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Credit Risk: Worst Case Scenarios for Swap Portfolios

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preliminary

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Abstract

The first objective of this paper is to apply the model of Barth (1999) to the numerical generation of credit loss distributions of a portfolio consisting entirely of interest rate swaps. The different possibilities for modelling the response function, which gives the impact of a interest rate change onto the credit default probability, is the main subject of this investigation. The second objective is the discussion of several measures for the risk-based capital, needed to back the portfolio. The focus is on the suitablility of these measures to an analysis of worst case scenarios. While two measures for the risk-based capital are based on percentiles, the third measure is a coherent measure. These measures are applied to the analysis of the data generated by the model in regard to the modelling of the response function.

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0 Introduction

Several crises in the worldwide financial markets during the last two years pointed out that the risk management of portfolios of financial contracts which are subject to credit risk is not working satisfactory in extreme market situations. One of the most crucial points in the credit risk management of such situations is the modelling of the risks involved. Though with CreditMetrics and CreditRisk+ (see J.P.Morgan (1997) and CreditSuisse (1997)) two theoretical frameworks are presented which address these problems, the emphasis is not on the analysis of worst cases. Furthermore in these approaches primarily the management of credit risks of portfolios of bonds is considered.

The situation is more difficult if one deals not only with simple contracts as bonds but also with instruments as derivatives which values depend highly on market variables. Not only the "pure" credit risk of such contracts has to be regarded but also the behaviour of the value of these contracts under changes of market variables. For example, a swap will have a positive or a negative value in dependency on the changes of the interest rate. The default of the counterparty in the swap contract might therefore lead to no loss or a severe loss dependent on the actual value of the interest rate. The credit risk of such market-driven contracts might get very high due to large changes in the market which could weak the financial standing of some counterparties. If the considered contracts depend on market variables the possible credit risk in unusual situations is often accounted with "rules of thumb". But the application of those rules of thumb appear as dangerous in the light of the distresses in the present markets.

Usually credit risk management models credit and market risk as independent stochastic variables. The subdivision into these two "kinds" of risk is deeply embodied in the thinking about risk management. However this "independence assumption" is not working satisfactory even in "normal" market situations (compare for example Duffee (1996a)), in extreme situations the failure of this assumption is unavoidable: Due to extraordinary changes of market variables financial institutions might get into difficulties and the credit risk of these institution will increase dramatically.

However the consideration of dependent credit and market risks is very difficult: In addition to the modelling of the credit risk for each counterparty one has to model also the correlation with the market variables. The resulting models are usually not analytically tractable for realistic portfolios. But even numerically there are very large requirements for computing power and time because the simulation of the default events in a large portfolio involves a high number of credit and market variables. More elegant procedures for simulating defaults (see Duffie and Singleton (1998)) are not always applicable for the analysis of the credit risk of market-driven instruments. And, at last, there is a lack of reliable data about correlations between market and credit risks.

One of the most important issues of the risk management concerned by these remarks is the determination of the risk-based capital. This capital should back the portfolio of an institution so that this institution is safe up with a high probability in very unfavourable situations. There are two critical problems with the determination of this capital: First, the amount of capital is strongly dependent on the kind of model used (and therefore dependent on the handling of the correlation between market and credit risk). Second, it is difficult to introduce and establish on the basis of the chosen model well founded procedures for determining this risk-based capital, because the model only generates data on default and losses. Especially if worst cases analysis is concerned, the lack of systematic procedures to measure the risk-based capital needed in such situations prevents the utilization of the data generated by the model. Mostly a combination of the Value-at-Risk approach and ad-hoc stress-testing procedures is used, but unfortunately these procedures do not feature a systematic approach.

This paper has two objectives: The first objective is to apply the model derived in Barth (1999) to circumvent some of these difficulties mentioned above concerning the correlation between credit and market risks. This model is used in the paper to generated data of loss distributions for differing portfolios and correlation specifications. What are the effects of the correlation of the default probability of the counterparties to the changes of the market variables in regard to the portfolio losses?

The "tools" which are necessary to measure these effects in terms of the risk-based capital are the second objective of this paper: Several different measures are provided and discussed. With these measures the risk-based capital needed to be safe (with a high probability) in worst cases is defined. These measures are used based on the data generated by the applied model.

Three measures for the risk-based capital will be introduced: Like the Value-at-Risk approach, two of these measures are based on percentiles, but use other mechanism of time aggregation. Both of them account for worst worst case by concentrating on the "maximal loss". The third measure is a coherent measure (cp. Artzner et al. (1998)) which is based on a kind of shortfall measure. Properties like the subadditivity are studied. The behaviour of these measures for loss distribution with "fat tails" are compared.

In the numerical investigation we consider a portfolio consisting only of interest rate swaps. Interest rate swaps are regarded because, first, their exposure depend very much on the development of the underlying interest rate and are therefore a good example for market-driven instruments. Second, they are the most important traded derivative contracts on interest rates. Third, the valuation of these swaps is straightforward and analytically easy to access. At last we use only the interest rate as the underlying market variables because of the accessibility of empirical data for the "response" of the interest rate to default rates. But these investigations might be extended to other contracts as currency swaps, bond options, caps and floors and other underlyings as currency rates or stocks as well.

The paper is organized as follows: The model of Barth (1999) is described shortly in section 1. In section 2, we discuss three measures of the risk-based capital suited for worst-case scenarios. In section 3, these measures are applied to the numerical investigation of a portfolio of swap contracts.

1 Short Description of the Model

In this section the model introduced in Barth (1999) is reviewed shortly. A financial institution is considered which holds a large portfolio with many different counterparties $a = 1, \ldots, N$. The number N of the counterparties is fixed. None of the counterparties a holds an excessive part of the portfolio (which is called therefore "homogenic"). Only one market variable is considered, called r, which will later be the interest rate. A model is developed where "individual" and "collective" compenents of the credit risk (i.e. the default probability) are distinguished: The individual component is not known to the regarded financial instution (and modeled as noise), but the collective component is given by the initial rating and a response function $S_a(r)$ which measures the impact of a change in an observable market variable r onto the default probability. Each counterparty might have a different response function $S_a(r)$, but this function is known to the financial institution (as it is the rating). Speaking roughly, in this model the market risks r influence the credit risks but the credit risks do not influence the market risks. We refer to this kind of interaction as "response" instead of correlation.

This approach leads to the following formula for the portfolio loss $dL(r(t), t)$ in the time interval $[t \Leftrightarrow dt, t)$ (with dt infinitesimal). The loss $dL(r(t), t)$ is due to default events of the counterparties $a = 1, \ldots, N$ *conditioned* to a fixed realisation of the short rate process $r(t)$, with $t \in [0, T]$:

$$dL(r(t), t) = \frac{1}{B(0,t)} \sum_{a=1}^{N} V_a^+(r(t), t) S_a(r(t), t) dt, \tag{1}$$

where $V_a(r(t), t)$ is the netted value of the contracts with counterparty $a = 1, \ldots, N$. $B(t_1, t_2)$ is the money market account i.e. the value of one dollar at time t_2 which is put to this account at time $t_1 < t_2$. In this paper we only consider the total loss of the exposure in the case of default. $V_a^+(r(t), t)$ describes the positive part of V_a, i.e. the exposure with respect to counterparty a in market situation $r(t)$. $S_a(r(t), t)$ is the default intensity given a realisation $r(t)$ of the short rate process. In the following we drop the notation of the explicit time dependency of dL, S_a, and V_a.

To give a clear idea of eq.(1): The exposures $V_a^+(r(t))$ at time t are "weighted" with the actual probability of default $S_a(r(t))dt$ in the short time interval $[t \Leftrightarrow dt, t)$. This corresponds to the *expected* loss[1] due to credit events in $[t \Leftrightarrow dt, t)$, conditioned on a market situation which is given by r. This formula eq.(1) is easy to handle by Monte-Carlo simulation because only the process of the short rate r has to be simulated. No explicit default processes has to be regarded. This approach is therefore especially suited for an investigation of the impact of changes of the market variables on the credit risk of a large portfolio: eq.(1) is a simplification because only the changes in the market risks have to be regarded as the "driving force" for changes in the credit risks via the response function S_a. The credit risks of different counterparties are correlated because they have the same underlying stochastic variable r.

This approach is complementary to the approach of J.P.Morgan (1997) to incorporate market risks into their framework of CreditMetrics: In CreditMetrics an expectation value of the market risks is taken to yield the average exposure. Or an upper percentile of the distribution of the market risks is taken to yield the maximum exposure. With this the average or the maximum exposure (where the stochasticity of the market risks is eliminated) is then treated as the exposure of a bond which is constant. Then the credit risks due to default or migration are studied. This approach is vice-versa to the approach described in this paper: Here the individual credit risks are eliminated and then the impact of the changes of the market variables onto the systematic component of the credit risk is studied. Only the stochasticity of the market risks remains here, in CreditMetrics remains only the stochasticity of the credit risks.

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[1] In the case that N is large unfortunately the Law of Large Numbers can not be applied to eq.(1) because the risks are *added* rather than *subdivided*.

In Barth (1999) furthermore the modelling of $S_a(r)$ is discussed: The exact analytic form of this function is somewhat arbitrary. Based on the idea of Hull (1989) who is (up to our knowledge) the only one who gives an explicit response function $S_a(r)$ the following functions are considered, see figure 1:

$$
\begin{aligned}
S_a^e(r(t)) &= S_a^e(r(0)) \exp\left[k_a \left(r(t) \Leftrightarrow r(0)\right)\right], & (2)\\
S_a^q(r(t)) &= S_a^q(r(0)) \left(1 + \max\left\{0, \operatorname{sgn}\left(k_a(r(t) \Leftrightarrow r(0))\right)\left(k_a(r(t) \Leftrightarrow r(0))\right)^2\right\}\right), & (3)\\
S_a^l(r(t)) &= S_a^l(r(0)) \max\left\{1, 1 + k_a \left(r(t) \Leftrightarrow r(0)\right)\right\}, & (4)\\
S_a^r(r(t)) &= S_a^r(r(0)) \sqrt{\max\left\{1, 1 + k_a \left(r(t) \Leftrightarrow r(0)\right)\right\}}. & (5)
\end{aligned}
$$

In Barth (1999) also empirical evidence is reviewed for calibrating the parameter k_a of these functions, see section 3.2.

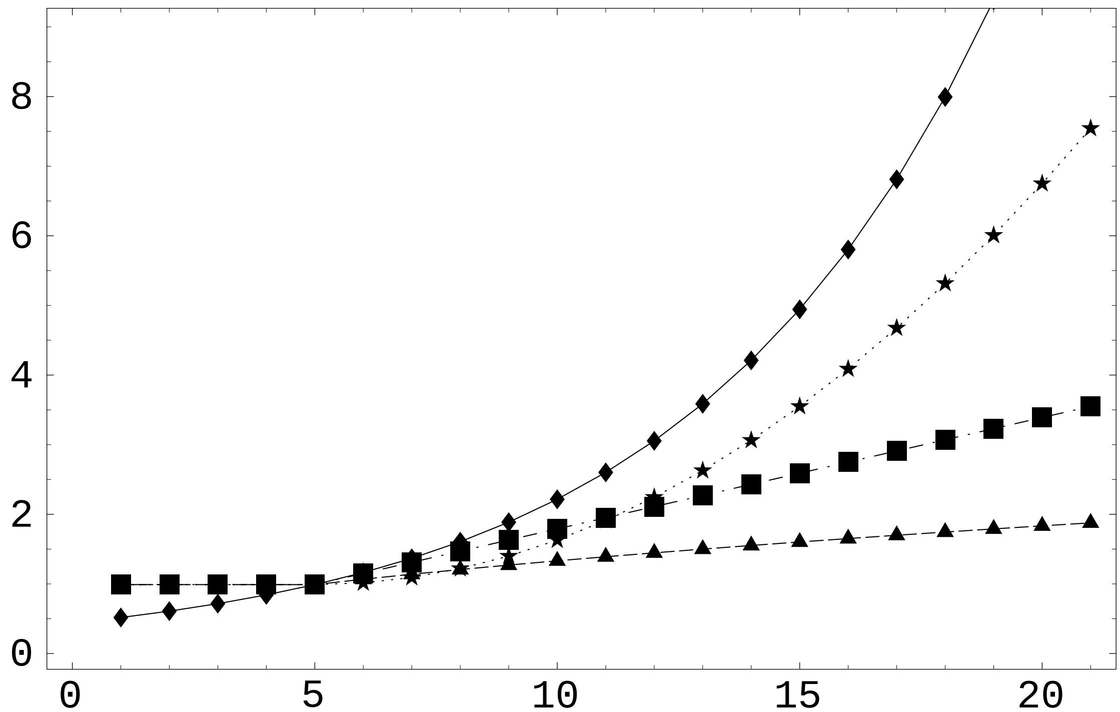

Figure 1: *Correlation functions for $k_a = 16$ and $r(0) = 0.05$ and $S_a(r(0)) = 1$: Exponential eq.(2) (checks), quadratic eq.(3) (stars), linear eq.(4) (squares), and square-root eq.(5) (triangles). The value of the response function S is plotted versus r.*

In the following this model is applied to the analysis of the impact of market risk worst case scenarios onto the credit risk. It is the goal to determine the risk-based capital which has to be hold back by the financial institution to be safe to a high probability. But how to measure this risk-based capital? If one deals with market risks only one usually applies the Value-at-Risk approach on a short time interval, say ten days. But due to the longer time horizonts in credit risk management this might not be the best suited approach. Furthermore, by regarding worst case scenarios the Value-at-Risk approach might lead to underestimates of the risk-based capital.

2 Worst Case Measures

First we will derive some measures which result of percentile-based approaches like the Value-at-Risk approach. After that we will discuss some other measures, which are not based on this approach. We have to stress again that all these measures concern the impact of the market variables (which are the only stochastic quantity here) on the expected credit loss given this market situation, which is described by these market variables. The market situation is stochastic, the credit risks are "eliminated" by the conditioned expectation value.

2.1 Percentile-based Measures

In the following we discuss different definitions of the risk-based capital which are based on the consideration of percentiles. We will choose two of them for being suited for a worst case analysis. After that we compare the properties of these two measures.

Definition

As the fundamental quantity acts the discounted credit loss eq.(1). One possibility to define the risk-based capital is the Value-at-Risk (VaR), which is given as the upper q-percentile[2] of the distribution of the cumulative loss over a given time-horizont $[0, T]$:

$$\text{Prob}\left[\int_0^T dL(s) > V_q^{\text{VaR}}[0,T]\right] \leq 1 \Leftrightarrow q. \tag{6}$$

This Value-at-Risk $V_q^{\text{VaR}}[0,T]$ is usually used for estimating market risks over short time-horizonts (say ten days). But credit risks have to be considered over much longer time-intervals, for example ten *years*. The risk-based capital could be "refilled", so that an aggregated credit loss like in eq.(6) is not more an useful measure for the immediate

[2] In the case of not well-behaved V_a it might be necessary to define the percentiles V_q as $V_q = \inf\{x|\text{Prob}[\text{loss} > x] > 1 - q\}$.

financial distress after a severe loss has occured. One could define a VaR on a shorter time interval $[t, t + \Delta t] \subset [0, T]$ with $\Delta t > 0$ and

$$\text{Prob}\left[\int_t^{t+\Delta t} dL(s) > V_q^{\text{VaR}}[t, t + \Delta t]\right] \leq 1 \Leftrightarrow q. \tag{7}$$

This Δt might describe a "refilling-period", but this would be a further unknown parameter which had to be chosen firm-specificly.

By refering to the long time horizont T one could assume an instantaneous refilling in an infinitesimal small time interval after the default event. This correspond to an infinitesimal[3] small $\Delta t \to dt$:

$$\text{Prob}\left[dL(t) > V_q^{\text{P}}[t]\right] \leq 1 \Leftrightarrow q, \tag{8}$$

where $V_q^{\text{P}}[t]$ is the q-percentile of the loss distribution at time t, which gives an upper boundary for losses at time t (with probability q). To be safe over the whole time interval $[0, T]$ up to probability q one has to chose the risk-based capital

$$V_q^{\text{MP}}[0, T] = \max_{t \in [0,T]} V_q^{\text{P}}[t]. \tag{9}$$

We refer to $V_q^{\text{MP}}[0, T]$ as the "peak loss", cf. Duffee (1996a). This peak loss can be calculated for a realistic portfolio by Monte-Carlo simulation. It is not used very frequently because more conservative capital requirements result than by applying measures like the "average loss". But V_q^{MP} might be a measure which is sensitive for rare events like a very large loss. Therefore it is with sense to see V_q^{MP} not only in the context of the risk-based capital but also in the context of stress testing procedures.

One measure which is not examined or applied in the context of risk management in financial markets up to now (cf. Albrecht (1997)) originates in the actuarial approach

$$\begin{aligned} & \text{Prob}\left[dL(t) > V_q^{\text{PM}}[0, T] \text{ for one } t \in [0, T]\right] \leq 1 \Leftrightarrow q \\ \Leftrightarrow \quad & \text{Prob}\left[\max_{t \in [0,T]} dL(t) > V_q^{\text{PM}}[0, T]\right] \leq 1 \Leftrightarrow q. \end{aligned} \tag{10}$$

Given a safety level $1 \Leftrightarrow q$ the loss should at no time be greater than V_q^{PM} for each realisation. In other words V_q^{PM} is the $(1 \Leftrightarrow q)$-percentile of the distribution of the pathwise maximum. Glancing over the formulas of this "ruintheoretic" approach eq.(10) and eq.(8) with eq.(9), one would recognize that (roughly spoken) the order of the percentile and the

[3] In the practical implementation on has to concider a discrete approximation, for instance $\Delta T = 1$ month.

maximum is reversed, though there are totally different mathematical objects addressed. This is the reason for the notation V_q^{PM} and V_q^{MP}, abbreviated PM and MP. These measures constitute a more systematic approach to worst case scenarios than ad-hoc stress testing procedures.

Properties

Contrary to the measures used in common practice (cf. for example J.P.Morgan (1997)), it not sufficient to regard the discounted credit loss only in yearly intervals for determining V_q^{PM} and V_q^{MP}. Far more points in time have to be simulated and analyzed.

The main difficulty by handling V_q^{PM} consists in the analytical inaccessibility of the distribution of the maximum in eq.(10). Only in the case of a Wiener process there is an analytically result, cf. for example Karatzas and Shreve (1988). There are attempts to apply the extreme value theory (cf. Embrechts et al. (1997)) on stochastical processes like the Cox/Ingersoll/Ross process (Borkovec and Klüppelberg (1997)), but up to now the results of these attempts are not applicable. In the case of the loss process of a realistic portfolio it is necessary to perform a numerical approximation with a “sufficient” high number of points in time, where “sufficient” has to be characterized in a quantitative way, cf. Barth (1998).

We will give some qualitative remarks on the relationship of V_q^{PM} and V_q^{MP}:

- The percentile of the pathwise maximum V_q^{PM} has to be larger than the maximum of the percentils V_q^{MP}: For V^{MP} the q-percentile is determined by the values of the loss process at the *fixed* time, at which these percentiles are maximal: The mass $1 \Leftrightarrow q$ is concerned to a marginal distribution. In opposite, in the case of the pathwise maximum the mass $1 \Leftrightarrow q$ is concerned to the distribution of the maxima over *all the time interval*. Speaking roughly: All the points in the maximum distribution are maxima. This is not the case for the marginal distribution, even if this distribution is regarded at the time at which the q-percentile is maximal.

- The same qualitative argument applies to proove[4]

  $$V_q^{\mathrm{PM}}[0,T] \geq \max\left\{V_q^{\mathrm{PM}}[0,t], V_q^{\mathrm{PM}}[t,T]\right\} \tag{11}$$

  for every $t \in [0,T]$.

[4] The notation is not clear in this equation, because there has to be some conditioning of the starting value in respect to $V_q^{\mathrm{PM}}[t,T]$. But this argument applies if the starting value is fixed at time 0 or t.

- Because all the marginal distributions V_q^{P} have to be given for the calculation of $V_q^{\mathrm{MP}}[0,T]$, it is no problem to derive the same measure $V_q^{\mathrm{MP}}[t_1,t_2]$ on a smaller time interval $[t_1,t_2] \subset [0,T]$ by limiting the range over which the maximum in eq.(9) is taken. This is not possible in the case of the pathwise maximum measure V_q^{PM}. Only the distribution of the maximum is regarded, there is no information on time. A new Monte-Carlo simulation is needed.

- The last point clearifies, that even by new simulations it is not possible to construct for V_q^{PM} a kind of "upper bound for the cumulative loss" on a given time interval $[t,t+\Delta t]$. In the case of V_q^{MP} this is easily done by integrating over the given V_q^{P}.

- A trivial connection between V_q^{PM} and V_q^{MP}: For infinitesimal long "paths" these measures are the same:

$$\lim_{\Delta t \to 0} V_q^{\mathrm{PM}}[t,t+\Delta t] = V_q^{\mathrm{P}}[t] = \lim_{\Delta t \to 0} V_q^{\mathrm{MP}}[t,t+\Delta t]. \tag{12}$$

- Though the pathwise maximum is subadditiv (in fact it constituts a norm in the space of all possible path realisations starting at 0), this feature can not be transfered to V_q^{PM}, because the percentiles do not support this property. Therefore the same criticism as applied to the VaR-measure as not being *coherent* has to be applied to V_q^{PM} too, cf. Artzner et al. (1998) and Artzner et al. (1997).

- In the case of a constant default intensity there is "usually" a decomposition of a measure for the credit loss into the product of some kind of "exposure" times default probability. For both V_q^{PM} and V_q^{MP} this decomposition is only possible if there is only one counterparty.

Because of the first of these remarks we expect that V_q^{PM} is more sensitive to extreme realisations of the loss process than V_q^{MP}. Therefore V_q^{PM} should be better suited for a worst case analysis. This will lead to more conservative requirements for the risk-based capital. This will be part of the following considerations. In the next chapter we want to apply both measures V_q^{PM} and V_q^{MP} to a portfolio of swaps. These results will be compared to the results of other measures which are presented first in the next section.

2.2 Other Measures

The measures presented in this section are not based on a percentile approach like the measures derived in the last section. In principle there are two mechanism of aggregation over time if a cumulative approach is excluded (refering to the derivation of eq.(8)): First, considering the marginal distribution of the discounted loss $dL(t)$ at all times t in the fixed time interval $[0,T]$ and applying some functional (for example the expectation value, variance, kurtosis, or, as it was done in the last section, the percentile). Afterwards one takes the maximum of this functional over all $t \in [0,T]$ and defines this as the risk-based capital. The other possibility is to consider the distribution of the pathwise maximum and apply the same functionals to this distribution.

In the next chapter we will not use the measures based on the expectation value, the variance, or the kurtosis because they are not sensitive to extreme events and not suited for an analysis of worst cases[5]. We will illustrate this by giving in all simulation studies in addition to the worst case measures additionally the maximum value of the expectation value of the marginal distribution at all $t \in [0,T]$, we refer to this measure by "EM":

$$V^{\mathrm{EM}}[0,T] = \max_{t\in[0,T]} \mathrm{E}\left[dL(t)\right]. \tag{13}$$

This measure EM gives an answer to the question: What is the largest expected loss which might occur at some $t \in [0,T]$? Because the expected loss is used frequently in risk management, EM gives one "conventional" measure to contrast the worst case measures.

But we will discuss one other measure which is very sensitive to such events: The Tail Conditional Expectation ("TCE") (cf. for example Embrechts et al. (1997)) resembles the shortfall risk measure, but instead of the excess the whole loss $dL(t)$ is considered. As with MP and ME, we take the maximum over $t \in [0,T]$:

$$V_q^{\mathrm{TCE}}[0,T] = \max_{t\in[0,T]} \mathrm{E}\left[dL(t)|dL(t) > V_q^{\mathrm{P}}[t]dt\right]. \tag{14}$$

The niveau q gives with $V_q^{\mathrm{P}}[t]$ (the q-percentile of the marginal loss distribution) the upper border for losses which are "normal". All losses greater than that niveau are defined as exceedances. The conditional expectation only refers to exceedances and is therefore very sensitive to "fat tails". Given a fixed t, the function $q \to V_q^{\mathrm{TCE}}[t] \Leftrightarrow V_q^{\mathrm{P}}[t]$ is the mean excess function which represents per definition the tail of $dL(t)$. This measure also has the advantage of being coherent, cf. Artzner et al. (1998). We will discuss the properties of the TCE with respect to MP and PM numerically in the next chapter.

[5] The kurtosis might give a hint if there are heavy tails, but gives no reliable measure.

3 Numerics: Swap Portfolio

This numerical simulation study investigates the interplay of the response of the default risk to interest rate moves and the worst case analysis. After giving a description of the specification and the simulation technique, We first treat the case without response to get a reference for the study with response included, which follows afterwards.

Due to a lack in computing power[6] these simulations should be viewed as studies which give an impression of the methods used, the measures applied, and the role of the interaction between market and credit risks. For obtaining results which are more accurate one has to perform a larger number of realisations. This is necessary to calculate for example 99.98% percentiles, here only 95% percentiles are determined. In a practical application one has also to consider larger portfolios and a larger number of different contracts which increases the demand for computing power further.

Specification and Simulation Technique

We investigate a portfolio which consists only of interest rate swaps on the same underlying interest rate r. The time period is $T = 8$ years, the starting time is defined as $t = 0$. We limit ourself to four swaps with different remaining maturities (3, 4, 6, 8 years). All these swaps have semi-annual paying dates, which are the same paying-dates for all four swaps. One of these swaps is settled at t=0, the others already exist at this time. We model this fact by off-par shifts of the swap rate at time 0. The principal of all these swaps is the same (the absolute value does not matter).

This interest rate is modeled by the process proposed in Cox et al. (1985):

$$dr(t) = \kappa(\theta \Leftrightarrow r(t))dt + \sigma\sqrt{r(t)}dW, \tag{15}$$

with $\kappa = 0.268, \theta = 0.063, \sigma = 0.082$, and a starting value of $r(0) = \theta$. W is a standard Wiener process. The model is closed by the specificaton of the market price of risk $\lambda = 0$. These parameters are given by an estimation of Duffee (1996a) for the U.S. three month treasury bills in the period 1959 to 1992.

Further we assume that the value of the swap is not dependent on the rating of the counterparty (and of the considered financial institution itself). Due to the fact that the principals are not exchanged and that it is not clear which of the two parties will be

[6] These simulations are performed with an IBM PC with Pentium 166 processor under Mathematica 3.0.

exposed at which time (as seen at $t = 0$), this is current market practice. Sometimes an upfront payment is demanded if the difference between the ratings is high. Further there are theoretical and empirical investigations (cf. Duffie and Huang (1996), see also Stephan (1992)) which determine the spread concerning the swap rate. This spread is usually small (in the order of 1 basispoint for interest rate swaps[7]). We will neglect this spread and possible upfront payments. Table 1 summarizes the specification of the four swaps.

Swap	Par rate	Offset	Maturity	Nom.
# 1	6.35%	+0.5%	4 years	1
# 2	6.32%	0	6 years	1
# 3	6.29%	−0.4%	8 years	1
# 4	6.36%	+0.2%	3 years	1

Table 1: *Specification of the swaps used in the numerical investigation of the portfolio.*

Table 2 lists the different measures applied to single exposures of these swaps, seen from the paying fixed side. 5223 simulation runs were performed (monthly discretized), the simulation technique is treated below. Additionally the time at which the ME, MP and TCE are evaluated (i.e. the time when the measure of the marginal distribution is maximal) is given. The calculation of the different measures are done for the same set of realisations of paths of the short rate r. The intervals of confidence given for the two percentile measures are calculated by the method described in Barth (1998). These confidence intervals are independent of any distribution assumptions. But without any distribution assumptions there are no confidence intervals for ME and TCE available. There is no Brownian Bridge correction applied (cf. Beaglehole et al. (1997)).

In figure 2 the paths of the 95% percentile of the marginal distributions are displayed, cf. Duffee (1996a). Because of the offset the swaps 1 and 4 have at all times an expected value lower than 0, with this ME is 0. Swap 3 has a negative offset which results in a small expexted exposure ME, swap 2 has no offset and ME fluctuates round 0. As expected, the MP provides lower values than PM. More interesting are the values for TCE: They are "close" to PM, and TCE is for two swaps smaller, for the other two larger than PM. The time at which the MP and the TCE are maximal is the last month before a paying

[7] Duffie and Huang (1996) report a higher spread (up to 10 bp) for currency swaps because the nominal value is exchanged.

Swap	ME	t	MP	t	conf.	PM	conf.	TCE	t
# 1	0.0	0	6.3	18	[5.9, 6.7]	8.6	[8.2 ,8.9]	8.9	18
# 2	0.1	6	9.8	18	[9.1, 10.2]	13.5	[13.1, 14.0]	14.0	24
# 3	0.3	6	12.8	24	[12.1, 13.2]	17.3	[16.9, 18.1]	16.2	24
# 4	0.0	0	5.3	18	[5.0, 5.7]	6.7	[6.4, 7.0]	7.3	18

Table 2: *Measures of the expousures of the considered swaps, seen from the paying fixed side. Values are given in % of the nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. The time t of the maximum is given in months.*

date, which is not very surprisingly, because in this month the exposure is higher than in the following month, where a payment has been made.

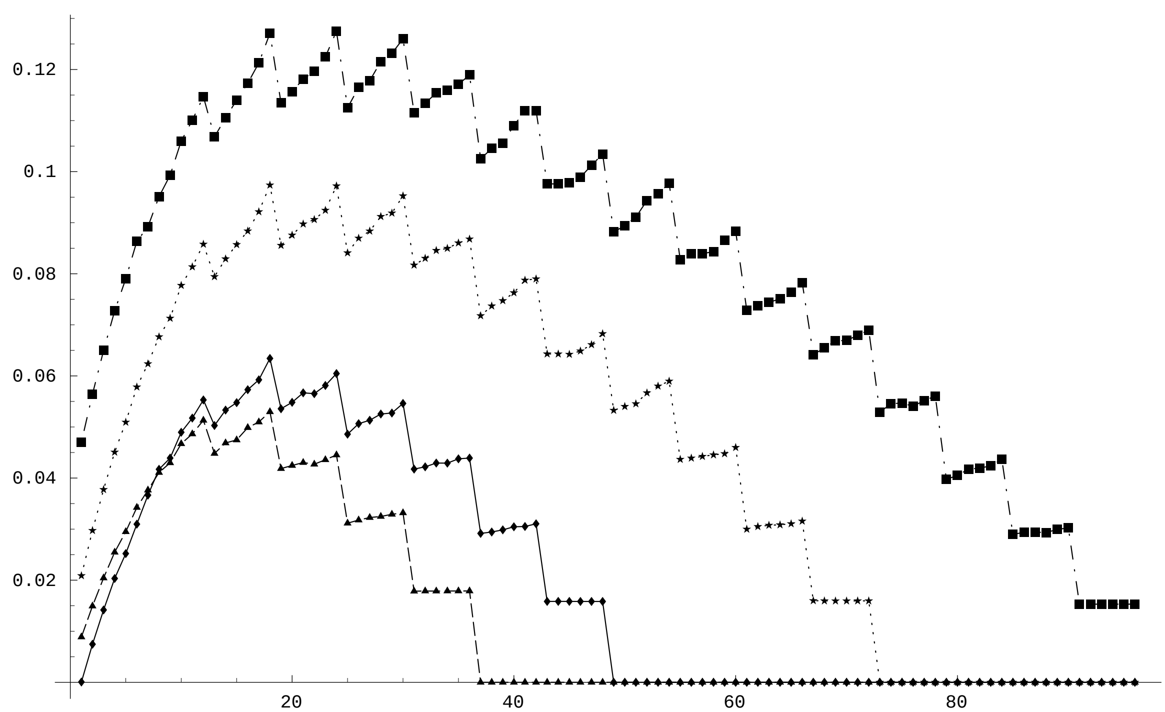

Figure 2: *"Paths" of the 95% percentile of the marginal distributions of the exposure of the swaps, see table 1. The exposure (in % of the nominal value) is plotted versus time (in months).*

The simulation technique is straightforward: A path of the short rate process r is simulated. We use a monthly discretization[8]. Based on this realisation of r the values of the

[8] For a proper treatment of the extremes of this process we use the Ito/Taylor-1.5-scheme as described in Kloeden and Platen (1992) for discretizing. Later on we discuss the correction methods described in

four different swaps are calculated at all (monthly) times. For each counterparty a the exposure V_a^+ and the actual default intensity $S_a(r)$ is determined for the specification ot the response function S_a. Subsequently the path of the loss in eq.(1) is calculated and the discounted maximum loss is stored, given in basispoints of the nominal value of all the contracts (which is the sum over all nominal values of all positions without netting), i.e. relatively to the size of the portfolio. After calculating a certain number of realisations the measures EM, PM, MP and TCE are determined.

3.1 Without Response

The aim of this section is to investigate the measures EM, PM, MP, and TCE without any response of the default intensities to the interest rate changes. These results will be refered to in the next section (with response) as the reference case. Moreover this section summarizes all results without such a response.

Setting

We consider a portfolio with 50 different counterparties. The different swap positions (each party is allowed to take up to seven units of the nominal value of each swap), the initial ratings and the response coefficients are randomly distributed at the beginning of the investigation, but after that they remain fixed. This mechanism leads to a "homogenic" portfolio: There are no counterparties who concentrate a large part of the value. We model the financial institution which holds the portfolio as an intermediary (which is not necessary): The only constraint to the random selection of the portfolio is given by the total balance of all the positions, so that the intermediary faces no market risk directly, but only credit risk. Due to the influence of the market variables onto the credit risk, there is an "indirect" dependency on the market variables. The study of this "indirect" dependency is one of the objectives of this study.

There are six rating classes with the default-intensities listed in table 3 (averaged default intensities in basispoints per year estimated on the time period 1920-1997 by Moody's and used in Duffie and Singleton (1998)). These default intensities are used as initial values for S_a at time 0. In this section we do not consider any response so that these default intensities of the counterparties will remain at this initial level for the whole time period.

Beaglehole et al. (1997) and Barth (1998) for taking account of difficulties in determining an extremum of a continuous time process by a discrete approximation.

Rating	Aaa	Aa	A	Baa	Ba	B
Intensity (bp/y)	0	9	9	32	146	442

Table 3: *Averaged default probabilities in basispoints per year estimated by Moody's for the time period 1920-1997.*

Dependency of the Realisations of the Short Rate Process

After generating the portfolio randomly, this portfolio is kept fixed. 1000 realisations of the short rate process are simulated, the credit loss for this portfolio is tracked and the measures ME, MP, PM, and TCE are calculated. This procedure is repeated three times to obtain three different sets of 1000 realisations of the same portfolio. The results are given in table 4. This investigation illustrated the stability of the results and the consistency with the estimated confidence intervals.

Real.	ME	t	MP	t	conf.	PM	conf.	TCE	t
# 1	0.16	18	0.38	23	[0.34, 0.43]	0.52	[0.49, 0.56]	0.50	24
# 2	0.15	18	0.37	24	[0.33, 0.40]	0.49	[0.45, 0.52]	0.48	24
# 3	0.15	24	0.36	24	[0.33, 0.39]	0.47	[0.46, 0.50]	0.47	24

Table 4: *Losses for different realisation of the same portfolio. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. Values are given in basispoints of the nominal value of the whole portfolio. The time t of the maximum is given in months.*

Why are these values so small? 0.38 basispoints does not seem to be much, but, first, these values are related to the total gross nominal value of the portfolio which is the sum over all nominal values of all the swaps (without any kind of netting). Second, the nominal value is not exchanged when dealing with interest rate swaps, therefore the credit risk is much smaller than if one is dealing with currency swaps or bonds themselves. Third, all the considered measures are related to the loss in a very short time interval (in this discretization one month) and not to the loss over all the time. This should not be confused with the elimination of the time dependency by taking the pathwise maximum or the maximum of the marginal distribution.

The values listed in the tabular above should be interpreted as follows: Given a fixed portfolio, the considered firm will lose in every unit Δt of time due to credit defaults on average not more than ME, at a 95% safety level not more than MP. The expected excess loss over a 95% percentile niveau in every time unit will not be larger than TCE, the largest loss in each time unit on a 95% niveau will for each realisation of the short rate process not be larger than PM.

One could interpret these measures as the amount of capital which might get lost at every point of time (calculated according to the chosen measure). While this interpretation addresses the risk-based capital, another interpretation addresses the spread of the swap rate, which might be also determined by analyzing the expected credit losses: If the considered firm is a financial intermediary who offsets each swap with one counterparty with an opposite position with another party to face no market risk, this firm should choose the "bid ask" spread of the swap rate by considering that measure which fulfills his demand for safety, cf. chapter 4 and Levis and Suchar (1994).

Dependency of the Realisations of the Portfolios

Port.	ME	t	MP	t	conf.	PM	conf.	TCE	t	# Real
# 1	0.15	18	0.33	18	[0.31, 0.34]	0.39	[0.37, 0.42]	0.38	24	1026
# 2	0.09	24	0.23	24	[0.21, 0.26]	0.28	[0.26, 0.30]	0.28	24	1010
# 3	0.16	18	0.38	23	[0.34, 0.43]	0.52	[0.49, 0.56]	0.50	24	1000
# 4	0.08	18	0.21	18	[0.20, 0.23]	0.25	[0.24, 0.27]	0.26	18	1067

Table 5: *Losses of different randomly generated portfolios (without considering any response). MP, PM, and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. Values are given in basispoints of the nominal value of the whole portfolio. The time t of the maximum is given in months.*

For four different portfolios of the same size, which are generated randomly by the same algorithm, the measures ME, MP, PM and TCE are calculated after doing more than 1000 simulation runs. Table 5 lists the results. We restricted ourself to a portfolio with only 50 counterparties. These results indicate that the size of the portfolio is not large enough to expect that due to the number of counterparties the individual contributions are averaged out. There exist large differences in the considered measures. We expect that for a portfolio with more than say 1000 counterparties the normalized absolute differences would vanish due to the Law of Large Numbers.

Size Effects, Marginal Effects and Subadditivity

Hull (1989) reports that the losses of small portfolio should have heavier tails than the losses of large portfolios due to diversification. He states that this is the reason why the capital to asset ratios are forced to be higher for small banks than for large banks. We want to see if this happens too in the model described in this paper:

For measuring the tail of a "small" portfolio (50 counterparties) we take the weighted average of the (relative) losses reported above for the portfolios 1, 2, and 3. As weights the gross nominal values are used. This is done for all measures ME, MP, PM, and TCE. We refer to these results with AME, AMP, APM, and ATCE. For a "large" portfolio we consider the (relative) loss path of the sum of the portfolios 1, 2, and 3, which constituts a portfolio with 150 counterparties. We calculate again the measures listed above for this portfolio (MEA, MPA, PMA, and TCEA[9]). In table 6 the results are listed (without response), losses are given in basispoints of the gross nominal value:

	MEA	AME	MPA	AMP	PMA	APM	TCEA	ATCE
no corr.	0.13	0.13	0.23	0.31	0.27	0.39	0.27	0.39

Table 6: *Size effects of the portfolio measures. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. Values are given in basispoints of the nominal value of the whole portfolio. The time t of the maximum is given in months.*

While the maximum of the expectation values ME is not influenced by the size of the portfolio, the measures MP, PM, and TCE indicate that the tail of the (relative) losses of the average of the small portfolios is heavier than the tail of the losses of the large portfolio. This result might be forced by the subadditivity of the maximum operation. But MPA (or PMA, TCEA) is even smaller than the smallest MP (or PM, TCE) for each single portfolio 1, 2, or 3. With this result Hulls observation is verified in the case without any response. We do not refer to these size effects as the "subadditivity" because *relative* losses are regarded.

Artzner et al. (1998) develop an axiomatic framework for "measures of risk" by proposing several general properties. The most "critical" axiom is given by the subadditivity, which is for example not fulfilled by the percentile-based Value-at-Risk approach. Measures which fulfill these axioms are called "coherent". Further they proove that the TCE is a

[9] Speaking roughly, the averaging procedure "A"is interchanged with the calculation of the measures ME, ... This is reflected in the nomenclature "AME" and "MEA", ...

coherent measure. By investigating the *absolute* losses (in units of the nominal value) we check in special cases if the subadditivity is fulfilled by the measures ME, MP, PM, and TCE.

This we do in the context of marginal effects (cf. Duffee (1996a)): What is the change in the measures ME, MP, PM, and TCE, if one more swap position is taken? We assume that the new swap position is taken by a new counterparty. We compare this marginal effect with the considered swap itself, for different cases in credit rating of the new counterparty. If the considered measure is subadditive, the marginal effect of adding one swap to the portfolio should be smaller than the stand-alone credit loss of this swap itself. The magnitudes of these effects will depend on the special structure of the given portfolio so that only an exemplary approach is possible.

Based on the investigation of portfolio 4 (see tabular above) we study the marginal effect of adding a new counterparty which holds a paying fixed position in swap 3 with one unit of nominal value. We vary the rating of this counterparty (Aa, Baa, B). In table 7, we label with "M(P)" the considered measure ME, MP, PM, or TCE applied to the portfolio without the new counterparty, with "M(P+S)" to the portfolio with the new counterparty, and with "M(S)" the measure applied to the credit loss of the new counterparty alone.

As expected, both the stand-alone value M(S) and the marginal effect M(P+S) $\Leftrightarrow$ M(P) are increased by a decrease in credit quality of the new counterparty. What is more important: The marginal effect M(P+S) $\Leftrightarrow$ M(P) is in all cases smaller than the the stand-alone value M(S). With this the subadditivity property M(P+S) $<$ M(P) + M(S) is fulfilled in these special cases. This might not always be the case, especially for the percentile-based measures MP and PM.

Brownian Bridge Correction

If a discretized stochastic process is considered instead of the "original" continuous time process, one has to be aware of the fact, that the maxima of the discrete version might differ from the maxima of the continuous time process. Beaglehole et al. (1997) propose an "interpolation" of the discrete points by a Brownian Bridge, which might be for these short time intervals a good approximation to the true stochastic process. Then a maximum could be chosen out of the (analytically known) distribution of the maximum of this Brownian Bridge between these points. In Barth (1998) this procedure is applied to the determination of the measures MP and PM in the cases of swaps. Here we apply this method to a portfolio of swaps. We perform 1241 realisations of the short rate process

Effect	Rat.	ME	MP	conf.	PM	conf.	TCE
M(P)	-	2.789	6.929	[6.490, 7.331]	8.215	[7.891, 8.888]	8.334
M(P)+M(S)	Aa	2.791	6.939	-	8.228	-	8.347
M(P+S)	Aa	2.791	6.929	[6.490, 7.331]	8.215	[7.891, 8.888]	8.342
M(S)	Aa	0.002	0.010	[0.008, 0.011]	0.013	[0.013, 0.014]	0.013
M(P+S)-M(P)	Aa	0.002	0.000	-	0.000	-	0.008
M(P)+M(S)	Baa	2.798	6.966	-	8.263	-	8.380
M(P+S)	Baa	2.797	6.942	[6.490, 7.344]	8.215	[7.920, 8.884]	8.360
M(S)	Baa	0.009	0.037	[0.031, 0.040]	0.048	[0.045, 0.051]	0.046
M(P+S)-M(P)	Baa	0.008	0.013	-	0.000	-	0.026
M(P)+M(S)	B	2.908	7.437	-	8.876	-	8.971
M(P+S)	B	2.904	7.227	[6.812, 7.631]	8.523	[8.028, 9.155]	8.707
M(S)	B	0.119	0.508	[0.429, 0.555]	0.661	[0.062, 0.070]	0.637
M(P+S)-M(P)	B	0.113	0.298	-	0.308	-	0.373

Table 7: *Marginal effects of adding a new swap position with a new counterparty rated Aa, Baa, B. Listed are absolute losses in one-thousandth units of the nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. The time t of the maximum is given in month.*

and calculate the measures ME, MP, PM, and TCE without any correction of the loss process of the portfolio, after that with the Brownian Bridge correction based on a "global" volatility estimation, and with a "local" volatility estimation, see for details Barth (1998). Results are given in table 8: In opposite to the effect of the correction applied to a single contract there is few influence of the Brownian Bridge correction method on the considered measures. This will change partially when the response is taken into account.

Method	ME	t	MP	t	conf.	PM	conf.	TCE	t
without	0.15	24	0.36	24	[0.33, 0.39]	0.47	[0.46, 0.50]	0.47	24
global	0.16	23	0.37	24	[0.35, 0.41]	0.47	[0.46, 0.51]	0.48	23
local	0.16	23	0.37	23	[0.35, 0.41]	0.47	[0.46, 0.51]	0.48	23

Table 8: *Brownian Bridge correction (with "global" and "local" volatility estimates) compared to the values without correction. Results are given in basispoints of the gross nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98 % niveau. The time t of the maximum is given in month.*

3.2 With Response

The no-response case has set a reference for the measures ME, MP, PM, and TCE. Now the role of the response between the default risk and the interest rate is explored. In the following first three sections only the exponential response function eq.(2) is considered, after that other response functions are studied. At last a portfolio with different response functions for different counterparties is investigated as the "most realistic" case of the study presented in this paper.

Dependency of the Strength of the Correlation: Exponential Correlation

The effects of the response strength k used in eq.(2) for the dependency of the (known) default intensity $S_a(r)$ on the interest rate r is one of the main subjects of this investigation. We assume that the counterparties could be divided into four "response classes": One class with a slight negative response (-1), one with a vanishing response (0), one with a slight positive response (+1) and the last one with a strong positive response (+4). We vary the absolute response strength with a scalar multiplicate k of this "response vector" $(-1, 0, +1, +4)$, where we choose $k = 0, 2, 4, 6, 8$. The case $k = 0$ corresponds to no response between the interest rate and the default probability. For example, if counterparty a responds with coefficient -1, the response coefficient k_a in eq.(2) writes $k_a = k(-1)$. In Barth (1999) empirical evidence (see Düllmann et al. (1998), Duffee (1996b), and Longstaff and Schwartz (1995)) is reviewed for calibrating the parameter k_a of these functions. It results that the parameter k_a should be chosen out of the interval $[-32, 32]$ which is done here.

As described in the last section, the portfolio is drawn randomly and then kept fixed. 1000 realisations of the short rate process r are performed. For each realisation the losses at all the points in time are calculated for the different response parameters $k = 0, 2, 4, 6, 8$. To give a qualitative impression of the impact of the response stength k on the resulting data, the discounted maximal loss (over the given time interval) of each realisation of r is plotted in figure 3. For each k, the data set gives the distribution of the maximal loss. The measure PM is the upper percentile of this distribution[10].

By analyzing figure 3, several observations can be done:

[10] The other measures ME, MP, and TCE are not based on this distribution but on the marginal distributions, which are not given here.

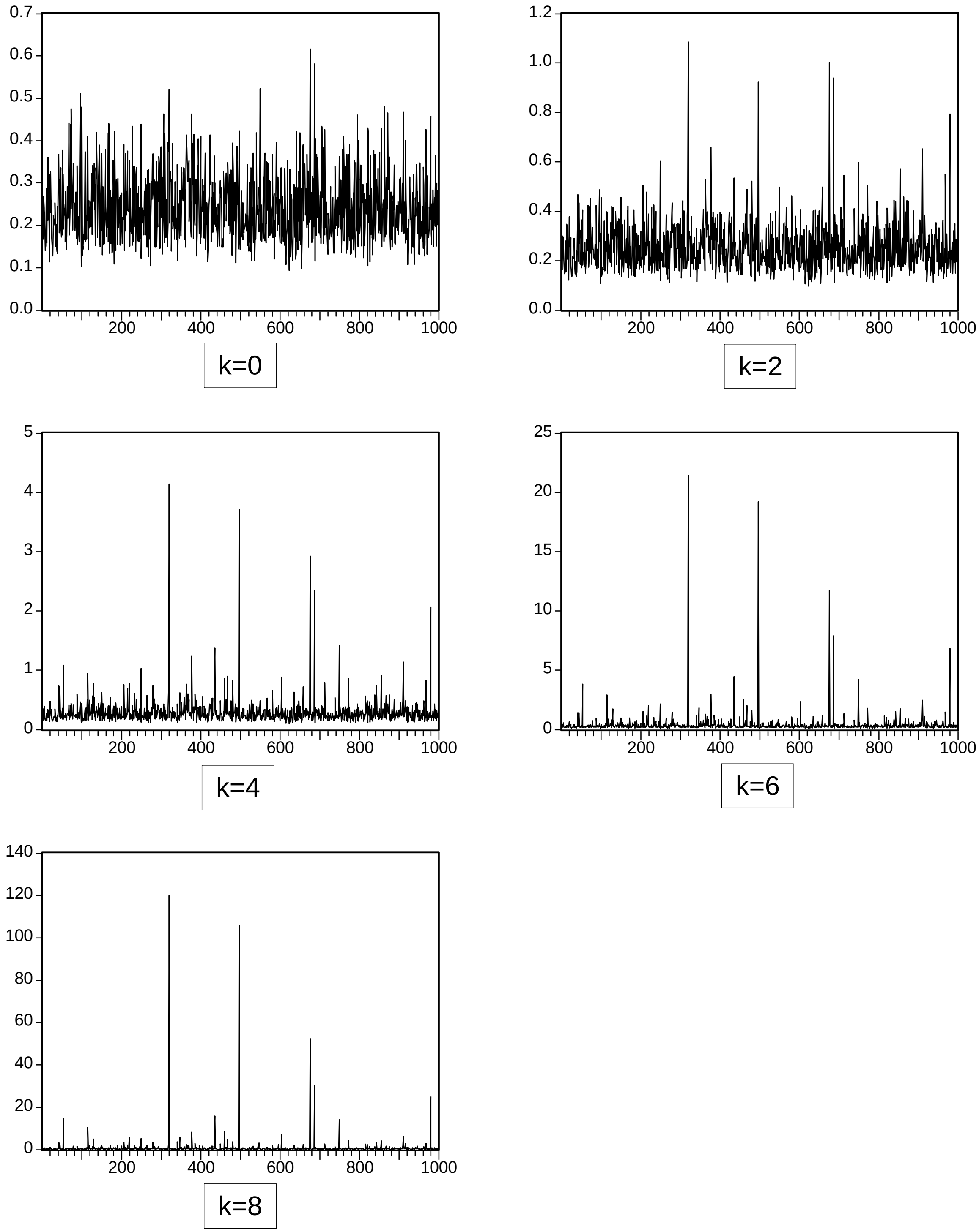

Figure 3: *Maximum loss (in basispoints of the nominal value) versus the number of the realisation under the exponential response function eq.(2). Each figure is based on the same realisations, but calculated with different response strength* $k = 0, 2, 4, 6, 8$.

- It exists a "ground level" of the maximal loss: This ground level ($\approx 0.2 \dots 0.3$ basispoints of the nominal value) might correspond to the average loss, if only the losses in ordinary market situations are regarded. This ground level seems to be unaffected by the response strength.

- Without response the losses seem to be distributed without "fat tails": There are some peaks, but these peaks did not exceed the threefold ground level. With response the situation changes: The ground level remains the same, but the peaks are much more distinct. The peaks grow with the response strength k.

- But not all the peaks which appear in the case of no response are getting very large, only a few. Therefore the peaks in the case with response are not only an "exponential rescaling" of the peaks in the case of no response. For example one could compare the losses for two realisationen, listed in table 9 and pictured in figures 4 and 5: Though there is a higher loss for realisation #603 than #604 for $k = 0$, the loss in #604 grows fastly in opposite to #603 for $k > 0$: In case of #603 there is a sudden drop of the interest rate at the beginning of the time-periode, which causes the height of the losses. But the height of these losses is less influenced by the response than the losses in case of #604, which is due to a rise of the interest rate at the end of the time period.

- The graphs for the case of strong response resemble some graphs which appear in the context of insurances, for instance fire insurance data: claime sizes plotted against time (compare Embrechts et al. (1997)).

Abs. corr. str. k	0	2	4	6	8
Max. loss (bp/nom): #603	0.34	0.32	0.32	0.31	0.36
Max. loss (bp/nom): #604	0.27	0.41	0.89	2.38	7.09

Table 9: *Comparison of the maximal losses of the same portfolio for two realisations with varying response strength k. Results are given in basispoints of nominal value.*

We did these simulations with three different realisations of the random portfolio. For each portfolio the qualitative observations remain the same, though the ground level shifts. There are larger differences for the measures PM, MP, and TCE for these three portfolios, results are listed in table 10. We refer to the portfolio which data are displayed in figure 3 as portfolio 1 in table 10.

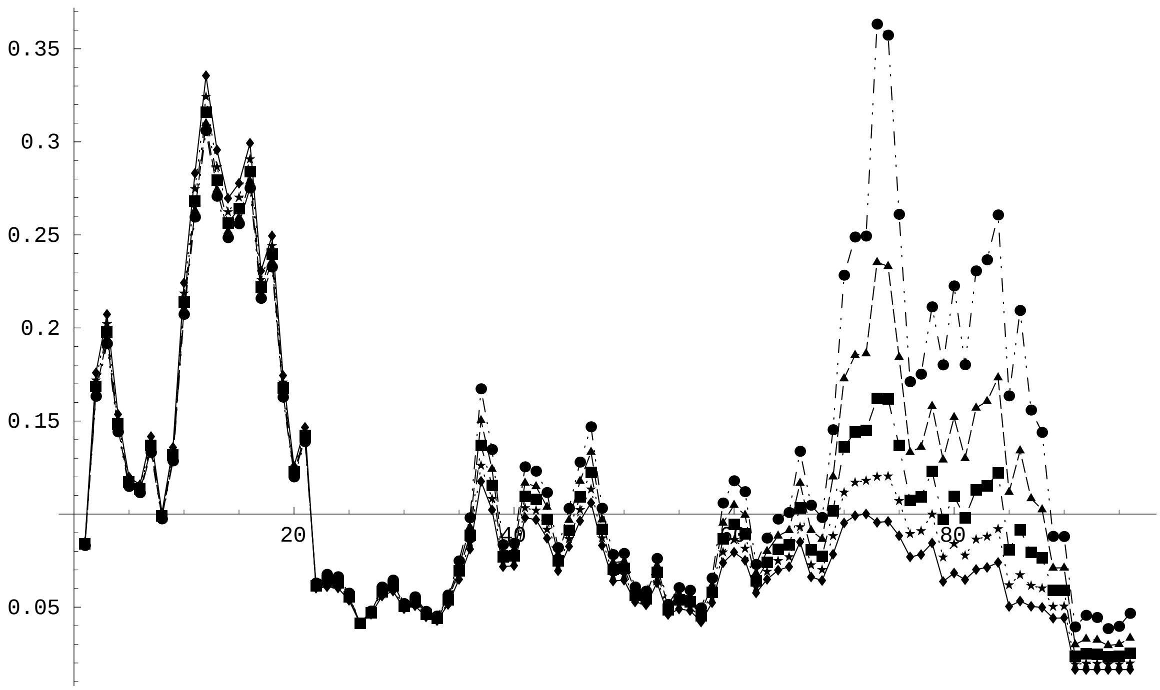

Figure 4: *Realisation #603: "Normal scenario": The loss (in basispoints of the gross nominal value) is plotted versus time (number of months), given an exponential response function with response strength $k = 0$ (checks), 2 (stars), 4 (squares), 6 (triangles), 8 (circles).*

These large differences are not surprising because it is to be expected that measures which are concerned by the tails of the marginal distributions or the maximum distribution are more sensitive to changes in the portfolio than an average value, which is represented in this investigation by the ground level. The most important result: The measure PM, which is based on the distribution of the pathwise maximum, and the measure TCE, which is based on exceedances over a threshold, are much more concerned by the heavy tails than the measure MP, which is based on the maximum of the percentiles of the marginal distributions. In other words: PM does not recognize the high peaks caused by the introduction of the response, MP and TCE do. For this reason we conclude that the measures MP and TCE are more suited as worst case measures than PM.

Size Effects, Marginal Effects and Subadditivity: Exponential Correlation

As it was done in the case without any response, we study now size and marginal effects in the case with response: Again we calculate the weighted average of the (relative) losses reported in table 10 for the portfolios 1, 2, and 3, based on the investigation with a

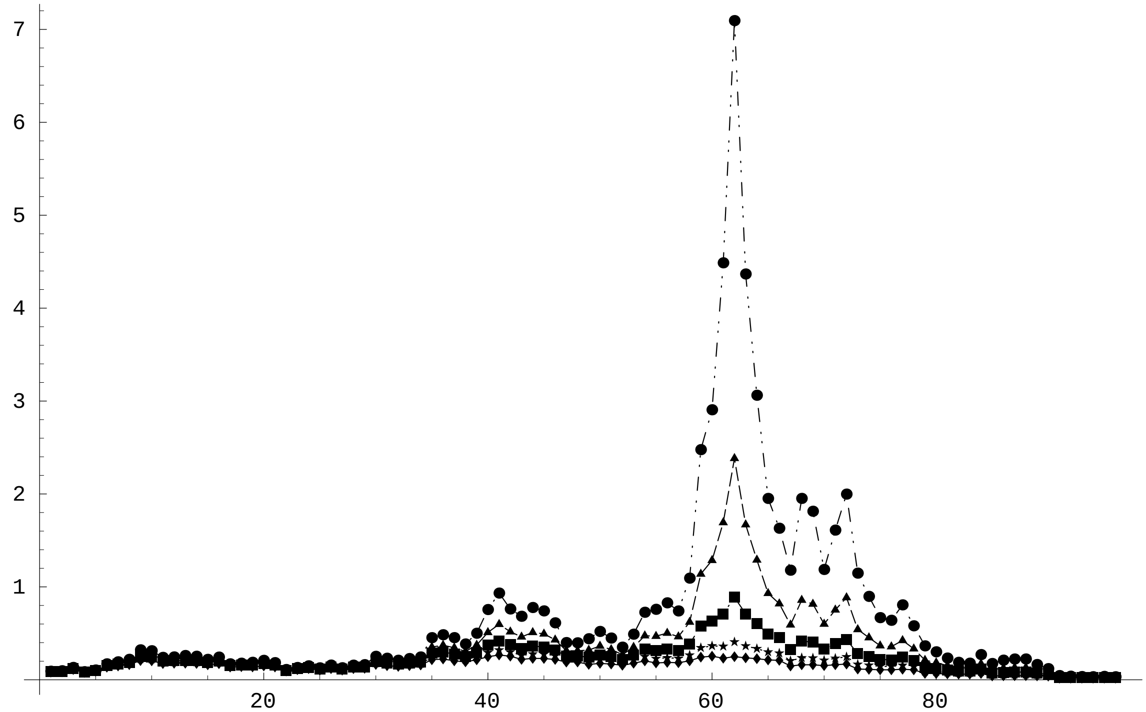

Figure 5: *Realisation #604: "Catastophe": The loss (in basispoints of the gross nominal value) is plotted versus time (number of months), given an exponential response function with response strength $k = 0$ (checks), 2 (stars), 4 (squares), 6 (triangles), 8 (circles).*

strong response ($k = 8$) modelled by the exponential function eq.(2), As weights the gross nominal values are used. As the loss of a "large" portfolio we consider again the (relative) loss of the sum of the portfolios 1, 2, and 3, which constituts a portfolio with 150 counterparties. This is done for several different response strengths $k = 0, 2, 4, 6, 8$, results are listed in table 11. For details and nomenclature see the corresponding section above (no response).

While the maximum of the expectation values ME differs only slightly, the measures MP, PM, and TCE differ to a larger extend when these measures are applied to the large portfolio instead of taking the average over the sub-portfolios (as it was done in the case without any response). But it is no longer true that MPA (or PMA, TCEA) is even smaller than the smallest MP (or PM, TCE) for each single portfolio 1, 2, or 3: This is due to the large differences in these measures between the single portfolios 1, 2, and 3. For example will portfolio 3 be mainly responsible for the heavy tail measured by PM and TCE for the large portfolio in the case $k = 8$. But diversification effects are still very strong even in this case, so that the statement of Hull (1989) can by supported in this study too.

k	Portf.	ME	t	MP	t	conf.	PM	conf.	TCE	t
0	#1	0.15	18	0.33	18	[0.31, 0.34]	0.39	[0.38, 0.42]	0.38	24
0	#2	0.09	24	0.23	24	[0.21, 0.26]	0.28	[0.26, 0.30]	0.29	24
0	#3	0.16	18	0.38	23	[0.34, 0.43]	0.52	[0.49, 0.56]	0.50	24
2	#1	0.15	18	0.33	18	[0.31, 0.35]	0.41	[0.40, 0.44]	0.41	30
2	#2	0.10	24	0.25	24	[0.23, 0.30]	0.37	[0.35, 0.42]	0.37	24
2	#3	0.17	24	0.48	22	[0.39, 0.55]	0.77	[0.70, 0.87]	0.70	24
4	#1	0.16	18	0.34	30	[0.31, 0.38]	0.54	[0.49, 0.63]	0.53	30
4	#2	0.11	24	0.32	36	[0.27, 0.40]	0.61	[0.56, 0.71]	0.56	30
4	#3	0.20	24	0.63	22	[0.49, 0.75]	1.35	[1.18, 1.68]	1.35	47
6	#1	0.17	24	0.38	54	[0.30, 0.45]	0.91	[0.77, 1.20]	1.06	48
6	#2	0.14	30	0.44	36	[0.34, 0.59]	1.14	[0.94, 1.48]	1.00	30
6	#3	0.32	46	0.85	47	[0.59, 1.15]	2.68	[2.24, 3.31]	3.96	46
8	#1	0.28	48	0.51	54	[0.39, 0.68]	1.79	[1.46, 2.81]	3.50	47
8	#2	0.20	30	0.65	40	[0.45, 0.82]	2.40	[1.77, 3.40]	2.22	66
8	#3	0.93	46	1.30	47	[0.84, 1.89]	5.32	[4.53, 7.11]	15.70	46

Table 10: *Three different portfolios, losses in dependency of the response strength k. In case of: portfolio 1 1026 runs are done, portfolio 2 1010, portfolio 3 1000. Results are given in basispoints of the gross nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. The time t of the maximum is given in months.*

k	MEA	AME	MPA	AMP	PMA	APM	TCEA	ATCE
0	0.13	0.13	0.23	0.31	0.27	0.39	0.27	0.39
2	0.14	0.14	0.26	0.35	0.34	0.50	0.33	0.48
4	0.15	0.16	0.33	0.42	0.57	0.80	0.54	0.78
6	0.19	0.21	0.46	0.54	1.14	1.51	1.51	1.90
8	0.43	0.45	0.74	0.79	2.87	3.03	5.92	6.73

Table 11: *Size effects for different response strengths. Results are given in basispoints of the gross nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau.*

Similar to the investigation of the subadditivity in the case without any response, We consider a new counterparty with varying credit rating (Aa, Baa, B), but with a fixed strong positive response (+4) of the default intensity to the interest rate, modelled by the exponential function eq.(2), see table 12.

Effect	Rat.	ME	MP	conf.	PM	conf.	TCE
M(P)	-	3.66	9.01	[7.21, 12.00]	22.57	[18.12, 34.39]	25.54
M(P)+M(S)	Aa	3.68	9.07	-	22.84	-	25.87
M(P+S)	Aa	3.67	9.07	[7.26, 12.10]	23.19	[18.40, 34.84]	25.74
M(S)	Aa	0.02	0.06	[0.04, 0.07]	0.28	[0.20, 0.44]	0.32
M(P+S)-M(P)	Aa	0.01	0.06	-	0.62	-	0.200
M(P)+M(S)	Baa	3.73	9.22	-	23.55	-	26.69
M(P+S)	Baa	3.71	9.22	[7.28, 12.30]	23.56	[18.84, 36.18]	26.23
M(S)	Baa	0.07	0.21	[0.13, 0.26]	0.98	[0.71, 1.55]	1.15
M(P+S)-M(P)	Baa	0.05	0.21	-	1.00	-	0.69
M(P)+M(S)	B	4.63	11.88	-	36.11	-	41.49
M(P+S)	B	4.33	11.86	[8.39, 14.30]	33.10	[27.58, 52.61]	35.07
M(S)	B	0.97	2.87	[1.84, 3.64]	13.54	[9.80, 21.45]	15.95
M(P+S)-M(P)	B	0.67	2.85	-	10.54	-	9.53

Table 12: *Marginal effects of adding a swap position (swap 3) with a new counterparty with rating Aa, Baa, B and a strong positive (+4) exponential response. Listed values are absolute losses in one-thousandth units of the nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau.*

There are two important results: First, the subadditivity is not valid in all cases: In the case of the measure PM the marginal effect is larger than the measure of the new counterparty alone. This implies that the capital requirement is lower if the new counterparty is booked separately from the portfolio, which does not make sense from the viewpoint of diversification. Artzner et al. (1998) have given other examples for the non-validity of the subadditivity of percentile-based risk measures. But they prooved that the TCE measure is coherent, i.e. that the marginal effects are always lower than the stand-alone effect. The results of this study do not contradict this statement. Second, the effect of one contract with one counterparty with a low credit rating and a large response coefficient gives an important contribution to the heavyness of the tail: If the new counterparty is rated “B”, more than one-third of the TCE of the extended portfolio is determined by this one contract.

Duffee (1996a) stresses the fact that the marginal impact of one contract to the credit risk of the portfolio is more important to consider than the credit risk of the contract itself, i.e. the contract should be seen only in the context of the portfolio. But he concentrats on percentile-based measures which are not coherent. Given a subadditive risk measure, the credit risk of the contract alone is an upper bound to the marginal effect to the credit risk of the portfolio by this contract. With this, the warning of Duffee (1996a) is not important when coherent measures of credit risk are regarded.

Brownian Bridge Correction: Exponential Correlation

As it was done in section 3.1 without any response, the Brownian Bridge correction method is now applied in the case with exponential response function. It is important to investigate the effects of this correction method for the worst case measures in the case with response between the interest rate and the default intensity, because due to the resulting higher volatility the effects of this correction method might lead to different results than in the "harmless" case of no response, see above. Therefore we choose portfolio 3 (see table 10) because this portfolio displays the most extreme behaviour of all the portfolios regarded in this investigation. As could be seen in the last two sections, the differences between the worst case measures are getting larger with the response strength. What role will the Brownian Bridge correction play? We performed 1240 realisations with portfolio 3, results are displayed in table 13.

Several observation can be done:

- The effects of the Brownian Bridge correction to the measures MP and TCE are getting larger with increasing response strength.
- There is nearly no effect of the correction to the percentile of the pathwise maximum PM. This is in opposite to the application of the Brownian Bridge correction of single swap exposures.
- Only for the strong response with $k = 8$ there is a large difference between the correction with the "local" or "global" estimation of the volatility (see Barth (1998)) for the measures MP and TCE. In the other cases there are nearly no differences.

The last item is one of the reasons why we will not apply the Brownian Bridge correction further: A more sophisticated method for estimating the volatility would be needed to give a unique result. Another reason is that it would be necessary to determine the volatility

k	Method	ME	t	MP	t	conf.	PM	conf.	TCE	t
0	without	0.15	24	0.36	24	[0.33, 0.39]	0.47	[0.46, 0.50]	0.47	24
0	global	0.16	23	0.37	24	[0.35, 0.41]	0.47	[0.46, 0.51]	0.48	23
0	local	0.16	23	0.37	23	[0.35, 0.41]	0.47	[0.46, 0.51]	0.48	23
2	without	0.16	30	0.44	30	[0.39, 0.48]	0.70	[0.66, 0.78]	0.67	36
2	global	0.18	29	0.46	29	[0.41, 0.51]	0.70	[0.66, 0.78]	0.68	35
2	local	0.18	29	0.46	29	[0.41, 0.51]	0.70	[0.66, 0.78]	0.69	35
4	without	0.19	36	0.60	42	[0.50, 0.70]	1.20	[1.07, 1.41]	1.20	40
4	global	0.21	29	0.64	41	[0.53, 0.76]	1.20	[1.07, 1.41]	1.30	41
4	local	0.21	29	0.64	41	[0.53, 0.76]	1.20	[1.07, 1.42]	1.30	41
6	without	0.28	36	0.86	42	[0.70, 1.08]	2.27	[2.02, 2.81]	2.69	39
6	global	0.31	41	0.95	41	[0.76, 1.19]	2.27	[2.02, 2.81]	2.92	39
6	local	0.38	43	0.96	41	[0.77, 1.20]	2.28	[2.02, 2.82]	2.92	39
8	without	0.51	39	1.27	42	[1.00, 1.66]	5.01	[4.04, 6.08]	7.05	39
8	global	0.59	41	1.46	41	[1.10, 1.93]	5.01	[4.04, 6.08]	7.71	39
8	local	1.26	43	2.20	43	[2.04, 2.35]	5.02	[4.11, 6.21]	7.72	38

Table 13: *Brownian Bridge correction (with "global" and "local" volatility estimates) compared to the values without correction for different response strengths k. Results are given in basis-points of the gross nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. The time t of the maximum is given in months.*

much more precise at the times in which the credit risk of the portfolio is very large. For doing this more computing power is needed. Otherwise the results are not stable. And because we are interested in the worst case we think it is better to have stable results which are maybe a little bit to low than unstable large results, which are not reliable.

Other Correlation Functions

Up to now only an exponential function was examined as response function S_a. But the response behaviour might not be described "correctly" by this function. In this section we regard three other response functions: As listed in section 1, we consider the quadratic eq.(3), linear eq.(4), and the square-root function eq.(5).

But in opposite to eq.(4) and figure 1, We let the linear function go to 0 instead of the "start level" $S_a^l(r(0))$ for large "favourable" interest rate moves $k_a(r(t) \Leftrightarrow r(0)) < 0$ (as seen from the counterparty), i.e. instead of eq.(4) we use

$$S_a^l(r(t)) = S_a^l(r(0)) \max\{0, 1 + k_a(r(t) \Leftrightarrow r(0))\}\,. \tag{16}$$

The reason for this is that the investigation of eq.(4) is received as a by-product of the mixed-function investigation of the next section. Here it is studied whether a "reversed symmetrical" response function as eq.(16) in regard to the start level (the absolute value of the effect on S_a of a favourable interest rate move has the same size as the unfavourable move of the same magnitude) leads to significant different results than without response.

As described above, we consider four different response coefficients k_a. The responsiveness to interest rate changes of the default risk of each counterparty in the portfolio is characterized by k_a. In opposite to the previous approach we consider the four coefficients k times ($\Leftrightarrow 4$, $\Leftrightarrow 1$, $+1$, $+4$) instead of k times ($\Leftrightarrow 1$, 0, $+1$, $+4$): Beside the investigation of the dependency of the reponse function it is interesting to see whether the response effects disappear in the case of a non-biased distribution of the response coefficients. We choose again $k = 0, 2, 4, 6, 8$. All the effects of these different responses are calculated on base of the same realisations of the short rate. After performing 1067 realisations the values in table 14 result.

The linear and square root response function lead to nearly the same values for all measures compared to these measures in the case without response. The linear function eq.(16) allows the counterparty to drop out of the credit risk calculation. In the following section the linear response function eq.(16) is replaced by eq.(4), which leads to larger values. This effect can also be seen at the results of the square-root response eq.(5),

function	ME	t	MP	t	conf.	PM	conf.	TCE	t
no corr.	0.09	18	0.21	18	[0.20, 0.23]	0.25	[0.24, 0.27]	0.26	18
root	0.09	18	0.24	18	[0.22, 0.25]	0.29	[0.27, 0.31]	0.30	18
linear	0.08	18	0.20	24	[0.17, 0.22]	0.26	[0.24, 0.31]	0.28	18
quadr.	0.10	18	0.28	24	[0.24, 0.33]	0.48	[0.41, 0.58]	0.52	24
expon.	0.11	24	0.28	24	[0.22, 0.37]	0.69	[0.56, 1.05]	0.78	24

Table 14: *Different response functions are applied to all counterparties of the same portfolio. We only report results without response and for the strongest response $k = 8$. Results are given in basispoints of the gross nominal value. MP, PM, and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98% niveau. The time t of the maximum is given in months.*

which are slightly higher than the linear results, though the square-root response has a slower increase for unfavourable interest rate moves than the linear response. But the square-root response functions is not allowed to drop under the start level.

In the case of the quadratic and exponential response MP changes only a little too, but PM and TCE change much more. In other words, the quadratic and exponential response function are responsible for heavier tails in the loss distribution, which is only recognized by the PM and TCE measure.

The response effects are not vanishing in consequence of the "non-biased" distribution of the response coefficients: There is some "asymmetry" of credit risk: In "good" cases (for the regarded counterparty) the credit risk will not disappear, but in "bad" cases it might increase dramatically.

Mix of Correlation Functions

In this section the most realistic numerical investigation of this paper is described. The counterparties are divided into several classes. Each of these classes is interpreted as one "sector" (for example industry, bank, insurance company, ...). The response behaviour of all the counterparties in one sector is qualitatively the same. But each sector has another response function. In this approximation it is only necessary to determine the response function for each sector instead of each counterparty. This is more realistic because there are more data available for each sector than for each counterparty. Estimations as in Düllmann et al. (1998), Duffee (1996b) and Longstaff and Schwartz (1995) could be improved and applied. We distinguish six different sectors by introducing six different sorts of response functions: Quadratic as in eq.(3), linear as described in eq.(4), and exponential as in eq.(2), each with positive and negative sign of the response coefficient k_a.

The absolute value of the response coefficient is not fixed within each sector: In each sector there are members with different credit quality as measured by the rating. Duffee (1996b) describes that the (absolute) responsiveness of the change in the yield spreads grows as the credit rating gets lower. This behaviour is modelled here for the (relative) responsiveness which is given by k_a in eq.(2), eq.(4), and eq.(3): k_a is dependent of the rating of the counterparty and not (as in the last sections) drawn randomly. We used again six rating classes, but we took another estimate of the default intensities. These default intensities are used as initial values for $S_a(r)$ at time 0. The chosen k_a and the default intensities are given in table 15.

Rating	Aaa	Aa	A	Baa	Ba	B
Intensity S_a (bp/y)	7	9	20	50	195	400
Resp. coeff. k_a	6	6	12	18	24	24

Table 15: *Averaged default intensities over ten years in basispoints per year estimated on the time period 1970-1993 by Moody's reported in Fons (1994) and response coefficients, dependent on the rating.*

We vary the mapping of the response functions to the different sectors. This is done with two purposes in mind: First, how important is the variation of the response function of one sector with the other sectors fixed? Second, is there a "dominance" of the sector with the strongest response dependency (i.e. the exponential response) over the other sectors (which might be only correlated linearly or quadratically), especially in worst case scenarios? If this should be the case, the credit risk management could concentrate on the counterparties which belong to this sector and ignoring the others sectors to get an impression of a worst case scenario.

After generating a fixed portfolio with each counterparty related with one sector, We investigate ten different mappings of reponse functions, as described in table 16. The first mapping corresponds to no response at all. In the other cases the sign gives the sign of the coefficient k, the absolute value of k is determined by the tabular above. "lin", "qua", and "exp" refer to the sort of reponse function.

#	sec. 1	sec. 2	sec. 3	sec. 4	sec. 5	sec. 6
0	-	-	-	-	-	-
1	⇔lin	⇔lin	+lin	+lin	+lin	+lin
2	⇔lin	⇔lin	+lin	+lin	+lin	+qua
3	⇔qua	⇔lin	+lin	+lin	+lin	+qua
4	⇔qua	⇔lin	+lin	+qua	+qua	+qua
5	⇔qua	⇔qua	+qua	+qua	+qua	+qua
6	⇔lin	⇔lin	+lin	+lin	+lin	+exp
7	⇔qua	⇔lin	+lin	+lin	+qua	+exp
8	⇔exp	⇔qua	+lin	+qua	+exp	+exp
9	⇔exp	⇔exp	+exp	+exp	+exp	+exp

Table 16: *Mapping of the reponse functions to the different sectors of the counterparties.*

To diminish the probability of special portfolio effects we run this simulation with 100 counterparties. Due to limited computing power only 551 realisations are done. Table 17 gives the results.

#	ME	t	MP	t	conf.	PM	conf.	TCE	t
0	0.11	18	0.24	18	[0.21, 0.26]	0.32	[0.30, 0.36]	0.30	24
1	0.14	18	0.40	22	[0.30, 0.47]	0.72	[0.60, 0.91]	0.61	36
2	0.14	18	0.40	22	[0.30, 0.47]	0.72	[0.60, 0.91]	0.61	36
3	0.14	18	0.40	22	[0.30, 0.47]	0.72	[0.60, 0.91]	0.61	36
4	0.15	36	0.42	22	[0.30, 0.53]	1.21	[0.86, 1.54]	1.10	60
5	0.15	36	0.43	22	[0.30, 0.54]	1.25	[0.90, 1.59]	1.10	60
6	0.14	18	0.39	22	[0.30, 0.47]	0.72	[0.60, 0.91]	0.60	36
7	0.14	18	0.39	22	[0.30, 0.49]	0.84	[0.67, 1.10]	0.71	36
8	0.16	60	0.43	22	[0.29, 0.56]	1.39	[0.94, 1.85]	1.87	64
9	0.32	64	0.48	22	[0.33, 0.65]	1.83	[1.19, 2.83]	5.18	64

Table 17: *Portfolio losses for different mappings of the response functions to the counterparties. Results are given in basispoints of the gross nominal value. MP, PM and TCE are related to a 95% percentile niveau, the confidence intervals are calculated at a 98 % niveau. The time t of the maximum is given in months.*

Several observations can be done, though they may depend on the special structure of the portfolio (we cannot do more than exemplary studies here, but the same method might be applied more generally):

- The introduction of the response leads to a strongly increased credit risk, regardless which measure is considered.
- Except in the case of mapping 9 the tail measures PM, MP and TCE are much more sensitive to changes in the response behaviour than the ground level ME.
- Regarding mappings 8 and 9, the measure TCE is much more concerned with the strong response than PM. Only in these two cases TCE exceeds PM.
- The variation of the response function of sector 6 (see mappings 1, 2, and 6) alone does not change the credit risk. Sector 6 does not seem to be very sensitive to the kind of the response.
- By regarding mappings 4, 5, 8, and 9 one might conclude that the counterparties in sector 4 are responsible to a large amount of the behaviour of the loss distribution, both for the ground level ME and the tails MP, PM and TCE. Somewhat surprising is the strong rise of the ground level ME in the case 9 of only exponential response functions. In the case of this special portfolio this observation might give a hint to the credit risk management to pay attention to sector 4.

A further idea is to take the average of each measure over all the different mappings. This might take account of the uncertainty about the representation of the response function of each sector. A possible practicable procedure could be: After sorting the counterparties into different sectors, one derives several possible response functions. The parameters might be taken from an estimation for small movements in the interest rate, like given in Duffee (1996b). Then a simulation like the one described above is run. By taking the (maybe weighted) average over the resulting measures one achieves a picture of the response without relying on one special response function.

4 Extensions

The points listed here are ideas for extending the work:

- As an extension to the percentile-based approach eq.(10) one could consider the following situation: If the credit loss is netted with the gain through a (deterministic) premium payment function $P_a(t)$, an extension of the approach eq.(10) could be given by:

$$\text{Prob}\left[\frac{1}{B(0,t)}\sum_{a=1}^{N}\left[\left(V_a^+(t)+P_a(t)\right)I_a(t)\Leftrightarrow P_a(t)\right]>V_0\,\text{for one}\,t\in[0,T]\right]\leq 1\Leftrightarrow q. \quad (17)$$

  This is the situation of the classical risk theory, cf. for example Bühlmann (1970). We think of the situation of a financial intermediary, who receives a premium P_a for each counterparty. This premium payment should be dependent on the credit rating of the counterparty (and of the contracts with this counterparty). The intermediary might balance his portfolio in the sense that he is not facing any market risk by offsetting every position with different counterparties. He earns the premium and is only exposed to credit risk, cf. Levis and Suchar (1994). Introducing deterministic premium functions as in eq.(17) might give a possibility to calculate a dependency of these premia and the risk-based capital, but will not change the model essentially. Given a fixed set of contracts which are traded by this intermediary: If this intermediary is able to vary over the positions with the different counterparties, he wishes to minimize the credit loss and maximize the gain through the premia payments under the constraint that there is no offset.

- It would be interesting to apply the techniques of the extreme value theory, for instance the Peaks-Over-Threshold fitting with generalized Pareto distributions, cf. Embrechts et al. (1997) to the data generated by this model.

5 Conclusion

Based on the model derived in Barth (1999), in chapter 2 several different measures of a worst case credit risk scenario were discussed. These measures were discussed qualitatively and numerically: The first measure (MP) is given by the maximum of the percentiles of the marginal loss distribution. The second measure (PM) is defined as an upper percentile of the distribution of the pathwise maximum of the loss. The third measure (TCE) is defined as the maximum of the conditioned expectation values of all the marginal loss distributions, given that the loss exceeds an upper percentile. Due to the fact that by determining numerically one measure the other measures are also given without much more expenses in computing power, a simultaneous use of all three will give well-founded informations about possible worst case scenarios.

This is explored in chapter 3 where the technique and results of a simulation study of a large portfolio of interest rate swaps is described. As implied by the model, an interaction between the interest rate and the default intensity is incorporated. Different functions which model this response of the default intensity to interest rate changes are investigated: The heavyness of the tails of the loss distribution is sensitive to the kind of response. It seems to be very important to incorporate such a response for describing worst cases. These "fat tails" could only be detected by the PM and the TCE measure. The MP measure is not sensitive enough.

In the last section of chapter 3 the most realistic investigation of this paper is presented: The counterparties are divided into several sectors. All the counterparties in one sector have the same response function, but different response strength due to their default probability. The mapping of several response function to the sectors is varied. It is demonstrated that this is a valuable tool to detect dangerous segments of the portfolio. With proposals of possible extensions the paper concludes.

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