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Dedicated to Professor Larry L. Schumaker on the occasion of his sixtieth birthday.

Abstract

We describe a general method for constructing triangulations Δ which are suitable for interpolation by $S_q^r(\Delta)$, $r = 1, 2$, where $S_q^r(\Delta)$ denotes the space of splines of degree q and smoothness r. The triangulations Δ are obtained inductively by adding a subtriangulation of locally chosen scattered points in each step. By using Bézier-Bernstein techniques, we determine the dimension and construct Lagrange and Hermite interpolation sets for $S_q^r(\Delta)$, $r = 1, 2$. The Hermite interpolation sets are obtained as limits of the Lagrange interpolation sets. The interpolating splines can be computed locally by passing from triangle to triangle. Several numerical results on interpolation of functions and scattered data are given.

Keywords : Bivariate Splines, Interpolation, Bézier-Bernstein techniques, Triangulation, Scattered data.

AMS Classification: 41A05, 41A15, 65D05, 65D07, 65D17, 41A63

1 Introduction

Let $\Delta = \{T^{[1]}, \ldots, T^{[N]}\}$ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$. For $0 \leq r < q$, the set

$$S_q^r(\Delta) = \{s \in C^r(\Omega) : \; s|_{T^{[l]}} \in \tilde{\Pi}_q, \; l = 1, \ldots, N\}$$

is called the space of **bivariate splines** of degree q and smoothness r on Δ. Here,

$$\tilde{\Pi}_q = span\{x^i y^j : \; i,j \geq 0, \; i+j \leq q\}$$

denotes the space of **bivariate polynomials** of total degree q and Π_q denotes the space of univariate polynomials of degree q.

A set $\{z_1, \ldots, z_m\}$ in Ω, where $m = dim \; S_q^r(\Delta)$ is called a **Lagrange interpolation set** for $S_q^r(\Delta)$ if for each function $f \in C(\Omega)$, a unique spline exists such that $s(z_i) = f(z_i), \; i = 1, \ldots, m$. If also partial derivatives of f are involved and the total number of Hermite conditions is m, then we speak of a **Hermite interpolation set** for $S_q^r(\Delta)$.

Lagrange and Hermite interpolation sets for $S_q^r(\Delta^c)$ were constructed for crosscut partitions Δ^c, in particular for rectangular partions with diagonals, in [1, 10, 32, 33, 43, 44, 49, 55, 56]. Results on the approximation order of these interpolation methods were given in [10, 18, 32, 42, 45, 47, 55, 56].

Much less is known about interpolation by $S_q^r(\Delta)$ for more general classes of triangulations Δ. Based on the results of Morgan & Scott [40] a Hermite interpolation scheme for $S_q^1(\Delta), \; q \geq 5$, where Δ is an arbitrary triangulation, was defined by Davydov [16]. In this case, Lagrange interpolation sets were constructed by Davydov & Nürnberger [17]. Their method can also be applied for $q = 4$, where Δ has to be slightly modified if exceptional constellations of triangles occur. Earlier, Gao [26] defined a Hermite interpolation scheme for $S_4^1(\Delta)$ in the special case when Δ is an odd degree triangulation. Interpolation sets for $S_3^1(\Delta)$, where Δ is a nested polygon triangulation, were given in Davydov, Nürnberger & Zeilfelder [19]. For $q \geq 3r + 2$, a Hermite interpolation set for $S_q^r(\Delta)$, Δ an arbitrary triangulation, was constructed by Chui & Lai [13]. In this case, a Hermite-Birkhoff type interpolation scheme was given by Davydov, Nürnberger & Zeilfelder [21] with detailed investigations of its approximation order (see also de Boor & Höllig [7], de Boor & Jia [8], Chui, Hong & Jia [14], Lai & Schumaker [38]). Results on almost interpolation (i.e. interpolation after small perturbations of the points) by $S_q^r(\Delta)$ were given by Davydov, Sommer & Strauß [22], and the references therein.

In this paper, we describe an inductive method for constructing triangulations Δ which are suitable for interpolation by $S_q^r(\Delta), \; r = 1, 2$. By starting with one triangle, in each step, we add locally chosen scattered points and obtain a larger subtriangulation (to which the splines can be extended). Simultaneously, in each step, we determine the dimension of the spline space on the resulting subtriangulation and construct Lagrange- respectively Hermite interpolation sets. In this way, we obtain interpolation sets for $S_q^1(\Delta), \; q \geq 3$ and $S_q^2(\Delta), \; q \geq 5$. For the space $S_q^2(\Delta)$ it is necessary to split some of the triangles. In addition, we describe a more general class of triangulations Δ_Q such that its vertices form an interpolation set for $S_2^1(\Delta_Q)$.

In contrast to global methods, the interpolating splines can be computed locally by passing from triangle to triangle and by solving small systems. We also note that our interpolation method can be used for the construction of smooth surfaces, where only data are used - and no derivative. For scattered data fitting the (approximative) data are computed by local methods. This in contrast to finite element methods for cubic splines, where all triangles have to be subdivided by a Clough-Tocher split and derivatives are involved. For details see Remark 7.2. Our numerical results show that the interpolation methods for functions and scattered data work efficiently, where for low degree splines some triangles have to be subdivided.

2 Construction of Triangulations

In the following, we construct a triangulation Δ for a set of finitely many points in the plane which is suitable for interpolation by $S_q^r(\Delta)$, $r = 1, 2$. The triangulation is constructed inductively as follows.

We first assume that in each step sufficiently many points can be added. In the first step, we choose three points and consider the corresponding triangle. Now, we assume that a simply connected triangulation $\tilde{\Delta}$ is already constructed. We denote the vertices on the boundary of $\tilde{\Delta}$ by $v_1, \ldots, v_n$ (in clockwise order). Now, we pass through the vertices $v_1, \ldots, v_n$ and add a subtriangulation of locally chosen scattered points to each vertex. More precisely, for $\mu = 1, \ldots, n$, we choose points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$, $\lambda_\mu \geq 1$ (in clockwise order) and consider the polyhedron P_μ formed by the points $v_\mu, w_{\mu-1,\lambda_{\mu-1}}, w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}, v_{\mu+1}$, where $w_{0,\lambda_0} := v_n$ and $v_{n+1} := w_{1,1}$ (see Figure 1.). We connect the points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$ with v_μ by line segments and denote the edges of P_μ with endpoint v_μ by $e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}$ (in clockwise order). (For details see Remark 2.1.) We choose enough points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$ such that $\lambda_\mu \geq 2$ if two edges in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$ have the same slope. Analogously, we choose $\lambda_\mu \geq 3$ if an edge in $\{e_{\mu,1}, \ldots, e_{\mu,\lambda_\mu}\}$ has the same slope as $e_{\mu,0}$, and a further edge in $\{e_{\mu,1}, \ldots, e_{\mu,\lambda_\mu}\}$ has the same slope as $e_{\mu,\lambda_\mu+1}$.

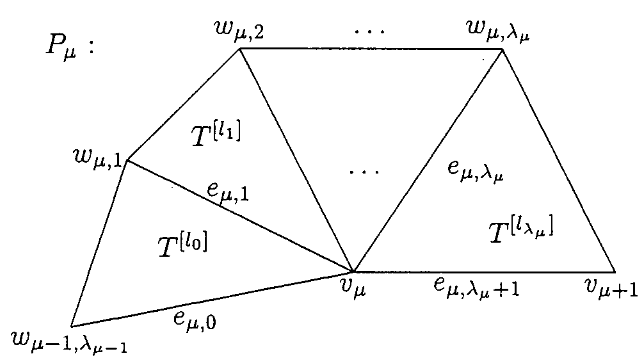

Figure 1. The polyhedron P_μ.

For the case, when $r = 2$, one triangle of P_μ has to be subdivided into three subtriangles (see Figure 2.) if there do not exist four consecutive edges in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$ with different slopes. In this case, a triangle of P_μ has to be subdivided which has an edge $e_{\mu,\nu}$ with slope different from all other edges in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$, or an arbitrary triangle of P_μ has to be subdivided if there does not exist such an edge $e_{\mu,\nu}$. We subdivide this triangle such that we obtain four consecutive edges with end point v_μ which have different slopes.

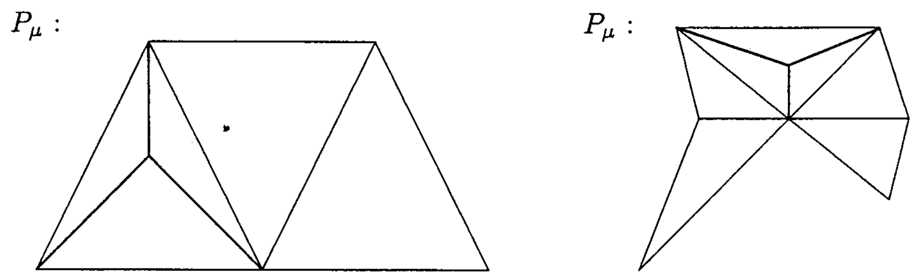

Figure 2. Subdivision of a triangle.

If there exist sufficiently many points such that for each $\mu \in \{1, \ldots, n\}$, a polyhedron P_μ with the above properties can be added, we obtain a larger triangulation. If for some $\mu \in \{1, \ldots, n\}$, such a polyhedron cannot be added, we choose some point and add a triangle with vertex v_μ which has exactly one common edge with the given subtriangulation and so forth. By proceeding with this method, we finally obtain the triangulation Δ.

Since in our method, there is some freedom in the choice of the polyhedrons P_μ, we briefly discuss some algorithmic aspects.

Remark 2.1 Our basic principle is to add a polyhedron P_μ to some boundary point v_μ of the subtriangulation $\tilde{\Delta}$ constructed so far. In order to obtain natural triangulations, it may be necessary to use the following variant of our method. Given $\tilde{\Delta}$, we add P_μ to that boundary point v_μ whose boundary edges $e_{\mu,0}$ and $e_{\mu,\lambda_\mu+1}$ form a minimal angle. In our computations, we choose the points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$ in a circular ring of the cone formed by $e_{\mu,0}$ and $e_{\mu,\lambda_\mu+1}$ such that $P_\mu \setminus \{e_{\mu,0}, e_{\mu,\lambda_\mu+1}\}$ does not intersect $\tilde{\Delta}$.

We note that by applying the spline method described in the subsequent sections we also obtain the interpolation sets for $S_q^r(\hat{\Delta})$, $r = 1, 2$, where $\hat{\Delta}$ is a convex quadrangulation with diagonals in [46], where different methods are used.

3 Construction of Admissible Sets

In this section, we construct admissible sets for spline spaces $S_q^r(\Delta)$, where $q \geq 3$ if $r = 1$, and $q \geq 5$ if $r = 2$. In order to describe admissible sets we need some notations (cf. [5, 6, 9, 23, 24]). Let $T^{[l]} = \Delta(v_1^{[l]}, v_2^{[l]}, v_3^{[l]})$, $l = 1, \ldots, N$, be the triangles of Δ. For $s \in S_q^r(\Delta)$, the polynomials $p^{[l]} = s|_{T^{[l]}} \in \tilde{\Pi}_q$, $l = 1, \ldots, N$, can be written as

$$p^{[l]}(x, y) = \sum_{i+j+k=q} a_{i,j,k}^{[l]} \tfrac{q!}{i!j!k!} \Phi_1^i(x, y) \Phi_2^j(x, y) \Phi_3^k(x, y), \ (x, y) \in T^{[l]}, \tag{1}$$

where $\Phi_\mu \in \tilde{\Pi}_1$, $\mu = 1, 2, 3$, is uniquely defined by $\Phi_\mu(v_\nu^{[l]}) = \delta_{\mu,\nu}$, $\nu = 1, 2, 3$. The representation (1) is called the **Bézier-Bernstein form** of $p^{[l]}$ and the real numbers $a_{i,j,k}^{[l]}$ are called the **Bézier-Bernstein coefficients** of $p^{[l]}$.

Given $\sigma = (i,j,k,l)$, we use the abbreviation $a(\sigma) = a^{[l]}_{i,j,k}$. A subset $\{\sigma_1, \ldots, \sigma_m\}$ of $I = \{(i,j,k,l) : \ i+j+k = q, \ l = 1, \ldots, N\}$ is called an **admissible set** for $S_q^r(\Delta)$ if for every choice of coefficients $a(\sigma_\mu) \in \mathbb{R}, \ \mu = l, \ldots, m$, a unique spline $s \in S_q^r(\Delta)$ exists with these coefficients in the representation (1) of s. We remark that the notion of admissible sets is closely related to the notion of minimally determining sets (cf. [3, 4, 31, 53, 54]). However, we need this notion for describing the interpolation sets in a unified way and for the argumentations in our proofs.

We need the following simple lemma on the connection of admissible sets and the dimension of $S_q^r(\Delta)$.

Lemma 3.1 *Let $\{\sigma_1, \ldots, \sigma_m\}$ be an admissible set for $S_q^r(\Delta)$ and for $\mu \in \{1, \ldots, m\}$, let $s_\mu \in S_q^r(\Delta)$ be the unique spline for which $a(\sigma_\mu) = \delta_{\mu,\nu}, \ \nu = 1, \ldots, m$, in (1). Then $\{s_1, \ldots, s_m\}$ forms a basis of $S_q^r(\Delta)$ and $m = dim \ S_q^r(\Delta)$.*

Proof of Lemma 3.1: Let a spline $s \in S_q^r(\Delta)$ with coefficients $a(\sigma_1), \ldots, a(\sigma_m)$ in its representation (1) be given. It follows from the definition of the splines $s_1, \ldots, s_m$ that the spline $\sum_{\mu=1}^m a(\sigma_\mu) s_\mu$ has the same coefficients in (1). Since $\{\sigma_1, \ldots, \sigma_m\}$ is an admissible set, we obtain $s = \sum_{\mu=1}^m a(\sigma_\mu) s_\mu$. A similar argument shows that $\sum_{\mu=1}^m \alpha_\mu s_\mu = 0$ implies $\alpha_1 = \ldots = \alpha_m = 0$. This proves Lemma 3.1. ∎

In the following, we construct admissible sets for $S_q^r(\Delta)$, where $r = 1, 2$. This is done by assigning a subset $M^{[l]}$ of $I = \{(i,j,k,l) : \ i+j+k = q\}$ to each triangle $T^{[l]}$ of Δ. In this case, for simplicity we say that $\{(i,j,k) : \ (i,j,k,l) \in M^{[l]}\}$ is assigned to $T^{[l]}$.

For $r = 1$, i.e. for the space $S_q^1(\Delta)$, we assign the following sets.

$$\begin{aligned}
Q &= \{(i,j,k) : \ i+j+k = q\} \\
A_1 &= \{(i,j,k) \in Q : \ k \geq 2\} \\
B_1 &= \{(i,j,k) \in Q : \ k \geq 2, \ i \neq q-2\} \\
C_1 &= \{(i,j,k) \in Q : \ j \geq 2, \ k \geq 2\} \ .
\end{aligned}$$

Case 1. $S_q^1(\Delta), \ q \geq 3$.

Here, we refer to the construction of the triangulation Δ (see Section 2). We recall that Δ is constructed by adding to each boundary point v_μ of the subtriangulation, constructed so far, a polyhedron P_μ (see Figure 1.). Therefore, in order to construct an admissible set for $S_q^1(\Delta)$, it essentially suffices to describe which sets are assigned to the triangles of P_μ. In Figure 1., we set $w_{\mu,0} = w_{\mu-1,\lambda_{\mu-1}}, \ w_{\mu,\lambda_\mu+1} = v_{\mu+1}$. By construction of Δ, three edges $e_{\mu,\nu}, \ e_{\mu,\nu+1}, \ e_{\mu,\nu+2}$ with different slopes exist. We now denote the triangles of P_μ by $T^{[l_{\nu_1}]} = \Delta(v_1^{[l_{\nu_1}]}, v_2^{[l_{\nu_1}]}, v_3^{[l_{\nu_1}]})$, where $v_1^{[l_{\nu_1}]} = v_\mu, \ v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1}, \ v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1+1}, \ \nu_1 = 0, 1, \ldots, \nu+1$, and $v_1^{[l_{\nu_1}]} = v_\mu, \ v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1+1} \ v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1}, \ \nu_1 = \nu+2, \ldots, \lambda_\mu$. We note that the sets which will be assigned to each $T^{[l_{\nu_1}]}$ are understood with respect to the representation (1) of $p^{[l_{\nu_1}]} \in \tilde{\Pi}_q$ on $T^{[l_{\nu_1}]}$.

We assign the set Q to the first triangle in the construction of Δ. Moreover, to each polyhedron P_μ, we assign the following sets: We assign the set B_1 to $T^{[l_\nu]}$, the set C_1

to $T^{[l_\nu+1]}$ and the set A_1 to the remaining triangles of P_μ (see Figure 3.). If for some μ such a polyhedron cannot be added, we assign the set A_1 to the triangle with vertex v_μ that has exactly one common edge with the given subtriangulation.

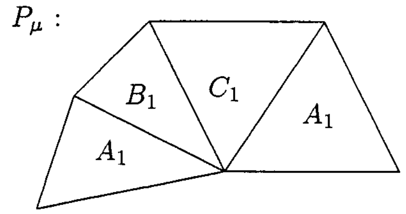

Figure 3. The sets A_1, B_1 and C_1 assigned to P_μ.

In this way, we assign to each triangle $T^{[l]}$ of Δ a set of indices (by adding the index l to the elements (i,j,k)). The union of all such sets yields a subset of $I = \{(i,j,k,l) : i+j+k = q,\ l = 1,\ldots,N\}$ denoted by $\mathcal{A}_1$.

Theorem 3.2 *For $q \geq 3$, the set $\mathcal{A}_1$ is an admissible set for $S_q^1(\Delta)$.*

For $r = 2$, i.e. for the space $S_q^2(\Delta)$, we assign the following sets.

$$\begin{aligned}
Q &= \{(i,j,k) :\ i+j+k = q\} \\
A_2 &= \{(i,j,k) \in Q :\ k \geq 3\} \\
B_2 &= \{(i,j,k) \in Q :\ k \geq 3,\ i \neq q-3\} \\
C_2 &= \{(i,j,k) \in Q :\ k \geq 3,\ i \neq q-3,\ (i,j,k) \neq (q-4,1,3)\} \\
D_2 &= \{(i,j,k) \in Q :\ j \geq 3,\ k \geq 3\}
\end{aligned}$$

In addition, if some triangle of Δ is subdivided, we assign one of the following sets.

$$\begin{aligned}
\tilde{C}_2 &= \begin{cases} \{(0,0,5),(1,0,4)\} & \text{, if } q = 5, \\ \{(i,j,k) \in Q :\ k \geq 3,\ i,j \neq q-3,\ (i,j,k) \neq (q-4,1,3), & \\ \quad (i,j,k) \neq (1,q-4,3)\} & \text{, if } q \geq 6, \end{cases} \\
\tilde{D}_2 &= \{(i,j,k) \in Q :\ i \geq 3,\ k \geq 3,\ i \neq q-3,\ (i,j,k) \neq (q-4,1,3)\}, \text{ if } q \geq 7.
\end{aligned}$$

Case 2. $S_q^2(\Delta),\ q \geq 5$.

As above, we refer to the construction of Δ (see Section 2). We recall that Δ is constructed by adding to each boundary point v_μ of the subtriangulation, constructed so far, a polyhedron P_μ (see Figure 1.). Therefore, in order to construct an admissible set for $S_q^2(\Delta)$, it essentially suffices to describe which sets are assigned to the triangles of P_μ. In Figure 1., we set $w_{\mu,0} = w_{\mu-1,\lambda_{\mu-1}},\ w_{\mu,\lambda_\mu+1} = v_{\mu+1}$.

We assign the set Q to the first triangle of Δ which we constructed. Moreover, to each polyhedron P_μ (see Figure 1.), we assign the following sets.

Case 2a. No triangle of P_μ is subdivided.

In this case, by construction of Δ, four edges $e_{\mu,\nu}, \ldots, e_{\mu,\nu+3}$ with different slopes exist. We now denote the triangles of P_μ by $T^{[l_{\nu_1}]} = \Delta(v_1^{[l_{\nu_1}]}, v_2^{[l_{\nu_1}]}, v_3^{[l_{\nu_1}]})$, where $v_1^{[l_{\nu_1}]} = v_\mu$, $v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1}$, $v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1+1}$, $\nu_1 = 0, 1, \ldots, \nu+2$, and $v_1^{[l_{\nu_1}]} = v_\mu$, $v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1+1}$ $v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1}$, $\nu_1 = \nu+3, \ldots, \lambda_\mu$. We assign the set B_2 to $T^{[l_\nu]}$, the set C_2 to $T^{[l_{\nu+1}]}$, the set D_2 to $T^{[l_{\nu+2}]}$ and the set A_2 to the remaining triangles of P_μ (see Figure 4.). We note that the sets which will be assigned to each $T^{[l_{\nu_1}]}$ are understood with respect to the representation (1) of $p^{[l_{\nu_1}]} \in \tilde{\Pi}_q$ on $T^{[l_{\nu_1}]}$.

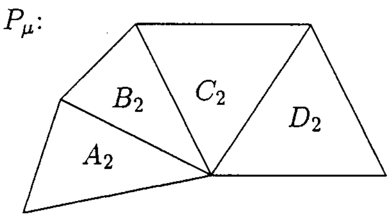

Figure 4. The sets A_2, B_2, C_2 and D_2 assigned to P_μ.

Case 2b. Some triangle of P_μ is subdivided.

Let $T^{[l_\nu]}$ be the triangle that is subdivided by the subdividing point $y_{\mu,\nu}$ from its interior into three subtriangles $T^{[(l_\nu,\sigma)]} = \Delta(v_1^\sigma, v_2^\sigma, v_3^\sigma)$, $\sigma \in \{0,1,3\}$. We now denote the triangles of P_μ by $T^{[l_{\nu_1}]} = \Delta(v_1^{[l_{\nu_1}]}, v_2^{[l_{\nu_1}]}, v_3^{[l_{\nu_1}]})$, where $v_1^{[l_{\nu_1}]} = v_\mu$, $v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1}$, $v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1+1}$, $\nu_1 = 0, 1, \ldots, \nu+1$, and $v_1^{[l_{\nu_1}]} = v_\mu$, $v_2^{[l_{\nu_1}]} = w_{\mu,\nu_1+1}$ $v_3^{[l_{\nu_1}]} = w_{\mu,\nu_1}$, $\nu_1 = \nu+2, \ldots, \lambda_\mu$. If $\nu < \lambda_\mu$, then we set $v_1^0 = v_1^1 = v_\mu$, $v_2^0 = v_2^3 = w_{\mu,\nu}$, $v_3^0 = v_2^1 = v_1^3 = y_{\mu,\nu}$, $v_3^1 = v_3^3 = w_{\mu,\nu+1}$ and $T^{[(l_\nu,2)]} = T^{[l_{\nu+1}]}$ (See Figure 5.).

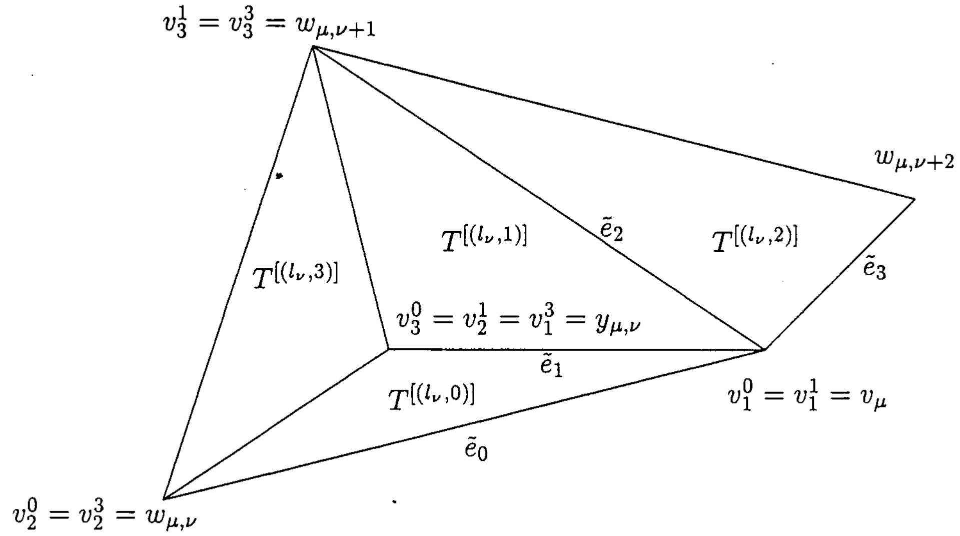

Figure 5. Notations for the subdivided triangle.

In this case, it follows from the choice of $T^{[l_\nu]}$ and $y_{\mu,\nu}$ that the edges $\tilde{e}_0 = e_{\mu,\nu}$, $\tilde{e}_1 = [v_\mu, y_{\mu,\nu}]$, $\tilde{e}_2 = e_{\mu,\nu+1}$ and $\tilde{e}_3 = e_{\mu,\nu+2}$ have different slopes. We assign the set B_2 to $T^{[(l_\nu,0)]}$, the set $\tilde{C}_2$ to $T^{[(l_\nu,1)]}$, the set D_2 to $T^{[(l_\nu,\sigma)]}$, $\sigma = 2,3$, and the set A_2 to the remaining triangles of P_μ. Alternatively, for $q \geq 7$, we assign the set B_2 to $T^{[(l_\nu,0)]}$, the set C_2 to $T^{[(l_\nu,3)]}$, the set $\tilde{D}_2$ to $T^{[(l_\nu,1)]}$, the set D_2 to $T^{[(l_\nu,2)]}$ and the set A_2 to the remaining triangles of P_μ (see Figure 6.). If $\nu = \lambda_\mu$, then we set $v_1^0 = v_1^1 = v_\mu$, $v_2^0 = v_2^3 = w_{\mu,\lambda_\mu+1}$, $v_3^0 = v_2^1 = v_1^3 = y_{\mu,\lambda_\mu}$, $v_3^1 = v_3^3 = w_{\mu,\lambda_\mu}$ and $T^{[(l_{\lambda_\mu},2)]} = T^{[l_{\lambda_\mu}-1]}$. The assignment of the sets A_2, B_2, $\tilde{C}_2$, D_2, respectively A_2, B_2, C_2, $\tilde{D}_2$, D_2, is analogous as above. We note that the sets which will be assigned to each $T^{[l_{\nu_1}]}$, respectively $T^{[(l_\nu,\sigma)]}$, are understood with respect to the representation (1) of the polynomial piece on $T^{[l_{\nu_1}]}$, respectively $T^{[(l_\nu,\sigma)]}$.

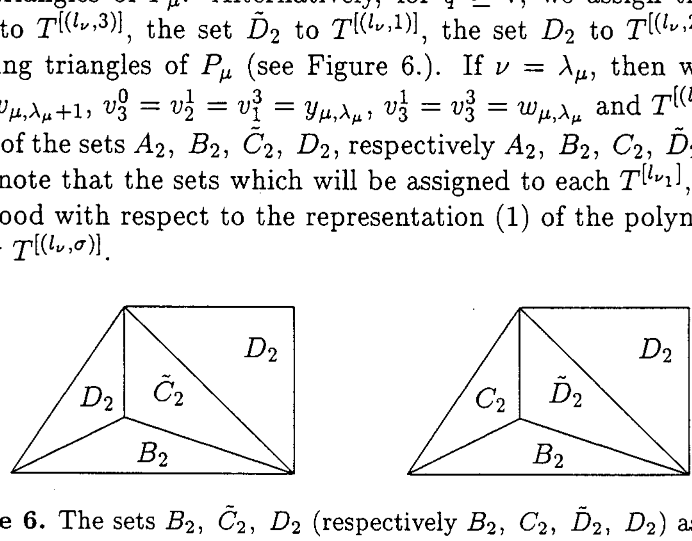

Figure 6. The sets B_2, $\tilde{C}_2$, D_2 (respectively B_2, C_2, $\tilde{D}_2$, D_2) assigned to $T^{[(l_\nu,\sigma)]}$.

If for some μ such a polyhedron cannot be added, we assign the set A_2 to the triangle with vertex v_μ that has exactly one common edge with the given subtriangulation.

In this way, we assign to each triangle $T^{[l]}$ of Δ a set of indices (by adding the index l to the elements (i,j,k)). The union of all such sets yields a subset $\mathcal{A}_2$ of $I = \{(i,j,k,l) : \; i+j+k = q, \; l = 1,\ldots,N\}$.

Theorem 3.3 *For $q \geq 5$, the set $\mathcal{A}_2$ is an admissible set for $S_q^2(\Delta)$.*

For proving our results on admissible and interpolation sets, we need the following well-known result (c.f. [6, 9, 23]) which expresses smoothness conditions between neighboring triangles. Let Δ^* be a triangulation consisting of the two triangles $T^{*[1]} = \Delta(v_1^*, v_2^*, v_3^*)$, $T^{*[2]} = \Delta(v_1^*, v_2^*, v_4^*)$ and let the polynomial pieces $p^{*[l]} = s|_{T^{*[l]}} \in \tilde{\Pi}_q$, $l = 1, 2$, of a spline $s \in S_q^0(\Delta^*)$ be given in the form (1) (with corresponding coefficients $a^{*[l]}_{i,j,k}$, $i + j + k = q$).

Lemma 3.4 *The following statements are equivalent.*
(i) $s \in S_q^r(\Delta^*)$
(ii) For all $\rho \in \{0, \ldots, r\}$:

$$a^{*[2]}_{i,j,\rho} = \sum_{i_1+j_1+k_1=\rho} a^{*[1]}_{i+i_1,j+j_1,k_1} \frac{\rho!}{i_1!j_1!k_1!} \Phi_1^{i_1}(v_4^*)\Phi_2^{j_1}(v_4^*)\Phi_3^{k_1}(v_4^*), \; i + j = q - \rho,$$

where $\Phi_\mu \in \tilde{\Pi}_1$, $\mu = 1, 2, 3$, *is uniquely determined by* $\Phi_\mu(v_\nu^*) = \delta_{\mu,\nu}$, $\nu = 1, 2, 3$.

It is well known (cf. [9, 23]) that for $r = 1$ the smoothness conditions *(ii)* of Lemma 3.4 have the geometric interpretation that the corresponding Bézier-Bernstein points lie in the same plane. Moreover, if the edges $[v_1^*, v_3^*]$, $[v_1^*, v_4^*]$ have the same slopes, then for $r = 1$ the geometric interpretation of these smoothness conditions is that this plane degenerates to a line that contains three of the corresponding Bézier-Bernstein points.

The next lemma will be needed in Section 6. If we assume that the edges $[v_1^*, v_3^*]$ and $[v_1^*, v_4^*]$ have different slopes, then the following result follows easily from Lemma 3.4 and some elementary computations.

Lemma 3.5 *Let* $s \in S_q^2(\Delta^*)$, $q \geq 5$, *and* $i + j = q - 2$. *If* $a^{*[2]}_{i,j,2}$, $a^{*[1]}_{i+i_1,j+j_1,k_1}$, $i_1 + j_1 + k_1 = 2$, $(i_1, j_1, k_1) \notin \{(0,1,1), (0,2,0)\}$ *and either* $a^{*[1]}_{i,j+1,1}$ *or* $a^{*[1]}_{i,j+2,0}$ *are given, then the coefficients* $a^{*[l]}_{i+i_1,j+j_1,k_1}$, $i_1 + j_1 + k_1 = 2$, $l = 1, 2$, *are uniquely determined.*

4 Construction of Interpolation Sets

By using the above results on admissible sets we construct Lagrange- and Hermite interpolation sets for the spline spaces $S_q^r(\Delta)$, where $q \geq 3$ if $r = 1$, and $q \geq 5$ if $r = 2$. For simplicity, we use the same symbols as in Section 3 for the interpolation sets.

In the following, we construct Lagrange interpolation sets for $S_q^r(\Delta)$, $r = 1, 2$ (simultaneously with the admissible sets constructed in Section 3).

Given a triangle $T = \Delta(v_1, v_2, v_3)$ in Δ, we choose one of the following point sets in T. For $r = 1$, i.e. for the space $S_q^1(\Delta)$, we consider the following sets.

Set Q: Choose $q+1$ disjoint line segments $p_1, \ldots, p_{q+1}$ in T. For $\mu = 1, \ldots, q+1$ choose $q + 2 - \mu$ points on p_μ.
Set A_1: Choose $q-1$ disjoint line segments $a_1, \ldots, a_{q-1}$ in T. For $\mu = 1, \ldots, q-1$ choose $q - \mu$ points on a_μ.

Set B_1: Choose $q-2$ disjoint line segments $b_1, \ldots, b_{q-2}$ in T. For $\mu = 1, \ldots, q-2$ choose $q-\mu$ points on b_μ.
Set C_1: Choose $q-3$ disjoint line segments $c_1, \ldots, c_{q-3}$ in T. For $\mu = 1, \ldots, q-3$ choose $q-2-\mu$ points on c_μ.

Note that we choose points and line segments according to the following general rules: the points should not lie on triangles considered before and the line segments should be parallel with respect to a certain direction and should have all a non-empty intersection with both of the edges $[v_1, v_2]$, $[v_1, v_3]$.

In Section 3, we described which index sets Q, A_1, B_1, C_1 are assigned to the triangles $T^{[l]}$, $l = 1, \ldots, N$, of Δ. Now, we choose point sets with exactly the same symbols Q, A_1, B_1, C_1 for the triangles $T^{[l]}$, $l = 1, \ldots, N$ (See Figure 3. and Figure 7.).

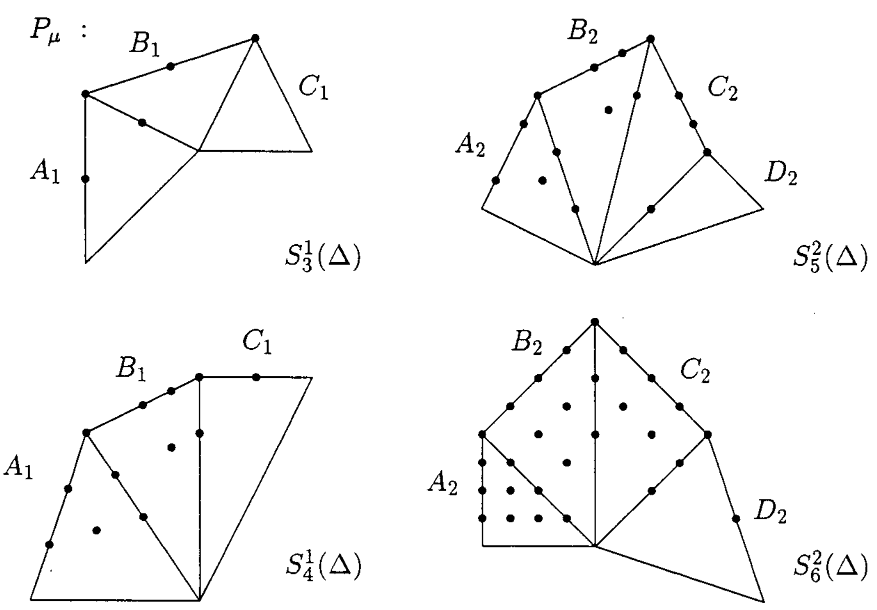

Figure 7. : Lagrange interpolation points.

The union of these points sets is denoted by $\mathcal{L}_1$ for Δ.

Theorem 4.1 *For $q \geq 3$, the set $\mathcal{L}_1$ is a Lagrange interpolation set for $S_q^1(\Delta)$.*

For $r = 2$, i.e. the space $S_q^2(\Delta)$, we consider the following sets.

Set Q: Choose $q+1$ disjoint line segments $p_1, \ldots, p_{q+1}$ in T. For $\mu = 1, \ldots, q+1$, choose $q+2-\mu$ points on p_μ.
Set A_2: Choose $q-2$ disjoint line segments $a_1, \ldots, a_{q-2}$ in T. For $\mu = 1, \ldots, q-2$, choose $q-1-\mu$ points on a_μ.

Set B_2: Choose $q-3$ disjoint line segments $b_1, \ldots, b_{q-3}$ in T. For $\mu = 1, \ldots, q-3$, choose $q-1-\mu$ points on b_μ.
Set C_2: Choose $q-3$ disjoint line segments $c_1, \ldots, c_{q-3}$ in T. For $\mu = 1, \ldots, q-4$, choose $q-1-\mu$ points on c_μ and choose the point on c_{q-3} which lies on the edge $[v_1, v_3]$.
Set D_2: Choose $q-5$ disjoint line segments $d_1, \ldots, d_{q-5}$ in T. For $\mu = 1, \ldots, q-5$, choose $q-4-\mu$ points on d_μ.

In addition, if T has to be subdivided, we consider the following sets for $q = 5, 6$.

Set $\tilde{C}_2$: If $q = 5$, then choose two distinct points on the edge $[v_1, v_3]$. If $q = 6$, then choose three distinct points on the edge $[v_1, v_3]$, two different distinct points on the edge $[v_2, v_3]$ and one point from the interior of T.

In this case, for $q \geq 7$, we choose the following set.

Set $\tilde{D}_2$: Choose $q-6$ disjoint line segments $d_1, \ldots, d_{q-6}$ in T. For $\mu = 1, \ldots, q-7$, choose $q-4-\mu$ points on d_μ and choose the point on d_{q-6} which lies on the edge $[v_1, v_3]$.

Note that we choose points and line segments according to the above general rules.
In Section 3, we described which index sets Q, A_2, B_2, C_2, $\tilde{C}_2$, D_2, $\tilde{D}_2$ are assigned to the triangles of Δ. Now, we choose point sets with exactly the same symbols for the triangles $T^{[l]}$, $l = 1, \ldots, N$ (see Figure 4., Figure 6. and Figure 7.). The union on these point sets is denoted by $\mathcal{L}_2$ for Δ.

Theorem 4.2 *For $q \geq 5$, the set $\mathcal{L}_2$ is a Lagrange interpolation set for $S_q^2(\Delta)$.*

In the following, we construct Hermite interpolations sets for $S_q^r(\Delta)$, $r = 1, 2$ (simultaneously with the admissible sets constructed in Section 3). For doing this we describe some basic Hermite interpolation conditions which we obtain by using the above Lagrange interpolation sets and taking limits, which means that certain points and line segments coincide. Roughly speaking, the corresponding Hermite interpolation conditions are obtained as follows. If certain points on a line segment coincide, then we pass to the directional derivatives along the line segment, and if certain line segments coincide, then we pass to the directional derivative of a unit vector which is not collinear to the directional derivative along the line segment.

For describing Hermite interpolation conditions, we denote by f_d the partial derivative in direction of the unit vector d. The higher partial derivatives are denoted by $f_{d_1^\alpha d_2^\beta}$, where the unit vectors d_1 and d_2 are not collinear. Given a point $z = (x, y) \in \Omega$ and ω a natural number, we set $D^\omega f(z) = (f_{d_1^\omega}(z), f_{d_1^{\omega-1} d_2}(z), \ldots, f_{d_2^\omega}(z))$.

For simplicity, we use the same symbols as in Section 3 for the Hermite interpolation conditions. Let $f \in C(\Omega)$ be a sufficiently differentiable function. For a given triangle $T = \Delta(v_1, v_2, v_3)$ in Δ, one of the following Hermite interpolation conditions is imposed to a polynomial $p \in \tilde{\Pi}_q$ on T at a point in T. Here, d_j denotes a unit vector in direction of the edge $[v_3, v_j]$, $j = 1, 2$. For $r = 1$, i.e. for the space $S_q^1(\Delta)$, we consider the following conditions.

Condition Q: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q$.
Condition A_1: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-2$.
Condition B_1: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-2$, except $p_{d_1^{q-2}}(v_3) = f_{d_1^{q-2}}(v_3)$.
Condition C_1: $D^\omega p(\overline{v}) = D^\omega f(\overline{v}),\ \omega = 0, \dots, q-4$, where $\overline{v} = \frac{1}{2}(v_2 + v_3)$.

Note, that v_3 and $\overline{v}$ should not lie on triangles considered before.

In Section 3 we described which index sets $Q,\ A_1,\ B_1,\ C_1$ are assigned to the triangles $T^{[l]},\ l = 1, \dots, N$, of Δ. Now, we choose Hermite interpolation conditions for the polynomials $p^{[l]}$ at a point of $T^{[l]},\ l = 1, \dots, N$, with exactly the same symbols $Q,\ A_1,\ B_1,\ C_1$. The union of these points is denoted by $\mathcal{H}_1$ for Δ.

Theorem 4.3 *For $q \geq 3$, the set $\mathcal{H}_1$ is a Hermite interpolation set for $S_q^1(\Delta)$.*

For $r = 2$, i.e. the space $S_q^2(\Delta)$, one of the following Hermite interpolation conditions is imposed to a polynomial $p \in \tilde{\Pi}_q$ on T at a point in T.

Condition Q: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q$.
Condition A_2: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-3$.
Condition B_2: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-3$, except $p_{d_1^{q-3}}(v_3) = f_{d_1^{q-3}}(v_3)$.
Condition C_2: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-3$, except $p_{d_1^{q-3}}(v_3) = f_{d_1^{q-3}}(v_3)$ and $p_{d_1^{q-4} d_2}(v_3) = f_{d_1^{q-4} d_2}(v_3)$.
Condition D_2: $D^\omega p(\overline{v}) = D^\omega f(\overline{v}),\ \omega = 0, \dots, q-6$, where $\overline{v} = \frac{1}{2}(v_2 + v_3)$.

In addition, if T has to be subdivided, we impose the following Hermite interpolation conditions.

Condition $\tilde{C}_2$: $D^\omega p(v_3) = D^\omega f(v_3),\ \omega = 0, \dots, q-3$, except $p_{d_1^\alpha d_2^\beta}(v_3) = f_{d_1^\alpha d_2^\beta}(v_3)$, where $\alpha + \beta = q - 3,\ \alpha,\ \beta = 0, 1$, if $q \geq 6$, and $p_{d_1^\alpha}(v_3) = f_{d_1^\alpha}(v_3),\ \alpha = 0, 1$, if $q = 5$.

Note, that v_3 and $\overline{v}$ should not lie on triangles considered before.

In Section 3 we described which index sets $Q,\ A_2,\ B_2,\ C_2,\ \tilde{C}_2,\ D_2$ are assigned to the triangles $T^{[l]},\ l = 1, \dots, N$ of Δ. Now, we choose Hermite interpolation conditions for the polynomials $p^{[l]}$ at a point of $T^{[l]},\ l = 1, \dots, N$, with exactly the same symbols $Q,\ A_2,\ B_2,\ C_2,\ \tilde{C}_2,\ D_2$. The union of these points is denoted by $\mathcal{H}_2$ for Δ.

Theorem 4.4 *For $q \geq 5$, the set $\mathcal{H}_2$ is a Hermite interpolation set for $S_q^2(\Delta)$.*

For later use, we discuss a fundamental connection of the partial derivatives of a polynomial (given in the form (1)) at a vertex and its Bézier-Bernstein coefficients (cf. [6, 13, 23]).

Let $p \in \tilde{\Pi}_q$ on $T = \Delta(v_1, v_2, v_3)$ be given in the form (1) and let $d_j,\ j = 1, 2$, be unit vectors in direction of the edge $[v_1, v_{j+1}],\ j = 1, 2$. For all $0 \leq \alpha + \beta \leq q$, we have

$$p_{d_1^\alpha d_2^\beta}(x, y) = \sum_{i+j+k=q} a_{i,j,k} \tfrac{q!}{i!j!k!} (\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha d_2^\beta}(x, y),\ (x, y) \in T.$$

Since $(\Phi_3^k)_{d_1^\mu} = 0,\ \mu \geq 1$, it follows from Leibniz' rule

$$(\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha} = \sum_{\mu=0}^{\alpha} \binom{\alpha}{\mu} (\Phi_1^i)_{d_1^{\alpha-\mu}} (\Phi_2^j)_{d_1^\mu} \Phi_3^k,\ i+j+k=q.$$

Analogously, since $(\Phi_2^j)_{d_2^\nu} = 0,\ \nu \geq 1$, we have

$$(\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha d_2^\beta} = \sum_{\mu=0}^{\alpha} \sum_{\nu=0}^{\beta} \binom{\alpha}{\mu} \binom{\beta}{\nu} (\Phi_1^i)_{d_1^{\alpha-\mu} d_2^{\beta-\nu}} (\Phi_2^j)_{d_1^\mu} (\Phi_3^k)_{d_2^\nu},\ i+j+k=q.$$

Thus,

$$(\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha d_2^\beta} = \sum_{\mu=0}^{\alpha} \sum_{\nu=0}^{\beta} \binom{\alpha}{\mu} \binom{\beta}{\nu} \frac{i!j!k!(\Phi_1)_{d_1}^{\alpha-\mu} (\Phi_1)_{d_2}^{\beta-\nu} (\Phi_2)_{d_1}^{\mu} (\Phi_3)_{d_2}^{\nu}}{(i-\alpha-\beta+\mu+\nu)!(j-\mu)!(k-\nu)!} \Phi_1^{i-\alpha-\beta+\mu+\nu} \Phi_2^{j-\mu} \Phi_3^{k-\nu}.$$

Since $\Phi_\mu(v_1) = \delta_{1,\mu},\ \mu = 1,2,3$, we get for $j \in \{0, \ldots, \alpha\},\ k \in \{0, \ldots, \beta\}$,

$$(\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha d_2^\beta}(v_1) = \binom{\alpha}{j} \binom{\beta}{k} \frac{i!j!k!}{(q-\alpha-\beta)!} (\Phi_1)_{d_1}^{\alpha-j} (\Phi_1)_{d_2}^{\beta-k} (\Phi_2)_{d_1}^{j} (\Phi_3)_{d_2}^{k},$$

and $(\Phi_1^i \Phi_2^j \Phi_3^k)_{d_1^\alpha d_2^\beta}(v_1) = 0$, if $j > \alpha$ or $k > \beta,\ i+j+k=q$. Therefore, we obtain

$$p_{d_1^\alpha d_2^\beta}(v_1) = \frac{q!}{(q-\alpha-\beta)!} \sum_{j=0}^{\alpha} \sum_{k=0}^{\beta} \binom{\alpha}{j} \binom{\beta}{k} (\Phi_1)_{d_1}^{\alpha-j} (\Phi_1)_{d_2}^{\beta-k} (\Phi_2)_{d_1}^{j} (\Phi_3)_{d_2}^{k} a_{q-j-k,j,k}. \tag{2}$$

It easily follows from (2) and induction that if the Bézier-Bernstein coefficients $a_{q-j-k,j,k},\ j = 0, \ldots, \alpha,\ k = 0, \ldots, \beta$, are determined, then all derivatives $p_{d_1^{\alpha_1} d_2^{\beta_1}}(v_1)$, $\alpha_1 = 0, \ldots, \alpha,\ \beta_1 = 0, \ldots, \beta$, are determined.

Conversely, if all these derivatives are given, then the Bézier-Bernstein coefficients $a_{q-j-k,j,k},\ j = 0, \ldots, \alpha,\ k = 0, \ldots, \beta$, are uniquely determined. This can be seen by induction and the following equation which is an immediate consequence of (2).

$$\begin{aligned} a_{q-\alpha-\beta,\alpha,\beta} &= \frac{(q-\alpha-\beta)!}{q!(\Phi_2)_{d_1}^{\alpha} (\Phi_3)_{d_2}^{\beta}} p_{d_1^\alpha d_2^\beta}(v_1) - \sum_{j=0}^{\alpha-1} \binom{\alpha}{j} \left(\frac{(\Phi_1)_{d_1}}{(\Phi_2)_{d_1}} \right)^{\alpha-j} a_{q-j-\beta,j,\beta} \\ &\quad - \sum_{j=0}^{\alpha} \sum_{k=0}^{\beta-1} \binom{\alpha}{j} \binom{\beta}{k} \left(\frac{(\Phi_1)_{d_1}}{(\Phi_2)_{d_1}} \right)^{\alpha-j} \left(\frac{(\Phi_1)_{d_2}}{(\Phi_3)_{d_2}} \right)^{\beta-k} a_{q-j-k,j,k}. \end{aligned} \tag{3}$$

5 Proof of the Main Theorems for S_q^1

In this section, we prove our main theorems for $S_q^1(\Delta)$ (Theorem 3.2, Theorem 4.1 and Theorem 4.3). We begin with the proof of our result on admissible sets. For doing this, we need Theorem 5.1.

Let $P = P_\mu$ be a polyhedron as in Figure 1. and Δ^* be a triangulation of a domain Ω^* such that P and Δ^* have common edges $[v, w_0]$, $[v, w_{\lambda+1}]$. (For simplicity, here we omit the index μ). By adding the triangles of P to Δ^* we obtain a triangulation Δ^*_P. We denote by $\mathcal{A}_P$ the union of sets chosen in Case 1 of Section 3.

Theorem 5.1 *Let $q \geq 3$. If $\mathcal{A}^*$ is an admissible set for $S^1_q(\Delta^*)$, then $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}_P$ is an admissible set for $S^1_q(\Delta^*_P)$.*

Proof. Let us first assume that $\lambda = 1$. We set $m_1 = \binom{q}{2} - 1$, $m_2 = \binom{q-2}{2}$ and $m = card(\mathcal{A})$. Since $\mathcal{A}^* = \{\sigma_1, \ldots, \sigma_{m-m_1-m_2}\}$ is an admissible set for $S^1_q(\Delta^*)$, $q \geq 3$, it follows that for every choice of coefficients $a(\sigma_\mu)$, $\mu = 1, \ldots, m-m_1-m_2$, a unique spline $s^* \in S^1_q(\Delta^*)$ exists with these coefficients in the representation (1) of s^*. Since e_0 and e_2 have different slopes, it follows from Lemma 3.4 that the coefficients $a^{[l_0]}_{i,j,\rho}$, $a^{[l_1]}_{i,\rho,j}$, $i+j = q-\rho$, $\rho = 0, 1$, of $p^{[l_\nu]} \in \tilde{\Pi}_q$ on $T^{[l_\nu]} = \Delta(v, w_\nu, w_{\nu+1})$, $\nu = 0, 1$, in the representation (1) are uniquely determined. Moreover, Lemma 3.4 implies that $a^{[l_0]}_{q-2,0,2}$ is uniquely determined. Now, it is easy to see that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-m_1-m_2+1, \ldots, m-m_2$, where $\{\sigma_{m-m_1-m_2+1}, \ldots, \sigma_{m-m_2}\} = B_1 = \{(i,j,k,l_0) : i+j+k = q,\ k \geq 2,\ i \neq q-2\}$, a unique polynomial $p^{[l_0]} \in \tilde{\Pi}_q$ on $T^{[l_0]}$ exists with these coefficients in the representation (1) of $p^{[l_0]}$. It follows from Lemma 3.4 that the coefficients $a^{[l_1]}_{i,j,\rho}$, $i+j = q-\rho$, $\rho = 0, 1$, of $p^{[l_1]}$ are uniquely determined. Therefore, for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-m_2+1, \ldots, m$, where $\{\sigma_{m-m_2+1}, \ldots, \sigma_m\} = C_1 = \{(i,j,k,l_1) : i+j+k = q,\ j \geq 2,\ k \geq 2\}$, a unique polynomial $p^{[l_1]} \in \tilde{\Pi}_q$ on $T^{[l_1]}$ exists with these coefficients in the representation (1) of $p^{[l_0]}$. Since all differentiability conditions for $r = 1$ at the edges e_0, e_1, e_2 have been involved, we get that for every choice of coefficients $a(\sigma_\mu)$, $\mu = 1, \ldots, m$, a unique spline s from $S^1_q(\Delta^*_P)$,

$$s(x,y) = \begin{cases} s^*(x,y) \text{ , if } (x,y) \in \Omega^*, \\ p^{[l_\nu]}(x,y) \text{ , if } (x,y) \in T^{[l_\nu]},\ \nu = 0, 1, \end{cases}$$

exists with these coefficients in the representation (1) of s. This shows the case $\lambda = 1$. If $\lambda > 1$, we may assume that the edges $e_{\lambda-1}$ and $e_{\lambda+1}$ have different slopes. It follows from Lemma 3.4 (applied to the edges $e_0, \ldots, e_{\lambda-2}$) that for every choice of coefficients corresponding to the sets $A_1 = \{(i,j,k,l_\nu) : i+j+k = q,\ k \geq 2\}$, $\nu = 0, \ldots, \lambda-2$, unique polynomials $p^{[l_\nu]} \in \tilde{\Pi}_q$ on $T^{[l_\nu]}$ in the representation (1) exist with these coefficients in the representation (1) of $p^{[l_\nu]}$, $\nu = 0, \ldots, \lambda-2$. Now, we argue as in the case $\lambda = 1$. This proves Theorem 5.1. ∎

Corollary 5.2 *For $q \geq 3$, we have*

$$dim\ S^1_q(\Delta^*_P) = dim\ S^1_q(\Delta^*) + \lambda\binom{q}{2} + \binom{q-2}{2} - 1.$$

Now, we prove Theorem 3.2.

Proof of Theorem 3.2: It is obvious that the set Q is an admissible set for the space defined on the triangle chosen in the first step of our construction. Let $\tilde{\tilde{\Delta}}$ be the triangulation that consist of the triangles of $\tilde{\Delta}$ and the triangles of the polyhedrons P_μ, $\mu = 1, \ldots, n$. It follows from induction and Theorem 5.1 that an admissible set for $S_q^1(\tilde{\Delta})$ and the union of sets assigned to the triangles of P_μ, $\mu = 1, \ldots, n$, yield an admissible set for $S_q^1(\tilde{\tilde{\Delta}})$. Moreover, it is obvious that if a polyhedron at v_μ cannot be added and there exists a triangle with vertex v_μ which has exactly one common edge with $\tilde{\Delta}$, then the assigned set A_1 leads to an admissible set. This proves Theorem 3.2. ■

Next, we prove Theorem 4.1. For doing this, we need Theorem 5.3 below. Let P, Δ^*, Ω^*, v, $w_0, \ldots, w_{\lambda+1}$, be defined as in the beginning of this section and denote by $\mathcal{L}_P$ the union of sets chosen in Section 4 for the case $r = 1$.

Theorem 5.3 *Let $q \geq 3$. If $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^1(\Delta^*)$, then $\mathcal{L} = \mathcal{L}^* \cup \mathcal{L}_P$ is a Lagrange interpolation set for $S_q^1(\Delta_P^*)$.*

Proof: Let us first assume that $\lambda = 1$. We set m_1, m_2 as in the proof of Theorem 5.1 and $m = dim\ S_q^1(\Delta_P^*)$. Moreover, let $\mathcal{L}^* = \{z_1, \ldots, z_{m-m_1-m_2}\} \subset \mathcal{L} = \{z_1, \ldots, z_m\}$ and a spline $s \in S_q^1(\Delta_P^*)$, $q \geq 3$, which satisfies $s(z_i) = 0$, $i = 1, \ldots, m$, be given. We will show that $s = 0$. Since $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^1(\Delta^*)$, it follows that $s|_{\Omega^*} = 0$. Since s is a C^1-spline the function values and all first derivatives of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ (respectively $p^{[l_1]} = s|_{T^{[l_1]}} \in \tilde{\Pi}_q$) vanish at e_0 (respectively e_2). Let d_1 be a unit vector in direction of e_1. Since e_0 and e_2 have different slopes, it follows from (2), (3) and the proof of Theorem 5.1 that $p_{d_1^2}^{[l_0]}(v) = p_{d_1^2}^{[l_1]}(v) = 0$. Thus,

$$D^\omega p^{[l_0]}(v) = 0, \ \omega = 0, 1, 2. \tag{4}$$

Let $b_\mu = \{(x, y) \in T^{[l_0]} : \ \alpha_\mu x + \beta_\mu y + \gamma_\mu = 0\}$, $\mu = 1, \ldots, q-2$, be the line segments chosen in $T^{[l_0]}$ such that $q - \mu$ points of $\{z_{m-m_1-m_2+1}, \ldots, z_{m-m_2}\}$ lie on b_μ, $\mu = 1, \ldots, q-2$. We claim that

$$p^{[l_0]}|_{b_\mu} = 0, \ \mu = 1, \ldots, q-2. \tag{5}$$

We prove (5) by induction on μ. We denote by $z_\mu^{[l_0]}$, the intersection points of b_μ, $\mu = 1, \ldots, q-2$, and e_0. Since the function value and the derivative (in direction of b_1) of $p^{[l_0]}|_{b_1} \in \Pi_q$ vanish at $z_1^{[l_0]}$, it follows from the interpolation conditions of $p^{[l_0]}$ on b_1 that the claim holds for $\mu = 1$. We assume that (5) holds for $\mu \in \{1, \ldots, \eta\}$, $\eta \leq q-3$, and show that (5) holds for $\eta + 1$. By induction hypothesis, a polynomial $q^{[l_0]} \in \tilde{\Pi}_{q-\eta}$ exists such that

$$p^{[l_0]}(x, y) = \prod_{\mu=1}^{\eta} (\alpha_\mu x + \beta_\mu y + \gamma_\mu) q^{[l_0]}(x, y), \ (x, y) \in T^{[l_0]}.$$

Since the function value and the derivative (in direction of $b_{\eta+1}$) of $q^{[l_0]}|_{b_{\eta+1}} \in \Pi_{q-\eta}$ vanish at $z_{\eta+1}^{[l_0]}$, it follows from the interpolation conditions of $p^{[l_0]}$ on $b_{\eta+1}$ that $q^{[l_0]}|_{b_{\eta+1}} = 0$, and $p^{[l_0]}|_{b_{\eta+1}} = 0$. This proves (5).

From (4), we conclude $p^{[l_0]} = 0$. Since s is a C^1-spline the function values and all the first derivatives of $p^{[l_1]}$ vanish at e_1. Thus,

$$D^\omega p^{[l_1]}(v) = 0, \ \omega = 0, \dots, 3. \tag{6}$$

Let c_μ, $\mu = 1, \dots, q-3$, be the line segments chosen in $T^{[l_1]}$ such that $q - 2 - \mu$ points of $\{z_{m-m_2+1}, \dots, z_m\}$ lie on c_μ, $\mu = 1, \dots, q-3$. Analogously as in the proof of (5), we can see that

$$p^{[l_1]}|_{c_\mu} = 0, \ \mu = 1, \dots, q-3.$$

From this and (6), we conclude that $p^{[l_1]} = 0$ and $s = 0$. This proves the case $\lambda = 1$.

If $\lambda > 1$, we may assume that the edges $e_{\lambda-1}$ and $e_{\lambda+1}$ have different slopes. Since s is a C^1-spline the function values and all first derivatives of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ vanish at e_0. Let a_μ, $\mu = 1, \dots, q-1$, be the line segments chosen in $T^{[l_0]}$ such that $q - \mu$ of the chosen points lie on a_μ, $\mu = 1, \dots, q-1$. Analogously as in the proof of (5), we can see that $p^{[l_0]}|_{a_\mu} = 0$, $\mu = 1, \dots, q-1$. Since $D^\omega p^{[l_0]}(v) = 0$, $\omega = 0, 1$, we have $p^{[l_0]} = 0$. By proceeding with these arguments, we obtain $s|_{T^{[l_\nu]}} = p^{[l_\nu]} = 0$, $\nu = 0, \dots, \lambda - 2$. Now, we can argue as in the case $\lambda = 1$. This proves Theorem 5.3. ■

Proof of Theorem 4.1: It is well known that the set Q is a Lagrange interpolation set for the space defined on the triangle chosen in the first step of our construction. Let $\tilde{\Delta}$ and $\tilde{\tilde{\Delta}}$ be defined as in the proof of Theorem 3.2. Then it follows from induction and Theorem 5.3 that a Lagrange interpolation set for $S_q^1(\tilde{\Delta})$ together with the points chosen on the line segments in the triangles of P_μ, $\mu = 1, \dots, n$, form a Lagrange interpolation set for $S_q^1(\tilde{\tilde{\Delta}})$. This proves Theorem 4.1. ■

Next, we prove Theorem 4.3. For doing this, we need Theorem 5.4 below. Let P, Δ^*, Ω^*, v, $w_0, \dots, w_{\lambda+1}$, be defined as in the beginning of this section and denote by $\mathcal{H}_P$ the union of the sets chosen in Section 4 for the case $r = 1$.

Theorem 5.4 *Let $q \geq 3$. If $\mathcal{H}^*$ is a Hermite interpolation set for $S_q^1(\Delta^*)$, then $\mathcal{H} = \mathcal{H}^* \cup \mathcal{H}_P$ is a Hermite interpolation set for $S_q^1(\Delta_P^*)$.*

Proof: Let us first assume that $\lambda = 1$. Let a spline $s \in S_q^1(\Delta_P^*)$ which satisfies the homogenous interpolation conditions be given. We will show that $s = 0$. Since $\mathcal{H}^*$ is a Hermite interpolation set for $S_q^1(\Delta^*)$, it follows that $s|_{\Omega^*} = 0$. By Lemma 3.4, $a_{i,j,\rho}^{[l_0]} = 0$, $i + j = q - \rho$, $\rho = 0, 1$, where $a_{i,j,k}^{[l_0]}$, $i + j + k = q$, are the coefficients of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ in the representation (1), where $T^{[l_0]} = \Delta(v, w_0, w_1)$. Since the slopes of e_0 and e_2 are different, Lemma 3.4 implies that $a_{q-2,0,2}^{[l_0]} = 0$. We claim that

$$a_{\mu,j,q-\mu-j}^{[l_0]} = 0, \ j = 0, \dots, q-2-\mu, \ \mu = 0, \dots, q-3. \tag{7}$$

We prove (7) by induction on μ and by using the homogeneous interpolation conditions at w_1. Let d be a unit vector in direction of the edge $[w_1, w_0]$. By (3) and the interpolation conditions $p^{[l_0]}_{d^j}(w_1) = 0,\ j = 0,\dots,q-2$, (7) holds for $\mu = 0$. We assume that (7) holds for $\mu \in \{0,\dots,\eta\},\ \eta \le q-4$, and show that (7) holds for $\eta+1$. Let d_1 be a unit vector in direction of the edge $e_1 = [w_1, v]$. It follows from (3) that

$$
\begin{aligned}
a^{[l_0]}_{\eta+1,j,q-\eta-1-j} &= \theta_j p^{[l_0]}_{d_1^{\eta+1} d^j}(w_1) + \sum_{j_1=0}^{j-1} \theta_{j_1,j} a^{[l_0]}_{\eta+1,j_1,q-j_1-\eta-1} \\
&\quad + \sum_{i_1=0}^{\eta} \sum_{j_1=0}^{j} \theta_{i_1,j_1,j} a^{[l_0]}_{i_1,j_1,q-i_1-j_1},\ j = 0,\dots,q-\eta-3, \qquad (8)
\end{aligned}
$$

where $\theta_j, \theta_{j_1,j}, \theta_{i_1,j_1,j}$ are suitable real numbers. By induction hypothesis the third term on the right hand-side of (8) vanishes. Since $p^{[l_0]}_{d_1^{\eta+1} d^j}(w_1) = 0,\ j = 0,\dots,q-\eta-3$, it follows from (8) and by induction on j that $a^{[l_0]}_{\eta+1,j,q-\eta-1-j} = 0,\ j = 0,\dots,q-\eta-3$. This proves (7). From this and $a^{[l_0]}_{i,j,k} = 0,\ (i,j,k) \in Q \setminus B_1$, we conclude that $p^{[l_0]} = 0$. Since s is a C^1-spline, the function values and all first derivatives of $p^{[l_1]} = s|_{T^{[l_1]}} \in \tilde{\Pi}_q$ vanish at e_1 and e_2. Thus,

$$
D^{\omega} p^{[l_1]}(v) = 0,\ \omega = 0,\dots,3. \qquad (9)
$$

Now, we claim that

$$
p^{[l_1]}_{d_1^{\mu}}|_{[w_1,w_2]} = 0,\ \mu = 0,\dots,q-4. \qquad (10)
$$

We prove (10) by induction on μ. Now, let d be a unit vector in direction of $[w_1, w_2]$. Since the function value and the first derivative in direction of d of $p^{[l_1]}|_{[w_1,w_2]} \in \Pi_q$ vanish at w_1 and w_2, it follows from the interpolation conditions of $p^{[l_1]}$ at $\overline{v} = \frac{1}{2}(w_1 + w_2)$ that the claim holds for $\mu = 0$. We assume that (10) holds for $\mu \in \{0,\dots,\eta\},\ \eta \le q-5$, and show that (10) holds for $\eta+1$. In the following, we use that for $g \in C^{\omega}(\Omega)$,

$$
g_{(\alpha_1\delta_1+\alpha_2\delta_2)^{\omega}} = \sum_{\sigma=0}^{\omega} \binom{\omega}{\sigma} \alpha_1^{\omega-\sigma} \alpha_2^{\sigma} g_{\delta_1^{\omega-\sigma}\delta_2^{\sigma}}, \qquad (11)
$$

where $\delta_1,\ \delta_2$ and $\alpha_1\delta_1+\alpha_2\delta_2$ are unit vectors and ω is a natural number. Let d_2 be a unit vector in direction of the edge $[w_2, v]$ and $\alpha,\ \beta \neq 0$ be given such that $d_1 = \alpha d_2 + \beta d$. By (11), we have

$$
p^{[l_1]}_{d_1^{\eta+1} d^{\rho}}(w_2) = \alpha^{\eta+1} p^{[l_1]}_{d_2^{\eta+1} d^{\rho}}(w_2) + \sum_{\sigma=1}^{\eta+1} \binom{\eta+1}{\sigma} \alpha^{\eta+1-\sigma} \beta^{\sigma} p^{[l_1]}_{d_2^{\eta+1-\sigma} d^{\rho+\sigma}}(w_2),\ \rho = 0,1. \qquad (12)
$$

Again by (11), we obtain

$$
p^{[l_1]}_{d_2^{\eta+1-\sigma} d^{\rho+\sigma}}(w_2) = \sum_{\tau=0}^{\eta+1-\sigma} \binom{\eta+1-\sigma}{\tau} \alpha^{\sigma-\eta-1} (-\beta)^{\tau} p^{[l_1]}_{d_1^{\eta+1-\sigma-\tau} d^{\rho+\sigma+\tau}}(w_2),\ \sigma = 1,\dots,\eta+1.
$$

From this and the induction hypothesis it follows that $p^{[l_1]}_{d_2^{\eta+1-\sigma}d^{\rho+\sigma}}(w_2) = 0,\ \sigma = 1,\ldots,\eta+1$, and therefore (12) implies

$$p^{[l_1]^*}_{d_1^{\eta+1}d^\rho}(w_2) = \alpha^{\eta+1} p^{[l_1]}_{d_2^{\eta+1}d^\rho}(w_2) = 0,\ \rho = 0,1. \tag{13}$$

Moreover, $p^{[l_1]}_{d_1^{\eta+1}d^\rho}(w_1) = 0,\ \rho = 0,1$. Then it follows from (13) and the interpolation conditions of $p^{[l_1]}_{d_1^{\eta+1}} \in \tilde{\Pi}_{q-\eta-1}$ at $\overline{v} = \frac{1}{2}(w_1+w_2)$ that $p^{[l_1]}_{d_1^{\eta+1}}|_{[w_1,w_2]} = 0$. This proves (10). From this and (9), we conclude $p^{[l_1]} = 0$ and $s = 0$. This proves the case $\lambda = 1$.

If $\lambda > 1$, we may assume that the edges $e_{\lambda-1}$ and $e_{\lambda+1}$ have different slopes. Analogously as in the proof of (7) the interpolation conditions of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ at w_1 imply that $a^{[l_0]}_{\mu,j,q-\mu-j} = 0,\ j = 0,\ldots,q-2-\mu,\ \mu = 0,\ldots,q-2$. Since s is a C^1-spline, we conclude $p^{[l_0]} = 0$. By proceeding with these arguments, we obtain $s|_{T^{[l_\nu]}} = p^{[l_\nu]} = 0,\ \nu = 0,\ldots,\lambda-2$, and $s = 0$. This proves Theorem 5.4. ∎

Proof of Theorem 4.3: The proof is similar to the proof of Theorem 4.1 by using Theorem 5.4 instead of Theorem 5.3. ∎

6 Proof of the Main Theorems for S_q^2

In this section we prove our main theorems for $S_q^2(\Delta)$ (Theorem 3.3, Theorem 4.2 and Theorem 4.4). We begin with the proof of our result on admissible sets. For doing this, we need Theorem 6.1, and Theorem 6.3. Let P, Δ^*, Ω^*, Δ_P^*, v, $w_0,\ldots,w_{\lambda+1}$ be defined as in the beginning of Section 5 and denote by $\mathcal{A}_P$ the union of sets chosen in Case 2a of Section 3.

Theorem 6.1 *Let $q \geq 5$. If $\mathcal{A}^*$ is an admissible set for $S_q^2(\Delta^*)$, then $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}_P$ is an admissible set of $S_q^2(\Delta_P^*)$.*

Proof: By our construction we have $\lambda \geq 2$. We first assume that $\lambda = 2$. We set $m_1 = \binom{q-1}{2} - 1$, $m_2 = \binom{q-1}{2} - 2$, $m_3 = \binom{q-4}{2}$ and $m = card(\mathcal{A})$. Since $\mathcal{A}^* = \{\sigma_1,\ldots,\sigma_{m-m_1-m_2-m_3}\}$ is an admissible set for $S_q^2(\Delta^*)$, $q \geq 5$, it follows that for every choice of coefficients $a(\sigma_\mu)$, $\mu = 1,\ldots,m-m_1-m_2-m_3$, a unique spline $s^* \in S_q^2(\Delta^*)$ exists with these coefficients in the representation (1) of s^*. Since e_0, e_1, e_2 and e_3 have different slopes, it follows from Lemma 3.4 that the coefficients $a^{[l_0]}_{i,j,\rho}$, $a^{[l_2]}_{i,\rho,j}$, $i+j = q-\rho$, $\rho = 0,1,2$, and $a^{[l_1]}_{q-\rho,j,k}$, $j+k = \rho$, $\rho = 0,1,2$, of $p^{[l_\nu]} \in \tilde{\Pi}_q$ on $T^{[l_\nu]} = \Delta(v,w_\nu,w_{\nu+1})$, $\nu = 0,1,2$, in the representation (1) are uniquely determined. We claim that the coefficients $a^{[l_1]}_{q-3,j,k}$, $j+k = 3$, (and $a^{[l_0]}_{q-3,0,3}$, $a^{[l_2]}_{q-3,3,0}$) are uniquely determined. We may assume that $v = (0,0)$, $w_\nu = \tau_\nu(\cos\omega_\nu, \sin\omega_\nu)$, $\nu = 0,\ldots,3$, where

$\tau_\nu > 0,\ \nu = 0, \ldots, 3$, and $2\pi > \omega_0 > \omega_1 > \omega_2 > \omega_3 = 0$. By Lemma 3.4 the vector $x^t = (a^{[l_1]}_{q-3,3,0}, a^{[l_1]}_{q-3,2,1}, a^{[l_1]}_{q-3,1,2}, a^{[l_1]}_{q-3,0,3})$ satisfies

$$\begin{pmatrix} \Phi_3^{[l_0]}(w_2) & -1 & 0 & 0 \\ (\Phi_3^{[l_0]}(w_2))^2 & 0 & -1 & 0 \\ 0 & 0 & -1 & \Phi_2^{[l_2]}(w_1) \\ 0 & -1 & 0 & (\Phi_2^{[l_2]}(w_1))^2 \end{pmatrix} x = \gamma,$$

where $\gamma \in \mathbb{R}^4$ is suitable chosen. Since $\Phi_2^{[l_2]}(w_1) = \frac{\tau_1 \sin(\omega_1)}{\tau_2 \sin(\omega_2)}$, $\Phi_3^{[l_0]}(w_2) = \frac{\tau_2 \sin(\omega_0 - \omega_2)}{\tau_1 \sin(\omega_0 - \omega_1)}$, it follows from some elementary computations that

$$D = -\frac{\sin(\omega_0)\sin(\omega_0 - \omega_2)\sin(\omega_1)\sin(\omega_1 - \omega_2)}{(\sin(\omega_2)\sin(\omega_0 - \omega_1))^2}.$$

(Here, D is the determinant of the above system.) Since $e_0,\ e_1,\ e_2$ and e_3 have different slopes, we have $\omega_0 - \omega_2,\ \omega_0 - \omega_3,\ \omega_1 - \omega_3 \neq \pi$. Thus, $D \neq 0$. Note that $\omega_0 - \omega_1,\ \omega_1 - \omega_2,\ \omega_2 - \omega_3 \neq \pi$. This shows that the coefficients $a^{[l_1]}_{q-3,j,k}$, $j + k = 3$ (and $a^{[l_0]}_{q-3,0,3}$, $a^{[l_2]}_{q-3,3,0}$) are uniquely determined.

Now, it is easy to verify that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m - m_1 - m_2 - m_3 + 1, \ldots, m - m_2 - m_3$, where $\{\sigma_{m-m_1-m_2-m_3+1}, \ldots, \sigma_{m-m_2-m_3}\} = B_2 = \{(i,j,k,l_0) :\ i + j + k = q,\ k \geq 3,\ i \neq q - 3\}$ a unique polynomial $p^{[l_0]} \in \tilde{\Pi}_q$ on $T^{[l_0]}$ exists with these coefficients in the representation (1) of $p^{[l_0]}$. By Lemma 3.4 the coefficients $a^{[l_1]}_{i,j,\rho}$, $i + j = q - \rho$, $\rho = 0, 1, 2$, of $p^{[l_1]}$ are uniquely determined. Since $(q - 4, 0, 4, l_1) \in C_2 = \{(i,j,k,l_1) :\ i + j + k = q,\ k \geq 3,\ i \neq q - 3,\ (i,j,k) \neq (q - 4, 1, 3)\}$, it follows from Lemma 3.5 that $a^{[l_1]}_{q-4,1,3}$ is uniquely determined. This implies that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m - m_2 - m_3 + 1, \ldots, m - m_3$, where $\{\sigma_{m-m_2-m_3+1}, \ldots, \sigma_{m-m_3}\} = C_2$, a unique polynomial $p^{[l_1]} \in \tilde{\Pi}_q$ on $T^{[l_1]}$ exists with these coefficients in the representation (1) of $p^{[l_1]}$.

Now, by Lemma 3.4 for every choice of coefficients $a(\sigma_\mu)$, $\mu = m - m_3 + 1, \ldots, m$, where $\{\sigma_{m-m_3+1}, \ldots, \sigma_m\} = D_2 = \{(i,j,k,l_2) :\ i + j + k = q,\ j \geq 3,\ k \geq 3\}$ a unique polynomial $p^{[l_2]} \in \tilde{\Pi}_q$ on $T^{[l_2]}$ exists with these coefficients in the representation (1) of $p^{[l_2]}$.

Since all differentiability conditions for $r = 2$ at the edges $e_0,\ e_1,\ e_2,\ e_3$ have been involved, we get that for every choice of coefficients $a(\sigma_\mu)$, $\mu = 1, \ldots, m$, a unique spline s from $S_q^2(\Delta_P^*)$,

$$s(x,y) = \begin{cases} s^*(x,y) \text{ , if } (x,y) \in \Omega^*, \\ p^{[l_\nu]}(x,y) \text{ , if } (x,y) \in T^{[l_\nu]},\ \nu = 0,1,2, \end{cases}$$

exists with these coefficients in the representation (1) of s. This proves the case $\lambda = 2$.

If $\lambda > 2$, we may assume that the edges $e_{\lambda-2},\ e_{\lambda-1},\ e_\lambda,\ e_{\lambda+1}$ have different slopes. It follows from Lemma 3.4 (applied to the edges $e_0, \ldots, e_{\lambda-3}$) that for every choice of coefficients corresponding to the sets $A_2 = \{(i,j,k,l_\nu) :\ i+j+k = q,\ k \geq 2\}$, $\nu = 0, \ldots, \lambda-3$, unique polynomials $p^{[l_\nu]} \in \tilde{\Pi}_q$ on $T^{[l_\nu]}$ exist with these coefficients in the representation (1) of $p^{[l_\nu]}$, $\nu = 0, \ldots, \lambda - 3$. Now, we can argue as in the case $\lambda = 2$. This proves Theorem 6.1. ■

Corollary 6.2 *For $q \geq 5$, we have*

$$dim\ S_q^2(\Delta_P^*) = dim\ S_q^2(\Delta^*) + \lambda\binom{q-1}{2} + \binom{q-4}{2} - 3.$$

The next theorem deals with the case when some triangle $T^{[l_\nu]}$ of the added polyhedron $\tilde{P} = P$ is subdivided (see Case 2b of Section 3). We denote by y_ν the point which subdivided the triangle $T^{[l_\nu]}$. Moreover, let $\tilde{e}_0,\ \tilde{e}_1,\ \tilde{e}_2$ and $\tilde{e}_3$ be as in Case 2b of Section 3, and denote by $\mathcal{A}_{\tilde{P}}$ the union of sets chosen in Case 2b of Section 3.

Theorem 6.3 *Let $q \geq 5$. If $\mathcal{A}^*$ is an admissible set for $S_q^2(\Delta^*)$, then $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}_{\tilde{P}}$ is an admissible set for $S_q^2(\Delta_{\tilde{P}}^*)$.*

Proof. We first assume that $\lambda = 1$. We set m_1 and m_3 as in the proof of Theorem 6.1, $m_2 = \binom{q-1}{2} - 4$ and let $m = card(\mathcal{A})$. We may assume that $T^{[l_0]}$ is subdivided. Since $\mathcal{A}^* = \{\sigma_1, \ldots, \sigma_{m-m_1-m_2-2m_3}\}$ is an admissible set for $S_q^2(\Delta^*)$, $q \geq 5$, it follows that for every choice of coefficients $a(\sigma_\mu)$, $\mu = 1, \ldots, m - m_1 - m_2 - 2m_3$, a unique spline $s^* \in S_q^2(\Delta^*)$ exists with these coefficients in the representation (1) of s^*. Since $\tilde{e}_0,\ \tilde{e}_1,\ \tilde{e}_2$ and $\tilde{e}_3$, have different slopes, it follows from Lemma 3.4 that the coefficients $a_{i,j,\rho}^{[(l_0,0)]}$, $a_{i,\rho,j}^{[(l_0,2)]}$, $i + j = q - \rho$, $\rho = 0, 1, 2$, and $a_{q-\rho,j,k}^{[(l_0,1)]}$, $j + k = \rho$, $\rho = 0, 1, 2$, of $p^{[(l_0,\sigma)]} \in \tilde{\Pi}_q$ on $T^{[(l_0,\sigma)]} = \Delta(v_1^\sigma, v_2^\sigma, v_3^\sigma)$, $\sigma = 0, 1, 2$, (cf. Case 2b of Section 3) in the representation (1) are uniquely determined. As in the proof of Theorem 6.1, it follows that the coefficients $a_{q-3,j,k}^{[(l_0,1)]}$, $j + k = 3$, (and $a_{q-3,0,3}^{[(l_0,0)]}$, $a_{q-3,3,0}^{[(l_0,2)]}$) are uniquely determined. Now, it is easy to see that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m - m_1 - m_2 - 2m_3 + 1, \ldots, m - m_2 - 2m_3$, where $\{\sigma_{m-m_1-m_2-2m_3+1}, \ldots, \sigma_{m-m_2-2m_3}\} = B_2 = \{(i, j, k, (l_0, 0)) : i + j + k = q,\ k \geq 3,\ i \neq q - 3\}$ a unique polynomial $p^{[(l_0,0)]} \in \tilde{\Pi}_q$ on $T^{[(l_0,0)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,0)]}$. By Lemma 3.4, the coefficients $a_{i,j,\rho}^{[(l_0,1)]}$, $a_{i,j,\rho}^{[(l_0,3)]}$, $i + j = q - \rho$, $\rho = 0, 1, 2$, are uniquely determined. Here, $a_{i,j,k}^{[(l_0,3)]}$, $i + j + k = q$, are the coefficients of $p^{[(l_0,3)]} \in \tilde{\Pi}_q$ on $T^{[(l_0,3)]} = \Delta(v_1^3, v_2^3, v_3^3)$ in the representation (1) (cf. Case 2b of Section 3). We claim that the coefficient $a_{0,q-3,3}^{[(l_0,1)]}$ is uniquely determined. Let d, respectively d_1, d_2, be unit vectors in direction of the edge $[y_0, w_1]$, respectively $[y_0, w_0]$, $[y_0, v]$, and let β_1, $\beta_2 \neq 0$ be given such that $d = \beta_1 d_1 + \beta_2 d_2$. It follows from the C^2-property and (11) that

$$\begin{aligned} p_{d^3}^{[(l_0,1)]}(y_0) &= \sum_{\tau=0}^{2} \binom{\tau}{2} \beta_1^{2-\tau} \beta_2^{\tau} p_{dd_1^{2-\tau} d_2^{\tau}}^{[(l_0,1)]}(y_0) \\ &= \beta_1^2 p_{dd_1^2}^{[(l_0,3)]}(y_0) + 2\beta_1\beta_2 p_{dd_1 d_2}^{[(l_0,0)]}(y_0) + \beta_2^2 p_{dd_2^2}^{[(l_0,0)]}(y_0) = p_{d^3}^{[(l_0,0)]}(y_0). \quad (14) \end{aligned}$$

On the other hand, it follows from the C^2-property and (11) that

$$p_{d^3}^{[(l_0,1)]}(y_0) = \beta_1 p_{d^2 d_1}^{[(l_0,3)]}(y_0) + \beta_2 p_{d^2 d_2}^{[(l_0,1)]}(y_0) = p_{d^3}^{[(l_0,0)]}(y_0).$$

From this and (14), we conclude that the derivative $p_{d^3}^{[(l_0,1)]}(y_0) = p_{d^3}^{[(l_0,3)]}(y_0)$ is uniquely determined. Then by (3) the coefficient $a_{0,q-3,3}^{[(l_0,1)]}$ (and $a_{q-3,0,3}^{[(l_0,3)]}$) is uniquely determined. Let us first consider the case when for $q \geq 5$, we assign $\tilde{C}_2$ to $T^{[(l_0,1)]}$ and D_2 to $T^{[(l_0,3)]}$. Since

$$(q-4,0,4) \in \tilde{C}_2 = \begin{cases} \{(0,0,5,(l_0,1)),(1,0,4,(l_0,1))\} & \text{, if } q=5, \\ \{(i,j,k,(l_0,1)) : \ i+j+k=q, \ k \geq 3, \ i,j \neq q-3, \\ \quad (i,j,k) \neq (q-4,1,3), \ (i,j,k) \neq (1,q-4,3)\} & \text{, if } q \geq 6, \end{cases}$$

by Lemma 3.5 the coefficient $a_{q-4,1,3}^{[(l_0,1)]}$ is uniquely determined.

Let us first consider the case $q \geq 6$. In this case, since $(0,q-4,4,(l_0,1)) \in \tilde{C}_2$, by Lemma 3.5 the coefficient $a_{1,q-4,3}^{[(l_0,1)]}$ is uniquely determined, and it is easy to verify that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-m_2-2m_3+1,\ldots,m-2m_3$, where $\{\sigma_{m-m_2-2m_3+1},\ldots,\sigma_{m-2m_3}\} = \tilde{C}_2$, a unique polynomial $p^{[(l_0,1)]} \in \tilde{\Pi}_q$, $q \geq 6$, on $T^{[(l_0,1)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,1)]}$.

The case $q=5$ is slightly different. In this case, since $a_{1,1,3}^{[(l_0,1)]}$ is determined, by Lemma 3.5 the coefficient $a_{0,1,4}^{[(l_0,1)]}$ is uniquely determined. Then for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-1,m$, where $\{\sigma_{m-1},\sigma_m\} = \tilde{C}_2$, a unique polynomial $p^{[(l_0,1)]} \in \tilde{\Pi}_5$ on $T^{[(l_0,1)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,1)]}$.

Then by Lemma 3.4 for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-2m_3+1,\ldots,m-m_3$, where $\{\sigma_{m-2m_3+1},\ldots,\sigma_{m-m_3}\} = D_2 = \{(i,j,k,(l_0,3)) : \ i+j+k=q, \ j \geq 3, \ k \geq 3\}$, a unique polynomial $p^{[(l_0,3)]} \in \tilde{\Pi}_q$ on $T^{[(l_0,3)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,3)]}$.

Now, we consider the case when for $q \geq 7$, we assign $\tilde{D}_2$ to $T^{[(l_0,1)]}$ and C_2 to $T^{[(l_0,3)]}$. Since $(q-4,0,4) \in C_2$ by Lemma 3.5 the coefficient $a_{q-4,1,3}^{[(l_0,3)]}$ is uniquely determined. Then for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-m_2-2m_3+1,\ldots,m-2m_3+2$, where $\{\sigma_{m-m_2-2m_3+1},\ldots,\sigma_{m-2m_3+2}\} = C_2 = \{(i,j,k,(l_0,3)) : \ i+j+k=q, \ k \geq 3, \ i \neq q-3, \ (i,j,k) \neq (q-4,1,3)\}$, a unique polynomial $p^{[(l_0,3)]} \in \tilde{\Pi}_q$, $q \geq 7$, on $T^{[(l_0,3)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,3)]}$. By Lemma 3.4 the coefficients $a_{\rho,j,k}^{[(l_0,1)]}$, $j+k=q-\rho$, $\rho=0,1,2$, of $p^{[(l_0,1)]}$ are uniquely determined. Since $(q-4,0,4,(l_0,1)) \in \tilde{D}_2$, by Lemma 3.5 the coefficient $a_{q-4,1,3}^{[(l_0,1)]}$ is uniquely determined. Therefore, for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-2m_3+3,\ldots,m-m_3$, where $\{\sigma_{m-2m_3+3},\ldots,\sigma_{m-m_3}\} = \tilde{D}_2 = \{(i,j,k,(l_0,1)) : \ i+j+k=q, \ i \geq 3, \ k \geq 3, \ i \neq q-3, \ (i,j,k) \neq (q-4,1,3)\}$, a unique polynomial $p^{[(l_0,1)]} \in \tilde{\Pi}_q$, $q \geq 7$, on $T^{[(l_0,1)]}$ exists with these coefficients in the representation (1) of $p^{[(l_0,1)]}$.

Now, in all cases, it follows from Lemma 3.4 that for every choice of coefficients $a(\sigma_\mu)$, $\mu = m-m_3+1,\ldots,m$, where $\{\sigma_{m-m_3+1},\ldots,\sigma_m\} = D_2 = \{(i,j,k,(l_0,2)) : \ i+j+k=q, \ j \geq 3, \ k \geq 3\}$, a unique polynomial $p^{[(l_0,2)]} \in \tilde{\Pi}_q$ on $T^{[(l_0,2)]}$, exists with these coefficients in the representation (1) of $p^{[(l_0,2)]}$.

The rest of the proof is similar to the proof of Theorem 6.1. This proves Theorem 6.3. ■

Corollary 6.4 *For $q \geq 5$, we have*

$$dim\ S_q^2(\Delta_P^*) = dim\ S_q^r(\Delta^*) + (\lambda+1)\binom{q-1}{2} + 2\binom{q-4}{2} - 5.$$

Now, we prove Theorem 3.3.

Proof of Theorem 3.3: The proof is similar to the proof of Theorem 3.2 for Δ. The only differences is that we use Theorem 6.1 and Theorem 6.3 (if some triangle of Δ has to be subdivided) instead of Theorem 5.1. ∎

Next, we prove Theorem 4.2. For doing this, we need Theorem 6.5 and Theorem 6.6. Let P, Δ^*, Ω^*, Δ_P^*, v, $w_0, \ldots, w_{\lambda+1}$ be defined as in the beginning of Section 5 and denote by $\mathcal{L}_P$ the union of the sets chosen in Section 4 which correspond to Case 2a of Section 3.

Theorem 6.5 *Let $q \geq 5$. If $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^2(\Delta^*)$, then $\mathcal{L} = \mathcal{L}^* \cup \mathcal{L}_P$ is a Lagrange interpolation set for $S_q^2(\Delta_P^*)$.*

Proof: By our construction we have $\lambda \geq 2$. We first assume that $\lambda = 2$. We set m_1, m_2, m_3, as in the proof of Theorem 6.1 and $m = dim\ S_q^2(\Delta_P^*)$. Moreover, let $\mathcal{L}^* = \{z_1, \ldots, z_{m-m_1-m_2-m_3}\} \subseteq \mathcal{L} = \{z_1, \ldots, z_m\}$ and a spline $s \in S_q^2(\Delta_P^*)$, $q \geq 5$, which satisfies $s(z_i) = 0$, $i = 1, \ldots, m$, be given. We will show that $s = 0$. Since $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^2(\Delta^*)$, it follows that $s|_{\Omega^*} = 0$. Since s is a C^2-spline the function value and all first and second derivatives of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ (respectively $p^{[l_2]} = s|_{T^{[l_2]}} \in \tilde{\Pi}_q$) vanish at e_0 (respectively e_3). Moreover, $D^\omega p^{[l_1]}(v) = 0$, $\omega = 0, 1, 2$, where $p^{[l_1]} = s|_{T^{[l_1]}} \in \tilde{\Pi}_q$. Since e_0, e_1, e_2 and e_3 have different slopes, it follows from (2) and the proof of Theorem 6.1 that

$$D^\omega p^{[l_0]}(v) = D^\omega p^{[l_1]}(v) = D^\omega p^{[l_2]}(v) = 0,\ \omega = 0, \ldots, 3. \tag{15}$$

Let $b_\mu = \{(x,y) \in T^{[l_0]} :\ \alpha_\mu x + \beta_\mu y + \gamma_\mu = 0\}$, $\mu = 1, \ldots, q-3$, be the line segments chosen in $T^{[l_0]}$ such that $q - 1 - \mu$ points of $\{z_{m-m_1-m_2-m_3+1}, \ldots, z_{m-m_2-m_3}\}$ lie on b_μ, $\mu = 1, \ldots, q-3$. We claim that

$$p^{[l_0]}|_{b_\mu} = 0,\ \mu = 1, \ldots, q-3. \tag{16}$$

We prove (16) by induction on μ. Denote by $z_\mu^{[l_0]}$, $\mu = 1, \ldots, q-3$, the intersection point of b_μ, $\mu = 1, \ldots, q-3$, and e_0. Since the function value, the first and second derivative (in direction of b_1) of $p^{[l_0]}|_{b_1} \in \Pi_q$ vanish at $z_1^{[l_0]}$, it follows from the interpolation conditions of $p^{[l_0]}$ on b_1 that (16) holds for $\mu = 1$. We assume that (16) holds for $\mu \in \{1, \ldots, \eta\}$, $\eta \leq q-4$, and show that (16) holds for $\eta + 1$. By induction hypothesis a polynomial $q^{[l_0]} \in \tilde{\Pi}_{q-\eta}$ exists such that

$$p^{[l_0]}(x,y) = \prod_{\mu=1}^{\eta} (\alpha_\mu + \beta_\mu y + \gamma_\mu) q^{[l_0]}(x,y),\ (x,y) \in T^{[l_0]}.$$

Since the function value, the first and the second derivative (in direction of $b_{\eta+1}$) of $q^{[l_0]}|_{b_{\eta+1}} \in \Pi_{q-\eta}$ vanish at $z_{\eta+1}^{[l_0]}$, it follows from the interpolation conditions of $p^{[l_0]}$ on $b_{\eta+1}$ that $q^{[l_0]}|_{b_{\eta+1}} = 0$, and $p^{[l_0]}|_{b_{\eta+1}} = 0$. This proves (16). From this and (15), we conclude $p^{[l_0]} = 0$. Let d_j be unit vectors in direction of the edge $e_j = [v, w_j]$, $j = 1, 2$. Since s is a C^2-spline, we get

$$p^{[l_1]}_{d_1^{4-\nu} d_2^{\nu}}(v) = 0, \;\; \nu = 0, 1, 2. \tag{17}$$

Let c_μ, $\mu = 1, \ldots, q-3$, be the line segments chosen in $T^{[l_1]}$ such that $q - 1 - \mu$ points of $\{z_{m-m_2-m_3+1}, \ldots, z_{m-m_3-1}\}$ lie on c_μ, $\mu = 1, \ldots, q-4$, and z_{m-m_3} lies on the intersection of c_{q-3} and $[v, w_2]$. As in the proof of (16) we obtain

$$p^{[l_1]}|_{c_\mu} = 0, \;\; \mu = 1, \ldots, q-4. \tag{18}$$

We denote by $z_\mu^{[l_1]}$, $\mu = 1, \ldots, q-3$, the intersection point of c_μ, $\mu = 1, \ldots, q-3$, and $[v, w_2]$. Then it follows from the interpolation condition of $p^{[l_1]}$ at $z_{m-m_3} = z_{q-3}^{[l_1]}$ and the above that $p^{[l_1]}(z_\mu^{[l_1]}) = 0$, $\mu = 1, \ldots, q-3$. Moreover, from (15) we obtain $p^{[l_1]}_{d_2^\nu}(v) = 0$, $\nu = 0, \ldots, 3$, and $p^{[l_1]}|_{[v,w_2]} = 0$. Therefore, $p^{[l_1]}_{d_2^4}(v) = 0$. Then by (3), (15) and (17) we have that the coefficients $a^{[l_1]}_{q-4,j,k}$, $j + k = 4$, $(j,k) \neq (1,3)$, in the representation (1) of $p^{[l_1]}$ on $T^{[l_1]} = \Delta(v, w_1, w_2)$ are zero. By Lemma 3.5, we obtain $a^{[l_1]}_{q-4,1,3} = 0$. It follows from (2) that $p^{[l_1]}_{d_1 d_2^3}(v) = 0$. Then we get $D^\omega p^{[l_1]}(v) = 0$, $\omega = 0, \ldots, 4$. It follows from (18) that $p^{[l_1]} = 0$.

Since s is a C^2-spline the function values, all the first and second derivatives of $p^{[l_2]}$ vanish at e_2. Thus,

$$D^\omega p^{[l_2]}(v) = 0, \;\; \omega = 0, \ldots, 5. \tag{19}$$

Let d_μ, $\mu = 1, \ldots, q-5$, be the line segments chosen in $T^{[l_2]}$ such that $q - 4 - \mu$ points of $\{z_{m-m_3+1}, \ldots, z_m\}$ lie on d_μ, $\mu = 1, \ldots, q-5$. As in the proof of (16) we obtain $p^{[l_2]}|_{d_\mu} = 0$, $\mu = 1, \ldots, q-5$. From this and (19) we conclude $p^{[l_2]} = 0$, and $s = 0$. This proves the case $\lambda = 2$.

If $\lambda > 2$, we may assume that the edges $e_{\lambda-2}$, $e_{\lambda-1}$, e_λ, and $e_{\lambda+1}$ have different slopes. Since s is a C^2-spline the function values, all the first and second derivatives of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ vanish at e_0. Thus,

$$D^\omega p^{[l_0]}(v) = 0, \;\; \omega = 0, 1, 2. \tag{20}$$

Let a_μ, $\mu = 1, \ldots, q-2$, be the line segments chosen in $T^{[l_0]}$ such that $q - 1 - \mu$ of the chosen points lie on a_μ, $\mu = 1, \ldots, q-2$. As in the proof of (16) we can see that $p^{[l_0]}|_{a_\mu} = 0$, $\mu = 1, \ldots, q-2$. From this and (20) we conclude $p^{[l_0]} = 0$. By proceeding with these arguments, we obtain $s|_{T^{[l_\nu]}} = p^{[l_\nu]} = 0$, $\nu = 0, \ldots, \lambda - 3$, and $s = 0$. This proves Theorem 6.5. ∎

Now, let $T^{[l_\nu]}$, $\tilde{P}$, y_ν, $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ be as in front of Theorem 6.3 and denote by $\mathcal{L}_{\tilde{P}}$ the union of sets chosen in Section 4 which correspond to Case 2b of Section 3.

Theorem 6.6 *Let $q \geq 5$. If $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^2(\Delta^*)$, then $\mathcal{L} = \mathcal{L}^* \cup \mathcal{L}_{\tilde{P}}$ is a Lagrange interpolation set for $S_q^2(\Delta^*_{\tilde{P}})$.*

Proof. We first assume that $\lambda = 1$. Let m_1, m_2, m_3 be as in the proof of Theorem 6.3, $m = dim\ S_q^2(\Delta^*_{\tilde{P}})$ and $\mathcal{L}^* = \{z_1, \ldots, z_{m-m_1-m_2-2m_3}\} \subseteq \mathcal{L} = \{z_1, \ldots, z_m\}$. We may assume that $T^{[l_0]}$ is subdivided. Let a spline $s \in S_q^2(\Delta^*_{\tilde{P}})$, $q \geq 5$, which satisfies $s(z_i) = 0$, $i = 1, \ldots, m$, be given. We will show that $s = 0$. Since $\mathcal{L}^*$ is a Lagrange interpolation set for $S_q^2(\Delta^*)$, it follows that $s|_{\Omega^*} = 0$.

Since s is a C^2-spline the function values and all first and second derivatives of $p^{[(l_0,0)]} = s|_{T^{[(l_0,0)]}} \in \tilde{\Pi}_q$ (respectively $p^{[(l_0,2)]} = s|_{T^{[(l_0,2)]}} \in \tilde{\Pi}_q$) vanish at $\tilde{e}_0$ (respectively $\tilde{e}_3$). Moreover, $D^\omega p^{[(l_0,1)]}(v) = 0$, $\omega = 0, 1, 2$, where $p^{[(l_0,1)]} = s|_{T^{[(l_0,1)]}} \in \tilde{\Pi}_q$. Since $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$ have different slopes, it follows from (2) and the proof of Theorem 6.1 that

$$D^\omega p^{[(l_0,0)]}(v) = D^\omega p^{[(l_0,1)]}(v) = D^\omega p^{[(l_0,2)]}(v) = 0, \ \omega = 0, \ldots, 3. \tag{21}$$

Let b_μ, $\mu = 1, \ldots, q-3$, be the line segments chosen in $T^{[(l_0,0)]}$ such that $q - 1 - \mu$ points of $\{z_{m-m_1-m_2-2m_3+1}, \ldots, z_{m-m_2-2m_3}\}$ lie on b_μ, $\mu = 1, \ldots, q-3$. As in the proof of Theorem 6.5, we can see that $s(z_i) = 0$, $i = m - m_1 - m_2 - 2m_3 + 1, \ldots, m - m_2 - 2m_3$, and (21) imply $p^{[(l_0,0)]} = 0$.

Since s is a C^2-spline the functional values and all the first and second derivatives of $p^{[(l_0,1)]} = s|_{T^{[(l_0,1)]}} \in \tilde{\Pi}_q$ (respectively $p^{[(l_0,3)]} = s|_{T^{[(l_0,3)]}} \in \tilde{\Pi}_q$) vanish at $[y_0, v]$ (respectively $[y_0, w_0]$). Moreover, it follows from the proof of Theorem 6.3 that

$$p_{d^3}^{[(l_0,3)]}(y_0) = p_{d^3}^{[(l_0,1)]}(y_0) = 0, \tag{22}$$

where d is a unit vector in direction of the edge $[y_0, w_1]$.

Let us first consider the case $q = 6$. Let z_{m-7}, z_{m-6}, z_{m-5}, be the points chosen on $(v, w_1]$, z_{m-4}, z_{m-3}, be the points chosen on (y_0, w_1) and z_{m-2} be the point chosen from the interior of $T^{[(l_0,1)]}$. Since $s(z_i) = 0$, $i = m - 7, \ldots, m - 5$, it follows from (21) that $p^{[(l_0,1)]}|_{[v,w_1]} = 0$. Since $s(z_i) = 0$, $i = m - 4$, $m - 3$, and $p^{[(l_0,1)]}(w_1) = 0$, it follows from (22) that $p^{[(l_0,1)]}|_{[y_0,w_1]} = 0$. As in the proof of Theorem 6.3, it follows from (3) that $a_{2,1,3}^{[(l_0,1)]} = a_{1,2,3}^{[(l_0,1)]} = 0$, where $a_{i,j,k}^{[(l_0,1)]}$, $i + j + k = 6$, are the coefficients of $p^{[(l_0,1)]} \in \tilde{\Pi}_6$ on $T^{[(l_0,1)]} = \Delta(v, y_0, w_1)$ in the representation (1). This shows that

$$p^{[(l_0,1)]}(x, y) = 30a_{1,1,4}^{[(l_0,1)]}\Phi_1(x, y)\Phi_2(x, y)\Phi_3^4(x, y), \ (x, y) \in T^{[(l_0,1)]}.$$

Now, it is easy to see that $s(z_{m-2}) = 0$ implies $a_{1,1,4}^{[(l_0,1)]} = 0$, and $p^{[(l_0,1)]} = 0$.

Now, we consider the case $q = 5$. Let z_{m-1}, z_m, be the points chosen on $(v, w_1]$. Since $s(z_i) = 0$, $i = m-1, m$, it follows from (21) that $p^{[(l_0,1)]}|_{[v,w_1]} = 0$. Analogously, as in the proof of Theorem 6.3, it follows from (3) that $a_{1,1,3}^{[(l_0,1)]} = 0$ and $a_{0,1,4}^{[(l_0,1)]} = 0$, where

$a_{i,j,k}^{[(l_0,1)]}$, $i+j+k=5$, are the coefficients of $p^{[(l_0,1)]} \in \tilde{\Pi}_5$ on $T^{[(l_0,1)]} = \Delta(v, y_0, w_1)$ in the representation (1). We conclude $p^{[(l_0,1)]} = 0$.

In these cases, as in the proof of Theorem 6.5 we get $p^{[(l_0,3)]} = 0$.

We finally consider the case $q \geq 7$. Let $\{z_{m-m_2-2m_3+1}, \ldots, z_{m-2m_3+2}\}$ be the points chosen in $T^{[(l_0,3)]}$. As in the proof of Theorem 6.5 it follows from (22) that $s(z_i) = 0$, $i = m - m_2 - 2m_3 + 1, \ldots, m - 2m_3 + 2$, implies $p^{[(l_0,3)]} = 0$. Since s is a C^2-spline the functional values and all the first and second derivatives of $p^{[(l_0,1)]} = s|_{T^{[(l_0,1)]}} \in \tilde{\Pi}_q$ vanish at $[y_0, w_1]$. We set $[y_0, w_1] = \{(x,y) \in T^{[(l_0,1)]} : \alpha x + \beta y + \gamma = 0\}$. It follows that a polynomial $q^{[(l_0,1)]} \in \tilde{\Pi}_{q-3}$ exists such that

$$p^{[(l_0,1)]}(x,y) = (\alpha x + \beta y + \gamma)^3 q^{[(l_0,1)]}(x,y), \; (x,y) \in T^{[(l_0,1)]}. \tag{23}$$

Let d_μ, $\mu = 1, \ldots, q-6$, be the line segments chosen in $T^{[(l_0,1)]}$ such that $q-4-\mu$ points of $\{z_{m-2m_3+3}, \ldots, z_{m-m_3}\}$ lie on d_μ, $\mu = 1, \ldots, q-7$, and z_{m-m_3} lies on the intersection of d_{q-6} with $[v, w_1]$. It is obvious that $q^{[(l_0,1)]}(z_i) = 0$, $i = m - 2m_3 + 3, \ldots, m - m_3$. Moreover, it follows from (21) that $D^\omega q^{[(l_0,1)]}(v) = 0$, $\omega = 0, \ldots, 3$. By using arguments as in the proof of Theorem 6.5, we get $q^{[(l_0,1)]} = 0$, and $p^{[(l_0,1)]} = 0$.

The rest of the proof is similar to the proof of Theorem 6.5. This proves Theorem 6.6. ∎

Proof of Theorem 4.2: The proof is similar as the proof of Theorem 4.1 for Δ. The only difference is that we use Theorem 6.5 and Theorem 6.6 (if some triangle of Δ has to be subdivided) instead of Theorem 5.3. ∎

Next, we will prove Theorem 4.4. For doing this, we need Theorem 6.7 and Theorem 6.8. Let P, Δ^*, Ω^*, Δ_P^*, v, $w_0, \ldots, w_{\lambda+1}$ be defined as in the beginning of Section 5 and denote by $\mathcal{H}_P$ the union of sets chosen in Section 4 which correspond to Case 2a of Section 3.

Theorem 6.7 *Let $q \geq 5$. If $\mathcal{H}^*$ is a Hermite interpolation set for $S_q^2(\Delta^*)$, then $\mathcal{H} = \mathcal{H}^* \cup \mathcal{H}_P$ is a Hermite interpolation set for $S_q^2(\Delta_P^*)$.*

Proof: By our construction we have $\lambda \geq 2$. We first assume that $\lambda = 2$. Let a spline $s \in S_q^2(\Delta_P^*)$ which satisfies the homogenous interpolation conditions be given. We will show that $s = 0$. Since $\mathcal{H}^*$ is a Hermite interpolation set for $S_q^2(\Delta^*)$, we have $s|_{\Omega^*} = 0$. By Lemma 3.4, $a_{i,j,\rho}^{[l_0]} = 0$, $i + j = q - \rho$, $\rho = 0, 1, 2$, where $a_{i,j,k}^{[l_0]}$, $i + j + k = q$, are the coefficients of $p^{[l_0]} = s|_{T^{[l_0]}} \in \tilde{\Pi}_q$ in the representation (1), where $T^{[l_0]} = \Delta(v, w_0, w_1)$. As in the proof of Theorem 6.1, we get $a_{q-3,0,3}^{[l_0]} = 0$. We claim that

$$a_{\mu,j,q-\mu-j}^{[l_0]} = 0, \; j = 0, \ldots, q - 3 - \mu, \; \mu = 0, \ldots, q - 4. \tag{24}$$

We prove (24) by induction on μ and by using the homogeneous interpolation conditions at w_1. Let d be a unit vector in direction of the edge $[w_1, w_0]$. By (2) and the interpolation conditions $p_{d^j}^{[l_0]}(w_1) = 0$, $j = 0, \ldots, q - 3$, (24) holds for $\mu = 0$. We assume that

(24) holds for $\mu \in \{0, \ldots, \eta\}$, $\eta \leq q-5$, and show that (24) holds for $\eta+1$. Let d_1 be a unit vector in direction of the edge $e_1 = [w_1, v]$. It follows from (3) that (8) now holds for $j = 0, \ldots, q-4-\eta$. As in the proof of Theorem 5.4 the interpolation conditions $p_{d_1^{\eta+1} d^j}(w_1) = 0$, $j = 0, \ldots q-\eta-4$, imply that (24) holds for $\eta+1$. This shows (24). From this and $a_{i,j,k}^{[l_0]} = 0$, $i+j+k \in Q \setminus B_2$, we conclude $p^{[l_0]} = 0$. By Lemma 3.4, $a_{i,j,\rho}^{[l_1]} = 0$, $i+j = q-\rho$, $\rho = 0,1,2$, where $a_{i,j,k}^{[l_1]}$, $i+j+k = q$, are the coefficients of $p^{[l_1]} = s|_{T^{[l_1]}} \in \tilde{\Pi}_q$ in the representation (1), where $T^{[l_1]} = \Delta(v, w_1, w_2)$. As in the proof of Theorem 6.1, we get $a_{q-3,0,3}^{[l_1]} = 0$. Moreover, we can see analogously as (24) that the interpolation conditions of $p^{[l_1]}$ at w_2 imply

$$a_{\mu,j,q-\mu-j}^{[l_1]} = 0,\ j = 0, \ldots, q-3-\mu,\ \mu = 0, \ldots, q-5, \text{ and } a_{q-4,0,4}^{[l_1]} = 0.$$

By Lemma 3.5, $a_{q-4,1,3}^{[l_1]} = 0$. This shows that $p^{[l_1]} = 0$.

Since s is a C^2-spline the function values and all first and second derivatives of $p^{[l_2]} = s|_{T^{[l_2]}} \in \tilde{\Pi}_q$ vanish at e_2 and e_3. Thus,

$$D^\omega p^{[l_2]}(v) = 0,\ \omega = 0, \ldots, 5. \tag{25}$$

Let d_2 be a unit vector in direction of the edge $e_2 = [w_2, v]$. We claim that

$$p_{d_2^\mu}^{[l_2]}|_{[w_2,w_3]} = 0,\ \mu = 0, \ldots, q-6. \tag{26}$$

We prove (26) by induction on μ. Now, let d be a unit vector in direction of $[w_2, w_3]$. Since the function value and the first and second derivative in direction of d of $p^{[l_2]}|_{[w_2,w_3]} \in \Pi_q$ vanish at w_2 and w_3, it follows from the interpolation conditions of $p^{[l_2]}$ at $\overline{v} = \frac{1}{2}(w_2 + w_3)$ that the claim holds for $\mu = 0$. We assume that (26) holds for $\mu \in \{0, \ldots, \eta\}$, $\eta \leq q-7$, and show that (26) holds for $\eta+1$. By induction hypothesis and similar arguments as in the proof of Theorem 5.4 we obtain

$$p_{d_2^{\eta+1} d^\rho}^{[l_2]}(w_3) = 0,\ \rho = 0,1,2. \tag{27}$$

Moreover, $p_{d_2^{\eta+1} d^\rho}^{[l_2]}(w_2) = 0$, $\rho = 0,1,2$. Then it follows from (27) and the interpolation conditions of $p_{d_2^{\eta+1}}^{[l_2]} \in \tilde{\Pi}_{q-\eta-1}$ at $\overline{v} = \frac{1}{2}(w_2 + w_3)$, that $p_{d_2^{\eta+1}}^{[l_2]}|_{[w_2,w_3]} = 0$. This proves (26). From this and (25) we conclude $p^{[l_2]} = 0$, and $s = 0$. This proves the case $\lambda = 2$.

If $\lambda > 2$, we may assume that the edges $e_{\lambda-2}$, $e_{\lambda-1}$, e_λ and $e_{\lambda+1}$ have different slopes. As in the proof of (24) the interpolation conditions of $p^{[l_0]} = s_{T^{l_0]}} \in \tilde{\Pi}_q$ at w_1 imply that $a_{\mu,j,q-\mu-j}^{[l_0]} = 0$, $j = 0, \ldots, q-3-\mu$, $\mu = 0, \ldots, q-3$. Since s is a C^2-spline we get $p^{[l_0]} = 0$. By proceeding with these arguments we obtain $s|_{T^{[l_\nu]}} = p^{[l_\nu]} = 0$, $\nu = 0, \ldots, \lambda-3$, and $s = 0$. This proves Theorem 6.7. ■

The next result is needed for the case when some triangle has to be subdivided. Let $T^{[l_\nu]}$, $\tilde{P}$, y_ν, $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ be as in front of Theorem 6.3 and denote by $\mathcal{H}_{\tilde{P}}$ the union of sets chosen in Section 4 which correspond to Case 2b of Section 3.

Theorem 6.8 *Let $q \geq 5$. If $\mathcal{H}^*$ is a Hermite interpolation set for $S_q^2(\Delta^*)$, then $\mathcal{H} = \mathcal{H}^* \cup \mathcal{H}_{\tilde{P}}$ is a Hermite interpolation set for $S_q^2(\Delta^*_{\tilde{P}})$.*

Proof: Let us first assume that $\lambda = 1$. We may assume that $T^{[l_0]}$ is subdivided. Let a spline $s \in S_q^2(\Delta^*_{\tilde{P}})$, $q \geq 5$, which satisfies the homogenous interpolation conditions be given. We will show that $s = 0$. Since $\mathcal{H}^*$ is an interpolation set for $S_q^2(\Delta^*)$, it follows that $s|_{\Omega^*} = 0$. Let $p^{[(l_0,\sigma)]} = s|_{T^{[(l_0,\sigma)]}} \in \tilde{\Pi}_q$, $\sigma = 0, \ldots, 3$, be given in the representation (1) (cf. Case 2b of Section 3). As in the proof of Theorem 6.1, we get $a^{[(l_0,0)]}_{q-3,0,3} = 0$. Then as in the proof of Theorem 6.7 it follows from the interpolation conditions of $p^{[(l_0,0)]}$ at y_0 that $p^{[(l_0,0)]} = 0$. By Lemma 3.4, $a^{[(l_0,1)]}_{i,j,\rho} = 0$, $i + j = q - \rho$, $\rho = 0, 1, 2$. Moreover, $a^{[(l_0,1)]}_{0,q-3,3} = a^{[(l_0,1)]}_{q-3,0,3} = 0$.

Let us first consider the case $q \geq 6$. By (2) and the interpolation conditions of $p|^{[(l_0,1)]}_{[v,w_1]} \in \Pi_q$ at w_1 we get $p|^{[(l_0,1)]}_{[v,w_1]} = 0$, and $p|^{[(l_0,1)]}_{[y_0,w_1]} = 0$. By Lemma 3.5, $a^{[(l_0,1)]}_{q-4,1,3} = a^{[(l_0,1)]}_{1,q-4,3} = 0$. As in the proof of Theorem 6.7 it follows from (3) and the remaining interpolation conditions at w_1 that $p^{[(l_0,1)]} = 0$.

Now, we consider the case $q = 5$. Again, we have $p|^{[(l_0,1)]}_{[v,w_1]} = 0$. By Lemma 3.5, $a^{[(l_0,1)]}_{1,1,3} = a^{[(l_0,1)]}_{0,1,4} = 0$. Thus, $p^{[(l_0,1)]} = 0$.

As in the proof of Theorem 6.7 we obtain $p^{[(l_0,2)]} = 0$ and $p^{[(l_0,3)]} = 0$. This shows the case $\lambda = 1$. The rest of the proof is similar as the proof of Theorem 6.7. This proves Theorem 6.8. ■

Proof of Theorem 4.4: The proof is similar to the proof of Theorem 4.2 by using Theorem 6.7 instead of Theorem 6.5 and, if some triangle of Δ is subdivided, Theorem 6.8 instead of Theorem 6.6. ■

7 Final Remarks and Numerical Examples

We finally discuss some variants of our basic principle of constructing triangulations Δ and interpolation sets for $S_q^r(\Delta)$, $r = 1, 2$, which result from our numerical experience. Moreover, we give some numerical examples.

We first consider the spaces $S_q^1(\Delta)$. By applying the above interpolation methods, we obtain good approximations for $q \geq 4$.

We first note, that we may use the following variant in the iterative construction of the triangulation Δ if small angles appear at the boundary of the subtriangulation $\tilde{\Delta}$ constructed so far. If two adjacent boundary edges form a small angle we may connect these edges and use a Clough-Tocher split of the resulting triangle. Now, for $S_q^1(\Delta)$, $q \geq 3$, interpolation schemes can be constructed analogously as in Section 4.

In order to obtain good approximations in the case $q = 3$ for non-uniform triangulations Δ it is necessary to modify the triangulation Δ, i.e. to subdivide some of the triangles of the polyhedron added in each step as follows. If a polyhedron is added such that two neighboring triangles form a convex quadrangle, then we add the second diagonal if possible. Otherwise, we subdivide one of the triangles of the polyhedron by using a Clough-Tocher split. The corresponding admissible sets are shown in Figure 8. (the admissible points added in one step are marked by filled circles), and the interpolation sets can be defined analogously as in Section 4.

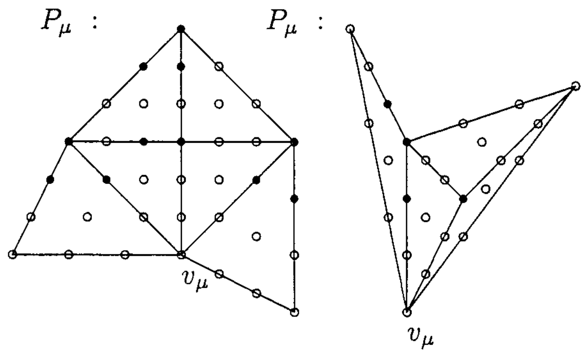

Figure 8. Admissible sets for $S_3^1(\Delta)$.

We finally consider the case $q = 2$. In this case, we consider triangulations Δ_Q of the following type. By starting with one triangle, we describe Δ_Q inductively as follows. Given a subtriangulation $\tilde{\Delta}_Q$, we add a triangle $\tilde{T}$ which has one common edge with $\tilde{\Delta}_Q$. Then in clockwise order, successively we add quadrangles (with two diagonals) having one common edge with $\tilde{\Delta}_Q$ and triangles having one common point with $\tilde{\Delta}_Q$, where the last quadrangle also has one common edge with $\tilde{T}$ (see Figure 9.). We denote the resulting subtriangulation again by $\tilde{\Delta}_Q$ and proceed with this method to obtain Δ_Q.

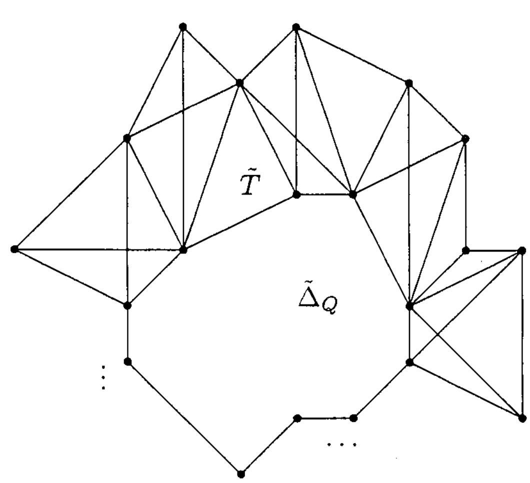

Figure 9. Construction of the triangulation Δ_Q.

In this case we obtain the admissible set shown in Figure 9. (the admissible points are marked by filled circles), since the intersection points of the diagonals of the quadrangles are singular (cf. [46]). We have the following result on interpolation by $S_2^1(\Delta_Q)$.

Theorem 7.1 *The vertices of Δ_Q (except the intersection points of the diagonals) together with three additional points in the starting triangle form a Lagrange interpolation set for $S_2^1(\Delta_Q)$.*

Theoretically, if we consider in Δ_Q instead of the quadrangles with two diagonals arbitrary quadrangles, then for the quadrangles with only one diagonal no interpolation point can be chosen. In this case, no good approximations can be expected, in general.

As a numerical test, we use our interpolation methods to approximate the test function of Franke

$$\begin{aligned} f(x,y) &= \frac{3}{4}e^{-\frac{(9x-2)^2+(9y-2)^2}{4}} + \frac{3}{4}e^{-\frac{(9x+1)^2}{49}-\frac{(9y+1)}{10}} \\ &\quad + \frac{1}{2}e^{-\frac{(9x-7)^2+(9y-3)^2}{4}} - \frac{1}{5}e^{-(9x-4)^2-(9y-7)^2}, \ (x,y) \in \mathbb{R}^2, \end{aligned}$$

by $S_q^1(\Delta)$, $q = 3, 4$, and $S_2^1(\Delta_Q)$. Here, Δ, respectively Δ_Q results from the above triangulation methods and the corresponding domain Ω contains $[0,1] \times [0,1]$. The results for the Hermite interpolating spline $s \in S_3^1(\Delta)$, respectively $s \in S_4^1(\Delta)$, are given in Table 1.

N	$dim\ S_3^1(\Delta)$	$\|f-s\|_\infty$	N	$dim\ S_4^1(\Delta)$	$\|f-s\|_\infty$
112	169	$3.31 * 10^{-2}$	32	131	$1.46 * 10^{-1}$
480	649	$1.03 * 10^{-2}$	211	652	$2.49 * 10^{-2}$
1984	2563	$1.24 * 10^{-3}$	745	2085	$1.30 * 10^{-3}$
8064	10224	$1.29 * 10^{-4}$	3257	8694	$1.33 * 10^{-4}$
32512	40725	$1.62 * 10^{-5}$	14495	38091	$7.80 * 10^{-6}$

Table 1. Interpolation by $S_q^1(\Delta)$, $q = 3, 4$.

Here, we note that there is some freedom in defining Hermite interpolation conditions. For example, we may only impose interpolation conditions at the vertices by replacing the condition C_1 for $r = 1$ as follows:

Condition C_1: $p_{d_1^\mu d_2^\nu}(v_3) = f_{d_1^\mu d_2^\nu}(v_3)$, $\mu = 0, \ldots, q-4$, $\nu = 2, \ldots, q-2-\mu$.

(Here, the unit vectors d_j, $j = 1, 2$, are chosen as in Section 4).

Table 2. contains our numerical results for the Lagrange interpolating spline $s \in S_2^1(\Delta_Q)$. Here, we use data which are rather uniformly distributed.

N	$dim\ S_2^1(\Delta_Q)$	$\|f-s\|_\infty$
24	16	$3.27 * 10^{-1}$
112	44	$1.51 * 10^{-1}$
480	148	$2.47 * 10^{-2}$
1984	548	$2.55 * 10^{-3}$
8064	2116	$2.68 * 10^{-4}$
32512	8324	$3.58 * 10^{-5}$
130560	33028	$5.10 * 10^{-6}$

Table 2. Interpolation by $S_2^1(\Delta_Q)$.

Numerical examples for $S_q^1(\hat{\Delta})$, $q = 2, \ldots, 7$, where $\hat{\Delta}$ is a given convex quadrangulation with diagonals, were given in [46].

Now, we consider the space $S_q^2(\Delta)$. By applying the above methods, we obtain good approximations for $q \geq 7$. We note that according to our numerical experience, for $q = 7$, it is advantageous to modify the admissible set (and the corresponding Hermite interpolation set) from the above sections as in Figure 10. (the admissible points added in one step are marked by filled circles).

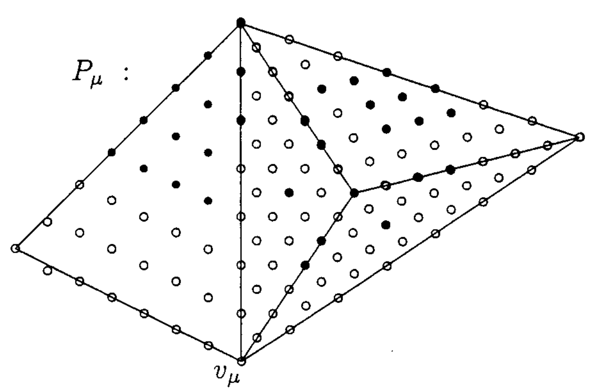

Figure 10. Admissible sets for $S_7^2(\Delta)$.

In order to obtain good approximations for $q = 6$, it is necessary to modify the triangulation Δ as follows. If in the construction of Δ a polyhedron P_μ is added with a triangle subdivided, then we also subdivide a neighboring triangle of P_μ. The corresponding admissible set is shown in Figure 11. (the admissible points added in one step are marked by filled circles), and the corresponding Hermite interpolation set can be defined analogously as in Section 4.

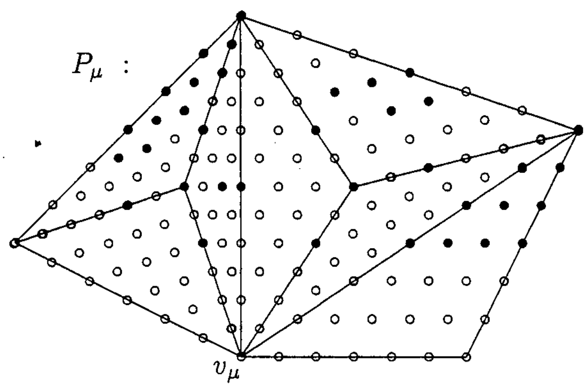

Figure 11. Admissible sets for $S_6^2(\Delta)$.

Again, we use our interpolation method to approximate the test function of Franke by $S_q^2(\Delta)$, $q = 6, 7$. The results for the Hermite interpolating spline $s \in S_6^2(\Delta)$, respectively $s \in S_7^2(\Delta)$, are given in Table 3.

N	$dim\ S_6^2(\Delta)$	$\|f - s\|_\infty$	N	$dim\ S_7^2(\Delta)$	$\|f - s\|_\infty$
72	418	$2.42 * 10^{-1}$	34	367	$5.44 * 10^{-1}$
567	2840	$3.87 * 10^{-3}$	333	2827	$5.61 * 10^{-2}$
2113	10204	$1.42 * 10^{-4}$	1325	10706	$1.18 * 10^{-4}$
9461	44996	$5.83 * 10^{-6}$	6073	48139	$5.30 * 10^{-6}$

Table 3. Interpolation by $S_q^2(\Delta)$, $q = 6, 7$.

Numerical examples for $S_q^2(\hat{\Delta})$, $q = 7, 8$, where $\hat{\Delta}$ is a given convex quadrangulation with diagonals, were given in [46]. Meanwhile we also computed examples for $S_6^2(\hat{\Delta})$ which give similar results.

Again, there is some freedom in defining Hermite interpolation conditions. For example, we may only impose interpolation conditions at the vertices by replacing the condition D_2 for $r = 2$ as follows:

Condition D_2: $p_{d_1^\mu d_2^\nu}(v_3) = f_{d_1^\mu d_2^\nu}(v_3)$, $\mu = 0, \ldots, q-6$, $\nu = 3, \ldots, q-3-\mu$.

(Here, the unit vectors d_j, $j = 1, 2$, are chosen as in Section 4).

We note that the complexity of the algorithm for computing the interpolating splines on the triangulation Δ is $\mathcal{O}(card\Delta)$.

After having written long computer programs for spline interpolation, we started with some tests on scattered data fitting. Let data be given at the vertices of a triangulation Δ constructed by our method. By using these data, we compute the interpolation conditions, needed for our spline method, approximatively by applying a local interpolation method for $\tilde{\Pi}_2$. With these approximative values, we compute splines from $S_q^r(\Delta)$, $r = 1, 2$ (See Table 4. and Table 5.).

N	$dim\ S_3^1(\Delta)$	$\|f-s\|_\infty$	N	$dim\ S_4^1(\Delta)$	$\|f-s\|_\infty$
112	169	$9.71 * 10^{-2}$	32	131	$2.67 * 10^{-1}$
480	649	$5.65 * 10^{-2}$	211	652	$1.48 * 10^{-1}$
1984	2563	$7.26 * 10^{-3}$	745	2085	$3.12 * 10^{-2}$
8064	10224	$2.22 * 10^{-3}$	3257	8694	$3.48 * 10^{-3}$
32512	40725	$3.87 * 10^{-4}$	14495	38091	$3.83 * 10^{-4}$

Table 4. Scattered data fitting by $S_q^1(\Delta)$, $q = 3, 4$.

N	$dim\ S_6^2(\Delta)$	$\|f-s\|_\infty$
72	418	$7.39 * 10^{-1}$
567	2840	$1.46 * 10^{-1}$
2113	10204	$3.75 * 10^{-2}$
9461	44996	$6.64 * 10^{-3}$

Table 5. Scattered data fitting by $S_6^2(\Delta)$.

Remark 7.2 Lagrange interpolation methods for $S_{2r+1}^r(\Delta)$, $r \geq 1$, were investigated by Gmelig Meyling & Pfluger [28] (see also Grandine [29]), where the solvability of the corresponding linear system has to be required. We also note that our interpolation methods are different from the finite element approach, where Hermite interpolation conditions are involved. In contrast to our method, all triangles of Δ have to be subdivided into at least three subtriangles while in our methods only some of the triangles have to be subdivided into three subtriangles. Moreover, there are no corresponding Lagrange interpolation schemes on Δ. For C^1-splines of degree $q = 2, 3$, there are the classical schemes of Clough & Tocher [15], Fraeijs de Veubeke and Sander [25, 51] (see also Lai [36]) and Powell & Sabin [48] on triangles, respectively quadrangles. For C^2-splines of degree $q = 5, 6, 7$, Alfeld [2], Gao [27], Laghchim-Lahlou and Sablonnière [34, 35], Sablonnière [50] and Wang [57] defined Hermite interpolation schemes of finite element type. We note that our Hermite interpolation schemes are different from those for $S_q^r(\Delta)$, $q \geq 3r + 2$ in Davydov, Nürnberger & Zeilfelder [21]. Quasi interpolation methods were developed by Chui & Hong [11, 12] for $S_4^1(\Delta)$ and by Lai & Schumaker [38] for $S_6^2(\Delta)$ (see also [39]) for certain classes of triangulations Δ.

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