{B}$ and $V = I\!R^n$ and use well known results from Hodge theory instead of solving the problems directly by tensor calculus.

To do so we introduce, in view of (3.3), the exterior derivative

$$\begin{aligned} d : \Omega^r(M;V) &\longrightarrow \Omega^{r+1}(M;V) \\ << v, d\omega >> &:= \mathbf{d} << v, \omega >> \qquad \forall\, v \in V \end{aligned} \tag{3.4}$$

where $\mathbf{d}$ is the exterior derivative on the algebra $\Omega(M, I\!R)$ of real-valued forms. Similarly the Hodge $*$-operator on $\Omega(M, I\!R)$ induces the operator

$$\begin{aligned} \star : \Omega^r(M;V) &\longrightarrow \Omega^{m-r}(M;V) \\ << v, \star\omega >> &:= * << v, \omega >> \qquad \forall\, v \in V \end{aligned} \tag{3.5}$$

and it makes sense to define by $\delta := (-1)^{mr+1} \star d \star$ the co-differential $\delta : \Omega^{r+1}(M;V) \rightarrow \Omega^r(M;V)$. Like the corresponding co-differential on $\Omega(M, I\!R)$ this is a nilpotent operator, i.e. $\delta^2 = 0$ on $\Omega(M;V)$. Furthermore it computes on a one form $\omega \in \Omega^1(M;V)$ as minus the divergence of the induced tensor $\omega^\sharp$, cf. [AMR], defined via a Riemannian metric G_M on M by

$$G_M(Y, \omega^\sharp) := \omega(Y) \qquad \forall\, Y \in \Gamma TM \qquad (3.6).$$

In generalization of that property the co-differential can be computed [Mat] by means of a local orthonormal frame $\{E_1, \dots E_m\}$ on TM as

$$(\delta\omega)(X_1, \dots X_r) := -\sum_{k=1}^{n} (\nabla_{E_k}\omega)(E_k, X_1, \dots X_r) \quad \text{with} \quad X_1, \dots X_r \in \Gamma TM \tag{3.7}$$

where ∇ is the Levi-Civita connection. Furthermore we can equip each space $\Omega^r(M;V)$ with a Riemannian structure, induced from the scalar product $<,>_V$ and the metric G_M by setting

$$\begin{aligned} <,>_{\Omega^r} : \Omega^r(M;V) \times \Omega^r(M;V) &\longrightarrow \Omega^0(M; I\!R) \\ <\omega, \eta>_{\Omega^r} &:= \sum_{j_1 < \dots < j_r} <\omega(E_{j_1}, \dots, E_{j_r}), \eta(E_{j_1}, \dots, E_{j_r})>_V \end{aligned} \tag{3.8}$$

where the (local) fields E_{j_r} run through the orthonormal frame on TM. This definition is frame independent and yields for $r = 0$ the scalar product $<,>_V$. It corresponds (for $V = I\!R$) to the usual inner product $\omega \wedge *\eta = <\omega, \eta>_{\Omega^r} \mu_M$ on $\Omega^r(M; I\!R)$ and one can prove [Ack] that (3.8) also given an expression for the "dot"-product, used in [BSS] to formulate the virtual work in terms of stress forms. With that scalar product the space

$\Omega^r(M;V)$ can be furnished with the structure of the Sobolev space $H^1\Omega^r(M;V)$, given as the completion of the space of C^1-differentiable forms $\omega \in \Omega^r(M;V)$ with respect to the norm

$$||\omega||^2 = \int\limits_M \Big(< \omega, \omega >_{\Omega^r} + < d\omega, d\omega >_{\Omega^{r+1}} \Big) \mu_M \qquad (3.9).$$

In that Sobolev space of H^1-forms over M the operators δ and d are adjoint to each other up to a boundary term. Especially for any pair of $\omega \in H^1\Omega^1(M;V)$ and $\eta \in H^1\Omega^0(M;V)$ we have

$$\int\limits_M < \omega, d\eta >_{\Omega^1} \mu_M = \int\limits_M < \delta\omega, \eta >_{\Omega^0} \mu_M + \int\limits_{\partial M} < \omega(\mathcal{N}), \eta >_{\Omega^0} \mu_\partial \qquad (3.10),$$

which is a consequence of the Stokes theorem. It yields the Gauß theorem in terms of differential forms by taking η to be constant.

Now we have introduced all stuctures, necessary to face the question of solving boundary value problems by means of Hodge theory. The Sobolev space $H^1\Omega^r(M;V)$ carries the same topology as the one, used in the book of Morrey [Mo62], hence we have :

Theorem 2

Let M be a compact C^k-manifold with boundary, where $k \geq 2$ or $k = \infty$ and let $\mathcal{N}$ denote the (outward pointing) unite normal field on $\partial M \subset M$. Call furthermore $\omega \in H^1\Omega^r(M;V)$ to be of class C^k_ν if its derivatives of order k are ν-Hölder smooth.

a) For any function $\Psi \in H^1\Omega^0(M;V)$, there is a decomposition

$$\Psi = \delta\beta_\Psi + c_\Psi \qquad (3.11)$$

where $\beta_\Psi \in H^1\Omega^1(M;V)$ is a one form obeying $\beta_\Psi(\mathcal{N}) = 0$ and $c_\Psi \in V$ is a constant. If $\Psi \in C^{k-2}_\nu(M;V)$ then β_Ψ is also of class C^{k-2}_ν.

b) Given a r-form $\beta \in H^1\Omega^r(M;V)$ with $\beta|_{\partial M} \in H^1\Omega^r(\partial M;V)$, there exists a $(r+1)$-form $\xi \in H^1\Omega^{r+1}(M;V)$ obeying the boundary conditions

$$\begin{aligned} \xi|_{\partial M} &\equiv 0 \\ (\delta\xi)|_{\partial M}(X_1,\dots,X_r) &= \beta|_{\partial M}(X_1,\dots,X_r) \qquad \forall\, X_1,\dots X_r \in \Gamma T\partial M \end{aligned} \qquad (3.12).$$

If $\beta|_{\partial M}$ is of class C^{k-2}_ν on ∂M then ξ can be chosen of class C^{k-2}_ν on M.

With this theorem we have reformulated some - at least for $V = \mathbb{R}$ - well established results. Part a) is usually refered to as Kodeira decomposition of the function Ψ and the solvability of the problem (3.12) is due to [Mo56]. This result is not quoted literally, but taken from the proof of the lemma 6.2 there, where the assertion is given and explicitely used. By means of the identification $\Omega(M;V) \cong \Omega(M;\mathbb{R}) \otimes V$ the generalization to $H^1\Omega(M;V)$ is obvious. Hence we obtain on a compact Riemannian C^k-manifold M with boundary :

Lemma 1

a) Given a pair of vector-valued functions $\Phi \in H^1\Omega^0(M;V)$ and $\varphi \in H^1\Omega^0(\partial M;V)$, which obey the integrability condition

$$\int_M \Phi\, \mu_M + \int_{\partial M} \varphi\, \mu_\partial = 0 \tag{3.13}$$

there is a one form $\alpha \in H^1\Omega^1(M;V)$ solving the boundary value problem

$$\begin{aligned} \delta\alpha &= \Phi \qquad \text{on } M \\ \alpha(\mathcal{N}) &= \varphi \qquad \text{on } \partial M \end{aligned} \tag{3.14}$$

If furthermore Φ and φ are of class C_ν^{k-2} on their respective domains then α is also of class C_ν^{k-2} on M.

b) Given a V-valued function $\Theta \in H^1\Omega^0(M;V)$, which obeys the integrability condition

$$\int_M \Theta\, \mu_M = 0 \tag{3.15}$$

there is a one form $\gamma \in H^1\Omega^1(M;V)$ solving the boundary value problem

$$\begin{aligned} \delta\gamma &= \Theta \qquad \text{on } M \\ \gamma|_{\partial M} &\equiv 0 \qquad \text{on } \partial M \end{aligned} \tag{3.16}$$

If Θ is of class C_ν^{k-2} then γ is of class C_ν^{k-2}, as well.

Proof :
By the Kodeira decomposition (3.11) some $\beta_\Phi \in H^1\Omega^1(M;V)$ is determined from Φ, obeying

$$\Phi = \delta\beta_\Phi + c_\Phi \qquad \text{and} \qquad \beta_\Phi(\mathcal{N}) = 0 \qquad (3.17).$$

On the other hand one can choose for any $\varphi \in H^1\Omega^0(\partial M;V)$ some one-form $\phi \in H^1\Omega^1(M;V)$ such that $\phi(\mathcal{N}) = \varphi$ on ∂M. Applying Kodeira's decomposition to the function $\delta\phi \in H^1\Omega^0(M;V)$ yields

$$\delta\phi = \delta\beta_\varphi + c_\varphi \qquad \text{with} \qquad \beta_\varphi(\mathcal{N}) = 0 \qquad (3.18).$$

Then the boundary value problem (3.14) is soved by the one-form

$$\alpha = \beta_\Phi + \phi - \beta_\varphi \qquad (3.19).$$

To see this it remains to show that the constants c_Φ and c_φ cancel each other by using the integrability condition (3.13) and the Gauß theorem, cf. (3.10) :

$$\int_M (c_\Phi - c_\varphi)\,\mu_M = \int_M (\Phi - \delta\alpha)\,\mu_M = \int_M \Phi\,\mu_M + \int_{\partial M} \varphi\,\mu_\partial = 0 \qquad (3.20).$$

To prove b) we start similar as above and decompose Θ by (3.11). From the integrabilty condition (3.15) the constant c_Θ has to vanish and hence

$$\Theta = \delta\beta_\Theta \qquad \text{with} \qquad \beta_\Theta(\mathcal{N}) = 0 \qquad (3.21).$$

Then there exists by part b) of theorem 2 some $\xi_\Theta \in H^1\Omega^2(M;V)$ such that

$$(\delta\xi_\Theta)|_{\partial M}(X) = \beta_\Theta|_{\partial M}(X) \qquad \text{and} \qquad \xi_\Theta|_{\partial M} \equiv 0 \quad \forall\, X \in \Gamma T\partial M \qquad (3.22).$$

We choose $\gamma = \beta_\Theta - \delta\xi_\Theta$ and obtain

$$\begin{aligned} \delta\gamma &= \Theta && \text{on } M \\ \gamma|_{\partial M}(X) &= 0 && \text{on } \partial M \end{aligned} \qquad (3.23),$$

holding for all vector fields X along ∂M. It remains to show that also $\gamma|_{\partial M}(\mathcal{N}) = 0$. To do so we use the tubular neighbourhood theorem [La], which guaranties the existence of a local orthonormal frame of the form $\{\widetilde{\mathcal{N}}, \widetilde{E}_2, \ldots, \widetilde{E}_m\}$ with $\widetilde{\mathcal{N}}|_{\partial M} = \mathcal{N}$ and $\widetilde{E}_i|_{\partial M}$ tangential to ∂M near any $p \in \partial M$. Then (3.7) yields

$$
\begin{aligned}
(\delta\xi_\Theta)(\mathcal{N}) &= -\Big((\nabla_{\widetilde{\mathcal{N}}}\xi_\Theta)(\widetilde{\mathcal{N}}, \mathcal{N}) + \sum_{k=2}^{m}(\nabla_{\widetilde{E}_k}\xi_\Theta)(\widetilde{E}_k, \mathcal{N})\Big)|_{\partial M} \\
&= -\sum_{k=2}^{m} \nabla_{\widetilde{E}_k}(\xi_\Theta(\widetilde{E}_k, \mathcal{N}))|_{\partial M}
\end{aligned}
\tag{3.24}
$$

since $(\xi_\Theta)|_{\partial M} \equiv 0$ by (3.22). Also due to that fact $(\xi_\Theta)|_{\partial M}$ is covariantly constant under the action of the vector fields $\widetilde{E}_k$ along ∂M, what proves $(\delta\xi_\Theta)|_{\partial M}(\mathcal{N}) = 0$. Since also $\beta_\Theta(\mathcal{N}) = 0$ by (3.21) the V-valued one form $\gamma|_{\partial M} = \beta_\Theta - \delta\xi_\Theta$ vanishs identically on ∂M. Finally the differentiability results directly read off from theorem 2. □

Both assertions of that lemma are not original. Part a) may also be derived using a solution theorem for the Neumann problem $\Delta\mathcal{H} = \Phi$ and $d\mathcal{H}(\mathcal{N}) = \varphi$, cf. [Hö], and setting $\alpha := d\mathcal{H}$. This (stronger) result has been applied in [Bi] to similar questions as we consider here. Also the (modified) Dirichlet boundary value problem of part b) has been considered elsewhere [vWa,Bo]. These results coincide with ours, however the authors are more restrictive in choosing M ($M \subset \mathbb{R}^n$ being a sum of starlike connected domains [Bo] or $M \subset \mathbb{R}^3$ [vWa], respectively) to obtain also estimates for the growth on the boundary. To relate their approach to the one used here we remark that the operator δ, acts on a one form, like the divergence of the corresponding vector field, induced from G_M by (3.6). Having this in mind the result b) of the lemma 1 reads in the language of classical boundary value problems :

Corollary

Let M be a (compact) Riemannian C^k-manifold ($k \geq 2$) with boundary and let $\Theta \in C^{k-2}_\nu(M; V)$ be given function, obeying the integrability condition $\int_M \Theta\,\mu_M = 0$. Then the boundary value problem

$$
\begin{aligned}
\operatorname{div} Z &= \Theta \qquad \text{on } M \\
Z &= 0 \qquad \text{on } \partial M
\end{aligned}
\tag{3.25}
$$

has a solution $Z \in \Gamma(TM \otimes V)$ of differentiability class C^{k-2}_ν. For $V = \mathbb{R}$ the section Z becomes a C^{k-2}_ν vector field on M.

4. Translational Invariance, Stress Tensors and the Piola Transformation

Now we have all technical tools at hand to apply Hodge theory to the boundary value problems we have in mind. To prove our central assertion, made in secction 2, we note that due to the product structure of the Euclidean group we can consider the translational and the rotational invariance separately. So we first use the invariance under global rigid translations as an integrability condition to show that any virtual work, given in the form (2.3) and obying a translational symmetry, also allows a tensorial description. Starting from Noll's axiom (2.7) of frame indifference of working such tensorial character of the stress is evident. Under the (weaker) global assumption (2.9), however, we need Hodge theory to obtain the local result, that any virtual work allows a description in terms of a two-point tensor on the body manifold $\mathcal{B}$.

Theorem 3

Let the body $\mathcal{B}$ be a Riemannian C^k-manifold with boundary and let the virtual work be determined from a pair $(\Phi(J), \varphi(J))$ of force densities, each of Sobolov class $H^1\Omega^0$ on its domain, as

$$F[J](\Lambda) = \int\limits_{\mathcal{B}} < \Phi(J), \Lambda >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int\limits_{\partial\mathcal{B}} < \varphi(J), \Lambda >_{\mathbb{R}^n} \mu_{\partial} \tag{4.1}$$

with the virtual displacement $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$ of Sobolev class $H^1\Omega^0(\mathcal{B}; \mathbb{R}^n)$. If global rigid translations cause no work, i.e.

$$F[J](z) = \int\limits_{\mathcal{B}} < \Phi(J), z >_{\mathbb{R}^n} \mu_{\mathcal{B}} + \int\limits_{\partial\mathcal{B}} < \varphi(J), z >_{\mathbb{R}^n} \mu_{\partial} = 0 \qquad \forall\, z \in \mathbb{R}^n \qquad (4.2),$$

there exists a $\mathbb{R}^n$-valued one form $\alpha(J) \in H^1\Omega^1(\mathcal{B}; \mathbb{R}^n)$, called the stress form of the system, such that the virtual work becomes

$$F[J](\Lambda) = \int\limits_{\mathcal{B}} < \alpha(J), d\Lambda >_{\Omega^1} \mu_{\mathcal{B}} \qquad (4.3).$$

Here $d\Lambda \in H^0\Omega^1(\mathcal{B}; \mathbb{R}^n)$ is the differential of the virtual displacement and $<,>_{\Omega^1}$ means the scalar product (3.8). $\alpha(J)$ is C^{k-2}_ν-differentiable if $\Phi(J)$ and $\varphi(J)$ were of class C^{k-2}_ν.

Proof :

Given the pair of functions $(\Phi(J), \phi(J))$ we observe that the invariance condition (4.2) is equivalent to the integrability condition (3.13) since $z \in \mathbb{R}^n$ is arbitrary. Hence part a) of lemma 1 above guaranties some $\alpha(J) \in H^1\Omega^1(\mathcal{B}; \mathbb{R}^n)$ to exist, such that the virtual work becomes

$$F[J](\Lambda) = \int_{\mathcal{B}} < \delta\alpha(J), \Lambda >_{\Omega^0} \mu_{\mathcal{B}} + \int_{\partial\mathcal{B}} < \alpha(J)(\mathcal{N}), \Lambda >_{\Omega^0} \mu_{\partial} \qquad (4.4).$$

By applying Stoke's theorem in the form (3.10) we shift the operator δ, acting on $\alpha(J)$, to its adjoint d acting on Λ, such that the boundary terms cancel, what proves (4.3). $\square$

This result enables us to link the description of the virtual work via force densities with the usual formulation of continuum mechanics. In terms of the stress form $\alpha(J)$, which is a (1,1)-type two-point tensor on $\mathcal{B}$, the principle of virtual work rewrites as :

J is an equilibrium configuration $\Leftrightarrow \int_{\mathcal{B}} < \alpha(J), d\Lambda >_{\Omega^1} \mu_{\mathcal{B}} = 0$ for all $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$.

As an immediate consequence of this weak problem we can derive the equilibrium equation for elastostatics in terms of the stress form, what yields the interpretation of $\alpha(J)$ as the 1^{st} Piola-Kirchhoff stress tensor of the system. Therefore we let $U \subset \mathcal{B}$ be some open (connected) subbody with boundary ∂U and assume that $\partial U \cap \partial\mathcal{B} = \emptyset$, for sake of simplicity. By U_ϵ we denote a family of open subset of $\mathcal{B}$, containing the closure of U, i.e. $\overline{U} \subset U_\epsilon \subset \mathcal{B}$, and require that the measure of the set $U_\epsilon \backslash U$ to be bounded by ϵ, i.e. $\int_{U_\epsilon \backslash U} \mu_{\mathcal{B}} < \epsilon$. Then infinitesimal displacement $\Lambda \in T_J E(\mathcal{B}, \mathbb{R}^n)$ we choose such that it takes an arbitrary constant value $\lambda \in \mathbb{R}^n$ on the subbody U, vanishes on $\mathcal{B}\backslash U_\epsilon$ and is smooth inbetween, i.e.

$$\Lambda(p) = \Lambda_\epsilon^\lambda(p) := \begin{cases} \lambda & \text{on } U \\ \widetilde{\lambda}_\epsilon(p) & \text{on } U_\epsilon \backslash U \\ 0 & \text{on } \mathcal{B}\backslash U_\epsilon \end{cases} \qquad (4.5).$$

Since $d\Lambda_\epsilon^\lambda = 0$ on U and on $\mathcal{B}\backslash U_\epsilon$, we obtain for the virtual work (4.3) done by Λ_ϵ^λ by using Stokes's theorem

$$F[J](\Lambda_\epsilon^\lambda) = \int_{U_\epsilon \backslash U} < \delta\alpha(J), \widetilde{\lambda}_\epsilon >_{\Omega^0} \mu_{\mathcal{B}} + \int_{\partial(U_\epsilon \backslash U)} < \alpha(J)(\widehat{\mathcal{N}}_\epsilon), \widetilde{\lambda}_\epsilon >_{\Omega^0} \mu_{\partial} \qquad (4.6)$$

where $\widehat{\mathcal{N}}_\epsilon$ denotes the outward pointing normal on $\partial(U_\epsilon)\backslash U$). By construction boundary splits into $\partial(U_\epsilon\backslash U) = \partial U_\epsilon \cup \partial U$ and we have $\widehat{\mathcal{N}}_\epsilon|_{\partial U} = -\widetilde{\mathcal{N}}$ where $\mathcal{N}$ is the outward pointing normal of U. Since $\widetilde{\lambda}_\epsilon(p)$ vanishs on ∂U_ϵ and takes the constant value λ on ∂U and furthermore $< \delta\alpha(J), \widetilde{\lambda}_\epsilon >_{\Omega^0}$ is bounded, we obtain

$$F[J](\Lambda_\epsilon^\lambda) = -\int_{\partial U} < \alpha(J)(\widetilde{\mathcal{N}}), \lambda >_{\Omega^0} \mu_\partial + o(\epsilon) \qquad (4.7).$$

Considering the limit $\epsilon \to 0$ and observing that $\lambda \in \mathbb{R}^n$ was chosen arbitrarily, the principle of virtual work in terms of the stress form $\alpha(J)$ yields

$$J \text{ is an equilibrium configuration } \Leftrightarrow \int_{\partial U} \alpha(J)(\widetilde{\mathcal{N}})\mu_\partial = 0 \text{ for any subset } U \subset \mathcal{B}.$$

So we have obtained a well established formulation for the integral equation of elastostatics, where the $\alpha(J)$ is to be considered as the 1^{st} Piola-Kirchhoff stress tensor. To make this interpretation more explicite and to invetigate the symmetry of the corresponding tensor, we have to study the Piola transformation in our framework.

Therefore we restrict our consideration to a n-dimensional body $\mathcal{B}$. Then any embedding $J \in E(\mathcal{B}; \mathbb{R}^n)$ is a regular map, saying that dJ, the principle part of the tangent map $TJ = (J, dJ)$ is an isomorphism. It makes sense to introduce the adjoint $dJ^\dagger$ of the tangent map, which depends on the Riemannian metric $G_\mathcal{B}$ as well as on the scalar product $<,>_{\mathbb{R}^n}$, by writing

$$G_\mathcal{B}(W\,,\,dJ^\dagger w) \;:=\; < dJW\,,\,w >_{\mathbb{R}^n} \qquad \forall\, W \in T_p\mathcal{B} \quad \forall\, w \in \mathbb{R}^n \qquad (4.8).$$

If Δ_J denotes the Jakobian determinant of the map J, then

$$\Delta_J{\cdot}\mathbf{A}_\alpha(J)|_{J(p)}(v) \;:=\; \alpha(J)|_p(dJ^\dagger v) \qquad \forall\, v \in \mathbb{R}^n \qquad (4.9)$$

yields a well defined tensor $\mathbf{A}_\alpha(J) : \mathbb{R}^n \to \mathbb{R}^n$ over each point $J(p)$ in the image of $\mathcal{B}$. It is the inverse of that transformation, sending the tensor $\mathbf{A}_\alpha(J)$ into the stress form $\alpha(J)$, which is denoted as the Piola tranformation [Ci] in continuum mechanics. To establish $\mathbf{A}_\alpha(J)$ as the Cauchy stress tensor we rewrite the virtual work (4.3) by pulling back the virtual displacement $\Lambda : \mathcal{B} \to \mathbb{R}^n$ to a $\mathbb{R}^n$-valued function $L = \Lambda \circ J^{-1}$ on $J(\mathcal{B}) \subset \mathbb{R}^n$. Then the differential becomes

$$d\Lambda|_p(E_i) = \big(\mathrm{grad}L|_{J(p)} \circ dJ|_p\big)(E_i) \qquad (4.10)$$

where grad L is the vector gradient in the usual sense [TrTo]. With (3.8) for the scalar product $<,>_{\Omega^1}$ we get

$$F[J](\Lambda) = \sum_{i=1}^{n} \int_{\mathcal{B}} \Delta_J < \mathbf{A}_\alpha(J) \circ (dJ^{-1})^\dagger(E_i), d\Lambda(E_i) >_{\mathbb{R}^n} \mu_{\mathcal{B}}$$
$$= \sum_{i=1}^{n} \int_{J(\mathcal{B})} G_{\mathcal{B}}\Big(E_i\,,\, dJ^{-1} \circ \mathbf{A}_\alpha^*(J) \circ \mathrm{grad}L \circ dJ(E_i)\Big)\, \mu_{\mathbb{R}^n} \tag{4.11}$$

where $\mathbf{A}_\alpha^*(J)$ is the adjoint with respect to $<,>_{\mathbb{R}^n}$ and we notice that the Riemannian volumes element on $\mathcal{B}$ and $\mathbb{R}^n$ are related to each other via the Jakobian by $J^*\mu_{\mathcal{B}} = \Delta_J^{-1} \cdot \mu_{\mathbb{R}^n}$. Observing finally that $\{E_1, \ldots, E_n\}$ is a orthonormal base on $T_p\mathcal{B}$ we use the cyclic property of the trace to obtain

$$F[J](L) = \int_{J(\mathcal{B})} \mathrm{trace}\Big(\mathbf{A}_\alpha^*(J) \cdot \mathrm{grad}L\Big) \mu_{\mathbb{R}^n} =: \int_{J(\mathcal{B})} \Big(\mathbf{A}_\alpha(J) : \mathrm{grad}L\Big)\, \mu_{\mathbb{R}^n} \tag{4.12}$$

Then the principle of virtual work becomes :

J equilibrium configuration $\Leftrightarrow \int_{J(\mathcal{B})} (\mathbf{A}_\alpha(J) : \mathrm{grad}L) \mu_{\mathbb{R}^n} = 0$ for all $L \in H^1\Omega^0(J(\mathcal{B}), \mathbb{R}^n)$

This is the standard form of the weak equilibrium equation of continuum mechanics, written in terms of the Cauchy stress tensor $\mathbf{A}_\alpha(J)$ and the gradient of the virtual displacement $L : J(\mathcal{B}) \to \mathbb{R}^n$, [HuMa].

To be able to reformulate the computations, made above, in terms of the Cauchy stress we further establish the celebrate Piola identity for one forms. To do so we observe that for $\kappa \in \Omega^1(\mathcal{B}, \mathbb{R})$ the induced vector fields $\kappa^\sharp$ and $(\kappa \circ dJ^\dagger)^\sharp$ on $\mathcal{B}$ and $J(\mathcal{B})$, respectively, are related by

$$< v, (\kappa \circ dJ^\dagger)^\sharp >_{\mathbb{R}^n} = \kappa(dJ^\dagger v) = < v, dJ\kappa^\sharp >_{\mathbb{R}^n} \qquad \forall\, v \in \mathbb{R}^n \tag{4.13}$$

We remark, that $\sharp$-operator(3.6) is defined with respect to the (different) metrics $<,>_{\mathbb{R}^n}$ and $G_{\mathcal{B}}$ on the left and right hand side, respectively. Then we can prove the Piola identity.

Lemma 2

The co-differential operator $\delta_{\mathcal{B}} : \Omega^1(\mathcal{B}, \mathbb{R}^n) \to \Omega^0(\mathcal{B}, \mathbb{R}^n)$ acting on the body manifold $\mathcal{B}$ and the corresponding operator $\delta_{\mathbb{R}^n} : \Omega^1(J(\mathcal{B}), \mathbb{R}^n) \to \Omega^0(J(\mathcal{B}), \mathbb{R}^n)$ on the embedded manifold $J(\mathcal{B})$ are related to each other via a Piola transformation by

$$\Delta_J \cdot \delta_{\mathbb{R}^n}\big(\mathbf{A}_\alpha(J)\big) = \delta_{\mathcal{B}}\big(\alpha(J)\big) \quad \text{where} \quad \Delta_J \cdot \mathbf{A}_\alpha(J) = \alpha(J) \circ dJ^\dagger \tag{4.14}$$

Proof :

Let $\mu_{I\!R^n} \in \Omega^n(J(\mathcal{B}), I\!R)$ be the Riemannian volume form on $J(\mathcal{B})$ and $v \in I\!R^n$ a constant vector. By using some standard properties of the Hodge $*$-operator [AMR] we obtain with (3.5) for the co-differential $\delta_{I\!R^n}$

$$(< v, \delta_{I\!R^n}\mathbf{A}_\alpha(J) >_{I\!R^n})\mu_{I\!R^n} = d\Big(* < v, \mathbf{A}_\alpha(J) >_{I\!R^n}\Big) = d\Big(\mathbf{i}_{K^\sharp_{\mathbf{A}}}\mu_{I\!R^n}\Big) \tag{4.15}$$

where $K_{\mathbf{A}} :=< v\,,\, \mathbf{A}_\alpha(J) >_{I\!R^n}$ is a $I\!R$-valued one form on $J(\mathcal{B})$. Replacing $\mathbf{A}_\alpha(J)$ by its Piola transformed we set $\kappa_\alpha :=< v\,,\, \alpha(J) >_{I\!R^n} \in \Omega^1(\mathcal{B}, I\!R)$ and obtain from (4.13)

$$K^\sharp_{\mathbf{A}} = \Delta_J^{-1}\cdot(< v, \alpha(J) >_{I\!R^n} \circ dJ^\dagger)^\sharp = \Delta_J^{-1}\cdot dJ(\kappa^\sharp_\alpha) \tag{4.16}.$$

Using $\Delta_J^{-1}\cdot\mu_{I\!R^n} = J^*\mu_{\mathcal{B}}$ for the pull back of the volume form $\mu_{\mathcal{B}}$ this yields

$$d(\mathbf{i}_{K^\sharp_{\mathbf{A}}}\mu_{I\!R^n}) = J^* d(\mathbf{i}_{\kappa^\sharp_\alpha}\mu_{\mathcal{B}}) \tag{4.17}.$$

and respelling (4.15) for $\kappa^\sharp_\alpha$ we finish the proof by observing that

$$(< v, J^*(\delta_{\mathcal{B}}\,\alpha(J)) >_{I\!R^n})\cdot(J^*\mu_{\mathcal{B}}) = J^* d\Big(\mathbf{i}_{\kappa^\sharp_\alpha}\mu_{\mathcal{B}}\Big) = (< v, \delta_{I\!R^n}\mathbf{A}_\alpha(J) >_{I\!R^n})\,\mu_{I\!R^n} \qquad \square$$

As mentioned above, cf. (3.6), the action of the co-differential on one forms and the divergence correspond to each other. Applied to the Cauchy stress tensor $\mathbf{A}_\alpha(J)$, which is a $I\!R^n$-valued one form on $J(\mathcal{B})$, this reads as

$$\delta_{I\!R^n}\mathbf{A}_\alpha(J) = \mathrm{div}_{I\!R^n}\mathbf{A}^\sharp_\alpha(J) \tag{4.18}.$$

Using the Gauß theorem and the Piola identity (4.14), the integral equation for the 1^{st} Piola-Kirchhoff stress tensor, cf. (4.7), yields $U \subset \mathcal{B}$

$$0 = \int_U \delta_{\mathcal{B}}\,\alpha(J)\,\mu_{\mathcal{B}} = \int_{J(U)} \mathrm{div}_{I\!R^n}\mathbf{A}^\sharp_\alpha(J)\,\mu_{I\!R^n} \qquad \forall U \subset \mathcal{B} \tag{4.19}$$

as the equilibrium equation of the system. This is the balance law of linear momentum for the Cauchy stress, as usually considered in contiumum mechanics. For a direct derivative of that equation from the virtual work (4.12) we refer to [AnOs], where also possible functional analytic suptilies are studied in detail.

Finally we remark that the stess form $\alpha(J)$, and hence also the Cauchy stress tensor $\mathbf{A}_\alpha(J)$ are not uniquely determined by the construction made above. The stress form may be redefined to any $\tilde{\alpha}(J)$, which correponds to the same phyiscal data $(\Phi(J), \varphi(J))$ by $\delta\tilde{\alpha}(J) = \Phi(J)$ and $\tilde{\alpha}(\mathcal{N}) = \varphi(J)$; a similar argument holds for $\mathbf{A}_\alpha(J)$. This gauge freedom corresponds to the fact, that only $\delta\alpha(J)$ or $\operatorname{div}\mathbf{A}_\alpha^\sharp(J)$, respectively, enter the equilibrium equation (4.19).

One can imagine several such modifications : From the mathematical point of view it seems natural to have $\alpha(J) \in \Omega^1(\mathcal{B}; I\!R^n)$ to be an exact one form, i.e. to be the gradient of some stress function $\mathcal{H}(J) \in \Omega^0(\mathcal{B}; I\!R^n)$. This is possible without further assumptions, as show in [Bi]. Considering continuum mechanics in its the classical formulation, however, a description of the Cauchy stress in terms of a symmetric tensor is needed : The virtual work (2.3) should admit some tensor $\widetilde{\mathbf{A}}_\alpha(J)$ on $J(\mathcal{B})$, which is symmetric.

5. Rotational Invariance and the Symmetry of the Stress Tensor

To investigate the symmetry of the Cauchy stress tensor we start from a translational invariant work of the form (2.3) for which theorem 3 guaranties the existence of a stress form $\alpha(J)$. Performing an inverse Piola transformation we know by using the Piola identity (4.14) that the Cauchy stress tensor solves the boundary value problem

$$\begin{aligned} \Delta_J \cdot \delta_{I\!R^n} \mathbf{A}_\alpha(J) &= \Phi(J) \qquad \text{on } J(\mathcal{B}) \\ \Delta_J \cdot \mathbf{A}_\alpha(J)\,\mathbf{n} &= \varphi(J) \qquad \text{on } J(\partial\mathcal{B}) \end{aligned} \tag{5.1}$$

where $\mathbf{n}$ is a normal field along $J(\partial\mathcal{B})$, defined by $dJ^\dagger\mathbf{n} := \mathcal{N}$. By means of Hodge theory we then can prove the existence of a symmetric stress tensor $\widetilde{\mathbf{A}}_\alpha(J)$, taking the (rigid) rotational invariance (2.9) of $F[J](\Lambda)$ as integrability condition.

Theorem 4

Let $\mathcal{B}$ be a n-dimensional Riemannian C^k-manifold with boundary and let the work done by any virtual displacement $L \in H^1\Omega^0(J(\mathcal{B}), I\!R^n)$ be

$$F[J](L) = \int_{J(\mathcal{B})} \Big(\mathbf{A}_\alpha(J) : \operatorname{grad} L\Big)\, \mu_{I\!R^n} \tag{5.2}$$

where the Cauchy stress tensor $\mathbf{A}_\alpha(J)$ is determined from the forces $(\Phi(J), \varphi(J))$ by (5.1). If infinitesimal rigid rotations of the whole body cause no work

$$\int\limits_{J(\mathcal{B})} \operatorname{trace}\Big(\mathbf{A}^*_\alpha(J)\cdot C\Big)\,\mu_{I\!R^n} = 0 \qquad \forall\, C \in so(n) \tag{5.3}$$

and there exists of a symmetric tensor $\widetilde{\mathbf{A}}_\alpha(J) : I\!R^n \to I\!R^n$ obeying also (5.1). If $\Phi(J)$ and $\varphi(J)$ were differentiable of class C^{k-2}_ν on their respective domains then $\widetilde{\mathbf{A}}_\alpha(J)$ is of class C^{k-2}_ν.

Proof :

With x and z vector fields on $J(\mathcal{B})$ we write the anti-symmetric part of the $\mathbf{A}_\alpha(J)$ as

$$< x, \mathbf{S}_\alpha(J)\, z >_{I\!R^n} := \frac{1}{2}\Big(< x, \mathbf{A}_\alpha(J) z >_{I\!R^n} - < z, \mathbf{A}_\alpha(J) x >_{I\!R^n}\Big) \tag{5.4}$$

and thus get a $so(n)$-valued zero form $\mathbf{S}_\alpha(J) \in \Omega^0(J(\mathcal{B}); so(n))$. Rewriting the $so(n)$-content of (2.9) in terms of the tensor $\mathbf{A}_\alpha(J)$ yields (5.3) and since $so(n)$ is the space of all anti-symmetric $n \times n$-matrices

$$\int\limits_{J(\mathcal{B})} \mathbf{S}_\alpha(J)\,\mu_{I\!R^n} = 0 \tag{5.5}.$$

This is an integrability condition to apply part b) of lemma 1 what yileds the existence of some $\sigma_{\alpha(J)} \in \Omega^1(J(\mathcal{B}); so(n))$, solving the boundary value problem

$$\mathbf{S}_\alpha(J) = \delta\sigma_{\alpha(J)} \qquad \text{with} \qquad \sigma_{\alpha(J)}|_{J(\partial\mathcal{B})} \equiv 0 \tag{5.6}.$$

Since $\sigma_{\alpha(J)}(x) \in so(n)$ is an anti-symmetric matrix at each point $q \in J(\mathcal{B})$, we can define an $I\!R^n$-valued two form $\Sigma_{\alpha(J)} \in \Omega^2(J(\mathcal{B}); I\!R^n)$ by

$$\begin{aligned} < x, \Sigma_{\alpha(J)}(y,z) >_{I\!R^n} \; &:= \\ &< x, \sigma_{\alpha(J)}(y)\, z >_{I\!R^n} - < x, \sigma_{\alpha(J)}(z)\, y >_{I\!R^n} - < z, \sigma_{\alpha(J)}(x)\, y >_{I\!R^n} \end{aligned} \tag{5.7}$$

where x, y and z are arbitrary vector fields over $J(\mathcal{B})$. The co-differential of the $I\!R^n$-valued two form $\Sigma_{\alpha(J)}$ computes according to (3.7) as

$$\begin{aligned} < x, \delta\Sigma_{\alpha(J)}(z) >_{I\!R^n} = -\Big(\sum_{i=1}^{n} \nabla_{e_i} < x, \Sigma_{\alpha(J)}(e_i, z) >_{I\!R^n} - < \nabla_{e_i} x, \Sigma_{\alpha(J)}(e_i, z) >_{I\!R^n} \\ - < x, \Sigma_{\alpha(J)}(e_i, \nabla_{e_i} z) >_{I\!R^n}\Big) \end{aligned} \tag{5.8}$$

where $\{e_1, \dots, e_n\}$ is a (local) orthonormal frame on $J(\mathcal{B})$. Expanding this by (5.7) yields

$$
\begin{aligned}
< x, \delta\Sigma_{\alpha(J)}(z) >_{\mathbb{R}^n} = &<x, \delta\sigma_{\alpha(J)}\, z >_{\mathbb{R}^n} \\
&+ \sum_{i=1}^{n} \Big(<x, \nabla_{e_i}(\sigma_{\alpha(J)}(z)) e_i >_{\mathbb{R}^n} - < x, \sigma_{\alpha(J)}(\nabla_{e_i} z)\, e_i >_{\mathbb{R}^n} \\
&+ <z, \nabla_{e_i}(\sigma_{\alpha(J)}(x)) e_i >_{\mathbb{R}^n} - < z, \sigma_{\alpha(J)}(\nabla_{e_i} x)\, e_i >_{\mathbb{R}^n} \Big)
\end{aligned} \qquad (5.9).
$$

We obtain a symmetric tensor on $J(\mathcal{B})$ by setting

$$
\widetilde{\mathbf{A}}_\alpha(J) := \mathbf{A}_\alpha(J) - \delta\Sigma_{\alpha(J)} \qquad (5.10),
$$

what is the symmetric part of $\mathbf{A}_\alpha(J)$, cf. (5.6), modified by a symmetric correction term. Due to the nilpotence of the co-differential on $J(\mathcal{B})$ we furthermore have $\delta\widetilde{\mathbf{A}}_\alpha(J) = \delta\mathbf{A}_\alpha(J)$. Hence it remains to study the behavior of $\widetilde{\mathbf{A}}_\alpha(J)$ on $\partial\mathcal{B}$. Therfore we argue similarly as in section 3, cf. (3.24), by chosing the (local) orthonormal frame as $\{\widetilde{\mathbf{n}}, \widetilde{e}_2, \dots, \widetilde{e}_n\}$ with $\widetilde{e}_i|_{J(\partial\mathcal{B})}$ tangential and $\widetilde{\mathbf{n}}|_{J(\partial\mathcal{B})}$ normal to $J(\partial\mathcal{B})$ near the surface of the body. By construction $\sigma_{\alpha(J)}$ vanishs on $J(\partial\mathcal{B})$ and we obtain

$$
\begin{aligned}
< x, \delta\Sigma_{\alpha(J)}(\widetilde{\mathbf{n}}) >_{\mathbb{R}^n} = &< \widetilde{\mathbf{n}}, \nabla_{\widetilde{\mathbf{n}}}(\sigma_{\alpha(J)}(x))\, \widetilde{\mathbf{n}} >_{\mathbb{R}^n} + \sum_{i=2}^{n} \Big(< x, \nabla_{\widetilde{e}_i}(\sigma_{\alpha(J)}(\widetilde{\mathbf{n}}))\, \widetilde{e}_i >_{\mathbb{R}^n} \\
&+ < \widetilde{\mathbf{n}}, \nabla_{\widetilde{e}_i}(\sigma_{\alpha(J)}(x))\, \widetilde{e}_i >_{\mathbb{R}^n} \Big)
\end{aligned} \qquad (5.11).
$$

But this expression vanishes on $J(\partial\mathcal{B})$ since $\sigma_{\alpha(J)}(x)$ is anti-symmetric and – as a consequence of $\sigma_{\alpha(J)}|_{J(\partial\mathcal{B})} \equiv 0$ – it is also covariantly constant under the fields $\widetilde{e}_i|_{J(\partial\mathcal{B})}$, which are vector fields along the boundary. We remark that in general $\widetilde{\mathbf{n}} \neq \mathbf{n}$ but $\delta\Sigma_{\alpha(J)}(\widetilde{\mathbf{n}})|_{\partial\mathcal{B}} = 0$ guaranties that $\widetilde{\mathbf{A}}_\alpha(J)(\widehat{\mathbf{n}}) = \mathbf{A}_\alpha(J)(\widehat{\mathbf{n}})$ for any field $\widehat{\mathbf{n}}$ normal to $J(\partial\mathcal{B})$. □

With that result on the existence of a symmetric Cauchy stress tensor $\widetilde{\mathbf{A}}_\alpha(J)$ the central assertion, we made in section 2, is proven. A local $so(n)$ invariance, as assumed by Noll's working axiom (2.7), would directly yield the symmetry of $\mathbf{A}_\alpha(J)$, but the global invariance (2.9) – and in consequence (5.3) – just gave the integrated symmetry result (5.5) and thus the construction, above was needed. Similar arguments as used in the proof are reputed to be due to Belinfante [MPP].

It is now is a matter of routine, to derive the balance law of angular momentum. We take $U \subset U_\epsilon \subset \mathcal{B}$ as in section 4 and construct the virtual displacement as

$$L_\epsilon^C(q) = \begin{cases} C\cdot q & \text{for } q \in J(U) \\ \widetilde{C}_\epsilon \cdot q & \text{for } q \in J(U_\epsilon \backslash U) \\ 0 & \text{for } q \in J(\mathcal{B}\backslash U_\epsilon) \end{cases} \qquad (5.12).$$

Here $C\cdot q$ denotes the action (2.5) of some constant matrix $C \in so(n)$ on $q \in J(\mathcal{B}) \subset \mathbb{R}^n$ and $\widetilde{C}_\epsilon$ means a $so(n)$-valued function, which is such that L_ϵ^C becomes smooth. Since $\mathrm{grad}(C\cdot q) = C$ and $\widetilde{\mathbf{A}}_\alpha(J)$ is symmetric we derive in analogy to (4.7) :

$$\begin{aligned} F[J](L_\epsilon^C) &= \int_{J(U_\epsilon \backslash U)} \mathrm{trace}(\,\widetilde{\mathbf{A}}_\alpha(J)\cdot \mathrm{grad}(\widetilde{C}_\epsilon \cdot q)\,)\, \mu_{\mathbb{R}^n} \\ &= - \int_{J(\partial U)} < \widetilde{\mathbf{A}}_\alpha(J)\widetilde{\mathbf{n}}, C\cdot q >_{\mathbb{R}^n} \mu_{\mathbb{R}^n} + o(\epsilon) \end{aligned} \qquad (5.13)$$

for any $U \subset \mathcal{B}$ with $\widetilde{\mathbf{n}}$ now denoting the unite normal field along $J(\partial U)$. In the limit $\epsilon \to 0$ this yields the usual form for the balance law of the angular momentum in equilibrium.

6. Remarks on the Constitutive Theory

Considering the Euclidean group as a symmetry in continuum mechanics also allows to face the constitutive question of the theory. This concerns the functional form of the stress tensor and means to figure out which mathematical information about the embedding J is necessary to determine $\widetilde{\alpha}(J)$ or $\widetilde{\mathbf{A}}_\alpha(J)$, respectively, and which is redundant. First we observe that taking $E(\mathcal{B}, \mathbb{R}^n)$ as the configuration space of the system is a primary – physically fundamental – constitutive assumption. For more general theories, e.g. for Cosserat media or for systems with defects, the virtual work in the form (2.3), depending only on the embedding J, will not yield a proper description.

To investigate the effect of the $E(n)$-symmetry on the constitutive question, we remark the tree different tensorial pictures for the stress. First there is the description in terms of the 1^{st} Piola-Kirchhoff stress, given by a $\mathbb{R}^n$-valued one form

$$\alpha(J) : T\mathcal{B} \longrightarrow \mathbb{R}^n \qquad (6.1),$$

which we derived in section 4. As a (1,1)-type two point tensor, having one leg in $\mathcal{B}$ and one in $\mathbb{R}^n$ it should transform like a vector under the $E(n)$-action (2.4) on an embedding J.

Second we worked in the spacial (Eulerian) picture by performing an inverse Piola transformation on the stress form $\alpha(J)$ which yields the tensor

$$\mathbf{A}_\alpha(J) = \Delta_J^{-1}\cdot\alpha(J)\circ dJ^t \qquad (6.2).$$

This is a proper tensor on $J(\mathcal{B}) \subset I\!R^n$ and should transform tensorial under the group $E(n)$. Third we can introduce the material (Lagrangian) picture, which is the most appropriate one to investigate the constitutive question. To do so we pull back the 1^{st} Piola-Kirchoff tensor $\alpha(J)$ under the embedding J and obtain the 2^{nd} Piola-Kirchhoff stress tensor $\mathcal{A}_\alpha(J)$ given by

$$dJ \circ \mathcal{A}_\alpha(J) = \alpha(J) \qquad \text{where} \qquad \mathcal{A}_\alpha(J) : T\mathcal{B} \longrightarrow T\mathcal{B} \qquad (6.3).$$

By the symmetry of $\widetilde{\mathbf{A}}_\alpha(J)$ it also can be chosen symmetric. Since the Euclidean group acts on the configuration space $E(\mathcal{B}, I\!R^n)$ only via a transformation in the ambient space $I\!R^n$, it should not touch $\mathcal{A}_\alpha(J)$ which is a proper tensor on the body $\mathcal{B}$. Hence the 2^{nd} Piola-Kirchhoff stress should behave as a scalar under the $E(n)$-action.

To make this explicite we investigate Noll's axiom of material frame indifference of forces (2.6), claiming that the forces densities $(\Phi(J), \varphi(J))$ obey a vectorial transformation law

$$\Phi(g_{(R,T)}[J]) = R\Phi(J) \qquad \text{and} \qquad \varphi(g_{(R,T)}[J]) = R\varphi(J) \quad \forall\, g_{(R,T)} \in E(n) \qquad (6.4)$$

under the action of the Euclidean group. In the virtual work approach this axiom appears quite naturally : Since the virtual work is defined as a linear functional on $T_J E(\mathcal{B}, I\!R^n)$, it transforms under the canonical lift [AbMa] of the $E(n)$-action (2.4) from the manifold $E(\mathcal{B}, I\!R^n)$ to its co-tangent bundle $T^*E(\mathcal{B}, I\!R^n)$. By definition (2.3) this functional $F[J]$ has a kernel, induced from the force densities $\Phi(J)$ and $\varphi(J)$ by the fixed scalar product $<,>_{I\!R^n}$. In consequence these force densities have to transform under the tangential lift of (2.4), saying they have to transform like vectors in $I\!R^n$.

By construction then also the stress form $\alpha(J)$ obeys a vectorial transformation law in its $I\!R^n$-argument under the $e(n)$-action on $E(\mathcal{B}, I\!R^n)$ and the axiom of the frame indifference of forces (6.4) becomes, written in terms of the 2^{nd} Piola-Kirchhoff tensor :

$$\mathcal{A}_\alpha(g_{(T,R)}[J]) = \mathcal{A}_\alpha(J) \qquad \forall\, g_{(T,R)} \in E(n) \qquad (6.5).$$

To obtain from this invariance the proper constitutive description for the stress we note, that (6.5) makes that the 2^{nd} Piola-Kirchhoff tensor into a (tensor-valued) functional on the quotient $E(\mathcal{B}, I\!R^n)/E(n)$, cf. [Bi]. Due to the product structure of the Euclidean group we again can investigate the translational and rotational part of the $E(n)$-action in sequence. Considering the translation group $I\!R^n$ there is the natural identification

$$E(\mathcal{B}, I\!R^n)/I\!R^n \cong \{dJ \mid J \in E(\mathcal{B}, I\!R^n)\} \tag{6.6}$$

where d is the exterior derivative acting on $E(\mathcal{B}, I\!R^n)$, which is an open subset of $\Omega^0(\mathcal{B}; I\!R^n)$. Hence an element of $E(\mathcal{B}, I\!R^n)/I\!R^n$ is to be indentified with the differential of an embedding. With the same argument as used for (4.10), that differential dJ corresponds to the deformation gradient. For the rotational symmetry we then introduce the Green deformation tensor as the pull-back of the scalar product $<,>_{I\!R^n}$ under the embedding J, defined explicitely by

$$\begin{aligned} C_J : T\mathcal{B} \times T\mathcal{B} &\longrightarrow I\!R \\ C_J(X,Y) &= < dJX, dJY >_{I\!R^n} \qquad \forall\, X,Y \in \Gamma T\mathcal{B} \end{aligned} \tag{6.7}.$$

It is a matter of routine, cf. [HuMa], to prove from (6.5) that the 2^{nd} Piola-Kirchhoff stress tensor $\mathcal{A}_\alpha(J)$, is a functional of the deformation tensor C_J only. So we may set in abust of notation $\mathcal{A}_\alpha(J) \equiv \mathcal{A}_\alpha(C_J)$ for the stress considered in the material picture as a functional on $E(\mathcal{B}, I\!R^n)/E(n)$. Hence we obtain as the final result concerning the principle of virtual work :

Theorem 5

Let a system in continuum mechanics be determined by an $E(n)$-invariant virtual work in the sense of (2.3), (2.6) and (2.9). Then an embedding J describes an equilibrium configuration of the system, iff the work integral

$$F[J](\Lambda) = \int_{\mathcal{B}} \Big(\mathcal{A}_\alpha(C_J) : D_\Lambda\Big)\, \mu_{\mathcal{B}} \tag{6.8}$$

vanishs for all virtual displacements $\Lambda \in T_J E(\mathcal{B}, I\!R^n)$. Here C_J is the Green deformation tensor, D_Λ denotes the symmetric part of the tensor $dJ^{-1} \circ d\Lambda$ on $T\mathcal{B}$ and $\mathcal{A}_\alpha(C_J)$ is the 2^{nd} Piola-Kirchhoff tensor.

Let us finally examine our results in the light of another standard approach to physical systems with symmetries. Therefore we specialize the treatment to a Lagrangian formulation, which it contained in our setting by considering the virtual work functional $F_J(\Lambda)$ to be the Frechet derivative of a (static) Lagrangian functional

$$F_J(J)(\Lambda) = \mathbf{D}\mathbb{L}(J)(\Lambda) \qquad \text{where} \qquad \mathbb{L}(J) = \int_{\mathcal{B}} \mathcal{L}(J)\,\mu_{\mathcal{B}} \tag{6.9}$$

Such an approach is equivalent to the consideration of the special case of a hyperelastic medium in continuum mechanics.

Symmetries of a Lagrangian (field) theory may be studied by means of Noether's theorem [Noe], which claims that each symmetry of $\mathcal{L}$ yields some conserved quantities. For an application to the Euclidean symmetry in elasticty we refer to [HuMa], where it is shown that Noether's conserved quantities for the $E(n)$-action in elasticity correspond to the existence of a symmetric stress tensor $\widetilde{\mathcal{A}}_\alpha(J)$. Hence for the hyperelastic case Noll's result may be understood as an application of Noether's theorem. We note, however, that assuming an $E(n)$ invariant Lagrangian (6.9) means to investigate a local symmetry, in contrast to our global treatment.

Equivalent to using Noether's theorem is – under certain assumptions – the momentum map technique [AbMa] in symplectic geometry. There the existence of conserved quantities in consequence of a symmetry is expressed as a constraint on the phase space of the system. The constraint subset $\mathcal{C}$ is determined from the momentum map of the given symmetry group $\mathcal{G}$ and the content of the Marsden-Weinstein reduction [MaWe] is to observe the quotient $\mathcal{C}/\mathcal{G}$ as a symplectic manifold. Furthermore all physical investigations for a $\mathcal{G}$-invariant system reduce to studies on that the reduced phase space.

Applied to our treatment of elasticity, the constraint for the Euclidean group action turns out to be the invariance (2.9) of the virtual work under rigid motions. The explicite construction of the coresponding momentum map will be given elsewhere [BiSc]. The discussions on Noll's theorem in section 4 and 5 then show that the constraint set $\mathcal{C}$ can be represented by the space of all symmetric tensors over $J(\mathcal{B})$, considered as functionals over the configuration space $E(\mathcal{B}, \mathbb{R}^n)$. On the other hand the transformation properties for the stress tensor, which were presented in this section, e.g. the relation (6.5), express the fact, that the proper reduced phase space is by $\mathcal{C}/E(n)$. In the material description this manifests itself in the fact, that the (symmetric) 2^{nd} Piola-Kirchhoff stress tensor is functional dependent on the embedding J only via the deformation tensor C_J.

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