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Abstract

Let Δ be a triangulation of some polygonal domain $\Omega \subset \mathbb{R}^2$ and let $S_q^r(\Delta)$ denote the space of all bivariate polynomial splines of smoothness r and degree q with respect to Δ. We present a Hermite type interpolation scheme for $S_q^r(\Delta)$, $q \geq 3r+2$, that possesses optimal approximation order $\mathcal{O}(h^{q+1})$. Furthermore, the fundamental functions of our scheme form a locally linearly independent basis for a superspline subspace of $S_q^r(\Delta)$.

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1 Introduction

Let $\Omega \subset \mathrm{R}^2$ be a polygonal domain, and let Δ denote a regular triangulation of Ω. The space of bivariate polynomial splines of degree q and smoothness r with respect to Δ is defined by

$$S_q^r(\Delta) := \{s \in C^r(\Omega) \,:\, s_{|_T} \in \Pi_q \quad \text{for all} \quad T \in \Delta\}, \quad 0 \leq r < q,$$

where

$$\Pi_q := \operatorname{span}\{x^i y^j \,:\, i \geq 0,\, j \geq 0,\, i + j \leq q\}$$

is the space of bivariate polynomials of total degree q.

In the literature, point sets that admit unique Lagrange and Hermite interpolation by spaces $S_q^r(\Delta)$ of splines of degree q and smoothness r were constructed for crosscut partitions Δ, in particular for Δ^1 and Δ^2-partitions [1, 7, 17, 22, 23, 24, 27, 28]. Results on the approximation order of these interpolation methods were given in [7, 13, 17, 21, 22, 25, 27, 28].

In the case of an abitrary triangulation Δ, the finite-element method provides a tool to construct Hermite type interpolation schemes for $S_q^r(\Delta)$ with optimal approximation order $\mathcal{O}(h^{q+1})$, where h is the maximal diameter of the triangles in Δ. However, as shown in [30], this technique only works if $q \geq 4r + 1$.

On the other hand, the approximation power of the spline space $S_q^r(\Delta)$ for $q \geq 3r + 2$ was studied in [4, 8, 9, 18]. Particularly, in [8, 18] it was shown that for a sufficiently smooth function f,

$$\operatorname{dist}(f, S_q^r(\Delta)) \leq K h^{q+1}, \tag{1.1}$$

where K is a constant that depends only on f, r, q and the smallest angle θ_Δ in Δ. (Great difficulties in the constructions and proofs of [8, 18] were caused by the desire to have this K independent on the geometry of Δ except the obviously unavoidable dependence on θ_Δ.) If $q < 3r + 2$, then the optimal approximation order fails for certain triangulations (see [5]).

In this paper we present a Hermite type interpolation scheme for $S_q^r(\Delta)$, $q \geq 3r+2$, that possesses optimal approximation order $\mathcal{O}(h^{q+1})$ in the same sense as in [8, 18], *i.e.*, the corresponding constant K does not depend on the geometric structure of Δ. Thus, we give a new proof of (1.1) that makes use of interpolation instead of quasi-interpolation methods developed in [8, 18]. The details of our construction are given in Section 2, whereas the main result of the paper, Theorem 3.1 about the approximation order, as well as its proof are presented in Section 3.

Let us emphasize that our technique is quite different from that of [8] and [18]. In each of these papers a stable local basis for a superspline subspace of $S_q^r(\Delta)$ was constructed first by using Bernstein-Bézier techniques, and then the basis functions were used to build up a quasi-interpolation operator that yielded the optimal approximation order. In contrast to this, we argue directly with nodal functionals, as it is common in the finite-element method. However, as mentioned above, the classical finite-element techniques could only work if $q \geq 4r+1$. In order to handle the case $q \geq 3r+2$, we had to develop a new approach that had its roots in the idea of "weak interpolation" introduced in [21] and further developed in [25] and [13]. Furthermore, we needed a new description of C^r smoothness across edges in terms of nodal functionals (see Lemma 3.2).

As a by-product of our construction, we get a nodal basis for the space of supersplines

$$S_q^{r,\rho}(\Delta) := \{s \in S_q^r(\Delta) \; : \; s \in C^\rho(v) \text{ for all vertices } v \text{ of } \Delta\},$$

where $\rho = r + \left[\frac{r+1}{2}\right]$ and $q \geq 3r+2$. The basis consists of the fundamental functions $s_1, \ldots, s_n$ of our interpolation scheme. Some properties of this basis are studied in Section 4. Namely, it is shown that $\{s_1, \ldots, s_n\}$ is *locally linearly independent* and thus *least supported*, i.e., the supports of the basis functions s_i are as small as possible, which is not the case for the basis functions constructed in [8, 18]. Moreover, we show that $\{s_1, \ldots, s_n\}$ is *stable* if Δ does not contain near-degenerate edges. (Although the basis is not stable in general, the norm of the interpolation operator $s_f : C^{2r}(\Omega) \to S_q^{r,\rho}(\Delta)$ of Section 3 is bounded by a constant that depends only on r, q and the smallest angle θ_Δ in Δ.) We note that there is some interrelation between our basis $\{s_1, \ldots, s_n\}$ and the basis for $S_q^{r,\rho}(\Delta)$ constructed in [16] by using Bernstein-Bézier techniques. Particularly, the supports of basis functions are the same. However, the minimal determining set of [16] cannot be transformed by standard Bernstein-Bézier arguments into a Hermite interpolation scheme of our type.

2 Nodal Functionals

Given a regular triangulation Δ, we denote by N the number of triangles, by V the number of vertices, by V_I and V_B the number of interior and boundary vertices respectively, $V_I + V_B = V$, by E the number of edges, and by E_I and E_B the number of interior and boundary edges respectively, $E_I + E_B = E$. It is well known that

$$
\begin{aligned}
E_B &= V_B\,, \\
E_I &= 3V_I + V_B - 3\,, \qquad (2.1)\\
N &= 2V_I + V_B - 2\,.
\end{aligned}
$$

In [16] it was shown that

$$
\begin{aligned}
\dim S_q^{r,\rho}(\Delta) &= \tbinom{\rho+2}{2}V + \left(\tbinom{q-3r-1}{2} - 3\tbinom{2r-\rho+1}{2}\right)N \\
&+ \frac{1}{2}(r+1)(2q-4\rho+r-2)E + \tbinom{2r-\rho+1}{2}\sigma\,,
\end{aligned}
\tag{2.2}
$$

with σ being the number of singular vertices of Δ, where a *singular vertex* v is a vertex which is formed by two lines which cross at v. It is easy to see that a vertex v is singular if and only if at least three edges are degenerate at v, where the degeneracy of an edge is defined as follows.

Definition 2.1 [2] Suppose e_1, e_2, e_3 are three consecutive edges attached to a vertex v. The edge e_2 is said to be *degenerate* at v whenever the edges e_1 and e_3 are collinear. An edge e attached to v is said to be *nondegenerate* at v if it is either a boundary edge or an interior edge which fails to be degenerate.

In the finite element method piecewise polynomial trial functions are usually determined by their values and derivatives at some points, so-called nodal values (see, e.g., [29, p. 101]). In [19, 12] and [26] this technique was applied to the study of spline spaces $S_q^1(\Delta)$, $q \geq 5$, and supersplines $S_q^{r,\rho}(\Delta)$ with $\rho \geq 2r$ and $q \geq 2\rho + 1$, respectively.

We set

$$
C^\mu(\Delta) := \{f \in C(\Omega) \;:\; f_{|_T} \in C^\mu(T) \text{ for all } T \in \Delta\}\,, \quad \mu = 0, 1, \ldots\,,
$$

and denote by D_τ the derivative operator in the direction of a unit vector $\tau = (\tau_x, \tau_y)$ in the plane, so that

$$
D_\tau f := \tau_x D_x f + \tau_y D_y f\,. \quad D_x f := \frac{\partial f}{\partial x}\,,\ D_y f := \frac{\partial f}{\partial y}\,.
$$

Definition 2.2 Given $f \in C^{\alpha+\beta}(\Delta)$, $\alpha, \beta \geq 0$, any number

$$
\nu f = D_{\tau_1}^{\alpha} D_{\tau_2}^{\beta}(f_{|_T})(z)\,, \tag{2.3}
$$

where $T \in \Delta$, $z \in T$, and τ_1, τ_2 are some unit vectors in the plane, is said to be a *nodal value* of f, and the linear functional $\nu : C^{\alpha+\beta}(\Delta) \to \mathrm{R}$ defined by (2.3) is a *nodal functional*, with $d(\nu) := \alpha + \beta$ being the *degree* of ν.

For some special choices of z, τ_1, τ_2 it is convenient to use the following simplified notation which goes back to [19]. 1) If v is a vertex of Δ and e is an edge attached to v, we set

$$D_e^\alpha f(v) := D_\tau^\alpha (f_{|_T})(v) , \quad \alpha \geq 1 ,$$

where τ is the unit vector in the direction of e away from v, and $T \in \Delta$ is one of the triangles with edge e. The notation is correct since in the case when there are two different triangles T_1, T_2 attached to e, $f_{|_{T_1}}$ and $f_{|_{T_2}}$ coincide along e, and hence

$$D_\tau^\alpha (f_{|_{T_1}})(v) = D_\tau^\alpha (f_{|_{T_2}})(v) .$$

2) If v is a vertex of Δ and e_1, e_2 are two consecutive edges attached to v, we set

$$D_{e_1}^\alpha D_{e_2}^\beta f(v) := D_{\tau_1}^\alpha D_{\tau_2}^\beta (f_{|_T})(v) , \quad \alpha, \beta \geq 1 ,$$

where $T \in \Delta$ is the triangle with vertex v and egdes e_1, e_2, and τ_i is the unit vector in the e_i direction away from v. 3) For every edge e of the triangulation Δ we choose a unit vector $\tau^\perp$ (one of two possible) orthogonal to e and set

$$D_{e^\perp}^\alpha f(z) := D_{\tau^\perp}^\alpha f(z) , \quad z \in e , \quad \alpha \geq 1 ,$$

provided $f \in C^\alpha(z)$.

We now associate with the superspline space $S_q^{r,\rho}(\Delta)$, with $q \geq 3r + 2$ and

$$\rho = r + \left[\frac{r+1}{2} \right] , \tag{2.4}$$

a set $\mathcal{N}$ of nodal functionals, as follows.

For every vertex v of Δ, let $T_v^1, \ldots, T_v^{n(v)}$ be all triangles attached to v and numbered counterclockwise (starting from a boundary triangle if v is a boundary vertex). Denote by e_i the common edge of T_v^{i-1} and T_v^i, $i = 2, \ldots, n(v)$. If v is an interior vertex, $e_1 = e_{n(v)+1}$ denote the common edge of T_v^1 and $T_v^{n(v)}$. Otherwise, e_1 and $e_{n(v)+1}$ are the boundary edges (attached to v) of T_v^1 and $T_v^{n(v)}$ respectively.

We define $\mathcal{N}(v)$ to be the set of nodal functionals assigning to every function $f \in C^\rho(v) \cap C^{2r}(\Delta)$ the following nodal values:

(v1) $D_x^\alpha D_y^\beta f(v)$ for all $(\alpha, \beta) \in A_1$, where

$$A_1 := \{(\alpha, \beta) \in \mathbb{Z}^2 \, : \, \alpha \geq 0, \;\; \beta \geq 0, \;\; \alpha + \beta \leq \rho\} ,$$

(v2) $D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} f(v)$ for all $(\alpha, \beta) \in A_2$, where

$$A_2 := \{(\alpha, \beta) \in \mathbb{Z}^2 \,:\, \alpha \le r,\ \beta \le r,\ \alpha + \beta \ge \rho + 1\},$$

and for each $i \in \{1, \ldots, n(v)\}$ such that e_i is nondegenerate at v,

(v3) $D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} f(v)$ for all $(\alpha, \beta) \in A_3$, where

$$A_3 := \{(\alpha, \beta) \in \mathbb{Z}^2 \,:\, \alpha \ge r + 1,\ 2\alpha + \beta \le 3r + 1.\ \alpha + \beta \ge \rho + 1\},$$

and for each $i \in \{1, \ldots, n(v)\}$ such that e_i is degenerate at v.

(v4) $D_{e_1}^{\alpha} D_{e_2}^{\beta} f(v)$ and $D_{e_{n(v)+1}}^{\alpha} D_{e_{n(v)}}^{\beta} f(v)$ for all $(\alpha, \beta) \in A_3$ if v is a boundary vertex, and

(v5) $D_{e_1}^{\alpha} D_{e_2}^{\beta} f(v)$ for all $(\alpha. \beta) \in A_2$ if v is a singular vertex.

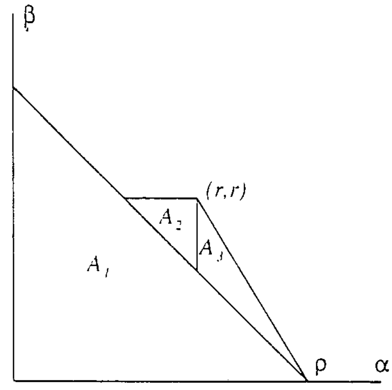

Fig. 2.1. The sets A_1, A_2 and A_3.

On every edge e of Δ, with vertices v_1 and v_2, we take points

(2.5) $$z_e^{\mu,i} := v_1 + \tfrac{i}{\kappa_\mu + 1}(v_2 - v_1), \quad i = 1, \ldots, \kappa_\mu, \quad \mu = 0, \ldots, r,$$

where

(2.6) $$\kappa_\mu := q - 3r - 1 - (r - \mu) \bmod 2 = q - 2r - 1 - \mu - 2\left[\tfrac{r+1-\mu}{2}\right],$$

and define $\mathcal{N}(e)$ to be the set of nodal functionals assigning to every function $f \in C^r(\Omega)$ the following nodal values:

(e) $D^{\mu}_{e^\perp} f(z_e^{\mu,1}), \ldots, D^{\mu}_{e^\perp} f(z_e^{\mu,\kappa_\mu})$ for all $\mu = 0, \ldots, r$.

In every triangle $T \in \Delta$, with vertices v_1, v_2 and v_3, we take uniformly spaced points

$$z_T^{i,j,k} := (iv_1 + jv_2 + kv_3)/q\,, \quad i + j + k = q\,, \tag{2.7}$$

and define $\mathcal{N}(T)$ to be the set of nodal functionals assigning to every function $f \in C(\Omega)$ the following nodal values:

(t) $f(z_T^{i,j,k})$ for all i, j, k such that $i + j + k = q$ and $r < i, j, k < q - 2r$.

We set

$$\mathcal{N} := \bigcup_v \mathcal{N}(v) \cup \bigcup_e \mathcal{N}(e) \cup \bigcup_T \mathcal{N}(T)\,.$$

Lemma 2.3 *We have*

$$\operatorname{card} \mathcal{N} = \dim S_q^{r,\rho}(\Delta)\,. \tag{2.8}$$

Proof. It is easy to see that

$$\begin{gathered} \operatorname{card} A_1 = \binom{\rho+2}{2}\,, \quad \operatorname{card} A_2 = \operatorname{card} A_3 = \binom{2r-\rho+1}{2}, \\ \operatorname{card} \mathcal{N}(e) = (r+1)(q-3r-1) - \left[\tfrac{r+1}{2}\right]\,, \quad \operatorname{card} \mathcal{N}(T) = \binom{q-3r-1}{2}\,. \end{gathered} \tag{2.9}$$

Therefore,

$$\begin{aligned} \operatorname{card} \mathcal{N} &= \binom{\rho+2}{2} V + \left(\binom{q-3r-1}{2} + 3\binom{2r-\rho+1}{2}\right) N + 2\binom{2r-\rho+1}{2} V_B \\ &+ \left((r+1)(q-3r-1) + r - \rho\right) E + \binom{2r-\rho+1}{2} \sigma\,. \end{aligned}$$

The lemma now follows from (2.1), (2.2) and a simple computation. ∎

3 Hermite Type Interpolation

Theorem 3.1 *Let $r \geq 1$, $q \geq 3r + 2$ and $\rho = r + \left[\frac{r+1}{2}\right]$. Given $f \in C^{2r}(\Omega)$, there exists a unique spline $s_f \in S_q^{r,\rho}(\Delta)$ satisfying the following Hermite type interpolation conditions*

$$\nu s_f = \nu f \quad \textit{for all} \quad \nu \in \mathcal{N}\,, \tag{3.1}$$

where $\mathcal{N}$ is defined above. Moreover, if $f \in C^m(\Omega)$ ($m \in \{2r, \ldots, q+1\}$) and $T \in \Delta$, then

$$\text{(3.2)} \qquad \|D_x^\alpha D_y^\beta (f - s_f)\|_{L_\infty(T)} \le K\, h_T^{m-\alpha-\beta} \max_{0\le\mu\le m} \|D_x^\mu D_y^{m-\mu} f\|_{C(T)},$$

for all $\alpha, \beta \ge 0$, $\alpha + \beta \le m$, where h_T is the diameter of T, and K is a constant which depends only on r, q and the smallest angle θ_Δ in Δ.

We will prove Theorem 3.1 at the end of this section, after establishing several lemmas.

In the first two lemmas we consider a simple triangulation consisting of two triangles and establish some relations between nodal values of two polynomials defined on each triangle and joined together with C^r smoothness across a common edge of the triangles.

Let T_1 and T_2 be two triangles sharing a common edge $e = [v_1, v_2]$, and let $e_i \ne e$ be the other edge of T_i with endpoint v_1, $i = 1, 2$. Denote by τ, τ_1, τ_2 the unit vectors applied at v_1 in the direction of edges e, e_1, e_2 respectively, and by θ_i the angle between τ and τ_i, $i = 1, 2$. (See Fig. 3.1.)

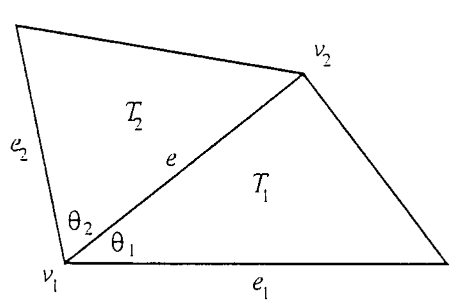

Fig. 3.1.

Furthermore, let s be a piecewise polynomial function on $T_1 \cup T_2$ such that

$$s|_{T_i} = p_i \in \Pi_q, \quad i = 1, 2.$$

Our first lemma characterizes C^r smoothness of s across e in terms of its nodal values.

Lemma 3.2 *Let $r \leq q$.*

1) If $\theta_1 + \theta_2 \neq \pi$, then $s \in C^r(T_1 \cup T_2)$ if and only if

$$\sin^\alpha\theta_1 \, D^\alpha_{\tau_2} D^{\gamma-\alpha}_{\tau} p_2(v_1) = \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \sin^{\alpha-\beta}(\theta_1+\theta_2) \sin^\beta\theta_2 \, D^\beta_{\tau_1} D^{\gamma-\beta}_{\tau} p_1(v_1), \tag{3.3}$$

for all $\alpha = 0, \dots, r$ and $\gamma = \alpha, \dots, q$.

2) If $\theta_1 + \theta_2 = \pi$, then $s \in C^r(T_1 \cup T_2)$ if and only if

$$D^\alpha_{\tau_2} D^{\gamma-\alpha}_{\tau} p_2(v_1) = (-1)^\alpha D^\alpha_{\tau_1} D^{\gamma-\alpha}_{\tau} p_1(v_1), \qquad \alpha = 0, \dots, r, \quad \gamma = \alpha, \dots, q\,. \tag{3.4}$$

Proof. Evidently, $s \in C^r(T_1 \cup T_2)$ if and only if for some unit vector τ' noncollinear with τ,

$$D^\mu_\tau D^\alpha_{\tau'} p_2(z) = D^\mu_\tau D^\alpha_{\tau'} p_1(z), \quad \text{for all } \alpha, \mu \geq 0, \; \alpha + \mu \leq r, \text{ and all } z \in e.$$

Since $(D^\alpha_{\tau'} p_i)|_e$, $i = 1, 2$, is a univariate polynomial of degree at most $q - \alpha$, this is equivalent to the condition

$$D^{\gamma-\alpha}_\tau D^\alpha_{\tau'} p_2(v_1) = D^{\gamma-\alpha}_\tau D^\alpha_{\tau'} p_1(v_1), \qquad \alpha = 0, \dots, r, \quad \gamma = \alpha, \dots, q.$$

We now choose $\tau' = \tau_2$. If $\theta_1 + \theta_2 = \pi$, then $\tau_1 = -\tau_2$, and we immediately get (3.4). Otherwise, if $\theta_1 + \theta_2 \neq \pi$, then the vectors τ, τ_1 and τ_2 stay in the relation

$$\tau \sin(\theta_1 + \theta_2) = \tau_1 \sin\theta_2 + \tau_2 \sin\theta_1\,,$$

which implies

$$\sin^\alpha\theta_1 \, D^\alpha_{\tau_2} D^{\gamma-\alpha}_{\tau} p_1(v_1) = \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \sin^{\alpha-\beta}(\theta_1+\theta_2) \sin^\beta\theta_2 \, D^\beta_{\tau_1} D^{\gamma-\beta}_{\tau} p_1(v_1),$$

and the first statement of the lemma follows. ■

Thus, the nodal values of $s \in C^r(T_1 \cup T_2)$ stay in relations (3.3). The same relations hold for every sufficiently smooth function f. By solving a linear system we can estimate some of the nodal values of $f - s$ at v_1 involved in (3.3) in terms of the others.

Lemma 3.3 *Suppose that s, as defined above, is in $C^r(T_1 \cup T_2)$, and let $f \in C^k(T_1 \cup T_2)$ for some $k \in \{\rho+1, \ldots, 2r\}$. If $\theta_1 + \theta_2 \neq \pi$, then for every $\beta = 2k-3r-1, \ldots, k-r-1$,*

$$
\begin{aligned}
|D^\beta_{\tau_1} D^{k-\beta}_\tau (f-p_1)(v_1)| \;\le\; & K \Bigg(\max_{0\le\alpha\le 2k-3r-2} |D^\alpha_{\tau_1} D^{k-\alpha}_\tau (f-p_1)(v_1)| \\
& + |\sin^{-r}(\theta_1+\theta_2)| \max_{\substack{k-r\le\alpha\le r \\ i=1,2}} |D^\alpha_{\tau_i} D^{k-\alpha}_\tau (f-p_i)(v_1)| \Bigg),
\end{aligned} \tag{3.5}
$$

where K depends only on r and θ_2.

Proof. Since $f \in C^k(v_1)$, we have

$$
\sin^\alpha\theta_1\, D^\alpha_{\tau_2} D^{k-\alpha}_\tau f(v_1) = \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \sin^{\alpha-\beta}(\theta_1+\theta_2) \sin^\beta\theta_2\, D^\beta_{\tau_1} D^{k-\beta}_\tau f(v_1)
$$

for all $\alpha = 0, \ldots, k$. This, together with (3.3), imply that

$$
a_{2,\alpha} = \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} a_{1,\beta}\,, \quad \alpha = 2k-3r-1, \ldots, r\,, \tag{3.6}
$$

where

$$
\begin{aligned}
a_{1,\beta} &:= \sin^{-\beta}(\theta_1+\theta_2) \sin^\beta\theta_2\, D^\beta_{\tau_1} D^{k-\beta}_\tau (f-p_1)(v_1)\,, \\
a_{2,\beta} &:= \sin^{-\beta}(\theta_1+\theta_2) \sin^\beta\theta_1\, D^\beta_{\tau_2} D^{k-\beta}_\tau (f-p_2)(v_1)\,.
\end{aligned}
$$

Consider (3.6) as a system

$$
Ax = b
$$

of $4r-2k+2$ linear equations in $4r-2k+2$ unknowns

$$
a_{i,\beta}\,, \quad \beta = 2k-3r-1, \ldots, k-r-1\,, \quad i = 1,2\,.
$$

Thus, we have

$$
x = (a_{1,2k-3r-1}, \ldots, a_{1,k-r-1}, a_{2,2k-3r-1}, \ldots, a_{2,k-r-1})^t,
$$

b is a $(4r-2k+2)$-vector whose components are some linear combinations of

$$
\begin{aligned}
& a_{1,\beta}\,, \quad \beta = 0, \ldots, 2k-3r-2, \quad \text{and} \\
& a_{i,\beta}\,, \quad \beta = k-r, \ldots, r, \quad i = 1,2,
\end{aligned}
$$

and

$$A = \begin{pmatrix} B & -I \\ C & O \end{pmatrix},$$

where

$$C = \left((-1)^{r+j} \binom{n+i}{n-m+j} \right)_{i,j=1}^{m}, \quad \text{with } n := k-r-1, \ m := 2r-k+1,$$

I is an $m \times m$ identity matrix, O is an $m \times m$ zero matrix, and B is a certain $m \times m$ matrix. Since the determinant of C is a nonzero constant multiple of

$$\det \left(\frac{1}{(m+i-j)!} \right)_{i,j=1}^{m} \neq 0,$$

A is nonsingular. Therefore,

$$\|x\|_\infty \leq \|A^{-1}\|_\infty \|b\|_\infty,$$

where $\|A^{-1}\|_\infty$ is bounded by a constant dependent only on r. Particularly, for all $\beta = 2k-3r-1, \dots, k-r-1$,

$$|a_{1,\beta}| \leq K_1 \left(\max_{0 \leq \alpha \leq 2k-3r-2} |a_{1,\alpha}| + \max_{k-r \leq \alpha \leq r} |a_{1,\alpha}| + \max_{k-r \leq \alpha \leq r} |a_{2,\alpha}| \right),$$

where K_1 depends only on r.

Recalling the definition of $a_{i,\beta}$, $i = 1, 2$, we obtain

$$\begin{aligned} |D_{\tau_1}^{\beta} D_{\tau}^{k-\beta}(f - p_1)(v_1)| &= |a_{1,\beta} \sin^{\beta}(\theta_1 + \theta_2) \sin^{-\beta}\theta_2| \\ \leq K_1 \Bigg(\max_{0 \leq \alpha \leq 2k-3r-2} &| \sin^{\beta-\alpha}(\theta_1 + \theta_2) \sin^{\alpha-\beta}\theta_2 | |D_{\tau_1}^{\alpha} D_{\tau}^{k-\alpha}(f - p_1)(v_1)| \\ &+ \max_{k-r \leq \alpha \leq r} | \sin^{\beta-\alpha}(\theta_1 + \theta_2) \sin^{\alpha-\beta}\theta_2 | |D_{\tau_1}^{\alpha} D_{\tau}^{k-\alpha}(f - p_1)(v_1)| \\ + \max_{k-r \leq \alpha \leq r} | \sin^{\beta-\alpha}&(\theta_1 + \theta_2) \sin^{\alpha}\theta_1 \sin^{-\beta}\theta_2 | |D_{\tau_2}^{\alpha} D_{\tau}^{k-\alpha}(f - p_2)(v_1)| \Bigg), \end{aligned}$$

and (3.5) follows. ■

We also need the following univariate "weak interpolation" lemma (compare [21, Remark 5ii] and [13, Lemma 4]).

Lemma 3.4 *Let $e \subset \mathbb{R}^2$ be an interval with endpoints v_1, v_2, and let $\mu \in \{0,\dots,r\}$ and $m \in \{r + \left[\frac{r+1-\mu}{2}\right], \dots, q+1-\mu\}$. Then for any $f \in C^m(e)$, any $p \in \Pi_{q-\mu}$ and every $\gamma = 0,\dots,m$,*

$$
\begin{aligned}
\|D_\tau^\gamma(f-p)\|_{C(e)} \;\le\; & K h^{-\gamma} \Bigg(h^m \|D_\tau^m f\|_{C(e)} + \max_{1\le i\le \kappa_\mu} |(f-p)(z_e^{\mu,i})| \\
(3.7) \qquad & + \max_{\substack{0\le\alpha\le r+\left[\frac{r+1-\mu}{2}\right] \\ i=1,2}} h^\alpha |D_\tau^\alpha (f-p)(v_i)| \Bigg),
\end{aligned}
$$

where h is the length of e, τ denotes the unit vector in the direction of e, $z_e^{\mu,i}$ and κ_μ are defined in (2.5) and (2.6), respectively, and K is a constant which depends only on q.

Proof. It is sufficient to consider the case $e = [0,h]$, i.e., $v_1 = (0,0)$, $v_2 = (0,h)$. Then $\tau = (1,0)$, $D_\tau = D_x$, $z_e^{\mu,i} = (\frac{ih}{\kappa_\mu+1}, 0)$, $i = 1,\dots,\kappa_\mu$.

Since $f \in C^m[0,h]$, we have

$$
(3.8) \qquad \|D_x^\gamma(f-\tilde{p})\|_{C[0,h]} \le \tfrac{h^{m-\gamma}}{(m-\gamma)!} \|D_x^m f\|_{C[0,h]}, \quad \gamma = 0,\dots,m,
$$

where $\tilde{p}$ is the (univariate) Taylor polynomial,

$$
\tilde{p}(x) := \sum_{\nu=0}^{m-1} \tfrac{D_x^\nu f(0)}{\nu!} x^\nu .
$$

Therefore,

$$
\begin{aligned}
\|D_x^\gamma(f-p)\|_{C[0,h]} &\le \|D_x^\gamma(f-\tilde{p})\|_{C[0,h]} + \|D_x^\gamma(\tilde{p}-p)\|_{C[0,h]} \\
&\le h^{m-\gamma}\|D_x^m f\|_{C[0,h]} + \|D_x^\gamma(\tilde{p}-p)\|_{C[0,h]};
\end{aligned}
$$

and we only need to estimate $\|D_x^\gamma(\tilde{p}-p)\|_{C[0,h]}$.

Let

$$
\lambda_\mu := r + \left[\tfrac{r+1-\mu}{2}\right].
$$

Since $\kappa_\mu + 2(\lambda_\mu + 1) = q - \mu + 1$, the following Hermite interpolation problem

$$
g(z_e^{\mu,i}) = a_i, \quad i = 1,\dots,\kappa_\mu, \quad D_x^\alpha g(v_j) = a_{j,\alpha}, \quad \alpha = 0,\dots,\lambda_\mu, \; j = 1,2,
$$

has a unique solution g among univariate polynomials of degree at most $q-\mu$, for any given data a_i, $i=1,\ldots,\kappa_\mu$, and $a_{j,\alpha}$, $\alpha=0,\ldots,\lambda_\mu$, $j=1,2$. Then

$$(\tilde{p}-p)(t)=\sum_{i=1}^{\kappa_\mu}(\tilde{p}-p)(z_e^{\mu,i})L_{i,h}(t)+\sum_{j=1,2}\sum_{\alpha=0}^{\lambda_\mu}D_x^\alpha(\tilde{p}-p)(v_j)L_{j,\alpha,h}(t), \quad t\in[0,h],$$

where $L_{i,h}$, $i=1,\ldots,\kappa_\mu$, and $L_{j,\alpha,h}$, $\alpha=0,\ldots,\lambda_\mu$, $j=1,2$, denote the fundamental polynomials of the above interpolation problem, *i.e.*, they are univariate polynomials of degree at most $q-\mu$, uniquely determined by the conditions

$$\begin{gathered}L_{i,h}(\tfrac{jh}{\kappa_\mu+1})=\delta_{i,j}, \quad i,j=1,\ldots,\kappa_\mu,\\ D_x^\alpha L_{i,h}(0)=D_x^\alpha L_{i,h}(h)=0, \quad \alpha=0,\ldots,\lambda_\mu, \quad i=1,\ldots,\kappa_\mu,\end{gathered}$$

and

$$\begin{aligned}&L_{i,\alpha,h}(\tfrac{jh}{\kappa_\mu+1})=0, \quad j=1,\ldots,\kappa_\mu, \ \alpha=0,\ldots,\lambda_\mu, \ i=1,2,\\ &D_x^\nu L_{1,\alpha,h}(0)=D_x^\nu L_{2,\alpha,h}(h)=\delta_{\alpha,\nu}, \quad \alpha,\nu=0,\ldots,\lambda_\mu,\\ &D_x^\nu L_{1,\alpha,h}(h)=D_x^\nu L_{2,\alpha,h}(0)=0, \quad \alpha,\nu=0,\ldots,\lambda_\mu,\end{aligned}$$

respectively. By a uniqueness argument, it is easy to check that

$$\begin{aligned}D_x^\gamma L_{i,h}(t) &= h^{-\gamma}D_x^\gamma L_{i,1}(\tfrac{t}{h}), \quad t\in[0,h],\\ D_x^\gamma L_{j,\alpha,h}(t) &= h^{\alpha-\gamma}D_x^\gamma L_{j,\alpha,1}(\tfrac{t}{h}), \quad t\in[0,h],\end{aligned}$$

and, consequently,

$$\begin{aligned}\|D_x^\gamma L_{i,h}\|_{C[0,h]} &= h^{-\gamma}\|D_x^\gamma L_{i,1}\|_{C[0,1]},\\ \|D_x^\gamma L_{j,\alpha,h}\|_{C[0,h]} &= h^{\alpha-\gamma}\|D_x^\gamma L_{j,\alpha,1}\|_{C[0,1]}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|D_x^\gamma(\tilde{p}-p)\|_{C[0,h]} \le& \sum_{i=1}^{\kappa_\mu}|(\tilde{p}-p)(z_e^{\mu,i})|\,h^{-\gamma}\|L_{i,1}\|_{C[0,1]}\\ &+\sum_{j=1,2}\sum_{\alpha=0}^{\lambda_\mu}|D_x^\alpha(\tilde{p}-p)(v_j)|\,h^{\alpha-\gamma}\|L_{j,\alpha,1}\|_{C[0,1]},\end{aligned}$$

Since $\tilde{p} - p = (\tilde{p} - f) + (f - p)$, (3.8) implies

$$\begin{aligned} |(\tilde{p} - p)(z_e^{\mu,i})| &\leq h^m \| D_x^m f \|_{C[0,h]} + |(f - p)(z_e^{\mu,i})|, \\ |D_x^\alpha(\tilde{p} - p)(v_j)| &\leq h^{m-\alpha} \| D_x^m f \|_{C[0,h]} + |D_x^\alpha(f - p)(v_j)|, \end{aligned}$$

and the lemma follows because $\|L_{i,1}\|_{C[0,1]}$ and $\|L_{j,\alpha,1}\|_{C[0,1]}$ are bounded by a constant dependent only on q. ■

Since our interpolation scheme is based on nodal values involving partial derivatives in various directions, we need a tool to recast the (weak) interpolation conditions in such a form that their interaction becomes tractable. As a "common unit" we will use derivatives of the type $D_e^\gamma D_{e^\perp}^\mu (f - s)$. The next two lemmas provide estimations of these derivatives in terms of nodal values of our scheme.

Consider first a single triangle $T_1 \in \Delta$, and let e be one of its edges, with vertices v_1 and v_2. (Note that e may be a boundary edge of Δ.) Denote by $e_{1,1}$ and $e_{1,2}$ two other edges of T_1, attached to v_1 and v_2, respectively, and by $\theta_{1,i}$ the angle between e and $e_{1,i}$, $i = 1, 2$. (See Fig. 3.2.)

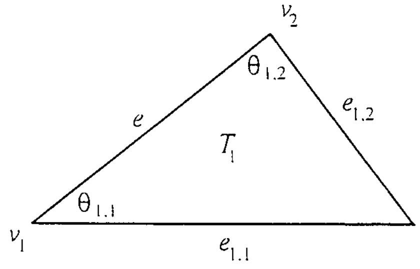

Fig. 3.2.

Lemma 3.5 *Let $s \in S_q^{r,\rho}(\Delta)$ and $f \in C^m(\Omega)$ ($m \in \{2r, \ldots, q+1\}$) be given. Then for all $\mu = 0, \ldots, r$ and $\gamma = 0, \ldots, m - \mu$,*

$$\begin{aligned}\|D_e^\gamma D_{e^\perp}^\mu (f-s)\|_{C(e)} &\le Kh^{-\gamma-\mu}\Bigg(h^m \max_{0\le\mu'\le\mu} \|D_e^{m-\mu'} D_{e^\perp}^{\mu'} f\|_{C(e)} \\ &+ \max_{\substack{0\le\mu'\le\mu\\ 0\le i\le\kappa_{\mu'}}} h^{\mu'} |D_{e^\perp}^{\mu'}(f-s)(z_e^{\mu',i})| \\ &+ \max_{\substack{(\alpha,\beta)\in A_1\cup A_2\cup A_3\\ \beta\le\mu,\ i=1,2}} h^{\alpha+\beta} |D_e^\alpha D_{e_{1,i}}^\beta (f-s)(v_i)|\Bigg),\end{aligned} \tag{3.9}$$

where h is the length of e, the sets A_1–A_3 are defined above, and K depends only on q and $\min\{\theta_{1,1},\theta_{1,2}\}$.

Proof. Since

$$(\alpha,\mu)\in A_1\cup A_2\cup A_3 \iff 0\le\alpha\le r+\left[\tfrac{r+1-\mu}{2}\right], \quad \mu=0,\dots,r,$$

Lemma 3.4 shows that there exists a constant K_1 dependent only on q, such that

$$\begin{aligned}\|D_e^\gamma(f-s)\|_{C(e)} &\le K_1 h^{-\gamma}\Bigg(h^m\|D_e^m f\|_{C(e)} + \max_{0\le i\le\kappa_\mu} |(f-s)(z_e^{0,i})| \\ &+ \max_{\substack{(\alpha,0)\in A_1\cup A_2\cup A_3\\ i=1,2}} h^\alpha |D_e^\alpha(f-s)(v_i)|\Bigg), \quad \gamma=0,\dots,m,\end{aligned}$$

which proves (3.9) for $\mu=0$. Proceeding by induction on μ, we suppose that (3.9) holds for $0,\dots,\mu-1$. Again by Lemma 3.4, applied to $D_{e^\perp}^\mu f\in C^{m-\mu}(e)$ and $p=D_{e^\perp}^\mu s$, we get for all $\gamma=0,\dots,m-\mu$,

$$\begin{aligned}\|D_e^\gamma D_{e^\perp}^\mu(f-s)\|_{C(e)} &\le K_1 h^{-\gamma}\Bigg(h^{m-\mu}\|D_e^{m-\mu} D_{e^\perp}^\mu f\|_{C(e)} + \max_{0\le i\le\kappa_\mu} |D_{e^\perp}^\mu (f-s)(z_e^{\mu,i})| \\ &+ \max_{\substack{(\alpha,\mu)\in A_1\cup A_2\cup A_3\\ i=1,2}} h^\alpha |D_e^\alpha D_{e^\perp}^\mu (f-s)(v_i)|\Bigg).\end{aligned}$$

Thus, we need to estimate $D_e^\alpha D_{e^\perp}^\mu(f-s)(v_i)$ in terms of $D_e^\alpha D_{e_{1,i}}^\beta(f-s)(v_i)$ with $\beta\le\mu$. To this end, we use the relation

$$\tau_{1,i} = \pm\tau\cos\theta_{1,i} \pm \tau^\perp \sin\theta_{1,i}, \quad i=1,2,$$

where $\tau_{1,i}$, τ and $\tau^\perp$ are the unit vectors in the directions of $e_{1,i}$, e and $e^\perp$ respectively, so that

$$(3.10)\quad D_e^\alpha D_{e_{1,i}}^\mu (f-s)(v_i) = \sum_{\mu'=0}^{\mu} \pm \binom{\mu}{\mu'} \cos^{\mu-\mu'}\theta_{1,i} \sin^{\mu'}\theta_{1,i} D_e^{\alpha+\mu-\mu'} D_{e^\perp}^{\mu'} (f-s)(v_i),$$

and hence,

$$\begin{aligned} |D_e^\alpha D_{e^\perp}^\mu (f-s)(v_i)| \;\le\; & \left| D_e^\alpha D_{e_{1,i}}^\mu (f-s)(v_i) \right| \\ & + K_2 \max_{0\le\mu'\le\mu-1} \left| D_e^{\alpha+\mu-\mu'} D_{e^\perp}^{\mu'} (f-s)(v_i) \right|, \quad i=1,2, \end{aligned}$$

where K_2 depends only on μ and $\min\{\theta_{1,1}, \theta_{1,2}\}$. Furthermore, by the induction hypothesis,

$$\begin{aligned} \left| D_e^{\alpha+\mu-\mu'} D_{e^\perp}^{\mu'} (f-s)(v_i) \right| \;\le\; & K h^{-\alpha-\mu} \Bigg(h^m \max_{0\le\mu''\le\mu'} \| D_e^{m-\mu''} D_{e^\perp}^{\mu''} f \|_{C(e)} \\ & + \max_{\substack{0\le\mu''\le\mu' \\ 0\le i\le\kappa_{\mu''}}} h^{\mu''} | D_{e^\perp}^{\mu''} (f-s)(z_e^{\mu'',i}) | \\ & + \max_{\substack{(\alpha',\beta)\in A_1\cup A_2\cup A_3 \\ \beta\le\mu',\; i=1,2}} h^{\alpha'+\beta} | D_e^{\alpha'} D_{e_{1,i}}^{\beta} (f-s)(v_i) | \Bigg), \end{aligned}$$

and (3.9) follows. ■

Under the notations of Lemma 3.5, suppose that e is an interior edge of Δ and denote by T_2 the triangle in Δ that share e with T_1. Let $e_{2,1}$ and $e_{2,2}$ be two other edges of T_2, attached to v_1 and v_2, respectively, and let $\theta_{2,i}$ be the angle between e and $e_{2,i}$, $i = 1, 2$. (See Fig. 3.3.)

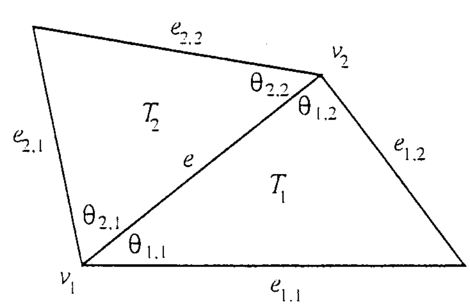

Fig. 3.3.

Furthermore, let h denote the length of e.

Lemma 3.6 *Let $s \in S_q^{r,\rho}(\Delta)$ and $f \in C^m(\Omega)$ ($m \in \{2r, \ldots, q+1\}$) be given.*
1) If $\theta_{1,2} + \theta_{2,2} \neq \pi$, then for all $\mu = 0, \ldots, r$ and $\gamma = 0, \ldots, m-\mu$.

$$
\begin{aligned}
&\|D_e^\gamma D_{e^\perp}^\mu (f-s)\|_{C(e)} \le K h^{-\gamma-\mu} \Bigg(h^m \max_{0\le\mu'\le\mu} \|D_e^{m-\mu'} D_{e^\perp}^{\mu'} f\|_{C(e)} \\
&+ \max_{\substack{0\le\mu'\le\mu \\ 0\le i\le\kappa_{\mu'}}} h^{\mu'} |D_{e^\perp}^{\mu'}(f-s)(z_e^{\mu',i})| + \max_{\substack{(\alpha,\beta)\in A_1\cup A_2\cup A_3 \\ \beta\le\mu}} h^{\alpha+\beta} |D_e^\alpha D_{e_{1,1}}^\beta (f-s)(v_1)| \\
&+ \max_{(\alpha,\beta)\in A_1,\ \beta\le\mu} h^{\alpha+\beta} |D_e^\alpha D_{e_{1,2}}^\beta (f-s)(v_2)| \\
&+ |\sin^{-r}(\theta_{1,2}+\theta_{2,2})| \max_{(\alpha,\beta)\in A_2,\ i=1,2} h^{\alpha+\beta} |D_e^\alpha D_{e_{i,2}}^\beta (f-s)(v_2)| \Bigg).
\end{aligned} \tag{3.11}
$$

where K depends only on q and $\min\{\theta_{1,1}, \theta_{1,2}, \theta_{2,2}\}$.
2) If both $\theta_{1,1} + \theta_{2,1} \neq \pi$ and $\theta_{1,2} + \theta_{2,2} \neq \pi$, then for all $\mu = 0, \ldots, r$ and $\gamma = 0, \ldots, m-\mu$.

$$
\begin{aligned}
&\|D_e^\gamma D_{e^\perp}^\mu (f-s)\|_{C(e)} \le K h^{-\gamma-\mu} \Bigg(h^m \max_{0\le\mu'\le\mu} \|D_e^{m-\mu'} D_{e^\perp}^{\mu'} f\|_{C(e)} \\
&+ \max_{\substack{0\le\mu'\le\mu \\ 0\le i\le\kappa_{\mu'}}} h^{\mu'} |D_{e^\perp}^{\mu'}(f-s)(z_e^{\mu',i})| + \max_{\substack{(\alpha,\beta)\in A_1,\ \beta\le\mu \\ j=1,2}} h^{\alpha+\beta} |D_e^\alpha D_{e_{1,j}}^\beta (f-s)(v_j)| \\
&+ \max_{j=1,2} |\sin^{-r}(\theta_{1,j}+\theta_{2,j})| \max_{(\alpha,\beta)\in A_2,\ i=1,2} h^{\alpha+\beta} |D_e^\alpha D_{e_{i,j}}^\beta (f-s)(v_j)| \Bigg),
\end{aligned} \tag{3.12}
$$

where K depends only on q and $\min\{\theta_{1,1}, \theta_{1,2}, \theta_{2,1}, \theta_{2,2}\}$.

Proof. 1) The essential difference between (3.11) and the already established inequality (3.9) is that the terms

$$h^{\alpha+\beta}|D_e^\alpha D_{e_{1,2}}^\beta (f-s)(v_2)|, \qquad (\alpha,\beta)\in A_3, \quad \beta\le\mu;$$

in the right hand side of (3.9) are substituted by

$$|\sin^{-r}(\theta_{1,2}+\theta_{2,2})|\, h^{\alpha+\beta}|D_e^\alpha D_{e_{i,2}}^\beta (f-s)(v_2)|, \qquad (\alpha,\beta)\in A_2, \quad i=1,2.$$

If $\mu = 0$, then $\{(\alpha,\beta)\in A_3 : \beta\le\mu\} = \emptyset$, and (3.11) is a straightforward consequence of (3.9). Moreover, in order to perform induction on μ, we only need an estimation of the form

$$
(3.13)\quad \begin{aligned} &\max_{\substack{(\alpha,\beta)\in A_3\\ \beta\le\mu}} h^{\alpha+\beta}|D_e^\alpha D_{e_{1,2}}^\beta (f-s)(v_2)| \le K_1 \Bigg(\max_{\substack{0\le\mu'\le\mu-1\\ 0\le\gamma\le m-\mu'}} h^{\gamma+\mu'} \|D_e^\gamma D_{e^\perp}^{\mu'}(f-s)\|_{C(e)} \\ &+ |\sin^{-r}(\theta_{1,2}+\theta_{2,2})| \max_{(\alpha,\beta)\in A_2,\ i=1,2} h^{\alpha+\beta}|D_e^\alpha D_{e_{i,2}}^\beta (f-s)(v_2)| \Bigg). \end{aligned}
$$

To this end we employ Lemma 3.3, which gives for all $(\alpha,\beta)\in A_3$, with $\beta\le\mu$,

$$
\begin{aligned} &|D_e^\alpha D_{e_{1,2}}^\beta (f-s)(v_2)| \le K_2 \Bigg(\max_{0\le\beta'\le 2(\alpha+\beta)-3r-2} |D_e^{\alpha+\beta-\beta'} D_{e_{1,2}}^{\beta'}(f-s)(v_2)| \\ &+ |\sin^{-r}(\theta_{1,2}+\theta_{2,2})| \max_{\alpha+\beta-r\le\beta'\le r,\ i=1,2} |D_e^{\alpha+\beta-\beta'} D_{e_{i,2}}^{\beta'}(f-s)(v_2)| \Bigg). \end{aligned}
$$

Since

$$
\beta' \le 2(\alpha+\beta)-3r-2 \Longrightarrow \beta' \le \mu-1,
$$

we obtain, by making use of (3.10),

$$
\begin{aligned} &\max_{0\le\beta'\le 2(\alpha+\beta)-3r-2} |D_e^{\alpha+\beta-\beta'} D_{e_{1,2}}^{\beta'}(f-s)(v_2)| \le \max_{0\le\mu'\le\mu-1} |D_e^{\alpha+\beta-\mu'} D_{e_{1,2}}^{\mu'}(f-s)(v_2)| \\ &\le K_3 \max_{0\le\mu'\le\mu-1} |D_e^{\alpha+\beta-\mu'} D_{e^\perp}^{\mu'}(f-s)(v_2)| \le K_3 \max_{0\le\mu'\le\mu-1} \|D_e^{\alpha+\beta-\mu'} D_{e^\perp}^{\mu'}(f-s)\|_{C(e)}. \end{aligned}
$$

Furthermore, since

$$
\alpha+\beta-r\le\beta'\le r \Longrightarrow (\alpha+\beta-\beta',\beta')\in A_2,
$$

we have

$$
\max_{\alpha+\beta-r\le\beta'\le r,\ i=1,2} |D_e^{\alpha+\beta-\beta'} D_{e_{i,2}}^{\beta'}(f-s)(v_2)| \le \max_{\substack{(\alpha',\beta')\in A_2\\ \alpha'+\beta'=\alpha+\beta,\ i=1,2}} |D_e^{\alpha'} D_{e_{i,2}}^{\beta'}(f-s)(v_2)|,
$$

and (3.13) follows.

2) This part can be established by exactly the same arguments, the only difference being that the terms

$$
h^{\alpha+\beta}|D_e^\alpha D_{e_{1,1}}^\beta (f-s)(v_1)|, \qquad (\alpha,\beta)\in A_3, \quad \beta\le\mu,
$$

now also have to be estimated. ■

Let $T \in \Delta$ and let v be a vertex of T. Then $T = T_v^i$ for some $i \in \{1, \dots, n(v)\}$, where $T_v^1, \dots, T_v^{n(v)}$ are all triangles attached to v and numbered counterclockwise, as in the definition of $\mathcal{N}(v)$ (see Section 2). We are going to define various subsets of $\mathcal{N}(v)$ and $\mathcal{N}$ that will be instrumental in the proof of Theorem 3.1 and the key Lemma 3.8.

We define $\mathcal{N}_T(v) \subset \mathcal{N}(v)$ to be the set of nodal functionals corresponding to the following nodal values:

(vt1) $D_x^\alpha D_y^\beta f(v)$ for all $(\alpha, \beta) \in A_1$,

(vt2) $D_{e_i}^\alpha D_{e_{i+1}}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if e_i is nondegenerate at v, or

$D_{e_{i-1}}^\alpha D_{e_i}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if e_i is degenerate at v, but e_{i-1} is nondegenerate at v, or

$D_{e_{i-2}}^\alpha D_{e_{i-1}}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if both e_i and e_{i-1} are degenerate at v, but e_{i-2} is nondegenerate at v, or

$D_{e_1}^\alpha D_{e_2}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if v is a singular vertex,

(vt3) $D_{e_{i+1}}^\alpha D_{e_{i+2}}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if e_{i+1} is a nondegenerate at v interior edge, or

$D_{e_{i+1}}^\alpha D_{e_{i+2}}^\beta f(v)$ for all $(\alpha, \beta) \in A_3$ if e_{i+1} is degenerate at v, or

$D_{e_{i+1}}^\alpha D_{e_i}^\beta f(v)$ for all $(\alpha, \beta) \in A_3$ if e_{i+1} is a boundary edge, and

(vt4) $D_{e_i}^\alpha D_{e_{i+1}}^\beta f(v)$ for all $(\alpha, \beta) \in A_3$ if either e_i is degenerate at v or e_i is a boundary edge, or

$D_{e_{i-1}}^\alpha D_{e_i}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if both e_i and e_{i-1} are nondegenerate at v, or

$D_{e_{i-2}}^\alpha D_{e_{i-1}}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if e_i is nondegenerate at v, e_{i-1} is degenerate at v, and e_{i-2} is again nondegenerate at v, or

$D_{e_{i-3}}^\alpha D_{e_{i-2}}^\beta f(v)$ for all $(\alpha, \beta) \in A_2$ if e_i is nondegenerate at v, both e_{i-1} and e_{i-2} are degenerate at v, and e_{i-3} is nondegenerate at v.

Furthermore, denote by

$$\mathcal{N}_{T,j}(v) \subset \mathcal{N}_T(v), \quad j = 1, 2, 3, 4,$$

the set of functionals corresponding to the nodal values listed in (vt1), (vt2), (vt3) and (vt4) respectively.

We also define

$$\widetilde{\mathcal{N}}_T(v) \subset \mathcal{N}_T(v)$$

as follows: if each of two edges e_i and e_{i+1} is either degenerate or lies on the boundary, then $\widetilde{\mathcal{N}}_T(v) := \emptyset$, if e_{i+1} is an interior nondegenerate at v edge, but e_i is not, then $\widetilde{\mathcal{N}}_T(v) := \mathcal{N}_{T,2}(v) \cup \mathcal{N}_{T,3}(v)$, if, conversely, e_i is an interior nondegenerate at v edge, but e_{i+1} is not, then $\widetilde{\mathcal{N}}_T(v) := \mathcal{N}_{T,2}(v) \cup \mathcal{N}_{T,4}(v)$, and, finally, if both e_i and e_{i+1} are interior nondegenerate at v edges, then $\widetilde{\mathcal{N}}_T(v) := \mathcal{N}_{T,2}(v) \cup \mathcal{N}_{T,3}(v) \cup \mathcal{N}_{T,4}(v)$.

For every triangle $T \in \Delta$ with vertices v_1, v_2, v_3 and edges e_1, e_2, e_3, let

$$\begin{aligned} \mathcal{N}_T &:= \bigcup_{i=1}^{3} \mathcal{N}_T(v_i) \cup \bigcup_{i=1}^{3} \mathcal{N}(e_i) \cup \mathcal{N}(T), \\ \widetilde{\mathcal{N}}_T &:= \bigcup_{i=1}^{3} \widetilde{\mathcal{N}}_T(v_i). \end{aligned}$$

Finally, we need a set of nodal functionals $\mathcal{N}_T^*$ of finite-element type. Let $T \in \Delta$, let v be a vertex of T, and let the edges e_1, e_2 of T be attached to v. Then $\mathcal{N}_T^*(v)$ is defined to be the set of nodal functionals corresponding to the nodal values

$$D_{e_1}^{\alpha} D_{e_2}^{\beta} f(v), \text{ for all } (\alpha, \beta) \in A_1 \cup A_2 \cup A_3 \cup \widetilde{A}_3,$$

where

$$\widetilde{A}_3 := \{(\alpha, \beta) \in \mathbb{Z}^2 : (\beta, \alpha) \in A_3\}.$$

Furthermore, for every edge e of T we define $\mathcal{N}_T^*(e)$ to be the set of nodal functionals assigning to every function $f \in C^r(\Omega)$ the following nodal values:

$$D_{\tau'}^{\mu} f(z_e^{\mu,1}), \ldots, D_{\tau'}^{\mu} f(z_e^{\mu,\kappa_\mu}) \quad \text{for all} \quad \mu = 0, \ldots, r,$$

where $z_e^{\mu,i}$ and κ_μ are defined in (2.5) and (2.6) respectively, and τ' is the unit vector in the direction from the middle point of e to the vertex of T opposite to e.

For every triangle $T \in \Delta$ with vertices v_1, v_2, v_3 and edges e_1, e_2, e_3, we set

$$\mathcal{N}_T^* := \bigcup_{i=1}^{3} \mathcal{N}_T^*(v_i) \cup \bigcup_{i=1}^{3} \mathcal{N}_T^*(e_i) \cup \mathcal{N}(T).$$

Lemma 3.7 *We have*

$$\operatorname{card}\mathcal{N}_T = \operatorname{card}\mathcal{N}_T^* = \binom{q+2}{2}. \tag{3.14}$$

Moreover, $\mathcal{N}_T^*$ is Π_q-unisolvent, i.e., for any real data a_ν, $\nu \in \mathcal{N}_T^*$, there exists a unique polynomial $p \in \Pi_q$ such that $\nu p = a_\nu$ for all $\nu \in \mathcal{N}_T^*$.

Proof. Obviously,

$$\operatorname{card}\mathcal{N}_T^* = 3\operatorname{card}A_1 + 3\operatorname{card}A_2 + 6\operatorname{card}A_3 + 3\operatorname{card}\mathcal{N}(e) + \operatorname{card}\mathcal{N}(T).$$

By (2.9) and some elementary computation, we obtain $\operatorname{card}\mathcal{N}_T^* = \binom{q+2}{2}$. Furthermore,

$$\operatorname{card}\mathcal{N}_T(v) = \sum_{j=1}^{4}\operatorname{card}\mathcal{N}_{T,j}(v) = \operatorname{card}A_1 + 3\operatorname{card}A_2 = \operatorname{card}\mathcal{N}_T^*(v).$$

Hence, $\operatorname{card}\mathcal{N}_T = \operatorname{card}\mathcal{N}_T^*$, which proves (3.14). Particularly, $\operatorname{card}\mathcal{N}_T^* = \dim\Pi_q$. Because of this, the second statement of the lemma will follow if we show that the only polynomial satisfying $\nu p = 0$ for all $\nu \in \mathcal{N}_T^*$ is the zero function. Following the lines of the proof of Lemma 3.5, with $f \equiv 0$ and $s = p$, we get

$$\|D_{e_i}^{\gamma} D_{e_i^\perp}^{\mu} p\|_{C(e_i)} = 0, \quad i = 1, 2, 3,$$

for all $\mu = 0, \ldots, r$ and $\gamma = 0, \ldots, q - \mu$, and every edge e_i of T. Therefore,

$$p = (l_1 l_2 l_3)^{r+1}\tilde{p},$$

where l_1, l_2 and l_3 are linear polynomials such that $e_i \subset \{(x,y) : l_i(x,y) = 0\}$, and $\tilde{p}$ is a polynomial in Π_{q-3r-3}. Then $\nu\tilde{p} = 0$, for all $\nu \in \mathcal{N}(T)$. Since $\mathcal{N}(T)$ is Π_{q-3r-3}-unisolvent, we have $\tilde{p} = 0$, and hence, $p = 0$. ∎

We also need some local geometric characteristics of the triangulation.

Let e be any interior edge of the triangulation Δ, and let v and v' be its vertices. Denote by e_1 and e_2 the adjacent edges of e at v, and by θ_i the angle between e and e_i, $i = 1, 2$. We set

$$\theta_{e,v} := \min\{\theta_1, \theta_2\}, \quad \tilde{\theta}_{e,v} := |\pi - \theta_1 - \theta_2|.$$

If e is a boundary edge, then $\theta_{e,v}$ denotes the angle between e and its unique adjacent edge at v. Furthermore,

$$\theta_e := \min\{\theta_{e,v}, \theta_{e,v'}\},$$

and for an interior edge e,

$$\tilde{\theta}_e := \begin{cases} \tilde{\theta}_{e,v}, & \text{if } e \text{ is degenerate at } v', \\ \min\{\tilde{\theta}_{e,v}, \tilde{\theta}_{e,v'}\}, & \text{if } e \text{ is nondegenerate at both } v \text{ and } v'. \end{cases}$$

(We note that no edge can be degenerate at both endpoints simultaneously.)

For every triangle $T \in \Delta$ we denote by

$$\theta_T \quad \text{and} \quad \tilde{\theta}_T$$

the minimum of θ_e over all edges of T, and the minimum of $\tilde{\theta}_e$ over all edges of T lying in the interior of Ω, respectively. Thus, θ_T denotes the smallest angle around T, whereas $\tilde{\theta}_T$ measures the "near-degeneracy" of the edges of T. Certainly,

$$\theta_T \geq \theta_\Delta.$$

The following key lemma shows that the nodal functionals in $\mathcal{N}_T^*$ can be estimated in terms of those in $\mathcal{N}_T$. Moreover, only the contribution of $\tilde{\mathcal{N}}_T$ to this estimation is influenced by $\tilde{\theta}_T$.

Lemma 3.8 *Let $T \in \Delta$, $s \in S_q^{r,\rho}(\Delta)$ and $f \in C^m(\Omega)$ ($m \in \{2r, \ldots, q+1\}$). Then for any $\nu^* \in \mathcal{N}_T^*$*

$$(3.15) \qquad \begin{aligned} |\nu^*(f-s)| \leq K h_T^{-d(\nu^*)} \Bigg(& h_T^m \max_{0 \leq m' \leq m} \|D_x^{m'} D_y^{m-m'} f\|_{C(\partial T)} \\ & + \max_{\nu \in \mathcal{N}_T \setminus \tilde{\mathcal{N}}_T} h_T^{d(\nu)} |\nu(f-s)| + \sin^{-r} \tilde{\theta}_T \max_{\nu \in \tilde{\mathcal{N}}_T} h_T^{d(\nu)} |\nu(f-s)| \Bigg), \end{aligned}$$

where h_T is the diameter of T, and K depends only on r, q and θ_T.

Proof. Since $\mathcal{N}(T) \subset \mathcal{N}_T^* \cap \mathcal{N}_T$, we do not need to estimate $|\nu^*(f-s)|$ for $\nu^* \in \mathcal{N}(T)$. Moreover, by simmetry, it is enough to consider $\mathcal{N}_T^*(v)$ for a vertex v of T, and $\mathcal{N}_T^*(e)$ for an edge e of T.

Let $T = T_v^i$ for some $i \in \{1, \ldots, n(v)\}$. Then $\mathcal{N}_T^*(v)$ corresponds to the nodal values

$$\nu^* g = D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} g(v)\,, \text{ for all } (\alpha, \beta) \in A_1 \cup A_2 \cup A_3 \cup \widetilde{A}_3\,.$$

We consider three cases.

Case 1: $(\alpha, \beta) \in A_1$.

Then

$$\begin{aligned} h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} (f - s)(v)| &\le 2^{\alpha+\beta} h_T^{\alpha+\beta} \max_{\alpha' \ge 0,\, \beta' \ge 0,\, \alpha'+\beta'=\alpha+\beta} |D_x^{\alpha'} D_y^{\beta'} (f-s)(v)| \\ &\le 2^{\rho} \max_{(\alpha', \beta') \in A_1} h_T^{\alpha'+\beta'} |D_x^{\alpha'} D_y^{\beta'} (f-s)(v)|\,. \end{aligned}$$

Therefore,

$$(3.16) \qquad h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} (f - s)(v)| \le 2^{\rho} \max_{\nu \in \mathcal{N}_{T,1}(v)} h_T^{d(\nu)} |\nu(f-s)|\,, \qquad (\alpha, \beta) \in A_1\,,$$

which proves (3.15).

Case 2: $(\alpha, \beta) \in A_2$.

If e_i is nondegenerate at v, then $\nu^* \in \mathcal{N}_{T,2}(v)$ and (3.15) trivially holds. If e_i is degenerate at v, but e_{i-1} is nondegenerate at v, then, by (3.4),

$$h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} (f - s)(v)| = h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i-1}}^{\beta} (f - s)(v)| \le \max_{\nu \in \mathcal{N}_{T,2}(v)} h_T^{d(\nu)} |\nu(f-s)|\,.$$

Similarly, if both e_i and e_{i-1} are degenerate at v, but e_{i-2} is nondegenerate at v, then a repeated application of (3.4) shows that

$$h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} (f - s)(v)| = h_T^{\alpha+\beta} |D_{e_{i-2}}^{\alpha} D_{e_{i-1}}^{\beta} (f - s)(v)| \le \max_{\nu \in \mathcal{N}_{T,2}(v)} h_T^{d(\nu)} |\nu(f-s)|\,.$$

Finally, if v is singular, then in the same manner we can see that

$$(3.17) \qquad h_T^{\alpha+\beta} |D_{e_i}^{\alpha} D_{e_{i+1}}^{\beta} (f - s)(v)| \le \max_{\nu \in \mathcal{N}_{T,2}(v)} h_T^{d(\nu)} |\nu(f-s)|\,, \qquad (\alpha, \beta) \in A_2\,,$$

which, hence, holds in either case and confirms (3.15).

Case 3: $(\alpha, \beta) \in A_3 \cup \widetilde{A}_3$.

By simmetry, assume without loss of generality that $(\alpha,\beta)\in A_3$.

If either e_i is degenerate at v or e_i is a boundary edge, then $\nu^*\in\mathcal{N}_{T,4}(v)$ and (3.15) trivially holds. If, otherwise, e_i is a nondegenerate at v interior edge, then analysis similar to that in Case 2 shows that

$$(3.18)\qquad h_T^{\alpha+\beta}|D_{e_i}^{\alpha}D_{e_{i-1}}^{\beta}(f-s)(v)|\le\max_{\nu\in\mathcal{N}_{T,4}(v)}h_T^{d(\nu)}|\nu(f-s)|,\qquad(\alpha,\beta)\in A_2\,.$$

Let us denote by v_{i-1}, v_i and v_{i+1} the vertices of e_{i-1}, e_i and e_{i+1} different from v, respectively, by e'_{i+1} the edge between v_i and v_{i+1}, and by e'_{i-1} the edge between v_i and v_{i-1}. The same argumentation as in the above shows that

$$(3.19)\qquad \begin{aligned}&h_T^{\alpha+\beta}|D_{e_i}^{\alpha}D_{e'_{i+1}}^{\beta}(f-s)(v_i)|\le 2^{\rho}\max_{\nu\in\mathcal{N}_{T,1}(v_i)}h_T^{d(\nu)}|\nu(f-s)|,\quad(\alpha,\beta)\in A_1\,,\\&h_T^{\alpha+\beta}|D_{e_i}^{\alpha}D_{e'_{i+1}}^{\beta}(f-s)(v_i)|\le\max_{\nu\in\mathcal{N}_{T,2}(v_i)}h_T^{d(\nu)}|\nu(f-s)|,\quad(\alpha,\beta)\in A_2\,.\end{aligned}$$

If now e_i is nondegenerate at v_i, then by the definition of $\mathcal{N}_T$,

$$(3.20)\qquad \mathcal{N}_{T,3}(v_i)=\{\nu g=D_{e_i}^{\alpha}D_{e'_{i-1}}^{\beta}g(v_i)\;:\;(\alpha,\beta)\in A_2\}\,.$$

In view of (3.16)–(3.20) and Lemma 3.6, 2), with $T_1=T$ and $T_2=T_v^{i-1}$, we have for every $\mu=0,\dots,r$ and $\gamma=0,\dots,m-\mu$,

$$(3.21)\qquad \begin{aligned}h_T^{\gamma+\mu}\|D_{e_i}^{\gamma}D_{e_i^{\perp}}^{\mu}(f-s)\|_{C(e_i)}\le K_1\Bigg(&h_T^{m}\max_{0\le\mu'\le\mu}\|D_{e_i}^{m-\mu'}D_{e_i^{\perp}}^{\mu'}f\|_{C(e_i)}\\&+\max_{\nu\in\mathcal{N}_T(e_i)}h_T^{d(\nu)}|\nu(f-s)|+\max_{\nu\in\mathcal{N}_{T,1}(v)\cup\mathcal{N}_{T,1}(v_i)}h_T^{d(\nu)}|\nu(f-s)|\\&+\sin^{-r}\tilde\theta_T\max_{\nu\in\mathcal{N}_{T,2}(v)\cup\mathcal{N}_{T,4}(v)\cup\mathcal{N}_{T,2}(v_i)\cup\mathcal{N}_{T,3}(v_i)}h_T^{d(\nu)}|\nu(f-s)|\Bigg)\,,\end{aligned}$$

where K_1 depends only on q and θ_T. Since

$$(3.22)\qquad |\nu^*(f-s)|=|D_{e_i}^{\alpha}D_{e_{i+1}}^{\beta}(f-s)(v)|\le 2^{\beta}\max_{0\le\mu\le\beta}|D_{e_i}^{\alpha+\beta-\mu}D_{e_i^{\perp}}^{\mu}(f-s)(v)|\,,$$

(3.15) follows from (3.21).

If e_i is degenerate at v_i, then

$$(3.23)\qquad \mathcal{N}_{T,3}(v_i)=\{\nu g=D_{e_i}^{\alpha}D_{e'_{i-1}}^{\beta}g(v_i)\;:\;(\alpha,\beta)\in A_3\}\,.$$

By (3.4),

$$|D_{e_i}^{\alpha} D_{e'_{i+1}}^{\beta} g(v_i)| = |D_{e_i}^{\alpha} D_{e'_{i-1}}^{\beta} g(v_i)|, \qquad (\alpha, \beta) \in A_3 . \tag{3.24}$$

In view of (3.16)–(3.19), (3.23), (3.24) and Lemma 3.6, 1), with $T_1 = T$ and $T_2 = T_v^{i-1}$, we have for every $\mu = 0, \ldots, r$ and $\gamma = 0, \ldots, m - \mu$,

$$\begin{aligned} h_T^{\gamma+\mu} \| D_{e_i}^{\gamma} D_{e_i^{\perp}}^{\mu} (f - s) \|_{C(e_i)} \leq K_2 \Bigg(& h_T^m \max_{0 \leq \mu' \leq \mu} \| D_{e_i}^{m-\mu'} D_{e_i^{\perp}}^{\mu'} f \|_{C(e_i)} \\ & + \max_{\nu \in \mathcal{N}_T(e_i)} h_T^{d(\nu)} |\nu(f - s)| + \max_{\nu \in \mathcal{N}_{T,1}(v) \cup \mathcal{N}_{T,1}(v_i)} h_T^{d(\nu)} |\nu(f - s)| \\ & + \max_{\nu \in \mathcal{N}_{T,2}(v_i) \cup \mathcal{N}_{T,3}(v_i)} h_T^{d(\nu)} |\nu(f - s)| + \sin^{-r} \tilde{\theta}_T \max_{\nu \in \mathcal{N}_{T,2}(v) \cup \mathcal{N}_{T,4}(v)} h_T^{d(\nu)} |\nu(f - s)| \Bigg), \end{aligned} \tag{3.25}$$

with K_2 being dependent only on q and θ_T. Therefore, (3.15) follows from (3.22).

Finally, let e be one of the edges of T, say $e = e_i$. Then for any $\nu^* \in \mathcal{N}_T^*$,

$$|\nu^*(f - s)| = |D_{\tau'}^{\mu}(f - s)(z_{e_i}^{\mu,j})| \leq 2^{\mu} \max_{0 \leq \mu' \leq \mu} |D_{e_i}^{\mu-\mu'} D_{e_i^{\perp}}^{\mu'} (f - s)(z_{e_i}^{\mu,j})|, \tag{3.26}$$

and (3.15) follows from (3.21) or (3.25) if e_i is an interior edge of Ω. If, otherwise, e_i is a boundary edge, then, similar to the above, Lemma 3.5 implies that

$$\begin{aligned} h_T^{\gamma+\mu'} \| D_{e_i}^{\gamma} D_{e_i^{\perp}}^{\mu'} (f - s) \|_{C(e_i)} \leq K_3 \Bigg(& h_T^m \max_{0 \leq \mu'' \leq \mu'} \| D_{e_i}^{m-\mu''} D_{e_i^{\perp}}^{\mu''} f \|_{C(e_i)} \\ & + \max_{\nu \in \mathcal{N}_T(e_i)} h_T^{d(\nu)} |\nu(f - s)| + \max_{\nu \in \mathcal{N}_{T,1}(v) \cup \mathcal{N}_{T,1}(v_i)} h_T^{d(\nu)} |\nu(f - s)| \\ & + \max_{\nu \in \mathcal{N}_{T,2}(v) \cup \mathcal{N}_{T,4}(v) \cup \mathcal{N}_{T,2}(v_i) \cup \mathcal{N}_{T,3}(v_i)} h_T^{d(\nu)} |\nu(f - s)| \Bigg), \end{aligned} \tag{3.27}$$

with K_3 being dependent only on q and θ_T, which, in view of (3.26), implies (3.15).
■

In the following lemma we use standard finite-element techniques to get an estimation of $\|s_{|_T}\|_{C(T)}$ in terms of the nodal functionals $\nu \in \mathcal{N}_T^*$.

Lemma 3.9 *If $s \in S_q^{r,\rho}(\Delta)$ and $T \in \Delta$, then*

$$\|s_{|_T}\|_{C(T)} \leq K \max_{\nu \in \mathcal{N}_T^*} h_T^{d(\nu)} |\nu s|, \tag{3.28}$$

where h_T is the diameter of T, and K depends only on q.

Proof. Let $\hat{T}$ be a fixed triangle in the plane, say, the triangle with vertices $\hat{v}_1 = (-\frac{1}{2}, 0)$, $\hat{v}_2 = (\frac{1}{2}, 0)$, $\hat{v}_3 = (0, \frac{\sqrt{3}}{2})$. Although $\hat{T}$ may be not in Δ, it is easy to see that the set of nodal functionals $\mathcal{N}^*_{\hat{T}}$ is well-defined for $\hat{T}$.

For every $T \in \Delta$, let $B_T : \mathrm{R}^2 \to \mathrm{R}^2$ be an affine mapping such that $B_T(\hat{T}) = T$. Then

$$B_T z = A_T z + b_T\,, \quad z \in \mathrm{R}^2\,,$$

where $A_T : \mathrm{R}^2 \to \mathrm{R}^2$ is an invertible linear mapping and $b_T \in \mathrm{R}^2$. Since $\hat{T}$ contains a disk of radius $\frac{\sqrt{3}}{6}$,

$$\|A_T\| \le 2\sqrt{3}\, h_T\,. \tag{3.29}$$

For every $\hat{\nu} \in \mathcal{N}^*_{\hat{T}}$, say of the form

$$\hat{\nu} g = D^{\alpha}_{\hat{\tau}_1} D^{\beta}_{\hat{\tau}_2} g(\hat{z}_0)\,,$$

let us define ν by

$$\nu g := D^{\alpha}_{\tau_1} D^{\beta}_{\tau_2} g(z_0)\,,$$

where

$$\tau_i = \|A_T \hat{\tau}_i\|^{-1} A_T \hat{\tau}_i\,, \quad i = 1, 2\,, \quad z_0 = B_T \hat{z}_0\,.$$

Then it is easy to check that

$$\hat{\nu} \in \mathcal{N}^*_{\hat{T}} \iff \nu \in \mathcal{N}^*_{T}\,. \tag{3.30}$$

Moreover, a standard computation shows that

$$D^{\alpha}_{\hat{\tau}_1} D^{\beta}_{\hat{\tau}_2} g(B_T \hat{z}_0) = \|A_T \hat{\tau}_1\|^{\alpha} \|A_T \hat{\tau}_2\|^{\beta} D^{\alpha}_{\tau_1} D^{\beta}_{\tau_2} g(z_0)\,. \tag{3.31}$$

By Lemma 3.7, $\mathcal{N}^*_T$ is Π_q-unisolvent. Therefore, for every $\nu \in \mathcal{N}^*_T$ there exists a unique fundamental polynomial $p_\nu \in \Pi_q$ such that

$$\nu^* p_\nu = \begin{cases} 1, & \text{if } \nu^* = \nu, \\ 0, & \text{if } \nu^* \in \mathcal{N}^*_T \setminus \{\nu\}. \end{cases}$$

Similarly, for every $\hat{\nu} \in \mathcal{N}^*_{\hat{T}}$ there exists a unique fundamental polynomial $\hat{p}_{\hat{\nu}} \in \Pi_q$ such that

$$\nu^* \hat{p}_{\hat{\nu}} = \begin{cases} 1, & \text{if } \nu^* = \hat{\nu}, \\ 0, & \text{if } \nu^* \in \mathcal{N}^*_{\hat{T}} \setminus \{\hat{\nu}\}. \end{cases}$$

It follows from (3.31) that

$$p_\nu(B_T z) \equiv \|A_T\hat{\tau}_1\|^\alpha \|A_T\hat{\tau}_2\|^\beta \hat{p}_{\hat{\nu}}(z)\,. \tag{3.32}$$

We are now ready to prove (3.28). Since $s_{|T}$ is a polynomial in Π_q, we have

$$s_{|T} = \sum_{\nu \in \mathcal{N}_T^*} (\nu s) p_\nu\,.$$

Therefore, by (3.32) and (3.29),

$$\begin{aligned}
\|s_{|T}\|_{C(T)} &\le \sum_{\nu \in \mathcal{N}_T^*} |\nu s| \cdot \|p_\nu\|_{C(T)} = \sum_{\nu \in \mathcal{N}_T^*} |\nu s| \cdot \|p_\nu \circ B_T\|_{C(\hat{T})} \\
&\le \sum_{\nu \in \mathcal{N}_T^*} |\nu s| \cdot \|A_T\|^{d(\nu)} \|\hat{p}_{\hat{\nu}}\|_{C(\hat{T})} \\
&\le \left(\sum_{\hat{\nu} \in \mathcal{N}_{\hat{T}}^*} \left(2\sqrt{3}\right)^{2r} \|\hat{p}_{\hat{\nu}}\|_{C(\hat{T})} \right) \max_{\nu \in \mathcal{N}_T^*} h_T^{d(\nu)} |\nu s|\,,
\end{aligned}$$

and (3.28) follows. ■

Remark 3.10 It is not difficult to see that the triples $(T, \Pi_q, \mathcal{N}_T^*)$, $T \in \Delta$, form an affine family of finite elements in the sense of [10, p. 87], and $(\hat{T}, \Pi_q, \mathcal{N}_{\hat{T}}^*)$ plays the role of the reference finite element for this family. Particularly, (3.30) shows that $(T, \Pi_q, \mathcal{N}_T^*)$ is affine-equivalent to $(\hat{T}, \Pi_q, \mathcal{N}_{\hat{T}}^*)$, with B_T being the corresponding affine mapping.

Proof of Theorem 3.1. It follows from Lemma 3.8 and Lemma 3.9 that the only spline $s \in S_q^{r,\rho}(\Delta)$ that satisfies $\nu s = 0$ for all $\nu \in \mathcal{N}$, is the zero function. In view of (2.8), a standard linear algebra argument shows that for any real data a_ν, $\nu \in \mathcal{N}$, there exists a unique spline $s \in S_q^{r,\rho}(\Delta)$ such that $\nu s = a_\nu$ for all $\nu \in \mathcal{N}$. Particularly, for every function $f \in C^{2r}(\Omega)$ the Hermite type interpolation problem

$$\nu s = \nu f \quad \text{for all} \quad \nu \in \mathcal{N}$$

has a unique solution $s_f \in S_q^{r,\rho}(\Delta)$, which proves the first statement of the theorem.

Let us fix a function $f \in C^m(\Omega)$ and a triangle $T \in \Delta$. Without loss of generality assume that $(0,0) \in T$. Then for the Taylor polynomial

$$\tilde{p}(x,y) := \sum_{j=0}^{m-1} \sum_{j'=0}^{j} \frac{D_x^{j'} D_y^{j-j'} f(0,0)}{j'!(j-j')!} x^{j'} y^{j-j'}$$

we have

$$\|D_x^\alpha D_y^\beta (f - \tilde{p})\|_{C(T)} \le \frac{2^{m-\alpha-\beta}}{(m+1-\alpha-\beta)!} h_T^{m-\alpha-\beta} \max_{0 \le m' \le m} \|D_x^{m'} D_y^{m-m'} f\|_{C(T)} \tag{3.33}$$

for all $\alpha, \beta \ge 0$, $\alpha + \beta \le m$. As a consequence, for every $\nu \in \mathcal{N}_T^*$,

$$|\nu(f - \tilde{p})| \le K_1 h_T^{m-d(\nu)} \max_{0 \le m' \le m} \|D_x^{m'} D_y^{m-m'} f\|_{C(T)}, \tag{3.34}$$

where K_1 is a constant depending only on q.

We have

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_{L_\infty(T)} \le \|D_x^\alpha D_y^\beta (f - \tilde{p})\|_{C(T)} + \|D_x^\alpha D_y^\beta (\tilde{p} - s_f)\|_{L_\infty(T)}. \tag{3.35}$$

By the bivariate Markov inequality (see, for example, [11]),

$$\|D_x^\alpha D_y^\beta p\|_{C(T)} \le K_2 (h_T \sin\theta)^{-\alpha-\beta} \|p\|_{C(T)} \quad \text{for all} \quad p \in \Pi_q, \tag{3.36}$$

where h_T and θ are the diameter and the smallest angle of T, respectively, and K_2 depends only on q.

Since $\tilde{p} - s_f \in S_q^{r,\rho}(\Delta)$ and $(\tilde{p} - s_f)_{|T} \in \Pi_q$, it follows from (3.36) and Lemma 3.9 that

$$\begin{aligned} \|D_x^\alpha D_y^\beta (\tilde{p} - s_f)\|_{L_\infty(T)} &\le K_2 (h_T \sin\theta)^{-\alpha-\beta} \|(\tilde{p} - s_f)_{|T}\|_{C(T)} \\ &\le K_3 h_T^{-\alpha-\beta} \max_{\nu^* \in \mathcal{N}_T^*} h_T^{d(\nu^*)} |\nu^*(\tilde{p} - s_f)|, \end{aligned} \tag{3.37}$$

where K_3 depends only on q and θ_T. By (3.34) and Lemma 3.8, since $\nu(f - s_f) = 0$ for all $\nu \in \mathcal{N}_T$, we have for every $\nu^* \in \mathcal{N}_T^*$

$$\begin{aligned} h_T^{d(\nu^*)} |\nu^*(\tilde{p} - s_f)| &\le h_T^{d(\nu^*)} |\nu^*(\tilde{p} - f)| + h_T^{d(\nu^*)} |\nu^*(f - s_f)| \\ &\le K_4 h_T^m \max_{0 \le m' \le m} \|D_x^{m'} D_y^{m-m'} f\|_{C(T)}, \end{aligned} \tag{3.38}$$

where K_4 depends only on r, q and θ_T.

Since $\theta_T \ge \theta_\Delta$, now (3.35), (3.33), (3.37) and (3.38) imply (3.2). ■

Remark 3.11 It is easy to see from the above proof that Theorem 3.1 in fact holds with θ_T in place of θ_Δ.

4 A Basis for $S_q^{r,\rho}(\Delta)$

Let $\mathcal{N} = \{\nu_i\}_{i=1}^n$, where $n = \dim S_q^{r,\rho}(\Delta)$ in view of (2.8). For every $f \in C^{2r}(\Omega)$, it follows from Theorem 3.1 that the interpolating spline $s_f \in S_q^{r,\rho}(\Delta)$ satisfying (3.1) can be represented as

$$s_f = \sum_{i=1}^{n} (\nu_i f) s_i \,, \tag{4.1}$$

where the *fundamental functions* $s_i \in S_q^{r,\rho}(\Delta)$, $i = 1, \ldots, n$, are uniquely determined by the conditions

$$\nu_i s_j = \delta_{ij} \,, \quad i, j = 1, \ldots, n \,. \tag{4.2}$$

Therefore, $\{s_1, \ldots, s_n\}$ is a basis for $S_q^{r,\rho}(\Delta)$. The following theorem establishes some useful properties of this basis.

Theorem 4.1 *The fundamental functions $s_1, \ldots, s_n$ form a basis for $S_q^{r,\rho}(\Delta)$ such that*

1) *$\{s_1, \ldots, s_n\}$ is locally linearly independent, i.e., for every open $B \subset \Omega$ the subsystem $\{s_i : B \cap \operatorname{supp} s_i \neq \emptyset\}$ is linearly independent on B,*

2) *$\{s_1, \ldots, s_n\}$ is least supported, i.e., for every basis $\{b_1, \ldots, b_n\}$ of $S_q^{r,\rho}(\Delta)$ there exists a permutation π of $\{1, \ldots, n\}$ such that*

$$\operatorname{supp} s_i \subset \operatorname{supp} b_{\pi(i)}, \quad \textit{for all} \quad i = 1, \ldots, n,$$

3) *for each $i = 1, \ldots, n$, $\operatorname{supp} s_i$ is either a triangle or the union of some triangles sharing one common vertex,*

4) *$K_1 h_i^{d(\nu_i)} \leq \|s_i\|_{C(\Omega)} \leq K_2 K_3\, h_i^{d(\nu_i)}$, where*

$$h_i := \max_{T \subset \operatorname{supp} s_i} h_T,$$

K_1, K_2 depend only on r, q and θ_Δ, and

$$K_3 = \begin{cases} 1, & \text{if } \nu_i \in \mathcal{N} \setminus \widetilde{\mathcal{N}}, \\ \sin^{-r} \tilde{\theta}_\Delta, & \text{if } \nu_i \in \widetilde{\mathcal{N}}, \end{cases}$$

with

$$\widetilde{\mathcal{N}} := \bigcup_{T \in \Delta} \widetilde{\mathcal{N}}_T, \qquad \tilde{\theta}_\Delta := \min_{T \in \Delta} \tilde{\theta}_T.$$

5) *the corresponding normalized basis* $\{s_1^*, \ldots, s_n^*\}$, *with* $s_i^* := h_i^{-d(\nu_i)} s_i$, *is stable in the sense that*

$$(4.3) \qquad K_4 \max_i |a_i| \le \| \sum_{i=1}^n a_i s_i^* \|_{C(\Omega)} \le \frac{K_5}{\sin^r \tilde{\theta}_\Delta} \max_i |a_i|,$$

where K_4 *and* K_5 *depend only on* r, q *and* θ_Δ.

Proof. 1) As shown in [12], a system of functions $\{s_1, \ldots, s_n\} \subset S_q^r(\Delta) \setminus \{0\}$ such that $\Pi_q \subset \operatorname{span}\{s_1, \ldots, s_n\}$ is locally linearly independent if and only if

$$(4.4) \qquad \operatorname{card}\{i : T \subset \operatorname{supp} s_i\} \le \binom{q+2}{2} = \dim \Pi_q, \qquad \text{for every } T \in \Delta.$$

Since $\Pi_q \subset S_q^{r,\rho}(\Delta)$, we only have to check (4.4). Fix a triangle T in Δ. By applying consecutively Lemma 3.9 and Lemma 3.8 (the latter with $f \equiv 0$), we get

$$(4.5) \qquad \|s_i|_T\|_{C(T)} \le K_6 \left(\max_{\nu \in \mathcal{N}_T \setminus \tilde{\mathcal{N}}_T} h_T^{d(\nu)} |\nu s_i| + \sin^{-r} \tilde{\theta}_T \max_{\nu \in \tilde{\mathcal{N}}_T} h_T^{d(\nu)} |\nu s_i| \right),$$

where K_6 depends only on r, q and θ_T. Therefore, $s_i|_T \equiv 0$ if $\nu_i \notin \mathcal{N}_T$, so that (4.4) follows from (3.14).

2) Least supportedness of $\{s_1, \ldots, s_n\}$ follows from its local linear independence in view of [6, Theorem 3.4].

3) As we have seen, $T \subset \operatorname{supp} s_i$ implies $\nu_i \in \mathcal{N}_T$. Therefore, it suffices to show that for each fixed $\nu \in \mathcal{N}$ the set

$$\mathcal{T}_\nu := \{T \in \Delta : \nu \in \mathcal{N}_T\}$$

consists either of a single triangle or of some triangles sharing one common vertex. Since

$$\mathcal{N} = \bigcup_v \mathcal{N}(v) \cup \bigcup_e \mathcal{N}(e) \cup \bigcup_T \mathcal{N}(T),$$

we consider several cases. First, if $\nu \in \mathcal{N}(T)$ for some $T \in \Delta$, then obviously $\mathcal{T}_\nu = \{T\}$. If $\nu \in \mathcal{N}(e)$ for some interior edge e of Δ, then $\mathcal{T}_\nu = \{T_1, T_2\}$, where T_1 and T_2 are the two triangels sharing the edge e. (If e is a boundary edge, then, of course, there is only one such triangle T, and $\mathcal{T}_\nu = \{T\}$.) Finally, assume that $\nu \in \mathcal{N}(v)$ for some vertex v of Δ. Then $\nu \in \mathcal{N}_T$ implies $\nu \in \mathcal{N}_T(v)$. The latter is possible only for the triangles T that are attached to v. Hence, $\mathcal{T}_\nu \subset \{T \in \Delta : v \in T\}$.

4) By (4.5),

$$\|s_i\|_{C(\Omega)} = \max_{T\subset \operatorname{supp} s_i} \|s_i|_T\|_{C(T)} \le K_6 K_3 \max_{T\subset \operatorname{supp} s_i} h_T^{d(\nu_i)} = K_6 K_3\, h_i^{d(\nu_i)},$$

which gives the upper bound. Furthermore, by Markov inequality (3.36),

$$1 = |\nu_i s_i| \le K_7 h_T^{-d(\nu_i)} \|s_i|_T\|_{C(T)}, \qquad \text{for some } T \subset \operatorname{supp} s_i\,,$$

where K_7 depends only on q and θ_Δ. It is not difficult to check that

$$\max_{T\subset \operatorname{supp} s_i} h_T \le K_8 \min_{T\subset \operatorname{supp} s_i} h_T$$

where K_8 depends only on q and θ_Δ (see, e.g., [18, Lemma 3.2]). Therefore,

$$\|s_i\|_{C(\Omega)} \ge K_7^{-1} K_8^{-2r} h_i^{d(\nu_i)},$$

and the lower bound is also shown.

5) We fix $\{a_i\}_{i=1}^n$ and set $s = \sum_{i=1}^n a_i s_i^*$. Let $z \in T \in \Delta$. By (4.4) and (4.5) we have

$$|\sum_{i=1}^n a_i s_i^*(z)| \le \max_i |a_i| \sum_{i=1}^n |s_i^*(z)| \le \max_i |a_i| \binom{q+2}{2} K_6 \sin^{-r}\tilde{\theta}_T,$$

which proves the upper bound for $\|s\|_{C(\Omega)}$. Moreover, let $|a_j| = \max_i |a_i|$. Then

$$\nu_j s = h_j^{-d(\nu_j)} a_j.$$

Therefore, in view of Markov inequality, we have for some $T' \subset \operatorname{supp} s_j$,

$$|a_j| = h_j^{d(\nu_j)} |\nu_j s| \le K_7 h_j^{d(\nu_j)} h_{T'}^{-d(\nu_j)} \|s|_{T'}\|_{C(T')} \le K_7 K_8^{2r} \|s\|_{C(\Omega)},$$

which completes the proof. ■

Remark 4.2 A similar interpolation scheme can be done for the superspline space $S_q^{r,\rho}(\Delta)$ with any ρ within the range

$$r + \left[\frac{r+1}{2}\right] \le \rho \le \min\left\{2r, \left[\frac{q-1}{2}\right]\right\}.$$

The only necessary change in the construction is that in the definition of $\mathcal{N}(e)$ one should take

$$\kappa_\mu := \min\{q - 2\rho - 1 + \mu\,,\ q - 3r - 1 - (r-\mu) \bmod 2\}\ .$$

All results of Section 3 and Section 4 remain valid.

We conclude the paper with a discussion of the results of Section 4.

First of all, it immediately follows from Theorem 3.1 that the norm of the interpolation operator $s_f : C^{2r}(\Omega) \to S_q^{r,\rho}(\Delta)$ is bounded by a constant which depends only on r, q and the smallest angle θ_Δ in Δ. On the other hand, it is easy to see that some of the fundamental functions s_i can grow unboundedly if the triangulation contains near-degenerate edges. (In Theorem 4.1 we could only estimate $\|s_i\|_{C(\Omega)}$ with a constant depending on $\tilde{\theta}_\Delta$.) This seems to be controversial at first glance. We will try to explain this phenomenon. Let v be a vertex of the triangulation Δ. The nodal values ν in the set $\mathcal{N}(v)$ are linearly independent, as we have shown, if they are considered as linear functionals on the spline space $S_q^{r,\rho}(\Delta)$. Contrary to this, the nodal values $\nu \in \mathcal{N}(v)$ corresponding to the partial derivatives of the same order k, with $\rho < k \leq 2r$, do stay in a linear relation as linear functionals on the space $C^{2r}(\Omega)$. (Recall that $S_q^{r,\rho}(\Delta)$ is not a subspace of $C^{2r}(\Omega)$.) Indeed, there exist exactly $k+1$ linearly independent partial derivatives $D_{\tau_1}^{\alpha} D_{\tau_2}^{\beta}(f)(v)$, with $\alpha + \beta = k$, for any k-times differentiable function f, and we certainly have in $\mathcal{N}(v)$ more than $k+1$ nodal values of this type. As a consequence, the coefficients $\nu_i f$ in (4.1) satisfy some linear relations reflecting the fact that f is $2r$-times differentiable at each vertex. This leads to some cancellations in the sum and makes possible estimation (3.2).

Let us also remark that, according to Theorem 4.1, 2), our basis is best possible for the space $S_q^{r,\rho}(\Delta)$ in regard to the size of the supports of the basis functions. It shares this property with the basis constructed in [16]. The bases in [8, 18] fail to be least supported, but they have the advantage that the stability inequality (4.3) holds for them without $\sin^r \tilde{\theta}_\Delta$ in the right hand side, *i.e.*, they enjoy stability even in the presence of near-degenerate edges.

Finally, we note that the property of local linear independence established for our basis in Theorem 4.1, 1), plays an important role in the theory of almost interpolation (see [12, 14, 15]).

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