# Banking Regulation with Value-at-Risk

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Raphael Paschke

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| Dekan: | Prof. Dr. Hans H. Bauer |
| Referent: | Prof. em. Dr. Dr. h.c. Wolfgang Bühler |
| Korreferent | Prof. Dr. Hans-Jochen Bartels |

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# List of Frequently used Symbols

| | |
|---|---|
| $A(\cdot), a(\cdot)$ | see equation (2.22) and (2.21) on p. 29 |
| $\alpha$ | VaR probability, regulatory control |
| $B$(Index) | economy with a regulated banking system and fixed prices |
| $b$(Index) | regulated banking sector |
| $\mathcal{B}$(Index) | economy with regulated banking system and endogenous prices |
| $\mathcal{B}(\cdot)$ | budget equation, see (2.17) on p. 25 |
| $\beta$ | heterogeneity of the banking sector, structural parameter |
| $C$ | value of costs (dollar amount), process |
| $c$ | consumption (stream), process |
| $CP$ | capital provision due to VaR, see (3.24) on p. 93 |
| $D = D^b$ | value of bank debt, process |
| $D^i$ | value of debt of the financial intermediary, process |
| $\delta$ | (exog. given) cash flow stream, see (2.43) on p. 45 |
| $d_1(\cdot), d_2(\cdot)$ | see equation (2.31) on p. 35. |
| $F$ | nominal debt level, structural parameter |
| $\vec{F}$ | transition nominal |
| $\hat{F}$ | debt capcity of the banking system, see (3.23) on p. 90 |
| $\phi(\cdot), (\Phi(\cdot))$ | (cumulative) density function of the standard normal distribution |
| $\gamma$ | coefficient of relative risk aversion |
| $h$ | cum/ex costs sensitivity, see (3.14) on p. 71 |
| $I$(Index) | economy with unregulated financial system and fixed prices |
| $i$(Index) | unregulated financial intermediary |
| $\mathcal{I}$(Index) | economy with unregulated financial system and endogenous prices |
| $\mathcal{I}(\cdot)$ | inverted marginal utility function, see (2.13) on p. 21 |
| $IRRA$ | implied relative risk aversion, process, see (3.28) on p. 100 |
| $J$ | jump size at boundary, see (3.19) on p. 76 |
| $\kappa$ | market price of risk, see (2.3) on p. 14 |
| $L$ | (debt over equity) leverage ratio in present value terms |
| $\mathcal{L}(\cdot;\cdot)$ | Lagrange function |
| $\lambda$ | cost fraction, structural parameter |
| $\mu_\delta$ | (exog. given) drift rate of cash flow $\delta$, parameter |
| $\mu$ | instantaneous return of the risky asset $P$, process |
| $n$ | Cooke ratio, regulatory control |
| $\omega$ | fraction of market held by investor $u$, structural parameter |

| | |
|---|---|
| $P$ | value of claim on cash flows / market value, see (2.10) on p. 18 |
| $p$ | probability of distress, regulatory control |
| $\psi$ | see equation (3.15) on p. 72 |
| $PD = PD^{\mathbf{P}}$ | probability of distress (under the real measure), process, see (3.25) on p. 95 |
| $PD^{\mathbf{Q}}$ | probability of distress (under the risk-neutral measure), process |
| $q = q^b$ | implied risk aversion of the banking system, process, see (3.28) on p. 100 |
| $q^i$ | implied risk aversion of the financial intermediary, process, see (3.28) on p. 100 |
| $R = R^{\mathbf{P}}$ | recovery rate (under the real measure), process, see (3.26) on p. 97 |
| $R^{\mathbf{Q}}$ | recovery rate (under the risk-neutral measure), process |
| $r$ | interest rate (process) |
| $\rho$ | see equation (2.20) on p. 29. |
| $s$ | yield spread, process, see (3.27) on p. 99 |
| $\sigma_\delta$ | volatility of cash flow $\delta$, parameter |
| $\sigma$ | instantaneous volatility of the risky asset $P$, process |
| $T$ | horizon of the economy / debt maturity |
| $t$ | arbitrary point in time |
| $\theta = \theta^b$ | portfolio strategy (fraction) of the banking system, process |
| $\theta^i$ | portfolio strategy (fraction) of the financial intermediary, process |
| $\theta^u$ | portfolio strategy (fraction) of the unrestricted investor, process |
| $u$(Index) | unrestricted investor |
| $u(\cdot)$ | utility function, see (2.12) on p. 21 |
| $V = V^b$ | value of total assets of the banking system, process |
| $V^i$ | value of total assets of the intermediated system, process |
| $\underline{V}$ | see equation (3.6) on p. 65 |
| $\check{V}$ | see equation (3.5) on p. 62 |
| $w$ | Brownian motion / underlying risk source |
| $W^b$ | equity value of the restricted banking sector, process |
| $W^i$ | equity value of the unrestricted financial intermediary, process |
| $W^u$ | wealth of the unrestricted investor, process |
| $\underline{W}$ | retention level for the VaR restriction, see (3.13) on p. 71 |
| $W_*$ | retention level to distress, see (3.13) on p. 71 |
| $\xi$ | state price process, see (2.5) on p. 16 |
| $y = y^b$ | Lagrange multiplier of static budget restriction of the banking system |
| $y^i$ | Lagrange multiplier of static budget restriction of the fin. intermediary |
| $y^u$ | Lagrange multiplier of static budget restriction of unrestricted investor |
| $\overline{\zeta}$ | VaR boundary on state price level, see (3.13) on p. 71 |
| $\zeta^*$ | distress boundary on state price level, see (3.13) on p. 71 |

| | |
|---|---|
| $\mathbf{E}[\cdot]$ | expectation (under real probability) |
| $\mathbf{P}[\cdot]$ | (real) probability |
| $\mathbf{Q}[\cdot]$ | risk-neutral probability |
| $\mathbf{Q}_t[\cdot] = \mathbf{Q}[\cdot\vert\mathcal{F}_t]$ | conditional risk-neutral probability |
| $\mathbf{V}[\cdot]$ | variance (under real probability) |
| $\cdot_t[\cdot] = \cdot[\cdot\vert\mathcal{F}_t]$ | conditional (real) expectation $\mathbf{E}$ / variance $\mathbf{V}$ / probability $\mathbf{P}$ |
| $\cdot_t^{\mathbf{Q}}[\cdot] = \cdot^{\mathbf{Q}}[\cdot\vert\mathcal{F}_t]$ | conditional risk-neutral expectation $\mathbf{E}$ / variance $\mathbf{V}$ |
| $f'(x)$ | (total) derivative with respect to $x$ |
| $\frac{\partial f}{\partial x} = f_x$ | partial derivative with respect to $x$ |
| $f^{(0,1)}$ | partial derivative with respect to the second variable |
| $\mathbf{1}$ | indicator function |
| $\breve{\cdot}$ | minimum(infimum) value of a variable |
| $\hat{\cdot}$ | maximum(supremum) value of a variable |
| $\cdot_*$ or $\cdot^*$ | variable attributable to distress |
| $\underline{\cdot}$ or $\bar{\cdot}$ | variable attributable to the VaR restriction |

# Chapter 1

# Introduction

## 1.1 Problem and Main Results

Recent developments vividly illustrate the importance of financial markets and financial institutions to a well-functioning modern economy. A seemingly small disruption in the sub-prime mortgage market has spread out to other financial markets and led to a severe crisis in the banking sector. Moreover, even the real sector of the economy is heavily impacted.
A central part of financial institutions, the banking sector, is heavily regulated and interventions in the banking sector undoubtedly have an impact on the economy far beyond the regulated banking institution or even the banking sector itself. Thus, the central question arises of whether regulation indeed supports financial stability or not?

The building block of the current banking regulation are risk-sensitive capital requirements, which are based upon the Value-at-Risk (VaR) as a measure for risk. This measure can be easily understood and it is proposed by many institutions such as the Basle Committee on Banking Supervision, the SEC, the G30, the ISDA, and the Derivatives Group. Moreover, it is implemented by risk management systems such as RiskMetrics and it is widely accepted in the industry.

In this thesis we do not question whether banks should be regulated or not[1], nor whether VaR is the optimal choice for a risk measure[2] to base regulation[3] on. Instead, we take the common banking regulation with VaR based capital requirements as given and analyze the impact of regulation on the stability of the banking sector.

First, we discuss how regulation affects the investment decision of the representative bank. The main results are the following. Banks are restricted if their nominal debt volume exceeds a certain threshold, and, if so, provide more equity capital to cover possible negative outcomes. There are two direct consequences. The representative bank is less able to maintain high nominal debt volumes. Moreover, debt holders are better protected in case of defaults due to regulation and charge a lower credit yield spread relative to an unregulated economy.
However, the indirect consequence of regulation is a higher probability of distress of the banking system and higher losses in case of defaults. This result can be traced back to the dynamic trading strategy, which replicates the change in the optimal profile that is directly attributable to regulation. This change consists of transferring wealth into all regulated states of the economy; the transfer is paid by the cheapest, unregulated states, namely the most extreme (negative) outcomes. The regulation-induced change in the trading strategy can be represented as a static derivatives position; hence, the optimal investment decision under regulation may serve as one explanation for the highly innovative derivatives market and the shifting of risk into the tails of the distribution by banks before the current crisis emerged.

Second, the impact of regulation on endogenous market prices of assets, debt, and equity as well as their dynamic structure is analyzed. Under endogenous prices, qualitatively the same results can be deduced as in the above discussed case with fixed prices. However, the increased demand by the trading strategy which replicates the adaption to regulation, shifts prices in a way that the implementation of the same

[1]See e.g. Dewatripont and Tirole (1994) or Freixas and Rochet (1999)
[2]See e.g. Artzner et al. (1999)
[3]See e.g. Danielsson et al. (2001)

trading strategy is more expensive. Consequently, the probability of distress soars even more in equilibrium and, if there is a crisis, the aggregate wealth of the economy is less than in a corresponding economy with an unregulated financial system.
In equilibrium, wealth is transferred between different states in order to fulfill the regulatory requirements. Yet, there exists a second transfer of wealth between the agents of the economy specific to the equilibrium formulation. Not only debt holders profit from the equity capital provisions of banks, but also the unrestricted investor since asset prices also move in favor of investors.
The volatility of assets is reduced due to regulation in most cases. Nevertheless, especially as the economic situation deteriorates, volatility is increasing substantially. Depending on the tightness of regulation, the volatility can be much higher than in an economy with unregulated financial intermediaries. Moreover, volatility is highly sensitive to a change in the underlying economic development, i.e. volatility changes quickly from low to high and back in adverse economic situations.

Third, we shed light onto the problem of how to improve the banking regulation, if banks actively manage both, assets and liabilities. The additional degree of freedom that arises if banks simultaneously manage their assets and liabilities, enables banks to mitigate the burden of regulation to some extent. This, in turn, results in a higher danger of defaults in the banking sector.
Since the central bank is able to control, at least in parts, the overall leverage of the economy and thereby the probability of distress, we propose to include the central bank into the set of regulatory authorities. We show, that in order to cope with the indirect incentives implied by the risk-sensitive capital requirements, leverage has to be decreased when a stricter regulatory regime is applied to the economy and the probability of distress is kept constant. This approach to regulation reduces the value of aggressively 'gaming' the VaR restriction.
Finally, one has to keep in mind that, if VaR-based capital requirements - which are imposed today - are not accompanied by state contingent costs - which arise after uncertainty has been resolved - regulation looses its power. To reach the same objectives with lower default costs, regulation must be much tighter. Hence,

credibly incorporating parts of systemic costs in case of distress is essential for a well-functioning regulatory regime.

In the following the organization of the thesis is outlined.

Chapter 2 introduces the general economic setting and the pricing of contingent claims in complete markets under no-arbitrage considerations. Then it reviews the well-known consumption-investment problem for an unrestricted investor as well as for a VaR restricted investor. Finally, the valuation of assets in a pure exchange equilibrium is discussed for an economy with both types of investors.

In Chapter 3 the optimal asset choice of the banking sector is analyzed by introducing banks into the setting of the previous chapter. The banking sector is characterized by having debt outstanding and, in order to capture the regulatory impact, a restriction by a VaR constraint. The banking sector is modelled in aggregate terms by specifying a pay-off structure dependent on the total assets at maturity. Assets are allocated using the first priority rule. However, if a bank fails, default costs accrue. Afterwards, the endogenous decision in terms of the optimal terminal wealth of the banking system under exogenously given prices is derived and analyzed. Furthermore, the resulting dynamic asset selection is discussed.

In Chapter 4, a competitive pure exchange equilibrium is formulated. This allows us to study the impact of the VaR regulation on market prices and their evolution over time. By comparing the regulated banking sector with $(a)$ an otherwise identical unrestricted financial intermediary and $(b)$ the banking sector under fixed prices, the wealth transfer consequences of regulation are deduced.

In contrast to previous chapters, where a fixed nominal debt volume was assumed, Chapter 5 studies the impact of the combined asset liability decision of the banking sector. It is first illustrated that the additional degree of freedom in the choice of their capital structure enables banks to mitigate regulatory constraints. Thereby, the banking system is even more susceptible to a financial crisis.
Based on the last results, a holistic regulatory approach is proposed. It includes the central bank in the regulatory authorities. The objective with regard to regulation

is to control the probability of distress; as an instrument the central bank utilizes the capability to restrict the overall leverage as a price-leading agent.

Chapter 6 summarizes the results, discusses the robustness of the results, and concludes with policy implications derived from the presented model.

## 1.2 Review of the Related Literature

Pyle (1971) and Hart and Jaffee (1974) were the first to apply a (mean-variance) portfolio selection approach to banks, while Kahane (1977) and Koehn and Santomero (1980) included capital constraints. They showed that a higher capital requirement may lead to a more risky portfolio selection in terms of the amount invested into risky assets. As there may exist some banks that are more risky and some that are less, the impact on the systemic risk is unclear. To remedy the problem of riskier investments due to regulation, they suggest risk-sensitive capital weights; Kim and Santomero (1988) find that this approach is indeed reducing risk, but only if the risk weights are set optimally. Alexander and Baptista (2004, 2006) refine those approaches, but still conclude that regulation in some circumstances increases fragility in the financial system.

Even though we are using a dynamic framework, we also recover similar results with respect to the portfolio selection problem. However, we are able to characterize the results more precise than in the static frameworks. In most economic situations, portfolios are indeed less risky than the ones of an unregulated economy. However, especially when the economic development deteriorates, regulation fails with respect to the portfolio decision, as banks substantially increase risk by investing more into risky assets. Moreover, the magnitude of this adverse portfolio decision is heavily dependent on the VaR horizon and on the tightness of regulation.

The importance of a dynamic model is illustrated by Blum (1999) in a two-period model. There exist four different cases, depending in which period regulation is binding; two of them are of special interest: If the capital constraint is active in the first period, whereas not in the second, there exists a risk reducing effect of regulation. However, if the case is reversed, i.e. the capital constraint is binding in the second period, whereas not in the first period, the bank increases its risk profile in the first period, in order to lessen the impact of the restriction in the second period by an increased capital basis, if the investment turns out to be successful. This simple argument in Blum's article shows that incorporating dynamic aspects into the analysis substantially adds to a better understanding of the effects of banking regulation. In the following we mainly focus on continuous time models.

Bodie et al. (2007) and Gray and Malone (2008) decompose the balance sheets of the main aggregated sectors of the economy into a set of derivatives in the style of Merton (1974) for the balance sheet of a levered firm. Lehar (2005) uses a similar approach to discuss empirically the probability of systemic crisis and the expected shortfall in a crisis.
Our approach has in common the modelling of the aggregate sectors, albeit less detailed than theirs. However, their modelling approach is not applicable to the question whether banking regulation contributes to financial stability, as there is no room for agents to adjust their portfolio decisions optimally due to the imposed restrictions, such as a VaR restriction. Banks do not actively manage their asset or liability side.

Basak and Shapiro (2001) introduced a VaR restriction into the optimization problem of an otherwise unrestricted investor in a complete market setting. They find that the VaR restriction induces gambling for resurrection, while the wealth of the VaR restricted agent is in the proximity of the VaR boundary. Leippold et al. (2006) set up a model with a VaR restriction in an incomplete market. The implicit incentive of a VaR restriction is an increased risk exposure in high volatility states. Kaplanski and Levy (2007) extend the analysis of Basak and Shapiro (2001) by additionally introducing a minimum capital requirement at the VaR horizon, which has to be fulfilled in any state. If regulation could in fact impose such a restriction to individual banks, systemic risk could be banned, as there never exist substantial defaults in the banking system. Obviously, this is in fact not the case.

We follow these models in explicitly considering a VaR restriction similar to theirs. However, since these models only contain individual investors without debt, they do not capture a central characteristic of financial intermediaries and/or banks.

In contrast, Basak and Shapiro (2005) present a structural model, incorporating the investment decision of a levered investor. They analyze the impact of costly default on credit spreads. Cuoco and Liu (2006) adopt insured debt in the form of a riskless zero bond and a VaR restriction. They focus on the impact of a simultaneous reporting and investment strategy with VaR. Since constantly reporting too low VaR values will increase the capital requirement, VaR in conjunction with a back-testing procedure reduces portfolio risk. Essential for deriving the results are the implicit costs associated with a violation of the reported VaR, namely an increased capital requirement in the next period.
We extend the model of Basak and Shapiro and include both, risky debt and a VaR-based capital requirement. In contrast to their focus on a single investor, we apply the results presented in Eisenberg and Noe (2001); they derive the payment vector (clearing vector) in a financial system, when firms default, obeying priority of debt claims and limited liability of equity. Their results allow us to model the banking sector on an aggregated level. A natural benchmark for comparison of the regulated banking sector with unregulated financial intermediaries can be easily obtained by relaxing the VaR restriction.

Decamps et al. (2004) and Dangl and Lehar (2004) discuss the impact of regulation with capital requirements under the endogenous decisions to deliberately violate the regulation. They conclude that there is (in most cases) no risk shifting due to capital requirements.
This strand of literature is of less interest to our work for three reasons. First, their focus is on problems of moral hazard in conjunction with regulation. Second, the structure of their modelling approach involves significant technical problems, when a VaR restriction with an inherently finite horizon is factored in. Third, and most important, deriving endogenous prices for assets can be hardly achieved in their setup.

Building on the pure exchange equilibrium formulation as in Cox and Huang (1991) and Karatzas et al. (1990), Basak and Shapiro (2001) derive the feedback effect of a VaR restricted investors on market prices and volatility. The excess demand, induced by portfolio decision of the VaR restricted investors, has a substantial impact on prices; in prosperous times and under very adverse situations, the value of assets is decreased, while in intermediate states assets are worth more. In equilibrium, the gambling for resurrection type of portfolio decision will increase the volatility of markets. Danielsson et al. (2004) deploy a similar setting in discrete time. They construct the solution by a sequence of myopic general equilibrium economies. They also find that there will be feedback effects of trading decisions which increase volatility and exacerbate financial stability. Leippold et al. (2006) apply an incomplete market framework. While the VaR restriction results in riskier portfolios, when the exogenous volatility is high, in equilibrium, effects are ambiguous.
We extend the model with fixed price dynamics to a pure exchange equilibrium using similar techniques as Basak and Shapiro. Thereby, we are able to analyze the impact of regulation on endogenous prices of an economy with a regulated banking sector. The core difference to their paper is the combination of the levered investment and the VaR-based capital requirements.

Finally, there are other strands of the literature discussing the problem that banks are not only able to choose their asset portfolio, but can also manage their liabilities, especially their deposit volume. Blum (1999), Calem and Rob (1999), Hellmann et al. (2000), Estrella (2004), and Repullo and Suárez (2004) analyze the impact of an endogenous deposit volume decision on a bank's stability under different assumptions. Their general conclusion is that capital requirements are (in many cases) effective in reducing the risk of failure, since a high leverage today induces large indirect costs tomorrow if, under adverse economic developments, regulatory constraint have to be fulfilled by substantially reducing the credit volume.
We undertake a first step of how to address the endogeneity of the liability side of the aggregate banking sector's balance sheet. Since the central bank is able to control,

at least in parts, the overall nominal debt level, we introduce the central bank as another regulatory authority. Its only goal is to keep the probability of distress in the financial system at a certain level. We abstract, thereby, from other objectives such as inflation targeting. The banking sector and the central bank form a market, where the central bank acts as a price-leading agent. While at the asset side of the banking sector, there still is a dynamic competitive equilibrium, the market for debt is open only once at the beginning.

# Chapter 2

# Individual Decision and Valuation in Equilibrium

This thesis is based upon an economy with two essential properties: first, the state variables describing the economy are diffusion processes. This includes both time as well as the driving economic variables. The second assumption concerns the security markets: only complete markets are considered. This restriction is important, as it enables agents in the economy to contract on any state of the world.

In this chapter we first review the standard framework. A complete security market is described and the fundamental equivalence between the concept of a representative agent, the notion of no-arbitrage, stochastic discount factors, and an equivalent martingale measure is introduced.

In the following part, the individual consumption and investment decisions of an unrestricted investor and of an investor who faces a Value-at-Risk (VaR) constraint are presented.

In the last section, a pure exchange equilibrium is defined and the conditions for its existence are specified. As a tool for solving equilibrium, a representative agent is introduced. Finally, using the optimal decisions of investors, asset prices are derived in economies with and without Value-at-Risk restrictions.

Most of the results in this chapter are well known, see for example Karatzas and Shreve (1998). The optimal consumption-investment decision under a VaR restriction, i.e. Section 1.2.3, is non-standard, but was derived by Basak and Shapiro (2001). The pure exchange equilibrium with agents that face a VaR restriction, i.e. Section 1.3.2, was in two ways modified relative to the one presented in Basak and Shapiro (2001). One modification was introduced in order to circumvent a jump of asset prices at maturity; the other one allows for a general utility function with constant relative risk aversion.

The formal setup is close to Karatzas and Shreve (1998), Chapters 1, 3, and 4; we also refer to Duffie (2001), Korn (1997), Bjoerk (2009), and Pliska (1997) for further reading.
However, we use some more restrictive assumptions than in the general framework of Karatzas and Shreve (1998) in order to keep the technical level as simple as possible without loosing economically relevant results. As a primer we state the main assumption used throughout the thesis:

- As only complete markets are of interest, we restrict, without loss of generality, the number of securities $N$ to the number of risk sources $D$, i.e. $N = D$. Furthermore, these securities are independent in the sense, that the covariance matrix $\Sigma$ has full rank and is, hence, invertible.
  Additionally, there are no frictions on financial markets such as short selling. Even though these two assumption are idealizing the world possibly too much, we first need to address the problem of how banking regulation impacts financial stability under perfect conditions. The question of how frictions or the incompleteness of markets affect financial stability is clearly of importance, but subject of future research.

- All stochastic variables and processes are measurable and adapted with respect to the filtration $\mathcal{F}$, if not specified otherwise.

- Furthermore, we assume that the processes for the interest rate $r$, the return rate $\mu$, and the covariance matrix $\Sigma$ as well as its inverse $\Sigma^{-1}$ are bounded. Relaxing boundedness is not difficult, but introduces a series of additional regularity conditions without adding to the economic content in our case. By this assumption, as an example, we exclude standard models for the interest rate process like the one of Cox et al. (1985b). Nevertheless, this poses no problem in our framework, since the dynamics of the riskless interest rate is not the focus of the thesis; in many cases it is for simplicity set to a constant or turns out to be in equilibrium constant.

The consumption process $c$ and the portfolio process $\pi$ satisfy the square-integrability conditions $\mathbf{E}[\int_0^T c_t^2 dt] < \infty$ and $\mathbf{E}[\int_0^T \pi_t^2 dt] < \infty$; the wealth at the horizon $W_T$ satisfies $\mathbf{E}[W_T^2] < \infty$. Thereby, arbitrage opportunities are dismissed (see Harrison and Pliska (1981)). In the pure exchange equilibria of later sections, these conditions are endogenously fulfilled.

- The security market will turn out to be a particular case of the standard financial market in the sense of Karatzas and Shreve (1998), Definitions (1.1.3) and (1.5.1).

- All agents optimize their decision using expected utility; they share the some coefficient of constant relative risk aversion CRRA.

## 2.1 Valuation of Securities under No-Arbitrage

Underlying the economy is a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the time period $[0, T]$. The set of states of the nature is denoted $\Omega$, $\mathcal{F}$ a $\sigma$-Algebra on $\Omega$, and $\mathbf{P}$ the real world probability measure defined on $\Omega$.

A $D$-dimensional standard Brownian motion $w = (w_1, \cdots, w_D)^\top$ is defined on $(\Omega, \mathcal{F}, \mathbf{P})$, which drives the uncertainty within the economy. $(\cdot)^\top$ denotes the transposed vector. Let $\mathcal{F}_t$ be the continuous filtration generated by the Brownian motion $w_{s\in[0,t]}$ up to time $t$, augmented by all null subsets. This filtration represents the information available to agents in the economy.

### Securities and their Dynamics

The security market consists of a money market account with dynamics

$$dB_t = r_t B_t dt, \qquad B_0 = 1 \ , \tag{2.1}$$

where $r$ is the locally risk-free rate.

In addition, there are $N = D$ risky assets with price dynamics

$$dP_t^{Y,n} = dP_t^n + \delta_t^n dt = P_t^n \left( \mu_t^n dt + \sigma_t^n dw_t \right), \qquad (n = 1, \dots, N) \ . \tag{2.2}$$

$P^n$ is the ex dividend price of the risky asset, whereas $P^{Y,n}$ is the price that includes the accumulated yield from the dividend stream $\delta^n$. The measurable and bounded processes for the interest rate $r$, the instantaneous expected return vector $\mu$, and the volatility matrix $\Sigma = \{(\sigma_d^1, \dots, \sigma_d^N) \ d = 1, \dots, D\}$ are adapted to the filtration $\mathcal{F}$ and fulfil regularity conditions, such that the stochastic differential equations (2.1) and (2.2) is well defined.

Most important for modelling a complete market is that there exists (almost surely) a unique solution vector $\kappa$ to the equation

$$\mu_t - r_t \mathbf{1} = \Sigma_t \kappa_t \ ,$$

which is true given the standing assumptions, as $\Sigma$ is invertible. The unique market price of risk is given by

$$\kappa_t = \Sigma_t^{-1} \left( \mu_t - r_t \mathbf{1} \right) \ . \tag{2.3}$$

$\mathbf{1}$ denotes the unit vector $(1, 1, \dots, 1)^\top$ of dimension $N$. The solution of the asset price dynamics (2.2) is

$$P_t^{Y,n} = P_0^n \exp \left( \int_0^t \left( \mu_s^n - \frac{1}{2} \parallel \sigma_s^n \parallel^2 \right) ds + \int_0^t \sigma_s^n dw_s \right) \ , \tag{2.4}$$

where the initial condition $P_0^{Y,n} = P_0^n$ is used and $\|x\| = \left( \sum_{i=1}^D x_i^2 \right)^{\frac{1}{2}}$.

Trading Strategies

Each agent has a (possibly state- and time-dependent) non-negative consumption process $c$ and a (possibly state- and time-dependent) trading strategy, $\pi = (\pi_1, \cdots, \pi_N)^\top$, where $\pi_n$ is the *dollar amount* held in the risky asset $n$. When investing in asset $n$ at timte $t$ and holding the asset over the next infinitesimal period, one has to pay the ex price $P_t^n$ in order to obtain the capital gain $dP_t^n$ as well as the cahs flow $\delta_t^n dt$. Thus, the total return can be written as

$$dR_t^n = \frac{dP_t^{Y,n}}{P_t^n} \ , \qquad (n = 1, \ldots, N) \ .$$

Let $W^\pi$ denote the wealth of an agent given a trading strategy $\pi$ and including dividends as inflow. The requirement that there is no outflow beside consumption over time $t \in (0, T]$ makes the portfolio strategy self-financing. Consequently, the amount $(W^\pi - \pi^\top \mathbf{1})$ is risklessly invested in the money market with instantaneous return $r$. Hence, the wealth process follows

$$\begin{aligned} dW_t^\pi &= \left(W_t^\pi - \pi_t^\top \mathbf{1}\right) r_t dt + \pi_t^\top dR_t - c_t dt \\ &= \left(W_t^\pi r_t - c_t\right) dt + \pi_t^\top \left(\mu_t - r_t \mathbf{1}\right) dt + \pi_t^\top \Sigma_t dw_t \\ &= \left(W_t^\theta r_t + W_t^\theta \theta_t^\top (\mu_t - r_t \mathbf{1}) - c_t\right) dt + W_t^\theta \theta_t^\top \Sigma_t dw_t \ . \end{aligned}$$

$\theta$ is the same portfolio strategy as $\pi$, however, expressed as the *fraction* of wealth invested in the risky assets, i.e. $\pi = W^\pi \theta$ and $W^\pi = W^\theta$.

To ensure the existence of a solution we need standard regularity conditions in order to have a well-defined solution.

However, this is not enough. In addition we have to exclude cases where $\int_0^T \pi_t^\top \Sigma_t dw_t$ looses the martingale property. This restriction is necessary as, with a portfolio violating this restriction, an investor will be able to profit from investing in a fair game, e.g. by doubling the bet every time on credit. One possible constraint to ensure that portfolio strategies with exploding variance such as a doubling strategies are not feasible, is to keep $\mathbf{V}[\int_0^T \pi_t^\top \Sigma_t dw_t]$ under control by the square integrability constraint

$$\mathbf{E}\left[\left(\int_0^T \|\pi_s\|^2 ds\right)\right] < \infty \ .$$

Another possibility to tame the portfolios is to restrict the ability to go short risky assets by a credit constraint.

Stochastic Discount Factors and Arbitrage

When using a self-financing portfolio strategy it is obvious that there also exists a self financing trading strategy in some other numeraire. More generally, when there exists a self-financing trading strategy for the price process $P = (P^1, \cdots, P^N)^\top$, there also exists a self-financing trading strategy for a deflated price process $\xi P$, when the deflator is strictly positive, $\xi > 0$. As a deflator may serve any one-dimensional and adapted process. Following Cox and Huang (1989), Cox and Huang (1991), Harrison and Kreps (1979), Karatzas et al. (1987), and Pliska (1986), the stochastic discount process is defined by

$$\xi_t = \xi_0 \exp\left(-\int_0^t (r_s + \frac{1}{2} \parallel \kappa_s \parallel^2) ds - \int_0 \kappa_s^\top dw_s\right) \quad (\xi_0 > 0) \ . \tag{2.5}$$

By construction, $\xi > 0$. Applying Ito's lemma, the dynamics of the stochastic discount factor is

$$d\xi_t = -\xi_t(r_t dt + \kappa_t^\top dw_t). \tag{2.6}$$

Without any risk ($w = 0$) or when risk is not priced ($\kappa = 0$), $\xi$ is the price of a zero bond. When priced risks are involved, the stochastic discount factor $\xi$ adjusts in addition by the market price of risk $\kappa$.

From Karatzas and Shreve (1998), Theorem (1.4.2), we know that there exists no arbitrage, if and only if the market price of risk $\kappa$ is well defined by

$$(\mu_t - r_t \mathbf{1}) = \Sigma_t \kappa_t \tag{2.7}$$

and fulfils the requirements

$$\int_0^T \|\kappa_s\|^2 ds < \infty \ ,$$
$$\mathbf{E}\left[\exp\left(-\frac{1}{2}\int_0^T \|\kappa_s\|^2 ds - \int_0^T \kappa_s^\top dw_s\right)\right] = 1 \ . \tag{2.8}$$

When the standing assumptions on page 12 are applied to the theorem, the market price of risk $\kappa$ exists by assumption. Moreover, as $\mu$, $r$, $\Sigma$, and $\Sigma^{-1}$ are bounded, $\kappa$ itself is bounded as well, which is sufficient to fulfil the stated requirements. Since $\kappa$ is unique by assumption (2.3), the stochastic discount factor is as well.
As the 'only if' part is true as well, we can also state that, if there exists a deflator under which security prices are martingales, than there exists no arbitrage; see also the Corollary in Duffie (2001) on page 110.

Market Completeness
If the solution to the market price of risk $\kappa$ in equation (2.7) is unique, the market is complete, see Karatzas and Shreve (1998) Theorem (1.6.6) or Corollary (1.6.8).
This property is essential for the following reason: in a complete market, every contingent claim $X_T$ that is a $\mathcal{F}_T$ measurable random variable with finite variance can be replicated by a self-financing trading strategy $\pi$ with a specific initial wealth $x_0$, see Karatzas and Shreve (1998), definition (1.6.1).
Furthermore, by no-arbitrage, the unique price at time $t$ is, see also Remark (1.6.3),

$$X_t = \mathbf{E}_t\left[\frac{\xi_T}{\xi_t} X_T\right] ,$$

where $\mathbf{E}_t[X_T] = \mathbf{E}[X_T|\mathcal{F}_t]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_t$. If $t = 0$, we refrain from indexing expectations (and variances).
If an exogenous cash-flow stream $c$ is added, then the value $X_t$ of the contingent claim $X_T$ is, under the previous assumptions and no-arbitrage,

$$X_t = \mathbf{E}_t\left[\int_t^T \frac{\xi_s}{\xi_t} c_s ds + \frac{\xi_T}{\xi_t} X_T\right] . \tag{2.9}$$

**Remark.** ***Relation between Stochstic Discount Factors and the Equivalent Martingale Measure (EMM)***

*Given some contingent claim $X_T$, the value at some time $t$ can be written in different forms*

$$\begin{aligned}
X_t &= \mathbf{E}_t^{\mathbf{P}}\left[\frac{\xi_T}{\xi_t}X_T\right] \\
&\stackrel{Def.\ (2.5)}{=} \mathbf{E}_t^{\mathbf{P}}\left[\exp\left(-\int_t^T (r_s + \frac{1}{2}\parallel \kappa_s \parallel^2)ds - \int_t^T \kappa_s^\top dw_s\right) X_T\right] \\
&= \mathbf{E}_t^{\mathbf{P}}\left[\exp(-\int_t^T r_s ds)\exp\left(-\int_t^T \frac{1}{2}\parallel \kappa_s \parallel^2 ds - \int_t^T \kappa_s^\top dw_s\right) X_T\right] \\
&= \mathbf{E}_t^{\mathbf{Q}}\left[\exp(-\int_t^T r_s ds) X_T\right] \\
&= \mathbf{E}_t^{\mathbf{Q}}\left[\frac{B_t}{B_T}X_T\right]
\end{aligned}$$

*The last equation represents the well-known risk-neutral pricing approach with a change of measure*

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\left(-\int_0^T \frac{1}{2}\parallel \kappa_s \parallel^2 ds - \int_0^T \kappa_s^\top dw_s\right) .$$

*The martingale requirement in equation (2.8) guarantees that* $\mathbf{Q}$ *is in fact an equivalent martingale measure. Again, when the market price risk is unique, there is a unique state price and, consequently, a unique equivalent martingale measure.*

Risky Asset Prices

With result (2.9) at hand we can derive the price of risky assets

$$\begin{aligned}
P_t^n &= \mathbf{E}_t\left[\frac{\xi_T}{\xi_t}P_T^n + \int_t^T \frac{\xi_s}{\xi_t}\delta_s^n ds\right] \quad (n = 1, \cdots, N). \qquad (2.10) \\
&= \mathbf{E}_t^{\mathbf{Q}}\left[\frac{B_t}{B_T}P_T^n + \int_t^T \frac{B_t}{B_s}\delta_s^n ds\right]
\end{aligned}$$

The proof can be found in Cox and Huang (1989), Lemma 2.4.

## 2.2 Optimal Consumption and Optimal Terminal Wealth

Investors trade securities in order to optimize their expected utility over consumption and terminal wealth. To study this optimization problem we first consider an investor whose portfolio strategy is not restricted by regulation. We refer to a method introduced into the finance literature by Cox and Huang (1989) and independently by Karatzas et al. (1987).

Merton (1969, 1971) derives the dynamic portfolio strategy by deducing the non-linear Hamilton-Jacobi-Bellman partial differential equation which more often than not can be attacked numerically only. The second difficulty adheres to the verification problem, if standard theorems are not applicable.

The advantage of the Cox and Huang approach is that it is much more flexible in the choice of processes and the handling of additional constraints. The optimization problem is first rewritten in a static variational form. Afterwards, one can further disintegrate it to separate state-wise problems which can be solved using standard deterministic optimization tools. The main drawback relative to the HJB approach is that it necessarily needs complete markets to achieve the first step of converting it to a static problem. However, as our setup is in a complete market setting, the Cox and Huang approach is better suited for our problem.

### 2.2.1 The Non-Regulated Investor

The unrestricted investor maximizes his expected utility by choosing an appropriate consumption process $c^u > 0$ and a self-financing trading strategy $\theta^u$ with initial wealth $W_0^u > 0$, which leads to a terminal wealth $W_T^u$. In addition, he can spend only his initial wealth $W_0^u$ on setting up the trading strategy, that is, $\theta^u$ must be (budget) feasible.

**Remark.** *In this subsection, that is from page 19 to 29, we refrain from using the u-indexed variables, as only the unrestricted investor is of interest.*

The optimization problem of the unrestricted investor is

$$\sup_{\{c_t,\theta_t\}_{t\in[0,T]}} U(c,W) = \mathbf{E}\left[\int_0^T u(c_t)\,dt + u(W_T)\right] \tag{2.11}$$

$$s.t. \quad \left\{ \begin{array}{rcl} \theta_t^\top P_t & + & (1-\theta_t^\top \mathbf{1})B_t \\ & = & W_0 + \int_0^t \theta_s^\top dP_s^Y + \\ & & + \int_0^t (1-\theta_s^\top \mathbf{1})dB_t \\ & & - \int_0^t c_s ds \geq 0 \quad \forall t \\ \theta_T^\top P_T & + & (1-\theta_T^\top \mathbf{1})B_T \\ & = & W_T \geq 0 \end{array} \right. .$$

The solutions for the unconstrained investor are derived in Cox and Huang (1989); Kramkov and Schachermayer (1999) deduct the solution in an even more general framework than the former authors used.

Utility Function

Each investor in the economy maximizes his decisions with respect to the expected utility (representation) $\mathbf{E}[u(W)]$, where the (von Neumann and Morgenstern (1944)) utility function $u(\cdot)$ of wealth $W$ and/or consumption $c$ fulfils the following properties. For analytical tractability, the utility function is *three times continuously differentiable*. In addition, agents are always *non-satiated*, i.e. more is better, all other things being equal. Consequently, we require $u' > 0$. This property is essential when no-arbitrage arguments are put forward. When there is satiation, there may be an arbitrage opportunity, but when all agents are satiated, no one will use this opportunity, because it decreases their utility level.

Furthermore, agents are required to be *risk averse*. This translates to a concave utility function, i.e. $u'' < 0$. Relative risk aversion is defined as (see e.g. Pratt (1964))

$$R(W) = -\frac{u''(W)W}{u'(W)} .$$

If $R'(W) > 0$, the agent has increasing relative risk aversion; decreasing relative risk aversion is defined analogously.

Essentially we restrict our analysis to the case of constant relative risk aversion $R(W) = \gamma > 0$. Integrating the previous equation twice results in the well-known utility function,

$$u(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma} & \text{if } \gamma \in (0,\infty)\backslash\{1\} \\ \ln W & \text{if } \gamma = 1 \end{cases} . \tag{2.12}$$

In the case of one risky and one riskless asset, this type of utility function exhibits decreasing absolute risk aversion $-\frac{u''}{u'}$; hence, the demand for the risky asset is normal, i.e. increasing with wealth, see Arrow (1970) or Pratt (1964). On the other hand, due to constant relative risk aversion, the fraction of wealth invested in the risky assets is independent of wealth.
Furthermore, in the case with $N$ securities and one period, two fund separation holds, since this type of utility function belongs to the HARA class, see Cass and Stiglitz (1970); for a similar result in the case of multiple securities in a continuous time financial market, see also Chamberlain (1988) or Schachermayer et al. (2009).

The utility is equipped with the regularity conditions

$$\lim_{W\to 0} u'(W) = \infty$$
$$\lim_{W\to \infty} u'(W) = 0 \ ,$$

consequently the utility function fulfils the Inada (1963) conditions. To shorten the notation,

$$\mathcal{I}(\cdot) = (u')^{-1}(\cdot) : (0,\infty) \to (0,\infty) \tag{2.13}$$

is the inverse function of marginal utility. Together with the previous assumption, $\mathcal{I}(\cdot)$ is continuous and strict monotonously decreasing over its domain.

Finally, we assume that all agents share the same utility function, in particular the same coefficient of relative risk aversion $\gamma$.

**Remark.** *The HARA class consists of utility functions, where the coefficient of risk tolerance* $-\frac{u'}{u''}$ *is of the form* $a_m + b_m W$*, satisfying the necessary conditions, such* *that a utility function is indeed defined; see Merton (1990).*

*Most results of this thesis can also be obtained by using a subset of utility functions of the HARA class, where investors wish to maintain at least some investor-specific minimum terminal wealth $K_m \geq 0$, but share the same coefficient $b$. Additionally, the adapted Inada conditions have to be fulfilled. In particular, these utility functions are of the form*

$$u(W) = \begin{cases} \frac{1}{1-\gamma}(W - K_m)^{1-\gamma} & \text{if } \gamma \in (0,\infty)\backslash\{1\} \\ \ln(W - K_m) & \text{if } \gamma = 1 \end{cases} .$$

*The requirement $b_m = b$ translates into $\gamma_m = \gamma$, $\forall m$, in the case of the CRRA utility function. This subset of HARA functions implies that the aggregation property in the sense of Rubinstein (1974) holds.*

### The Growth Optimal Portfolio: A Special Case

Before we discuss the general problem (2.11) we consider the special case where all investors are equipped with the log utility ($\gamma = 1$) and maximize terminal wealth only ($c_t = 0, \forall t$). We discuss the case separately, because this portfolio enables us in later sections to reduce dynamic trading strategies to much simpler static option portfolios with a scaled growth optimal portfolio as an underlying.

Integrating the dynamic budget equation (2.5) we obtain the following representation of terminal wealth for log utility

$$W_T = W_0 \exp\left(\int_0^T r_s + \theta_s^\top(\mu_s - r_s\mathbf{1}) - \frac{1}{2}\theta_s^\top \sigma_s \, ds + \int_0^T \theta_s^\top \Sigma_s dw_s\right).$$

Inserting into expected utility, we obtain

$$\mathbf{E}[\ln(W_T)] = \ln(W_0) + \mathbf{E}\left[\int_0^T r_s + \theta_s^\top(\mu_s - r_s\mathbf{1}) - \frac{1}{2}\|\theta_s^\top \Sigma_s\|^2 \, ds\right].$$

The optimization of the expected log utility is equivalent to maximizing the expected growth rate of wealth (expressed as percentage return p.a.) $\mathbf{E}\left[\frac{\ln \frac{W_T}{W_0}}{T}\right]$.

The optimal portfolio can be determined by maximizing pointwise

$$\theta_t^\top(\mu_t - r_t\mathbf{1}) - \frac{1}{2}\|\theta_t^\top\Sigma_t\|^2 \ .$$

There exists a unique solution, since $\Sigma$ is invertible by assumption. After substituting the market price of risk in equation (2.3), the optimal portfolio is

$$\theta_t^{GOP} = \Sigma_t^{-1}\kappa_t \ . \tag{2.14}$$

Latane (1959) and Hakansson (1971) call this specific portfolio the growth optimal portfolio (GOP). The value dynamics of the GOP, $G = P^\top\theta^{GOP}$, is

$$\frac{dG_t}{G_t} = (r_t + \parallel \kappa_t \parallel^2)dt + \kappa_t^\top dw_t \ . \tag{2.15}$$

Comparing the dynamics of the GOP (2.15) and the dynamics of the stochastic discount factor (2.6), one can deduce the relation

$$\frac{G_t}{G_0} = \frac{\xi_0}{\xi_t},$$

that is, deflating prices with $\xi$, i.e. $\xi P$, can be interpreted as discounting with the mutual fund $G$, i.e. $\frac{P}{G}$, whose portfolio policy is the growth optimal portfolio.

Revisiting the connection between the stochastic discount factor and the equivalent martingale measure (EMM), we obtain for any given contingent claim $X_T$, the initial value

$$\begin{aligned} X_0 &= \mathbf{E}^{\mathbf{P}}\left[\frac{\xi_T}{\xi_0}X_T\right] \\ &= \mathbf{E}^{\mathbf{P}}\left[\frac{G_0}{G_T}X_T\right] \\ &= \mathbf{E}^{\mathbf{Q}}\left[e^{(\int_0^T r_s ds)}X_T\right] \\ &= \mathbf{E}^{\mathbf{Q}}\left[\frac{B_0}{B_T}X_T\right] \ , \end{aligned}$$

i.e., as $\mathbf{Q}$ is the EMM with respect to the money market account as a numeraire, whereas the numeraire that makes $\mathbf{P}$ an EMM is the growth optimal portfolio $G$.

Solving the Optimization of the Unregulated Investor

By virtue of definition, complete markets ensure that any consumption process $c_t, t \in [0, T]$, and terminal wealth $W_T$ can be attained by a self-financing trading strategy with a given (unique) initial value. Instead of searching for a optimal consumption-trading strategy $(c_t, \theta_t), t \in [0, T]$, it is, therefore, sufficient to search for an optimal consumption $c_t, t \in [0, T]$, an terminal wealth $W_T$ for a given static budget equation. On the other hand, every trading strategy has a certain pay-off profile. This one-to-one correspondence between pay-off and trading strategies are difficult to proof in a continuous time and continuous state framework (see Cox and Huang (1989)); however, they are intuitive in the standard binomial tree approach of Cox et al. (1979), as the connection between the state contingent pay-offs and the trading strategy is a linear system of equations with a unique solution.

The budget dynamics in equation (2.5) can be transformed into a static budget equation by (2.9). After theses changes the optimization problems is

$$\begin{aligned} \sup_{\{c_t, W_T\}_{t\in[0,T]}} U(c, W) &= \mathbf{E}\left[\int_0^T u(c_t)\, dt + u(W_T)\right] \\ s.t. \quad & \mathbf{E}\left[\int_0^T \xi_t c_t \, dt + \xi_T W_T\right] \leq \xi_0 W_0 \\ & c_t \geq 0 \quad \forall t \\ & W_T \geq 0 \ . \end{aligned} \tag{2.16}$$

Hence, it is in some cases more convenient to split the solution of the problem into two parts.

1. Find an optimal $(c_t, t \in [0, T], W_T)$ pair that can be financed with initial wealth $W_0$. This is a static problem, since the solution to the process $c$ can be achieved by pointwise solving the problem.

2. If needed, recover the unique self-financing portfolio strategy $\theta$ using the market completeness. This is the representation problem.

The two-step procedure will be discussed now in detail:

STEP 1: SOLVING THE STATIC PROBLEM

From the previous analysis, the budget feasibility of the dynamic budget constraint in equation (2.5) can be simplified by no-arbitrage considerations in equation (2.9) to the static budget equation

$$\mathcal{B}(c,W) = \mathbf{E}\left[\int_0^T \xi_s c_s ds + \xi_T W_T\right] = \xi_0 W_0 \tag{2.17}$$

For convenience, the pair $(c, t \in [0,T], W_T)$ is abbreviated by $(c, W)$.

Let us to define the Lagrangian, with Lagrange multiplier $y$,

$$\begin{aligned} \mathcal{L}(c,W;y) &= U(c,W) - y(\mathcal{B}(c,W) - \xi_0 W_0) \\ &= \mathbf{E}\left[\int_0^T \big(u(c_s) - y\xi_s c_s\big)ds + \big(u(W_T) - y\xi_T W_T\big) + y\xi_0 W_0\right], \end{aligned}$$

which depends only on $(c, W)$, but not explicitly on the trading strategy $\theta$ any more.

When applying the saddle point theorem, we obtain that $(c^*, W^*, y^*)$ solves the optimization problem (2.11), if it is a saddle point, i.e. if

$$\mathcal{L}(c^*,W^*;y) \geq \mathcal{L}(c^*,W^*;y^*) \geq \mathcal{L}(c,W;y^*) \quad \forall\ (c,W)\ , y \geq 0\ . \tag{2.18}$$

Thus, the procedure is:

1. Consider the Lagrangian multiplier $y$ as parameter and find the optimal solution $(\hat{c}, \hat{W})(y), \forall y > 0$, as a function of $y$. The case $y = 0$ can be disregarded due to the utility's non-satiation property.

2. Substitute $(\hat{c}, \hat{W})(y)$ into the budget equation (2.17) such that

$$\mathcal{B}\big((\hat{c}, \hat{W})(y^*)\big) = \xi_0 W_0$$

holds. Afterwards obtain the candidate optimal solution $(c^*, W^*, y^*) = (\hat{c}(y^*), \hat{W}(y^*), y^*)$.

3. Check, whether the candidate solution $(c^*, W^*, y^*)$ is indeed a correct solution to the optimization problem.

Step 1.1: Find Parametric Solution

The first-order condition (FOC) of the Lagrangian with respect to $(c, W)$ can be 'split' into the pointwise optimization problem

$$\begin{aligned} u'(c_t) &= y\xi_t \ , \qquad \forall t \\ u'(W_T) &= y\xi_T \ . \end{aligned}$$

By the strict monotonicity of the utility function $u$ the first-order conditions can be inverted to obtain $(\hat{c}, \hat{W})$ as a function of the Lagrange multiplier $y > 0$,

$$\begin{aligned} \hat{c}_t(y) &= \mathcal{I}(y\xi_t) \ , \qquad \forall t \\ \hat{W}_T(y) &= \mathcal{I}(y\xi_T) \ . \end{aligned}$$

The boundary cases $c_t = 0$ or $W_T = 0$ do not exist by the Inada conditions, as the marginal utility approaches $\infty$ for these boundaries. From the strict concavity follows that the solution corresponds to a maximum. Furthermore, the assumptions of the stochastic discount factor $\xi$, namely being measurable and adapted, translate into the solution $(\hat{c}, \hat{W})$ as well.

Step 1.2: Solve for the Lagrange Multiplier

The parametric solution $(\hat{c}, \hat{W})$ naturally fulfils the following conditions

$$\begin{aligned} \mathbf{E}[\textstyle\int_0^T \xi_s \hat{c}_s(y) ds] < \infty \ , \qquad \forall y > 0 \\ \mathbf{E}[\xi_T \hat{W}_T(y)] < \infty \ , \qquad \forall y > 0 \ , \end{aligned}$$

since $(\hat{c}, \hat{W})$ is bounded due to the corresponding property of the process $\xi$. The budget equation $\mathcal{B}(\hat{c}(y), \hat{W}(y)) =: \mathcal{B}(y) = \xi_0 W_0$, which will be always hold with equality due to the non-satiation property of the utility function, has a unique solution $y^*$, if $\mathcal{B}(y)$ inherits the properties of $\mathcal{I}(\cdot)$, namely being continuous and strictly decreasing in $y$. $\mathcal{B}(\cdot)$ maps from $(0, \infty)$ into $(0, \infty)$. These properties can be easily checked by actually inserting the corresponding functions and applying the properties of $\mathcal{I}(y)$. Consequently, the static budget constraint as a function of the Lagrange multiplier, $\mathcal{B}(y)$, is invertible.

$\mathcal{B}^{-1}(W_0)$ maps from $(0,\infty)$ to $(0,\infty)$, so that there is a unique Lagrange multiplier for each level of initial wealth,

$$y^* = \mathcal{B}^{-1}(W_0) \ , \qquad \forall W_0 > 0 \ .$$

We thus arrive at the candidate solution

$$\begin{aligned} c_t^* &= \hat{c}_t(y^*) = \mathcal{I}(\mathcal{B}^{-1}(W_0)\xi_t) \\ W_T^* &= \hat{W}_T(y^*) = \mathcal{I}(\mathcal{B}^{-1}(W_0)\xi_T) \ . \end{aligned}$$

Step 1.3: Check the Candidate Solution

Finally, we have to check whether the candidate solution is indeed a correct solution to the optimization problem. Let $(\bar{c},\bar{W})$ be an arbitrarily consumption-wealth pair that satisfies the constraint $\mathcal{B}(\bar{c},\bar{W}) \le \xi_0 W_0$. Then

$$\begin{aligned} & \mathbf{E}\left[\int_0^T u(c_s^*)ds + u(W_T^*)\right] - \mathbf{E}\left[\int_0^T u(\bar{c}_s)ds + u(\bar{W}_T)\right] \\ = & \ \mathbf{E}\left[\int_0^T u(c_s^*)ds + u(W_T^*)\right] - \mathbf{E}\left[\int_0^T u(\bar{c}_s)ds + u(\bar{W}_T)\right] - y^*\xi_0 W_0 + y^*\xi_0 W_0 \\ \ge & \ \mathbf{E}\left[\int_0^T u(c_s^*)ds + u(W_T^*)\right] - \mathbf{E}\left[\int_0^T u(\bar{c}_s)ds + u(\bar{W}_T)\right] \\ & -y^*\mathbf{E}\left[\int_0^T \xi_s c_s^* ds + \xi_T W_T^*\right] + y^*\mathbf{E}\left[\int_0^T \xi_s \bar{c}_s ds + \xi_T \bar{W}_T\right] \\ \ge & \ 0 \ , \end{aligned}$$

where the first inequality follows from the static budget constraint holding with equality for $(c^*, W^*)$ while with inequality for any strategy $\theta$ corresponding to an arbitrarily $(\bar{c},\bar{W})$ pair. The second inequality follows from the specific property of the utility function

$$u(\mathcal{I}(y)) - y\mathcal{I}(y) \ge u(x) - yx \ , \qquad \forall x \ge 0, \ y > 0 \ .$$

Step 2: Recovering the Portfolio

In order to obtain an explicit characterization of the portfolio strategy, some additional assumptions are needed. As an example consistent with the results in the equilibrium of later sections, we require both the interest rate $r$ and the market price of risk $\kappa$ to be constants, whereas $\mu_t$ and $\sigma_t$ are not necessarily constant; notwithstanding, the stochastic discount factor $\xi$ follows a geometric Brownian motion, as can be seen in equation (2.6).

**Remark 1.** *There will be no essential difference in the results when assuming a time-dependent, but deterministic market price of risk $\kappa(t)$. In the multidimensional case, an equivalent restriction sets $||\kappa||^2$ as constant (or deterministically time-depended). The individual market prices of risk $\kappa_d$ are not necessarily deterministic functions and may stochastically evolve over time which adds far more degrees of freedom than in the case of a constant opportunity set $(r, \mu, \sigma)$ in the sense of Merton (1973a) or even with constant $\kappa_d$, $\forall d \in \{1, \ldots, D\}$ , see Nielsen and Vassalou (2006).*
*The restriction effectively implies that the instantaneous capital market line is at most a deterministic function of time. Consequently, there is no hedge motive and the $(D + 2)$ funds separation of Merton (1990) or Ross (1978) reduces to a two-fund separation, the money market account and the growth optimal portfolio $G$, (more specifically, a fund proportional to the GOP). Furthermore, the GOP $\theta^{GOP}$ in equation (2.14) on p. 23 is in general stochastic.*

With these additional structural assumptions, one can easily compute the wealth of agent $u$

$$
\begin{aligned}
\xi_t W_t &= \mathbf{E}_t \left[ \int_t^T \xi_s c_s^* ds + \xi_T W_T^* \right] \\
&= \mathbf{E}_t \left[ \int_t^T \xi_s \mathcal{I}(y^* \xi_t) ds + \xi_T \mathcal{I}(y^* \xi_T) \right] \\
&= \int_t^T \mathbf{E}_t \left[ \xi_s^{-\frac{1}{\gamma}+1} \right] \mathcal{I}(y^*) ds + \mathbf{E}_t \left[ \xi_T^{-\frac{1}{\gamma}+1} \right] \mathcal{I}(y^*) \\
&= \int_t^T \xi_t^{-\frac{1}{\gamma}+1} \mathcal{I}(y^*) a(s-t) ds + \xi_t^{-\frac{1}{\gamma}+1} a(T-t) \mathcal{I}(y^*) \ , \\
W_t &= \big( A(T-t) + a(T-t) \big) \mathcal{I}(y^* \xi_t) \ , \qquad (2.19)
\end{aligned}
$$

with

$$\rho = \frac{1-\gamma}{\gamma}\left(r + \frac{1}{2\gamma}\kappa^2\right) \begin{cases} > 0 & \text{if } 0 < \gamma < 1 \\ = 0, & \text{if } \gamma = 1 \\ < 0 & \text{if } \gamma > 1 \end{cases} \tag{2.20}$$

$$a(\tau) = e^{\rho\tau} > 0 \tag{2.21}$$

$$A(\tau) = \int_t^{t+\tau} e^{\rho s} ds = \frac{1-e^{\rho\tau}}{\rho} > 0 \ . \tag{2.22}$$

By applying Ito's lemma on optimal wealth (2.19), the dynamics of wealth

$$\begin{aligned} dW_t &= (\dots)\, dt + \frac{\partial W}{\partial \xi}(\xi_t, t)\ (-\xi_t \kappa^\top) dw_t \qquad (2.23) \\ &= (\dots) dt \ + \ (-\frac{W_t}{\gamma \xi_t})\ (-\xi_t \kappa^\top)\, dw_t \end{aligned}$$

are derived and, by equating the diffusive coefficient, with the one from the dynamic budget constraint (2.5), $W_t \theta_t \Sigma_t dw_t$,

$$(-\frac{W_t}{\gamma \xi_t})(-\kappa^\top \xi_t) = W_t \theta_t^\top \Sigma_t \tag{2.24}$$

we finally obtain the solution for the optimal portfolio

$$\theta_t = \frac{1}{\gamma}\ (\Sigma)_t^{-1} \kappa = \frac{1}{\gamma} \theta_t^{GOP} \ .$$

This interesting result illustrates that the two-fund separation holds, because agent invests in a multiple of the GOP and the riskless asset, as outlined in Remark 1. Moreover, the portfolio decision is independent of wealth or the horizon.

### 2.2.2 The Regulated Investor

Following Artzner et al. (1999) or Föllmer and Schied (2002) we define risk measure as a map $\varrho$ from the set of random variables, describing the wealth distribution at the terminal date $T$, to the extended reals, $\mathbb{R} \cup \infty$, which satisfies the (inverse) monotonicity property

$$\text{if } W_T^1 \leq W_T^2, \text{ then } \varrho( W_T^1) \geq \varrho(W_T^2)$$

and the translation invariance property (add some cash $m$ into the position)

$$\text{if } m \in \mathbb{R}, \text{ then } \varrho(W_T + m) = \varrho(W_T) - m \; .$$

A further useful property is that diversification is appropriately mapped, which results in the convexity property

$$\varrho(\varpi W_T^1 + (1-\varpi) W_T^2) \leq \varpi \varrho(W_T^1) + (1-\varpi)\varrho(W_T^2), \text{ for } 0 \leq \varpi \leq 1 \; .$$

If, in addition, the risk measure is positive homogeneous

$$\text{if } \varpi \geq 0 \text{ then } \varrho(\varpi W_T) = \varpi \varrho(W_T) \; ,$$

then it is a also a coherent measure.

The Value-at-Risk Measure

The Value-at-Risk (VaR) at the level $\alpha \in (0,1)$ is defined by

$$\begin{aligned} VaR_\alpha(W_T) &= \inf\{w \in \mathbb{R} : \; \mathbf{P}[W_T \leq w] > \alpha\} \\ &= \sup\{w \in \mathbb{R} : \; \mathbf{P}[W_T < w] \leq \alpha\} \; . \end{aligned}$$

The VaR is the terminal wealth level, that is undershot only with some given probability $\alpha$; in 'usual' situations, i.e. in $(1-\alpha)$ percent of the cases, terminal wealth will be greater than the VaR.

The VaR is always finite, as compared e.g. to the variance. It is well known that the VaR can be very sensitive to changes in the underlying probability $\alpha$ due to its left continuity. Moreover, the VaR is in general not a convex risk measure that is, there are cases where a better diversified portfolio has a higher VaR than a less diversified one. Some authors question, whether the absence of subadditivity of the

VaR indeed poses a problem, see Danielsson et al. (2005) and the references therein.

When an investor is optimizing his investment strategy $\theta$, the distribution of terminal wealth $W_T$ is endogenous, hence the $VaR_\alpha$ of his strategy. For regulatory reasons, however, the VaR from the investment strategy has to comply with an exogenous given wealth level $\underline{W}$. For technical reasons based on the later optimization problem, the VaR restriction is reformulated to

$$VaR_\alpha(W_T) \leq \underline{W} \Leftrightarrow \mathbf{P}[W_T < \underline{W}] \leq \alpha \Leftrightarrow \mathbf{P}[W_T \geq \underline{W}] \geq (1-\alpha) \ . \tag{2.25}$$

The Investment-Consumption Problem with VaR Restriction

The optimization problem of the VaR-restricted investor $v$ with intial wealth $W_0^v$ is different to the unrestricted investor, as he has to obey, in addition, the regulatory risk constraint (2.25).

**Remark.** *As we discuss in the following only the VaR restricted investor $v$, there arises no confusion about which agent is meant. Hence, we simplify the notation and omit the index $v$ from page 31 to 36.*

The dynamic problem recast in its static (variational) form reads

$$\sup_{\{c_t, W_T\}_{t\in[0,T]}} U(c,W) = \mathbf{E}\left[\int_0^T u(c_t)\,dt + u(W_T)\right] \tag{2.26}$$

$$s.t. \quad \begin{cases} \mathbf{E}\left[\int_0^T \xi_s c_s ds + \xi_T W_T\right] = \xi_0 W_0 \\ \mathbf{P}[W_T \geq \underline{W}] \geq (1-\alpha) \\ c_t \geq 0 \quad \forall t \\ W_T \geq 0 \ . \end{cases} .$$

Again, as for the unregulated investor, we apply the two step procedure and discuss the static problem first and recover afterwards the optimal portfolio decision.

Step 1: The Static Problem

In order to have a continuous cumulative distribution function (CDF) of $\frac{\xi_t}{\xi_0}$, the condition

$$\int_0^T \|\kappa_s\|^2 ds \neq 0$$

is needed additionally. It excludes trivial changes of measure $\mathbf{Q} = \mathbf{P}$ and/or risk neutrality. Consequently, there exists a unique $\overline{\zeta} = CDF^{-1}_{\frac{\xi_t}{\xi_0}}(1-\alpha)\ \xi_0$, such that $\mathbf{P}[\xi_T > \overline{\zeta}] = \alpha$.

There exists a feasible solution to the problem (2.26), if

$$\frac{W_0}{\underline{W}} > \mathbf{E}\left[\frac{\xi_T}{\xi_0}\mathbf{1}_{\{\xi_T \leq \overline{\zeta}(\alpha)\}}\right] \tag{2.27}$$

holds. In order to proof the sufficiency of this condition, consider the following feasible trading strategy: choose a very small constant consumption level with value $\epsilon$. Furthermore, implement a trading strategy that results in the constant payment $\underline{W}$ for all $\xi_T < \overline{\zeta}$. Invest in all other states of the world $\xi_T \geq \overline{\zeta}$ such that $W_T$ is constant and equal to $w$. Obviously this trading strategy is feasible as long as $w \geq 0$. The necessary budget $\xi_0\epsilon + \underline{W}E[\xi_T \mathbf{1}_{\xi_T \leq \zeta}] + wE[\xi_T \mathbf{1}_{\xi_T > \zeta}]$. Finally, letting $w \to 0$ and $\epsilon \to 0$, the minimum initial wealth which is needed to finance $\underline{W}$ equals $\underline{W}E[\xi_T \mathbf{1}_{\xi_T \leq \zeta}]$, which proves (2.27).

**Remark.** *We simplify the previously used notation where the optimal solution was explicitly indicated by star and omit this indication to enhance better readability.*

The solution of the optimization problem is, as derived by Basak and Shapiro (2001),

$$\begin{aligned} c_t(y) &= \mathcal{I}\left(y\xi_t\right) \\ W_T(y) &= \begin{cases} \mathcal{I}(y\xi_T) & \text{if } \xi_T \leq \underline{\zeta} \\ \underline{W} & \text{if } \underline{\zeta} < \xi_T \leq \overline{\zeta} \\ \mathcal{I}(y\xi_T) & \text{if } \overline{\zeta} < \xi_T \end{cases} \end{aligned} \tag{2.28}$$

with boundaries

$$\underline{\zeta} = \tfrac{u'(\underline{W})}{y} \qquad \overline{\zeta} : \mathbf{P}[\xi_T \geq \overline{\zeta}] = \alpha \ ,$$

where the Lagrange multiplier $y > 0$ is the solution to the (static) budget equation

$$\mathbf{E}\left[\int_0^T \xi_t c_t(y)dt + \xi_T W_T(y)\right] = \xi_0 W_0. \tag{2.29}$$

Note that the relation $\underline{\zeta} < \overline{\zeta}$ holds, if the VaR restriction is binding. If not, $\mathcal{I}(y\underline{\zeta}) \geq \underline{W}$ and the unrestricted optimal solution already fulfills the VaR constraint. In this case, the Lagrange multipliers of the unrestricted and the restricted investor are the same. If the VaR constraint is binding, this is not true and the Lagrange multiplier of the restricted investor is larger than the one of the unrestricted, since the budget is tighter with a binding restriction (see Basak and Shapiro (2001), Proposition 1).

In the following, low discount factors $\xi$ are associated with a 'good' economic situation and vice versa; this wording is backed by the equilibrium consideration put forward in the next Section 2.3.1, especially in equation (2.40), where low discount factors are associated to a high aggregate consumption of the representative agent.

The solution consist of three regions in the state space:

1. 'Good' states $\xi_T \leq \underline{\zeta}$: In this region the agents behaves as if being unrestricted.

2. 'Intermediate' states $\xi_T \leq \underline{\zeta}$: In this region the VaR-constrained investor just maintains the sufficient wealth level $\underline{W}$ to fulfill regulatory constraint.

3. 'Bad' states $\overline{\zeta} < \xi_T$: In this region the investor is effectively unrestricted, as the loss his strategy is incurring is not limited by the risk measure, but only by his risk aversion.

Berkelaar et al. (2002) point out that the solution under a VaR restriction is qualitatively similar to that of an unrestricted agent, who exhibits loss aversion in the sense of Kahneman and Tversky (1979). Agents are more cautious in 'good' states and take more risks, when losses arise.

The optimal wealth $W_T$ jumps downward at $\overline{\zeta}$. With this time $T$ solution at hand, it is obvious, that the classical approach of Merton using the HJB equation poses significant technical problem, when applied to this problem, as terminal wealth $W_T$ is neither continuous nor convex in $\xi$; standard verification theorems are no applicable. Note that the state $\overline{\zeta}$, where the VaR restriction becomes binding, is independent of the preferences or the initial endowment of the investor; it depends only on the distribution of the stochastic discount factor $\xi_T$ and on the VaR probability $\alpha$. Hence, aggregating multiple investors with different initial wealth, but the same regulatory constraint, into a single representative agent with the same solution structure can be easily achieved. Ahn et al. (1999) study the use of a finite number of standard options for risk management with VaR. They find an analogous results, namely, that $\overline{\zeta}$ is independent of individual agents' characteristics.

To illustrate the solution, if binding, in an economically more intuitive way, we replicate it by

1. the *scaled* GOP fund $G_T = \mathcal{I}(y\xi_T)$ of the unrestricted investor, and
2. a put long with maturity $T$, strike price $\underline{W}$, and down-and-out barrier $K = \mathcal{I}(y\overline{\zeta}),\ K < \underline{W}$ .

The optimal solution $W_T$ of the restricted investor can be reformulated as a static derivative position

$$W_T = \underbrace{G_T}_{(1.)} + \underbrace{\max\{(\underline{W} - G_T), 0\}\mathbf{1}_{\{G_T \geq K\}}}_{(2.)} . \tag{2.30}$$

Note that the VaR restricted investor is less invested in the mutual fund, since the additional option has a positive value, which in turn tightens the budget as expressed by a larger Lagrange multiplier $y$ for the VaR restricted investor.

In the extreme case $\alpha = 0$, the investor in fact prevents terminal wealth from falling beyond the boundary $\underline{W}$ as the knock-out barrier $K(\alpha) \to 0$. With $\alpha > 0$, there always exist states, where wealth is below the exogenous boundary $\underline{W}$.

Step 2: Recovering the Portfolio Decision

To derive the optimal strategy analytically we again assume that the interest rate $r$ and the market price of risk $\kappa$ are constant. The result also can be also found in

Basak and Shapiro (2001). If the optimal Lagrange multiplier $y$ satisfies $y \leq \frac{u(\underline{W})}{\bar{\zeta}}$, then the optimal strategy of the unrestricted investor emerges, as the unrestricted optimal profile already fulfills the VaR restriction. In case of $y > \frac{u(\underline{W})}{\bar{\zeta}}$ the optimal wealth reads

$$
\begin{aligned}
W_t \;=\; & \underbrace{\big(a(T-t) + A(T-t)\big)\mathcal{I}(y\xi_t)}_{(i)} \\
& + \underbrace{a(T-t)\mathcal{I}(y\xi_t)\left(\Phi(-d_1(\bar{\zeta})) - \Phi(-d_1(\underline{\zeta}))\right)}_{(ii)} \\
& - \underbrace{e^{-r(T-t)}\underline{W}\left(\Phi(-d_2(\bar{\zeta})) - \Phi(-d_2(\underline{\zeta}))\right)}_{(iii)},
\end{aligned}
$$

where $\Phi$ is the standard normal distribution and

$$
\begin{aligned}
d_1(\zeta) &= \frac{\left(r - \frac{\kappa^2}{2}\right)(T-t) + \ln\left(\frac{\zeta}{\xi_t}\right) + \frac{\kappa^2}{\gamma}(T-t)}{\sqrt{T-t}\;\kappa} \\
d_2(\zeta) &= \frac{\left(r - \frac{\kappa^2}{2}\right)(T-t) + \ln\left(\frac{\zeta}{\xi_t}\right)}{\sqrt{T-t}\;\kappa}\,.
\end{aligned}
\qquad (2.31)
$$

The deviation of $W_t$ can be found in the Appendix to Chapter 3.

As shown in (2.30), $W_T$ consists of two components. The first part $(i)$ corresponds to investment in the growth optimal fund with terminal value $G_T$, see (1.) in (2.30). The second and third parts $(ii)$ and $(iii)$ represent together the value of the put option with strike price $\underline{W}$ and knock-out barrier $K$ at point in time $t$, compare (2.) in (2.30).

We want to state that the budget equation $\mathcal{B}(c(y), W(y)) = \xi_0 W_0$ cannot be inverted analytically to obtain the optimal Lagrange multiplier $y$. Thus (2.29) has to be inverted numerically for $y$. Nevertheless, by the implicit function theorem, the derivatives with respect to exogenous variables, say $\chi$, $y'(\chi) = \frac{d\mathcal{B}^{-1}}{d\chi}$ are known in the neighborhood of the optimal solution $y$. It enables us to derive analytically the sign of effects in the comparative static.

Applying the portfolio comparison as in equation (2.23) on p. 29, the optimal strategy turns out to be

$$\begin{aligned}\theta_t &= q_t\theta_t^u = q_t\frac{1}{\gamma}\theta_t^{GOP} \\ q_t &= \underbrace{1}_{(i)}\underbrace{-e^{-r(T-t)}\frac{\underline{W}}{W_t}\left(\Phi(-d_2(\overline{\zeta}))-\Phi(-d_2(\underline{\zeta}))\right)}_{(ii)} \\ &\quad \underbrace{+e^{-r(T-t)}\frac{J}{W_t}\frac{\gamma}{\|\kappa\|\sqrt{T-t}}\phi(-d_2(\overline{\zeta}))}_{(iii)}\ ,\end{aligned} \tag{2.32}$$

where $\theta^u$ is the optimal strategy of the unrestricted investor and $J = \underline{W} - \mathcal{I}(y\overline{\zeta}) = \underline{W} - K$ is the jump's size. The derivation of the optimal strategy can be accomplished along the lines presented in the Appendix to Chapter 3.

The portfolio consists of investing in the growth optimal fund, part $(i)$, in a put option, part$(ii)$, and in the replication of the knock-out feature, part $(iii)$. Parts $(i)$ and $(ii)$ of (2.32) sum to less than 1, as the investment into the fund $(i)$ has a delta hedge of 1, and the put option $(ii)$ a delta hedge of (0,-1).

The portfolio decision is a multiple of the GOP, that is, the VaR restriction does not change the (relative) structure of the portfolio itself, but only how much is invested in the risky asset in total. Basak and Shapiro (2001) shows that there exists a (deterministic) $\xi(t)^*$, where $q_t > 1,\ \forall \xi_t > \xi(t)^*$, that is, a point where the portfolio with VaR constraint (2.32) is more risky then the unrestricted one (2.24). This behaviour shows the incentive of a VaR restriction to 'gamble for resurrection' in adverse situations.

Finally note, that the optimal strategy now depends on the current wealth level, even though the utility function has constant relative risk aversion.

**Remark 2.** *This section can be interpreted as production economy as well. Assume that there is a single perishable good, which must be consumed or invested in the production. There exist different production technologies with constant returns to scale. If a quantity of the physical good $P^n$ is invested in the production $n$, its pay-off in the next time instant is $P^n + dP^{Y,n}$. The coefficient functions $\mu$ and $\Sigma$ are, in this case, exogenously given and are possibly driven by $\mathcal{F}$-measurable state variables.*

*The production technology is perfectly elastic.*
*The difference between our model and the one of Cox et al. (1985a) is that the locally risk-less production technology is perfectly elastic, whereas in Cox et al. (1985a) the riskless asset in zero net supply. Our model is in line with Constantinides (1990) and Obstfeld (1994).*

## 2.3 Asset Prices in a Pure Exchange Equilibrium

This section describes asset pricing in a pure exchange economy of the Lucas (1978) type. First, we characterize the equilibrium in terms of aggregate demand, state prices (stochastic discount factor), and the marginal utility of the representative agents. Then, the results are applied to an economy, where some agents face a VaR restriction.

### 2.3.1 Definition and Characterization of the Equilibrium

There is an aggregate 'tree' which produces the aggregate cash-flow stream $\delta_t = \sum_{n=1}^{N} \delta_T^n$, the 'appels'. The aggregate cash-flow stream follows the exogenously given dynamics

$$\frac{d\delta_t}{\delta_t} = \mu_\delta(\delta_t, t)dt + \sigma_\delta(\delta_t, t)^\top dw_t \quad ,$$

where $\mu_\delta$ is the instantaneous return process and $\sigma_\delta$ the volatility process. Moreover, $\delta_0 > 0$. The number $N$ of trees is fixed (totally inelastic) and normalized to one. In a production economy like the one of Cox et al. (1985a), the decision on how much to invest in the different production technologies is endogenous as opposed to the fixed supply in the pure exchange economy of the Lucas type.

There are $m \in \{1, \ldots, M\}$ agents in the economy, each with a percentage initial endowment $e_0^m > 0$, $(e_0 = (e_0^1, \ldots, e_0^M \top)$, $\mathbf{1}^\top e_0 = 1$, in the tree. Agents can trade continuously in the (complete) financial market, as outlined in the previous section. It consists of the risk-less investment with dynamics (2.1) and of the risky securities with dynamics (2.2).

Definition of Equilibrium

An equilibrium is defined as a collection of *optimal* consumption, *optimal* portfolio and resulting *optimal* wealth processes $(c^m, \theta^m, W^m)$, as well as the investment opportunity processes of the market $(r, \mu^M, \sigma^M)$, such that the market for the consumption good with supply $\delta_t$, the markets for the risky assets and the market for the money market clear for all $t$,

$$\begin{aligned} \textstyle\sum_{m=1}^M c_t^m &= \delta_t \\ \textstyle\sum_{m=1}^M \theta_t^m W_t^m &= P_t^M \\ \textstyle\sum_{m=1}^M (1-\theta_t^m) W_t^m &= 0 \ , \end{aligned} \tag{2.33}$$

where $P = (P^1, \ldots, P^N)^\top$ is the vector of security prices and $P^M = P^\top \mathbf{1}$ denotes the market value.

Characterization of the Equilibrium in Terms of the Aggregate Demand

From the definition of equilibrium we obtain, together with the optimal consumption, that in any equilibrium

$$\delta_t = \sum_{m=1}^M c_t^m = \sum_{m=1}^M \mathcal{I}(y^m \xi_t)$$

holds, where the optimal Lagrange multiplier $y^m$ for the agent $m$ must satisfy the static budget equations (2.17)

$$\mathbf{E}\left[\int_0^T \xi_s \left(\mathcal{I}(y^m \xi_s)\right) ds + \xi_T W_T^m(y^m)\right] = e_0^m \mathbf{E}\left[\int_0^T \xi_s \delta_s ds + \xi_T \mathbf{1}^\top P_T\right] \ , \quad \forall m \ .$$

On the other side, if all budgets under optimal consumption are fulfilled and the consumption market clears, it is an equilibrium, see Theorem (4.5.2) in Karatzas and Shreve (1998).

In the following, the characterization of the equilibrium is reformulated in such a way that the individual decision can be recovered from aggregate quantities, namely from $\delta$ and the initial distribution of endowments.

Let us define the aggregate demand function for consumption $\mathcal{D} : (0,\infty) \to (0,\infty)$ by

$$\mathcal{D}(\xi; y) = \sum_{m=1}^{M} \mathcal{I}(y^m \xi) \text{ with } y = (y^1, \ldots, y^M)^\top ,$$

where $\mathcal{D}(\xi; y)$ is a continuous, strictly decreasing, and convex function in $\xi$, for all $y \in \mathbb{R}^m_+$. Thus, an inverse demand function $\mathcal{D}^{-1}$ exists. Consumption clearing in equilibrium can be rewritten as

$$\mathcal{D}(\xi_t; y) = \delta_t \ , \quad \forall t \ .$$

Inverting this equation, the stochastic discount factor corresponds to the inverse demand function

$$\xi_t \;\; = \;\; \mathcal{D}^{-1}(\delta_t; y) \ , \quad \forall t, \tag{2.34}$$

and, by inserting back into the static budget equation, the system of equations

$$\begin{aligned} \mathbf{E}_0 \quad & \left[ \int_0^T \mathcal{D}^{-1}(\delta_s; y) \left( \mathcal{I}(y^m \mathcal{D}^{-1}(\delta_s; y)) \right) ds + \mathcal{D}^{-1}(\delta_T; y) W_T^m(y^m) \right] \\ = \quad & e_0^m \mathbf{E} \left[ \int_0^T \mathcal{D}^{-1}(\delta_s; y) \delta_s ds + \mathcal{D}^{-1}(\delta_T; y) P_T^M \right] \ , \quad \forall m \end{aligned} \tag{2.35}$$

emerges. Again, if all budgets under optimal consumption are fulfilled, if the consumption market clears, and if prices calculated by the inverse demand function as a discount factor, it is an equilibrium (see Corollary (4.5.4) in Karatzas and Shreve (1998)). This result is of importance, since, if we find the 'right' discount factors, such that the market for consumption clear and all budget constraints are fulfilled, we also find an equilibrium.

In the previous derivation, we used the important property that clearing only the consumption good market also clears asset markets and, according to Walras's law, the riskless asset market as well, see Karatzas et al. (1990) (zero net supply securities) and Basak (1995) (positive net supply securities). Therefore, we need not to consider separately the situation at time $t = T$.

Individual consumption can be now characterized with respect to the (inverse) demand function

$$c_t^m = \mathcal{I}\big(y^m \mathcal{D}^{-1}(\delta_t; y)\big) . \tag{2.36}$$

From this equation a linear sharing rule follows as, for two agents $m$ an $n$

$$\frac{c_t^m}{c_t^n} = \frac{\mathcal{I}(y^m)}{\mathcal{I}(y^n)} , \quad \forall t$$

holds.

**Remark 3.** *There are slight differences how to model the terminal date in the economy. Our model uses utility at the horizon which can most easily be understood as a bequest function; at time $T$, the market value $P_T^M$ is transferred to the heirs of the representative agent, and the economy stops. With this approach, there are no jumps in prices from $T-$ to $T$. This approach was also chosen by Basak (2002) and Berkelaar et al. (2002).*
*Alternatively, the utility at the horizon of restriction can be modelled as the indirect utility of an ongoing economy, as in Basak (1995) or Basak and Shapiro (2001). In this case, a single predictable jump in asset prices $P$ occurs at the horizon of the restriction between $T-$ and $T$, since, all market participant know, that the restriction is relaxed and all market participant can follow their unrestricted investment strategy from then on; notwithstanding discounted prices $\xi P_t$ will still be continuous, as to prevent arbitrage opportunities. To maintain the easiest framework, the first variant is chosen.*

### Characterization of the Equilibrium in Terms of a Representative Agent

Now, we make use of two characteristics of the model to simplify the solution to the system of budget equations (2.35). $(i)$ Complete markets enable us, to define a representative agent, where we can isolate the impact of the wealth distribution between agents from the pricing kernel in equilibrium. $(ii)$ As all agents share the same coefficient of relative risk aversion $\gamma$, only the initial wealth distribution will be affecting equilibrium prices. In the special case of an economy with only unrestricted investors, prices are independent of the wealth distribution, which is a standard result of the utility function with constant relative risk aversion.

In the following the representative agent is constructed, as in Huang (1987) or Karatzas et al. (1990). Afterwards, the connection between the (inverse) demand function, the representative agent's marginal utility, state prices, and the stochastic discount factor is illustrated. Finally, we illustrate, how to apply the concept on an representative agent to solve for the equilibrium.

Define the representative agent as the weighted sum of individual utilities

$$\mathcal{U}(C_t, \lambda) = \max_{c^m : \sum_{m=1}^{M} c_t^m = C_t} \sum_{m=1}^{M} \lambda_m u(c_t^m) \ , \qquad t \in [0, T] \tag{2.37}$$

with positive weights $\lambda_m > 0$; the vector of weights is denoted by $\lambda = (\lambda_0, \dots, \lambda_M)^\top$. In the later part of this section, the connection between the vector of weights $\lambda$ and the vector of Lagrange multipliers $y$ which codes the initial wealth distribution, will become evident. $C$ is the aggregate consumption. The analogous definition holds for the representative agent's utility function for terminal wealth. Using $z$ as the Lagrange multiplier for consumption clearing and solving the utility function, one obtains the utility function of the representative agent[1]

$$\begin{aligned} \mathcal{U}(C, \lambda) &= u(C)\|\lambda\|_\gamma \\ \mathcal{U}(W, \lambda) &= u(W)\|\lambda\|_\gamma \end{aligned} \tag{2.38}$$

with

$$z = \mathcal{U}_C(C, \lambda) = u'(C)\|\lambda\|_\gamma \ ,$$

[1] The utility function is normalized in the level by a constant $g > 0$ which is, however, irrelevant for the following analysis.

where $\|\lambda\|_\gamma = \sum_{m=1}^{M} \left( \lambda_m^{\frac{1}{\gamma}} \right)^\gamma$ and $\mathcal{U}_C(C, \lambda) = \frac{\partial \mathcal{U}}{\partial C}$.

When differentiating the utility function of the representative agent with respect to aggregate consumption, the relation

$$\mathcal{U}_C(C_t = \delta_t; \lambda) = \mathcal{D}^{-1}(\delta_t; y), \qquad \forall t$$

holds, if and only if

$$\lambda_m = \frac{1}{y^m} \ , \qquad (m = 1, \ldots, M) \tag{2.39}$$

holds. If and only if the relation between the weights attributed by the representative agent to the individual investor and the Lagrange multiplier of the individual budget constraints is given by (2.39), the individual consumptions, expressed in dependence of the inverse demand function from equation (2.36), are indeed the optimal solution for the representative agent's utility maximization problem (2.37). Thus, the representative agent attributes, in particular, constant weights to the various agents.

Note that in equilibrium the following relations are valid

$$\begin{aligned} \xi_t &= \mathcal{U}_C \left( \delta_t; \frac{1}{y} \right) = u'(\delta_t) \|1/y\|_\gamma \\ &= \mathcal{D}^{-1}(\delta_t; y) \\ &= z_t \ . \end{aligned} \tag{2.40}$$

The stochastic discount factor $\xi$ can thus alternatively be interpreted as the representative's agent marginal utility $\mathcal{U}_C$, or as the value of the inverse demand function of the market $\mathcal{D}^{-1}$, or as the running shadow price of market clearing constraint $z$ of the representative agent, given the state of the economy $\delta$. The last interpretation motivates the name state price.

Note, that the aggregate consumption and the marginal aggregate consumption are linear homogenous in the second argument $\lambda$

$$\begin{aligned}\mathcal{U}(C, a\lambda) &= a\mathcal{U}(C, \lambda)\\ \mathcal{U}_C(C, a\lambda) &= a\mathcal{U}_C(C, \lambda)\ ,\end{aligned}$$

which gives one degree of freedom for the parameter vector $\lambda$; in our applications, we will use this homogeneity property to normalize the utility function of the representative agent with the condition $\|\lambda\|_\gamma = 1$.

Now equilibrium values for the instantaneous risk-free rate (process) $r$ and market price of risk (process) $\kappa$ can be easily derived by applying Ito's lemma to $\xi$ as characterized in equation (2.6) and to $\mathcal{U}_C(\delta; 1/y)$ in equation (2.40) and equalizing the coefficients of $dt$ and $dw_t$. The interest rate and the market price of risk are found by comparing coefficients

$$\begin{aligned}r_t &= \left(-\frac{\mathcal{U}_{CC}}{\mathcal{U}_C}\right) \cdot \mu_\delta \delta_t - \frac{1}{2}\left(\left(-\frac{\mathcal{U}_{CC}}{\mathcal{U}_C}\right)\left(-\frac{\mathcal{U}_{CCC}}{\mathcal{U}_{CC}}\right)\right)\|\sigma_\delta \delta_t\|^2\\ &= \gamma\mu_\delta\left(\delta_t, t\right) - \frac{1}{2}\gamma(1+\gamma)\|\sigma_\delta\left(\delta_t, t\right)\|^2\\ \kappa_t &= \left(-\frac{\mathcal{U}_{CC}}{\mathcal{U}_C}\right)\sigma_\delta\delta_t\\ &= \gamma\sigma_\delta\left(\delta_t, t\right)\ ,\end{aligned} \tag{2.41}$$

where $-\frac{\mathcal{U}_{CC}}{\mathcal{U}_C}C$ is the relative risk aversion of the representative agent and $-\frac{\mathcal{U}_{CCC}}{\mathcal{U}_{CC}}C$ is the coefficient of relative prudence.

Using that the utility function for the representative agent exhibits also CRRA, we further obtain that the relative risk aversion is $-\frac{\mathcal{U}_{CC}}{\mathcal{U}_C}C = \gamma$ and the coefficient of relative prudence $-\frac{\mathcal{U}_{CCC}}{\mathcal{U}_{CC}}C = (1+\gamma)$. Consequently, neither the interest rate process $r$ nor the market price of risk process $\kappa$ depend on the initial distribution of wealth of individual investors, as represented by the vector $y$. If, in addition, the coefficients of the cash-flow process of the economy, $\mu_\delta$ and $\sigma_\delta$, are constant, $r_t$ and $\kappa_t$ are also constant.

Inserting the state price $\xi = \mathcal{D}^{-1}(\delta, 1/y)$ together with the interest rate $r$ and the market price of risk $\kappa$ back into the system of static budget equations (2.35), the determination of equilibrium finally boils down to finding the Lagrange multipliers $y$ that satisfy the system of equations (2.35).

**Remark.** ***Relation to the (Consumption)CAPM of Breeden (1979)***
*By using the definition of the market price of risk $\kappa$ and its functional form in equation (2.41), the excess return of any security, say $X$, with diffusion coefficient vector $\sigma_t^X$, is in equilibrium*

$$\mu_t^X - r_t \quad = \quad \gamma \sigma_\delta^\top \sigma_t^X \ . \tag{2.42}$$

*Alternatively, a similar result, the Euler equation, can be obtained using the marginal utility of an agent along its optimal individual consumption path*

$$\mu_t^X - r_t = \gamma \ \mathbf{C}_t \left[ \frac{dX_t}{X_t}, \frac{dc_t^m}{c_t^m} \right] \Big/ \ dt \ .$$

*In order to obtain a result more in line with the CAPM, let us define a self-financing portfolio $\theta^M$ with value $P^M$ which contains the exact risk structure as the aggregate dividend*

$$\left(\theta_t^M\right)^\top D\left(P_t\right) \Sigma_t = \delta_t \sigma_\delta^\top \ ,$$

*where $D(P)$ is the diagonal matrix of the vector of prices $P$. This can be interpreted as the 'market' portfolio. This portfolio leads, under the assumed dynamics*

$$dP_t^M = P_t^M \mu_t^M dt + P_t^M \left(\sigma_t^M\right)^\top dw_t \ ,$$

*to an instantaneous volatility of*

$$\sigma_t^M = \frac{1}{P_t^M} \cdot D\left(P_t\right) \Sigma_t \theta_t^M = \frac{\delta_t}{P_t^M} \sigma_\delta$$

*by construction. Substituting $\sigma_\delta$ of previous equation into the CCAPM relation (2.42) and using the instantaneous excess return in equilibrium (2.42) on the* *portfolio itself, the equations for the instantaneous expected excess return*

$$\begin{aligned}\mu_t^X - r_t &= \gamma \frac{\delta_t}{P^M{}_t} \left(\sigma_t^M\right)^\top \sigma_t^X \\ \mu_t^M - r_t &= \gamma \frac{\delta_t}{P^M{}_t} \left(\sigma_t^M\right)^\top \sigma_t^M\end{aligned}$$

*emerge which result in a typical CAPM relation*

$$\mu_t^X - r_t = \beta_t^X (\mu_t^M - r_t)$$

*where*

$$\beta_t^X = \frac{\left(\sigma_t^M\right)^\top \sigma_t^X}{\left(\sigma_t^M\right)^\top \sigma_t^M} = \frac{\mathbf{C}_t \left[\frac{dX_t}{X_t}, \frac{dP^M{}_t}{P^M{}_t}\right]}{\mathbf{V}_t \left[\frac{dP^M{}_t}{P^M{}_t}\right]} .$$

## 2.3.2 Equilibrium with VaR-constrained Agents

In this section, we analyze in addition the specific assumption that the aggregate cash-flow follows a geometric Brownian motion with only one risk source,

$$\frac{d\delta_t}{\delta_t} = \mu_\delta dt + \sigma_\delta dw_t \quad (\delta_0 > 0) . \tag{2.43}$$

First, we solve analytically the pure exchange equilibrium with only unrestricted agents. Afterwards, we consider an economy with two groups of agents, one group of unrestricted agents and a second group of VaR constrained agents.

In both cases, the techniques to solve for the equilibrium are similar: The system of budget equations (2.35) for the individual investor together with the equilibrium state prices in equation (2.34) have to be solved. This will be done in three steps. First, the state prices can be determined by applying equation (2.40). Second, inserting these state prices into equation (2.35), all variables beside the Lagrange multipliers will be known, i.e. finding the equilibrium is reduced to solving for the vector of Lagrange multipliers. Third, one needs to show existence and uniqueness of a solution vector.

### Unconstrained Equilibrium

The derivation of the unconstrained equilibrium follows closely the steps in Section 2.2.1, in which we recovered the portfolio from the optimal terminal wealth. Due to the homogeneity property of the representative agent's utility function $\mathcal{U}$ in $\lambda$, the normalization $\|\lambda\|_\gamma = 1$ can be applied. Applying Ito's lemma on $\xi_t = \mathcal{U}_C(\delta_t, \lambda)$ and equating the term $d\xi_t$ from (2.6)

$$\begin{aligned} -rdt + \kappa dw_t \overset{(2.6)}{=} \frac{d\xi_t}{\xi_t} &= \frac{d\,\mathcal{U}_C(\delta_t, \lambda)}{\mathcal{U}_C(\delta_t, \lambda)} \\ &\overset{(2.38)}{=} \frac{d\ u'(\delta_t)}{u'(\delta_t)} \overset{(2.43)}{=} -\left(\gamma\mu_\delta - \frac{1}{2}\gamma(1+\gamma)\sigma_\delta^2\right) dt - (\gamma\sigma_\delta)\, dw_t \end{aligned} \tag{2.44}$$

allows us to identify the equilibrium interest rate and market price of risk. Comparing the LHS with the RHS

$$r = \gamma\mu_\delta - \frac{1}{2}\gamma(1+\gamma)\sigma_\delta^2 \tag{2.45}$$

$$\kappa = \gamma\sigma_\delta\ , \tag{2.46}$$

and substituting the equilibrium state prices (2.44) into equation (2.19), evaluated at $t = 0$, we obtain

$$W_0^m = \big(a(T) + A(T)\big)\mathcal{I}(y^m)\delta_0\ . \tag{2.47}$$

On the other hand, the vale of the market $P^M = \mathbf{1}^\top P_t$ in (2.10), i.e. the aggregate wealth at time $t$, using the equilibrium state prices, is given by

$$P_t^M = \mathbf{E}_t\left[\int_t^T \frac{\xi_s}{\xi_t}\delta_s ds + \frac{\xi_T}{\xi_t} P_T^M(\delta_T)\right] = \big(a(T-t) + A(T-t)\big)\delta_t\ . \tag{2.48}$$

The functions $a$ and $A$ are as in (2.21) and (2.22), in which the interest rate and market price of risks in equilibrium, (2.45) and (2.46), are substituted in. Inserting (2.47) and (2.48), evaluated at $t = 0$, into the budget equation of investor $m$,

$$\big(a(T) + A(T)\big)\mathcal{I}(y^m)\delta_0 = W_0^m = e_0^m P_0^M = e_0^m\big(a(T) + A(T)\big)\delta_0\ , \quad (m = 1, \ldots, M),$$

the Lagrange multipliers in equilibrium read

$$\begin{aligned} y^m &= u'(e_0^m) \\ &= \frac{1}{\lambda_m} \ . \end{aligned} \tag{2.49}$$

The last equation uses the former result (2.39), that the weight of the representative agent $\lambda_m$ and the Lagrange multiplier are inversely related (see also Karatzas et al. (1990), Theorem 10.2 and Theorem 10.3).

In order to solve for the drift and the volatility of the market portfolio, we apply Ito's lemma on the market value $P^M$ in equation (2.48) and obtain

$$\begin{aligned} dP_t^M &= (\cdot)dt + \big(a(T-t) + A(T-t)\big)(\sigma_\delta \delta_t)dw_t \\ &= (\cdot)P_t^M dt + \sigma_\delta P_t^M dw_t \ . \end{aligned}$$

Comparing the diffusion coefficients with the dynamic budget (2.5), where $W^{\theta=\mathbf{1}}$ is replaced by $P^M$, the instantaneous market volatility and drift are

$$\begin{aligned} \sigma_t^M = \sigma^M &= \Sigma\mathbf{1} = \sigma_\delta \\ \mu_t^M - r = \mu^M - r &= \mathbf{1}^\top \Sigma\kappa = \gamma\|\sigma_\delta\|^2 \ , \end{aligned} \tag{2.50}$$

where the drift term can be more easily derived from the equation $\mu_t^M = r + \kappa\sigma_t^M$.

#### Equilibrium with VaR-Constrained Agents

Now, the set of investors is split into two subsets. The first $R$ investors are unconstrained as before, $(1, \ldots, m_R)$, in contrast, the remaining investors $(m_R + 1, \ldots, M)$ face a VaR restriction.

The VaR restrictions for the agents are assumed to be homogenous in the VaR probability $\alpha$ and the resistance level $\underline{W}$, i.e. they are identical for all the restricted investors. Thus, the investors can be aggregated to two representative agents $u$ (unrestricted) and $v$ (VaR restricted), where the unrestricted one holds as initial endowment the fraction $\omega \geq 0$ of the market, whereas the restricted agent $v$ holds $(1-\omega)$.

By introducing the VaR constrained agents, the structure of the utility function of the (total) representative agent does not change. Only the level depends on the new weights $(\lambda_u, \lambda_v)$, as can be seen in equation (2.38). From (2.44) follows that the state prices are not changed if VaR constrained agents are added.

From the homogeneity property of the representative agent's utility function $\mathcal{U}(C, \lambda)$ in $\lambda$ and the normalization

$$1 \;=\; \|\Lambda\|_\gamma = \left(\lambda_u^{\frac{1}{\gamma}} + \lambda_v^{\frac{1}{\gamma}}\right)^\gamma$$

follows

$$\begin{aligned} 1 &= \lambda_u^{\frac{1}{\gamma}} + \lambda_v^{\frac{1}{\gamma}} \\ &= \left(\frac{1}{y^u}\right)^{\frac{1}{\gamma}} + \left(\frac{1}{y^v}\right)^{\frac{1}{\gamma}} \\ &= \mathcal{I}(y^u) + \mathcal{I}(y^v) \; . \end{aligned}$$

The last equation enables us to solve for the Lagrange multiplier of the unrestricted agent $u$ as a function of the Lagrange multiplier of the restricted agent $v$,

$$y^u(y^v) = u'\big(1 - \mathcal{I}(y^v)\big) \; . \tag{2.51}$$

Finally we need to solve the system of budget equations (2.35), analogously to the unrestricted case. To do this we insert into the RHS and the LHS of (2.35) $(i)$ the equilibrium state prices $\mathcal{D}^{-1}(\delta_t, y) = \xi_t = u'(\delta_t)$ from (2.44), $(ii)$ the equilibrium interest rate $r$ from (2.45) and the market price of risk $\kappa$ from (2.46), and $(iii)$ the initial endowment $(e_0^u = \omega, e_0^v = (1-\omega))$. As results we obtain the following two equations for the two groups of investors

$$\begin{aligned} W_0^u(y^u) &= \omega P_0(y^v, y^u) \; , \\ \underbrace{W_0^v(y^v)}_{LHS} &= \underbrace{(1-\omega) P_0(y^v, y^u)}_{RHS} \; . \end{aligned} \tag{2.52}$$

Substituting the representation of $y^u(y^v)$ from (2.51), we are able to reduce equations (2.52) to the single equation

$$\frac{W_0^u(y^v)}{P_0(y^v)} = \omega \ . \tag{2.53}$$

(2.53) can be solved for $y^v$. In our case of a VaR restriction, this has to be done numerically. Nevertheless, existence and uniqueness can be shown. As the same technique will be used later in a more complex setting, the following part illustrates, how to show the existence and uniqueness of the equilibrium. We do this by proving that there exists a unique $y^v$ satisfying the above-stated equilibrium equation.

**Proof:**

Since

$$\mathcal{I}(y^u) + \mathcal{I}(y^v) = 1 \wedge y^u > 0 \wedge y^v > 0$$

hold, the Lagrange multipliers have to satisfy $y^v > 1$ and $y^u > 1$.

To simplify the notation we replace $W_0^u(y^v), W_0^v(y^v)$, and $P_0(y^v)$ by $W_u, W_v$, and $P$. Since $W_v, W_u$ and $P$ are function of $y^v$ only, the derivative with respect to $y^v$ is denoted by $(\cdot)'$.

The fraction held by the unrestricted investors $\frac{W_u}{P}$ is strictly increasing in $y^v$ since

$$\begin{aligned}
\frac{d\frac{W_u}{P}}{dy^v} &= \frac{W_u' P - W_u(P')}{P^2} \overset{P=W_u+W_v}{=} \frac{W_u' P - W_u(W_u' + W_v')}{P^2} \\
&= \frac{W_u'(P - W_u) - W_u W_v'}{P^2} \overset{P-W_u=W_v}{=} \frac{W_u' W_v - W_u W_v'}{P^2} \\
&> 0 \ . 
\end{aligned} \tag{2.54}$$

In the derivation of (2.54) we need $(i)$ $W_u > 0, W_v > 0$, which is obvious, and $(ii)$ $W_v' < 0, W_u' > 0$. $W_v' < 0$ directly follows from $y^v$ being the Lagrange multiplier; $W_u' = \big(W_u(y^u(y^v))\big)' = W_u'(y^u) y_u'(y^v) > 0$ holds, as $W_u'(y^u) < 0$ and $y_u'(y^v) > 0$.

Furthermore, the domain of $y^v$ is

$$\begin{aligned} \lim_{y_v \to \infty} \frac{W_u}{P} &= \hat{\omega} \\ \lim_{y_v \to 1} \frac{W_u}{P} &= 0 \ , \end{aligned}$$

where $\hat{\omega}$ is defined by

$$\frac{1-\hat{\omega}}{\hat{\omega}} = \frac{\overbrace{\underline{W}\mathbf{E}\left[\frac{\xi_T}{\xi_0}\mathbf{1}_{\{\delta_T \leq \underline{\delta}\}}\right]}^{W_v/P: y_v \to \infty}}{\underbrace{\mathbf{E}\left[\int_0^T \frac{\xi_s}{\xi_0}\delta_s ds + \frac{\xi_T}{\xi_0}\delta_T\right]}_{W_u/P: y_v \to \infty}} \ .$$

The second assertion follows from

$$W_u = \mathcal{I}(y^u)\mathbf{E}\left[\int_0^T \frac{\xi_s}{\xi_0}\delta_s ds + \frac{\xi_T}{\xi_0}\delta_T\right] \overset{y^v \to 1 \Leftrightarrow y^u \to \infty}{\longrightarrow} 0 \ .$$

Consequently, there is a unique $y^v$ satisfying the equilibrium condition, if the initial endowment of the unrestricted investor $\omega$ is in $[0, \hat{\omega})$.

□

In contrast to the unrestricted case, where any wealth distribution was possible, the restricted agent $v$ needs to hold at least a market share of $(1-\hat{\omega})$ in order to finance the minimum wealth requirement of the VaR restriction in equilibrium, which is

$$\underline{W}\mathbf{E}\left[\frac{\xi_T}{\xi_0}\mathbf{1}_{\{\delta_T \leq \underline{\delta}\}}\right] \ .$$

Finally, we need to find the drift and the volatility of the market portfolio. The drift will be $\mu_t^M = r + \kappa\sigma_t^M$, once we have derived the volatility. The market value

$P^M = \mathbf{1}^\top P$ is

$$\begin{aligned} P_t^M &= \big(a(T-t) + A(T-t)\big)\delta_t \\ &\quad + a(T-t)\mathcal{I}(y^v)\delta_t \left(\Phi(-d_1(\underline{\delta})) - \Phi(-d_1(\overline{\delta}))\right) \\ &\quad - \underline{W} e^{-r(T-t)} \left(\Phi(-d_2(\underline{\delta})) - \Phi(-d_2(\overline{\delta}))\right) \end{aligned}$$

where the functions $a$ and $A$ are as in (2.21) and (2.22), using in addition the following variables for the equilibrium, corresponding to the variables in partial equilibrium in the first column,

$$\begin{array}{lclcl} r & \to & r & = & \gamma\mu_\delta - \frac{1}{2}\gamma(1+\gamma)\|\sigma_\delta\|^2 \\ \kappa & \to & \kappa & = & \gamma\sigma_\delta \\ \zeta_* & \to & \delta^* & = & \frac{W_*}{\mathcal{I}(y^v)} \\ \zeta^* & \to & \delta_* & = & \frac{W_*}{\mathcal{I}(hy^v)} \\ \underline{\zeta} & \to & \overline{\delta} & = & \frac{\underline{W}}{\mathcal{I}(y^v)} \\ \overline{\zeta} & \to & \underline{\delta} & = & \delta_0 e^{\left(\mu_\delta - 1/2\|\sigma_\delta\|^2\right)T + \|\sigma_\delta\|\Phi^{(-1)}(\alpha)\sqrt{T}} \end{array},$$

with $\Phi^{(-1)}(\alpha)$ being the inverse cumulative distribution of the standard normal density. By applying Ito's lemma to the market value $P^M$ and comparing the diffusive coefficients with the dynamic budget constraint (2.5), which has to hold for the market portfolio as well, i.e. $P^M = W^{\theta=\mathbf{1}}$, the market volatility and, hence, the drift are

$$\begin{aligned} \sigma_t^M &= q_t^M \sigma_\delta \\ \mu_t^M - r &= \gamma q_t^M \sigma_\delta \end{aligned} \tag{2.55}$$

where

$$\begin{aligned} q_t^M &= 1 - e^{-r(T-t)} \frac{\underline{W}}{P_t^M} \left(\Phi(-d_2(\underline{\delta})) - \Phi(-d_2(\overline{\delta}))\right) \\ &\quad + e^{-r(T-t)} \frac{J}{P_t^M} \frac{\gamma}{\|\kappa\|\sqrt{T-t}} \phi(-d_2(\underline{\delta})) \,. \end{aligned}$$

with

$$J = \underline{W} - \mathcal{I}(hy^v)\underline{\delta}$$

and $\phi$ being the standard normal density. Note the close resemblance to the investment strategy in (2.32).

# Chapter 3

# Regulating the Banking Sector: The Banks' Optimal Decision

This chapter introduces a banking sector into the economy. Banks are characterized as having debt outstanding and being under the supervision of a regulator. We discuss neither the existence of banks nor the optimality of regulation. Moreover, banks are not analyzed on an individual level and aggregated afterwards. Instead, the entire banking sector is directly modelled on an aggregate level. This new modelling approach builds on the results of Eisenberg and Noe (2001). They show, starting at the micro level, what the cash-flow consequences of an interconnected financial system with many firms/banks under default are.

We proceed as follows. First, the consequences of the banking sector's strategy are modelled by specifying payments to outside claim holders at the planning horizon. Second, the corresponding optimization problem for the aggregate banking sector is stated and the solution is characterized by comparative statics. In deriving the results the methodological approach of Basak and Shapiro (2001) and Basak and Shapiro (2005) is adopted and generalized by considering simultaneously debt financing and VaR regulation at the same time. Third, the consequences of regulating on the debt capacity and capital provisioning of the banking sector is discussed as well as the impact of regulation on the banks' debt market. Finally, the dynamics of the portfolio decision and its implications are illustrated.

## 3.1 The Banking Sector within the Economy

In order to keep the model as parsimonious as possible, the dimensionality of the underlying risk factors is set to $D = 1$. Consequently, there is only one risky asset necessary for a complete market. The whole analysis can easily be performed with $D$ factors driving the macro-economic risk, as long as the market remains complete. The $D$-factor model yields almost no further insights, but comes with an additional notational burden.

The Aggregate Debt Portfolio to the Real Sector

As we model the entire banking sector, the (aggregate) assets of the banking sector are represented by the intermediated share of the total outstanding loans to the real sector of the economy (borrowers). We assume, that the value $P$ of total loans given to the borrowers of the real sector follows the dynamics

$$\big(dP_t + \delta_t dt\big)/P_t = \mu_t dt + \sigma_t dw_t \ , \tag{3.1}$$

where $\delta$ is the cash-flow process associated with the aggregate loan portfolio. The cash-flow stream $\delta$ represents the (dollar) flow of interest (coupon) payments of the borrowers. It is stochastic as companies within the real sector refinance their capital needs depending on the state of nature. Note, that there is no redemption of the aggregate loan portfolio at time $t = T$. $\mu$ is the instantaneous gross expected return of this aggregated loan portfolio and $\sigma$ its volatility. In the following, we discuss $(i)$ why a continuous process for the value of aggregate outstanding loan $P$ to borrowers is a feasible modelling choice and $(ii)$ why the value of the loan portfolio is only driven by a single factor. At first sight, the assumption of a continuous value process is usually problematic, if a portfolio contains credit risk, since, when a company within the loan portfolio defaults, there is a jump in value. However, as we do not explicitly model each individual loan disbursed to a company of the real sector, it is feasible to model the aggregate loan portfolio as a continuous process by two reasons.

First, a loan portfolio is exposed to systematic and idiosynchratic risks. It is reasonable to assume that the systematic risk is driven by macro-economic factors, and that their dynamics can be modelled by a diffusion. We do not model these factors explicitly, but represent it in a reduced form by a standard Brownian motion. Second, our real economy consists of many companies, rolling over their loans with different maturities and coupons. Assuming that the loans of these companies build an infinite granular portfolio, the idiosynchratic default risk is perfectly diversified (see e.g. Gordy (2003)), and only systematic risk matters.

The total loans to the real economy are for brevity usually called the risky asset.

THE ECONOMIC AGENTS OF THE ECONOMY

The economy is populated by two types of agents. The

**unrestricted investor** $u$ represents all those investors who can directly invest in the market. He optimizes his expected utility over consumption $c^u$ and terminal wealth $W^u_T$ by his decision on the allocation of wealth $W^u$. He can invest in the (locally) riskless asset with interest rate $r$ and the risky asset with payment stream $\delta$ and price $P$. Since the trading portfolio $\theta^u$ (fraction of wealth invested in the risky asset) is self-financing, the wealth of the unrestricted investor follows the dynamics,

$$dW^u_t = (W^u_t r_t + W^u_t \theta^u_t(\mu_t - r_t) - c^u_t)\, dt + W^u_t \theta^u_t \sigma_t dw_t \ , \tag{3.2}$$

with initial wealth $W^u_0 > 0$. The

**regulated banking sector** $b$ is partly financed by equity holders with an initial equity value $W^b_0 > 0$. In addiction, the banking system issues at time $t = 0$ a zero bond with exogenously given nominal $F > 0$ and maturity $T$ at the fair value $D_0$. Both, equity and debt, are held by external agents, which are a not a part of the set of participants in the model. Additionally, the supervising authority imposes a VaR based equity requirement for the banking system. The banking system invests the total amount $V = D + W^b$ in the same investment opportunities as the unrestricted investor $u$, namely the locally riskless asset and the aggregate loan portfolio $P$. The resulting portfolio strategy fulfils the same budget dynamics (3.2) as the one of the unrestricted investor $u$, where $u$ is replaced by $b$.

Next, we characterize the claims of the banking system at the horizon $T$. In the classical Merton (1974) approach a levered firm defaults, if the value of the firm's total assets $V_T$ at maturity $T$ is lower than the nominal debt $F$. The equity position $W_T$ is zero and the external debt holder is entitled to all the remaining assets, $D_T = V_T$. Otherwise, the debt is repaid, $D_T = F$, and equity holders are entitled to all the remaining asset value, $W_T = (V_T - F)$.
The characteristics of the equity and debt claims at $T$ for a single firm are unrealistic, when the entire banking system is considered. $(i)$ The model excludes cases where one bank is in default, while another one is still solvent, as $D_T < F$ holds iff the equity $W_T^b$ of the whole banking system is zero. This result implies, that all banks default simultaneously, an unrealistic consequence. $(ii)$ Another drawback, when applied to the aggregate case , is that even if there are substantial defaults within the banking system, no costs for the solvent banks in a distressed banking system arise. As the banking system is vital to any modern economy, this assumption is unreasonable. Furthermore, without costs of distress, there is no room for an endogenous default decision, as there are no incentives to deviate from the optimal solution without debt. $(iii)$ Due to the importance of the banking sector to the economic development, banks are regulated. The core of any regulatory approach is to restrict the business activities of banks by requiring an underlying amount of equity capital.
Consequently, three additional components have to be included in a model of the Merton type, if the aggregate banking sector is considered:

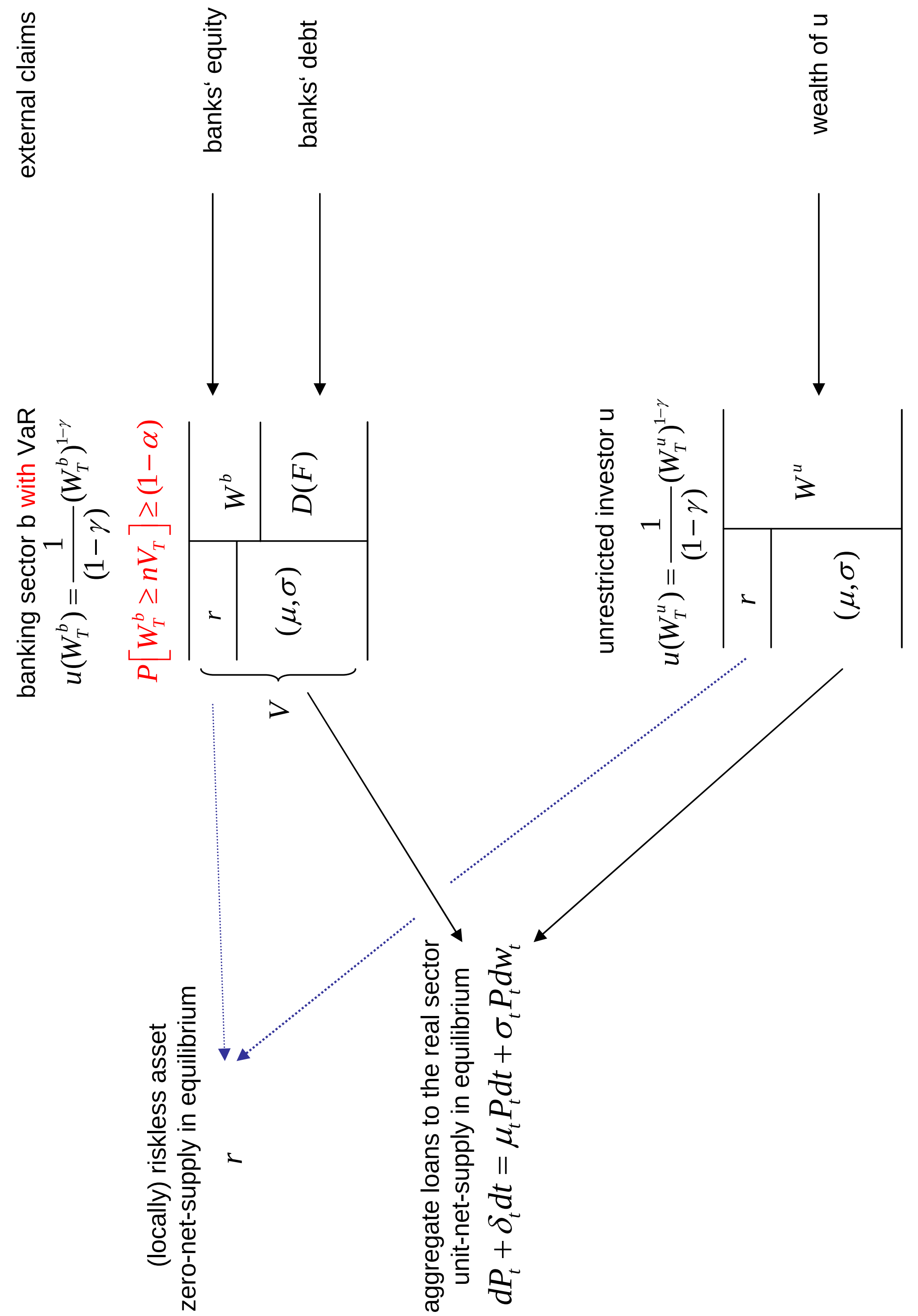


Figure 3.1: **Structure of the Economy**

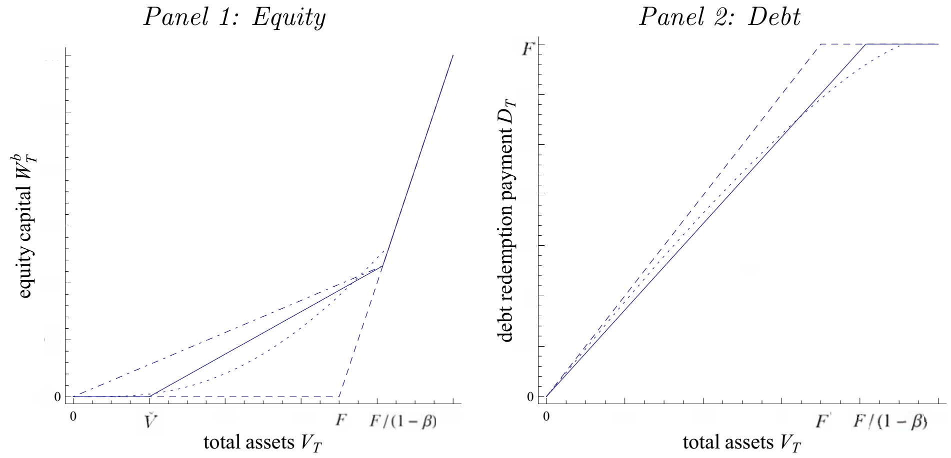


Figure 3.2: **Payments at Horizon**

*This figure shows the payments at maturity depending on the value of total asset $V_T$. The aggregate equity position $W_T^b$ is displayed in Panel 1; aggregate debt redemption payment $D_T$ on nominal $F$ in Panel 2. The dashed line represents the Merton (1974) case, whereas the dotted line the Eisenberg and Noe (2001) model. The dotted-dashed lines are the payment as used in the model **before** costs, whereas the solid is **after** costs.*

1. The payments to the claimants of the aggregate banking system at the planing horizon have to be modified, in order to gain a better image of heterogenous banks within the banking system.

2. Direct and indirect distress costs have to be modelled.

3. It has to be considered that banks are regulated.

### Modelling Heterogeneous Banks

When aggregating individual banks' balance sheets into a single aggregate balance sheet of the banking sector, it is at first sight unclear how the interbank market influences the repayments to the outside claim holders at the horizon. The problem appears as each individual bank holds on the assets side of the balance sheet liabilities of other banks and vice versa. At the horizon, all these different contractual relations have to be cleared. By applying the results of Eisenberg and Noe (2001), Lemma 5, the clearing vector (the repayments on the different debt titles) is a *concave and increasing* (non-expansive) function of operating cash-flows. Including the claim of the outside debt holders as an additional agent in the model of Eisenberg and Noe (2001), the redemption payment $D_T$ must inherit in particular these properties as well, see also Shin (2008), Lemma 1. In our modelling framework, the relevant operating cash-flow is total assets $V_T$. Hence, we assume that the default relations within the aggregate banking sector result in the following linear approximation of redemption payments for external debt with nominal $F$ as

$$D_T = \min\{(1-\beta)V_T, F\} \ . \tag{3.3}$$

The parameter $\beta \in (0, 1/2]$ is a measure of heterogeneity, where larger values correspond to a more heterogenous banking system. The restriction to this interval is for technical reasons. Economically it excludes cases, where the equity value of the total banking system will be higher than the debt value, when some banks default.

Figure 3.2 illustrates the differences in modelling by showing equity $W_T^b$ (Panel 1) and debt $D_T$ (Panel 2) at the planning horizon $T$, depending on the total assets (cash-flows) of the banking system $V_T$. The Merton (1974) solution is depicted by the dashed line, a function of the Eisenberg and Noe (2001) type is plotted as a dotted line, and our linear approximation by the dotted-dashed line.

In order to motivate, why $\beta$ is related to heterogeneity in the banking system, two aspects of heterogeneity are presented, namely the leverage and the business model of banks. We illustrate these two aspects by considering two extreme cases.
In the first case, banks have an homogenous business model and each bank invests (directly or by the use of the interbank market) in the well-diversified portfolio of loans to the real sector. Consequently, there are no idiosyncratic risks in each of the banks loan portfolios. The probability of default due to an idiosyncratic event is zero. However, in aggregate, because the banks' assets are perfectly correlated (an implication of diversification) on an individual level, the systemic risk to the banking sector is large. Banks default sequentially, starting at the bank with the highest leverage ratio. Banks default, even if the aggregate cash-flow $V_T$ is still larger than the aggregate nominal debt $F$. This property implies $\beta \gg 0$. In contrast, if all

banks are homogenous with respect to their leverage, they default at the same time and we recover in aggregate the Merton (1974) solution, i.e. $\beta = 0$.
As an example, Allen and Carletti (2006) formalize this argument and show in their model that credit risk transfers between banks, i.e. the construction of the well-diversified portfolios in our model context, can lead to contagion via the interbank market and effectively increase the risk of systemic crisis. Duffie (2008) also intensely discusses the problems arising from credit risk transfers.

In the second case banks set up very heterogenous business models: each bank invests in a 'single sector' of the economy and hence the loan portfolio of each bank contains sector risks. Notwithstanding, the aggregate credit portfolio of banks is again well diversified. In this case, the probability of default due to sector risks is larger than in the previous example. However, even with identical banks in terms of leverage, there will be no common default of the banking system.
From the viewpoint of the aggregated balance sheet of the banking sector, both types of heterogeneity have at least two consequences: debt may not be fully paid, $D_T < F$, while at the same time there is still equity capital left, $W_T^b > 0$. Moreover, the more the business models of banks differ, the earlier the first default of a bank within the system occurs. Thus, the specific value of total assets $V_T$, where debt will not fully be paid, is increasing in the heterogeneity parameter $\beta$.
On a theoretical level, the above argument is also put forward e.g. by the works of Wagner (2006, 2008) and Acharya (2009). The impact of theses two cases is in line with the empirical findings of e.g. Baele et al. (2007).

Costs, when Banks Default

First, we discuss the different components of costs that arise, if there are defaults within the banking system. Afterwards, we show that a part of these costs is borne by the banking system. In our modelling framework, we again use a linear formulation for the part of costs that is attributed to the banking system

Default costs may consist not only of direct costs, such as administrative costs and costs of liquidation (takeover), but also of indirect losses such as reputational considerations and the loss of future business possibilities (bank charter). These losses can be substantial parts of the value of defaulted assets as the Committee on Banking, Finance, and Urban Affairs of the United States Congress (1984) reports. James (1991) estimates 10% for administrative costs and a total loss in assets of 30%.

From an economic perspective, some of these costs might have *systemic* relevance in the sense that they impact market participants that are not directly connected to the defaulted banking institution: The forced liquidation of assets may depress their market prices and affect other banks, see e.g. Hombert (2007) or Acharya et al. (2009); damages to the reputation of the banking (group) may spread through the system and generate informational contagion. In extreme cases, there might even arise a breakdown of some markets, see Leland and Pyle (1977).

Since individual banks in general cannot invest into the well-diversified portfolio directly, banks will not only hold position of pure credit risk, but are additionally engaged in the interbank and derivatives markets in order to transfer risk and diversify the idiosyncratic risks' components they obtained due to their business model. This risk-sharing function of the interbank markets is obviously welfare enhancing, when comparing a banking system in autarky, where every bank holds its own portfolio including large idiosyncratic risks, with a banking system, where any risk can be transferred, i.e. complete markets. When aggregating over all banks, the interbank market is in zero net supply, as long as there are no defaults, which is in our model framework over the period $t \in [0, T)$ and in $t = T$ in case of no defaults. If, however, an individual bank defaults in $T$, not only are external debt holders confronted with costs due to a default, but also 'internal' debt holders, i.e. other banks by their contractual relations through the interbank markets. Hence, a part of total costs of a banking system in distress, i.e. where some banks default, transmits to the aggregate equity position of the banking system.

Beside these contagion effects within the banking sector, there might be a transmission into the real sector of the economy, e.g. via a credit crunch, with large costs to the economy, see e.g. Honohan and Klingebiel (2000) and Caprio and

Klingebiel (2002).

Parts of these default costs have to be covered by the holders of the aggregate equity position. They are modelled as

$$C_T \quad = \quad \begin{cases} \lambda(F - D_T) & V_T > \check{V} \\ \beta V_T & V_T \leq \check{V} \end{cases} , \tag{3.4}$$

with

$$\check{V} = \frac{\lambda}{\beta + \lambda(1-\beta)} F \ . \tag{3.5}$$

The loss given default in aggregate terms, $(F - D_T)$, proxies the severity of a banking system in distress. It seems reasonable that costs are increasing with the volume under distress. The cost share[1] $\lambda \in (0, 1]$ guarantees that the banking system covers a non-negative amount of costs and at most 100% of the loss given default. The case $V_T \leq \check{V}$ is introduced in order to enforce limited liability, as otherwise equity $W_T^b$ can be negative; $\check{V}$ is defined as the terminal value of assets, where equity is for the first time zero. The second part of the cost function is chosen such that $W_T = 0$, if $V_T \leq \check{V}$. Finally, the model we use is (up to now) formally a particular case of the one presented by Basak and Shapiro (2005).

In Panel 1 of Figure 3.2, the costs can be seen as the difference between the dotted-dashed line $W_T^b + C_T$ and the solid line $W_T^b$. In addition, Panel 1 in Table 3.1 summarizes the cash-flows at maturity, dependent on $V_T$. For further reference, Panel 2 shows the redemption payment structure after inversion to $W_T^b$, if $V_T \geq \check{V}$.

Regarding the *direct* cost components, the proportionality factor $\lambda$ can be seen as a proxy of intensity of the contractual interbank relations. We illustrate the argument with two examples:

First, think of a bank that has no relations to other banks; consequently, other banks face no loss due to their contractual relations with this bank, irrespective of the volume of total losses in the loan portfolio of this specific bank.

[1] Even though $\lambda$ was used in the previous chapter, no confusion should arise, as in the following $\lambda$ will denote the cost share.

In contrast, imagine a bank that is primarily financed using the interbank market. If this bank defaults - and assuming that the default raises the same costs as in the previous example and that the bank holds a loan portfolio of the same size - almost all the costs are effectively borne by the equity of other banks, since interbank liabilities make up a high proportion of the total debts and have the same seniority as external debt (senior unsecured).
In these two examples, the degree of linkages between banks affects, which part of the total costs stays within the banking system (and how the costs are distributed within the banking sector); this argument can be made more precise within a network type of model like Allen and Gale (2000), Freixas et al. (2000), Nier et al. (2007), and Elsinger et al. (2006).

When using a strict interpretation, the model presented above allows only for costs that results in leakage of cash-flows, i.e. direct costs. All other costs components without a direct cash-flow consequence are not included.
However, following Remark 3 on p. 40, alternative interpretations are possible. Instead of cash-flows, one can also think in continuation values for equity and debt in a post-horizon economy, if it were specified. Within such an environment, other cost components such as indirect costs, systemic costs and costs from feedback effects can be incorporated indeed.
The *systemic* parts of the costs of a banking system in distress cannot be attributed easily: by the fair value principle in accounting, depressed prices due to (cascades of fire) sales obviously spread into the equity value of other banks. However, it is not obvious to what extend this transmission channel is linked to the structure of the interbank markets. Even more complex is the question of how a credit crunch impairs the *real economy* and thereby feeds back into the portfolio of total loans. Still, it seems reasonable to assume that systemic and feed back effects are positively related to the volume under distress; for the case of system effects see e.g. Nier et al. (2007) and Elsinger et al. (2006).

**Remark.** *Costs arising to other economic agents, in particular the bank's debt holders, do not directly affect the optimal portfolio decisions, as bank managers solely*

Table 3.1: **Redemption Payments at Maturity**

*This table reports the decompostion of redemption payments to external debt holder $D_T$, to the outsiders, which receive the costs of distress $C_T$, and to equity holders $W_T$, in the case of distress and full repayment. Panel 1 shows the decomposition as a function of total assets $V_T$, whereas Panel 2 displays the same decomposition as a function of the equity position $W_T$.*

| | | *Panel 1* | | *Panel 2* | |
|---|---|---|---|---|---|
| | | *Distress* $(1-\beta)V_T < F$ | *Full Repayment* $(1-\beta)V_T \geq F$ | *Distress* $W_T^b < \frac{\beta}{1-\beta}F$ | *Full Repayment* $W_T^b \geq \frac{\beta}{1-\beta}F$ |
| Total Assets | $V_T$ | $V_T$ | $V_T$ | $\frac{1}{\beta+\lambda(1-\beta)}\left(W_T^b + \lambda F\right)$ | $W_T^b + F$ |
| Debt | $D_T$ | $(1-\beta)V_T$ | $F$ | $\frac{(1-\beta)}{\beta+\lambda(1-\beta)}\left(W_T^b + \lambda F\right)$ | F |
| Equity (cum) | $W_T^b + C_T$ | $\beta V_T$ | $V_T - F$ | $\frac{\beta}{\beta+\lambda(1-\beta)}\left(W_T^b + \lambda F\right)$ | $W_T^b$ |
| Costs | $C_T$ | $\lambda(F-(1-\beta)V_T)$ | 0 | $\frac{\lambda}{\beta+\lambda(1-\beta)}(\beta F-(1-\beta)W_T^b)$ | 0 |
| Equity (ex) | $W_T^b$ | $(\beta+\lambda(1-\beta))V_T - \lambda F$ | $V_T - F$ | $W_T^b$ | $W_T^b$ |

*act in the interest of equity holders in this model framework. Problems regarding managerial agency problems are discussed e.g. in Rochet (1992) and Jeitschko and Jeung (2005). In order to keep the model parsimonious, valuation is being conducted without costs or benefits to the external debt holder.*

Regulation and Risk Management of Banks

The failure of some small banking institutions will not constitute a crisis to the system as a whole. Notwithstanding, there is no clear understanding of how to differentiate on an economic basis between harmless losses and those critical losses that are threatening to the survival of the banking system as a whole.

However, one can characterize two critical levels. At $\check{V}$, given in equation (3.5), all the equity capital is needed to cover costs $C_T$, that is, $W_T^b = 0$ or all the banks are in default.

The second level arises due to regulation. Most regulatory policies with respect to the banking system are based on the principle, that, for any business activity on the active side of the balance sheet, there has to be a certain, possibly risk-weighted, amount of equity on the passive side, i.e. regulation restricts the size of the business in which a bank is able to engage. Consequently, one can *implicitly* define critical states depending on regulation, where there is 'too much' aggregate business activity per aggregate equity, i.e. $\frac{W^b}{V} \leq n$, where the Cooke ratio $n \in [0, 1]$ is the regulatory control. The boundary $\underline{V}$ can be determined by using the results from Table 3.1; one obtains

$$\underline{V} = \frac{\lambda}{\beta + \lambda(1-\beta) - n} F \ . \tag{3.6}$$

Hence, we are able to identify on a formal basis four regions by their special characteristics, namely a

**normal economic environment,** $V_T \geq \frac{F}{(1-\beta)}$, where there is no individual default in the banking system; debt is fully paid and the remaining value is paid as equity capital, i.e. $W_T^b = V_T - F$. The second region constitutes a

**distress** of the financial system, $\underline{V} \leq V_T < \frac{F}{(1-\beta)}$, where some banks default, but losses are not critical in a regulatory sense, i.e. on the aggregate level $W_T^b \geq nV_T$. The third region is called a

**crisis** of the financial system, $\check{V} < V_T < \underline{V}$, where the banking system is operating with less than the required equity capital on average, but some banks are still solvent, i.e. $0 < W_T^b < nV_T$. The last region encompasses the

**breakdown** of the financial system, $0 < V_T \leq \check{V}$, where all banks are in default, i.e. $W_T^b = 0$.

The regulatory policy within our model framework requires that the banking system initially holds enough capital such that there is currently no crisis,

$$W_0^b \geq nV_0 \ . \tag{3.7}$$

Moreover, in order to react to events that may bring forth a violation of the regulation constraint in the future, regulation enforces risk management systems such that the banking system behaves as if it uses a VaR-based capital requirement of the form

$$\mathbf{P}[W_T^b \geq nV_T] \geq (1-\alpha) \ . \tag{3.8}$$

By this restriction, the aggregate equity capital $W_T^b$ will be more than the total assets weighted by the Cooke ratio, $nV_T$, under 'normal market conditions', meaning in at least $(1-\alpha)$ percent of cases; crisis happens in at most $\alpha$ per cent of the cases.

We do not intend to discuss whether a VaR rule for risk management can be obtained as a solution to an optimal contracting problem, e.g. Adrian and Shin (2008a). This rule is meaningful for different reasons. Risk management by Value-at-Risk has become the industry standard, since the VaR idea emerged into markets from JPMorgan's RiskMetrics Group. Finally, it gained its seal of approval, when it became the fundamental idea behind regulation in most countries due to the 1996 Market Risk Amendment of the Basel Accord and the Basel II Regulation.

We want to stress two aspects of this risk-sensitive capital requirement:
First, it is not obvious that a VaR-based risk management of an individual bank translates into this form for the aggregate case. From a purely theoretical viewpoint, the VaR restriction is not an impediment to aggregation if the VaR probability is identical across banks (homogenous regulation) and if the additional assumption holds that the behaviour of the market participants does not influence neither the interest rate nor the market price of risk (valuation irrelevance of regulation). Then, the state of the economy where the VaR restriction becomes binding at an individual level is independent of the agent's characteristics and, hence, the same for every decision maker, see Basak and Shapiro (2001). Therefore, aggregation is simplified, as the individual restriction can be replaced by the one formulated on aggregate variables. It turns out that the pure exchange equilibrium in Chapter 4 and 5 is in compliance with this condition.

Second, the combined capital requirements (3.7) and (3.8) are (time-)inconsistent. Rational bank managers should take account in their decision strategy today that they have to comply at least with some probability with regulation again as time goes by. In a perfect framework, the regulatory restrictions should hold at any point in time. Instead, we choose to incorporate a single restriction over the period $[0, T]$. An intuitive argument runs as follows: most large banks have invested substantial amounts of money and knowledge in their risk management systems. This system influences the bank behaviour on at least two levels. $(i)$ On the executive board level, the management chooses its decision based on the information supplied by the risk management tools; annual reports of most banking institutions extensively discuss their VaR estimates. Furthermore, many banks define a range for their regulatory capital ratios within they attempt to operate, c.f. Annual Report (2007) of Commerzbank AG, p. 225 'Comfort Zone' for Tier 1 ($6.5\% - 7.5\%$) and Tier 2 ($10.5\% - 11.5\%$). $(ii)$ The second level can be attributed to the incentives of compensation schemes within banks. For the evaluation which activities are the most profitable ones, the adjustments for the risk and the costs of (regulatory) equity heavily depend on the risk management system as well. Extrapolating these arguments, one can imagine that, once these systems are installed, banks choose their portfolios quasi-'automatically'. Even though, these systems provide a 'state' dependent answer, they are slow to adapt to structural changes. Therefore, we prefer to use the commitment solution over the time-consistent solution with an continuously updated VaR restriction. For a discussion on time consistency see Strotz (1955) or in an investment decision framework Basak and Chabakauri (2009).

**Remark.** *Section 2 of the concluding Chapter 6 discusses the robustness of the results, when generalizing the structure of redemption payments, the costs structure, or the homogeneity assumption within the VaR restriction. Moreover, it argues that the VaR restriction is merely the one with the most pronounced impact, but many other risk measures will in fact generate similar results.*

**Remark.** *There are three different time horizons to consider in the model: the VaR horizon, the maturity of the bond, and the lifetime of the agents. We have chosen to use the same horizon for all of them. This choice induces a non-path-dependent solution. Furthermore, a more complex model with different time horizons will have a smoothing effect, which is documented in similar models: for the case of a VaR restriction see Basak and Shapiro (2001) and for debt see Basak and Shapiro (2005). Regarding amplifications of volatility due to regulatory impact, our model represents the worst case.*

## 3.2 Characterization of the Optimal Solution

In this section, we formulate the optimization problem of the unrestricted agent $u$ and of the banking sector $b$. The solution requires no assumption in addition to the ones outlined in the previous chapter, mainly a complete market and an adapted investment opportunity set $(\mu, r, \sigma)$. Furthermore, in their optimal decision, agents take the prices as given. Consequently, the wealth of the unrestricted agent $W_0^u$ and equity value $W_0^b$ are exogenous.
On this general level, we are able to discuss the impact of regulation at two points in time: first, the direct effect of a changing regulation at time $t = T$ on the *optimal* profile of the banking system $W_T^b$; second, by virtue of the budget constraint in $t = 0$, there is also an indirect impact of regulation, as these direct effects have to be financed by a modification of the portfolio decision.

### 3.2.1 Formulation of the Optimization Problems

The unrestricted agent $u$ as well as the banking system $b$ optimize the expected utility

$$\mathbf{E}\left[\int_0^T u(c_s^n)\,ds + u(W_T^n)\right] \quad n \in \{u, b\}\ ,$$

with a CRRA utility function as defined in (2.12) on p. 21. As heterogeneity in agents' characteristics introduces superimposing effects, economic agents are as similar as possible. Therefore, both representative agents share the same coefficient of constant relative risk aversion $\gamma$.

Since the market is complete, the problem can be solved using the martingale techniques of Cox and Huang (1989) and Karatzas et al. (1987), which convert the dynamic optimization problem into a static variational one. The state-price density process is defined as in the previous chapter,

$$d\xi_t = -\xi_t(r_t dt + \kappa_t dw_t)\ ,$$

where $r$ is the locally risk-free interest rate and $\kappa = \frac{\mu - r}{\sigma}$ the market price of risk process, both adapted to $\mathcal{F}$.

The resulting static-variational optimization of the aggregate banking sector reads as follows

$$\max_{\{c_t, W_T\}} \quad \mathbf{E}\left[\int_0^T u(c_t^b)dt + u(W_T^b)\right] \tag{3.9}$$
$$s.t. \begin{cases} \mathbf{E}[\int_0^T \xi_t c_t^b dt + \xi_T(W_T^b + C_T)] \leq \xi_0 W_0^b & \textsc{Budget with Costs} \\ \text{Eq. (3.3) and (3.4)} & \textsc{Aggregate Behavior} \\ \text{Eq. (3.7) and (3.8)} & \textsc{Regulation} \\ c_t^b \geq 0,\ W_T^b \geq 0 & \textsc{Non-Negativity.} \end{cases}$$

This new optimization encompasses the particular case of an

**unrestricted agent** $u$ as in Cox and Huang (1989). He is not indebted and has no other exogenous restrictions; i.e. $(F = 0, \alpha = 1, n = 1)$. Furthermore, the case of an

**VaR restricted agent** $v$ as in Basak and Shapiro (2001) is nested. The agent uses a VaR management, which aims at maintaining a certain wealth level with at least a probability of $\alpha$, but no debt; i.e. $(F = 0, n = 1)$. Finally, the optimization problem of the

**unrestricted financial intermediary** $i$ as in Basak and Shapiro (2005) can be recovered by setting $(\alpha = 1, n = 1)$. This agent has debt outstanding, but has no regulation and VaR management.

This case is also of further interest since it represents the case where the banking sector is not regulated. To facilitate an easy comparison of different solutions, we denote this unregulated banking sector in the following as *financial intermediary* $i$. Furthermore, regulation is in some cases *effectively* not binding, meaning that, even though there exists regulation, initial equity $W_0^b$ is sufficient such that the VaR restriction does not bind.

### 3.2.2 Optimal Solution

First, the solution to the unrestricted representative agent $u$ is repeated. Afterwards, the solution of the restricted banking system is given and characterized.

Solution of the Unrestricted Investor

The solution to the problem of the unrestricted investor $u$ is, as outlined in the previous chapter,

$$c_t^u(y^u) = \mathcal{I}\left(y^u \xi_t\right), \quad W_T^u(y^u) = \mathcal{I}(y^u \xi_T) \ , \tag{3.10}$$

where the Lagrange multiplier of the unrestricted agent $y^u > 0$ is the solution to the static budget equation

$$\mathbf{E}\left[\int_0^T \xi_s c_s^u(y^u) ds + \xi_T W_T^u(y^u)\right] = \xi_0 W_0^u. \tag{3.11}$$

$\mathcal{I}(y)$ is the inverse function of marginal utility, see equation (2.13) on p. 21.

Optimal Solution of the Banking Sector

We characterize the solution when the VaR restriction is binding

$$\begin{aligned} c_t^b(y^b) &= \mathcal{I}\left(y^b \xi_t\right) \\ W_T^b(y^b) &= \begin{cases} \mathcal{I}(y^b \xi_T) & \text{if } \xi_T \le \zeta_* \\ W_* & \text{if } \zeta_* < \xi_T \le \zeta^* \\ \mathcal{I}(h y^b \xi_T) & \text{if } \zeta^* < \xi_T \le \underline{\zeta} \\ \underline{W} & \text{if } \underline{\zeta} < \xi_T \le \overline{\zeta} \\ \mathcal{I}(h y^b \xi_T) & \text{if } \overline{\zeta} < \xi_T \end{cases} \end{aligned} \tag{3.12}$$

with the boundaries

$$\begin{array}{lll} W_* = \frac{\beta}{(1-\beta)} F & \zeta_* = \frac{u'(W_*)}{y^b} & \zeta^* = \frac{\zeta_*}{h} \\ \underline{W} = \psi(n) \frac{\beta}{1-\beta} F & \underline{\zeta} = \frac{u'(\underline{W})}{h y^b} & \overline{\zeta} : \mathbf{P}[\xi_T \ge \overline{\zeta}] = \alpha \ , \end{array} \tag{3.13}$$

the costs sensitivity

$$h = \frac{\beta}{\beta + \lambda(1-\beta)} \in (0,1) \ , \tag{3.14}$$

and a variable, representing the strictness of regulation

$$\psi(n) = \frac{1-\beta}{\beta} \frac{n\lambda}{\beta + \lambda(1-\beta) - n} \in (0,1) \ , \tag{3.15}$$

where again the Lagrange multiplier of the banking sector $y^b > 0$ is the solution to the static budget equation

$$\mathbf{E}\left[\int_0^T \xi_s c_s^b(y^b)ds + \xi_T(W_T^b(y^b) + C(W_T^b(y^b))\right] = \xi_0 W_0^b \ . \tag{3.16}$$

The proof is provided in the Appendix. If $\underline{\zeta} > \overline{\zeta}$, the VaR restriction is effectively not binding. In this case or when the solution of the financial intermediary $i$ is of interest, it can be recovered by formally setting $\underline{\zeta} = \overline{\zeta}$.

The feasibility constraint for the restricted problem is

$$\begin{aligned} \frac{W_0^b}{F} \geq \ & \frac{\beta\lambda}{\beta + \lambda(1-\beta) - n}\mathbf{E}_0\left[\frac{\xi_T}{\xi_0} 1_{\{\xi_T \leq \overline{\zeta}(\alpha)\}}\right] \\ & + \frac{\beta\lambda}{\beta + \lambda(1-\beta)}\mathbf{E}_0\left[\frac{\xi_T}{\xi_0} 1_{\{\xi_T > \overline{\zeta}(\alpha)\}}\right] \ . \end{aligned} \tag{3.17}$$

If capital requirements $n$ (given $\alpha$) are too restrictive, the solution ceases to exist, as a foreclosure of any banking business activity is optimal. Accordingly, the condition

$$n \leq \beta \tag{3.18}$$

represents an incentive compatibility constraint.

**Remark.** *$h$ can be seen as cum-ex sensitivity of equity in case of distress or crisis,* $\frac{\partial(W_T^b + C_T)/\partial V_T}{\partial W_T^b/\partial V_T}|_{V_T < F/(1-\beta)}$*, see Table 3.1, Panel 1. A* decrease *in $h$ measures the loss in equity return due to bankruptcy costs,* $\frac{W_T^b + C_T}{W_T}$*, with respect to a loss in $V_T$. Under optimality, the agent balances state-by-state these losses with the marginal loss, paid to the outsiders, by setting up a portfolio policy to increase (resp. decrease) $V_T$ , thereby providing more (resp. less) equity wealth net of fees in this particular state.*

Table 3.2: **Standard Parameter Set**

*This table reports the set of parameters that will be used, if not stated otherwise.*

| | | | |
|---|---|---|---|
| *partial equilibrium* | | | |
| riskless rate | market price of risk | initial state | initial wealth (distr.) |
| $r = 5\%$ | $\kappa = 40\%$ | $\xi_0 = 3.69$ | $W_0^u = 2$<br>$W_0^b = 2$ |
| *pure exchange equilibrium* | | | |
| growth c.f. | vola. c.f. | initial c.f. | initial endowment |
| $\mu_\delta = 8.5\%$ | $\sigma_\delta = 20\%$ | $\delta_0 = 0.52$ | $1 - \omega = \frac{7}{9}$ |
| *agents' data* | | | |
| CRRA coefficient | horizon | | |
| $\gamma = 2$ | $T = 5$ | | |
| nominal | heterogeneity | prop. costs | |
| $F = 7$ | $\beta = 14.10\%$ | $\lambda = 5.34\%$ | |
| *regulation* | | | |
| Cooke ratio | VaR probability | | |
| $n = 13.3\%$ | $\alpha = 1\%$ | | |

*resulting economy in a balance sheet (present value)*

| | | |
|---|---|---|
| tot. debt to real sector<br>$P_0 = 9$ | total assets<br>$V_0 = 7$ | banks' equity<br>$W_0^b = 2$ |
| | | banks' debt<br>$D_0 = 5$ |
| | wealth unr. agent<br>$W_0^u = 2$ | |

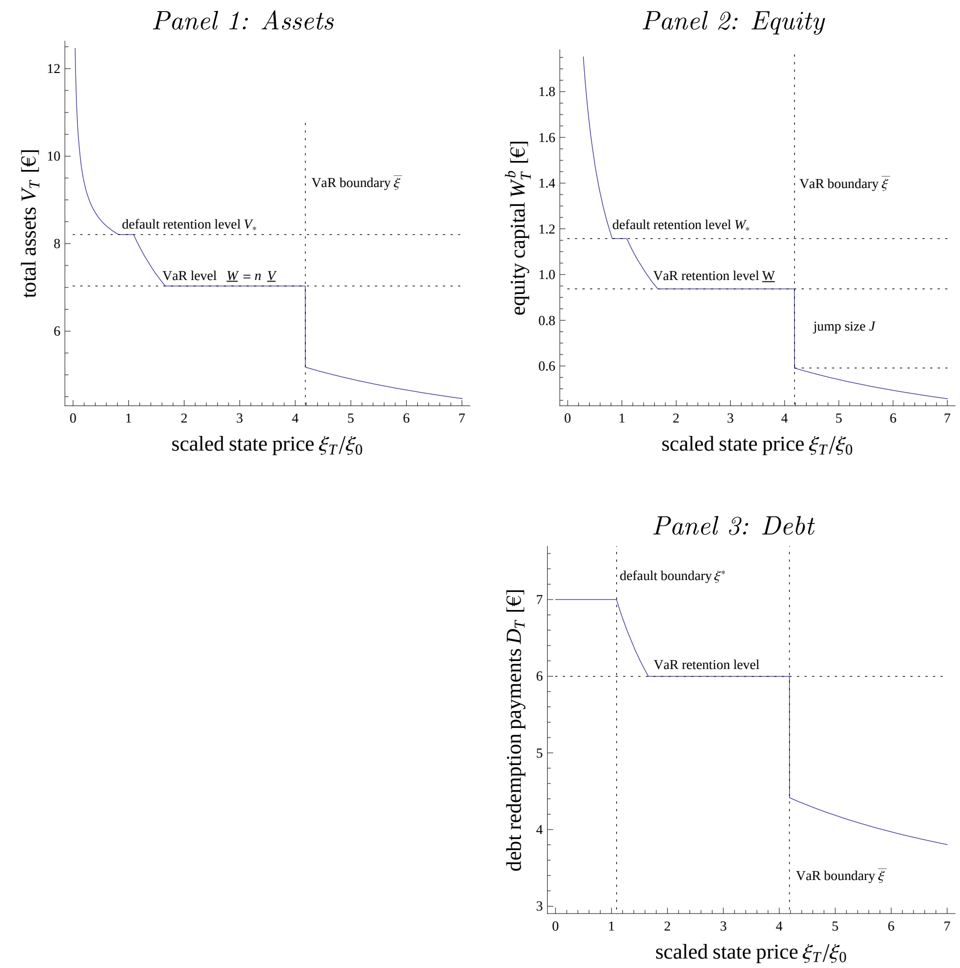


Figure 3.3: **The Bank Sector's Optimal Solution for Terminal Wealth**

*This figure shows the solution of the optimization problem using the set of standard parameters. Each graph is plotted over the scaled state price $\xi_T/\xi_0$, where low values represent 'good' times and vice versa. Panel 1 shows the total value $V_T$, whereas Panel 2 the equity capital $W_T^b$, and Panel 3 the debt redemption payment $D_T$. The parameters are as defined in the standard parameter set in Table 3.2.*

Figure 3.3 illustrates the optimal solution in the form of a balance sheet: the resulting total value $V_T$ is displayed in Panel 1, debt value $D_T$ in Panel 2, and terminal equity value $W_T^b$ in Panel 3, each plotted against the scaled state price $\xi_T/\xi_0$. High scaled state prices correspond to economically 'bad' times, whereas low state prices indicate a favourable economic development. Due to the fixed default procedure, total assets $V_T$ and equity $W_T^b$ are structurally identical. For those states in which default occurs, debt $D_T$ also inherits a similar structure. The graphs are plotted using the parameters of the model as given in Table 3.2 (standard parameter set).

The optimal solution splits the entire state space into the same regions mentioned on p. 65, namely a

**normal economic environment** with $\xi_T \leq \zeta_*$ and $\zeta_* < \xi_T \leq \zeta^*$. This region can be further separated into two parts. In economically 'good' time, the banking systems behaves as if being unrestricted. However, at $\zeta_*$ the retention level to distress $W_*$ starts, corresponding to $F/(1-\beta)$ on the total asset level. Here, the agent avoids the first default, as long as the default costs are higher than the costs of obviation, i.e. up to the distress boundary $\zeta^*$. A

**distress** of the financial system will occur in the regions $\zeta^* < \xi_T \leq \underline{\zeta}$ and $\underline{\zeta} < \xi_T \leq \overline{\zeta}$. In the first part, the banking system behaves like the unregulated financial intermediary $i$.

The second part with the retention level $\underline{W}$ is induced by the VaR requirement. The least wealth $\underline{W}$ that is required to maintain the VaR restriction, is reliant on regulation through its dependence on the regulation parameter $n$. The VaR boundary $\overline{\zeta}$ is the $(1-\alpha)$ quantile of the state price density and thus independent of preferences and endowments. The last region corresponds to a

**crisis** of the financial system that occurs in the states $\overline{\zeta} < \xi_T$. In these remaining states, wealth $W_T^b$ is not restricted and is proportional to the unrestricted profile $\mathcal{I}(\xi_T)$. This property is especially noteworthy in the 'tail' of the distribution, as VaR does not restrict wealth in these states.

Finally, at the VaR boundary $\overline{\zeta}$, a jump from the retention level to the unrestricted policy appears,

$$J = \underline{W} - \mathcal{I}(hy^b\overline{\zeta}) \geq 0 \ . \tag{3.19}$$

A total

**breakdown** of the financial system is not sustainable due to the form of the utility, which prevents the banking sector from reaching a breakdown, as $u'(W) \to \infty$ for $W \to 0$.

The optimal wealth at time $T$ can be replicated by a static portfolio of derivatives. As in the previous chapter, one can define a mutual fund $G = \mathcal{I}\left(y^b \frac{\xi_T}{\xi_0}\right)$ and decompose the optimal solution into

1. $\mathcal{I}(h)$ units of the fund $G$,
2. $\mathcal{I}(h) > 1$ calls options short with strike $\frac{W_*}{\mathcal{I}(h)}$,
3. 1 call option long with strike $W_*$,
4. $\mathcal{I}(h) > 1$ puts long with strike $\frac{\underline{W}}{\mathcal{I}(h)}$ and knock-out barrier $\frac{K(\alpha)}{\mathcal{I}(h)}$,

where $K(\alpha) = \mathcal{I}(hy^b\overline{\zeta})$. Positions (1., 2., 3.) correspond to the solution of the unrestricted financial intermediary $i$ and position (4.) is necessary to generate the deviation from optimal policy due to the VaR constraint. The static decomposition in more formal terms is

$$\begin{aligned} W_T^b &= \underbrace{\mathcal{I}(h)G_T - \max\{\mathcal{I}(h)G_T - W_*, 0\} + \max\{G_T - W_*, 0\}}_{\text{Solution of } i} \\ &\quad \underbrace{+ \max\{\underline{W} - \mathcal{I}(h)G_T, 0\}\mathbf{1}_{\{\mathcal{I}(h)G_T \geq K(\alpha)\}}}_{\text{VaR - effect}} \ . \end{aligned} \tag{3.20}$$

Note the resemblance to the static derivatives position of the VaR restricted agent in (2.30) on p. 34. This static derivative portfolio is of importance for an attempt to explain the risk taking behaviour of banks and the success of derivatives' innovations

before the current financial crisis emerged, based on a regulatory side effect[2]. The completeness of markets, be it based on dynamic trading or on a sufficient set of derivatives existent on markets, enables banks to exactly fulfill the requirements of regulation. However, it also permits to make full use of all states, where regulation is not binding. Within any kind of VaR based measure, these states are inherently in the tails of the distribution, as is illustrated in the above derivative position through the knock-out options. In a more general understanding, which overcomes our specific model framework, these states may cover also developments, where markets participants expect regulatory authorities to lessen their policy and/or to intervene, in order to stabilize the system. Finally note, that this side effect of regulation is increasing with the nominal debt level of the banking system.
As our model framework provides a close link between dynamic trading strategies and static derivative positions via market completeness, this side effect cannot be obtained in a standard static framework with linear contracts.

### 3.2.3 Reaction of the Banking System to Regulation

This model framework facilitates the separation of the reaction to a change of a parameter by comparative statics of the solution into three effects:

**Step 1** results in the *direct effect* on the optimal decision in time $T$, i.e. how do boundaries change and what are the implications for equity wealth in these states? The parameters of interest $\chi$ are the VaR probability $\alpha$, the aggregate capital requirements $n$, nominal debt $F$, and the parameters that structure the model, $\lambda$ (costs fraction borne by the banking sector) and $\beta$ (heterogeneity within the banking sector).
These parameters may have an impact on the relevant boundaries $\overline{\zeta}$, $W_*$, and $\underline{W}$ as well as on the cost sensitivity $h$. Thereby, terminal wealth reacts to changes in parameters, $\frac{\partial W_T^b}{\partial \chi}$.

[2]This certainly covers only one aspect of the crisis. Other ones, such as compensation schemes or problems of moral hazard due to securitization, are not the focus of this thesis.

**Step 2** results in the indirect reactions due to *wealth transfers* at time $t = T$, i.e. how are the previously mentioned changes in wealth financed? In general, the budget equation does not hold any more after the first step. The only way to fulfil the budget equation optimally is to adjust wealth in these states, where a solution of the unregulated type $\mathcal{I}(\cdot)$ prevails, i.e. in the regions $(0, \zeta_*]$, $(\zeta^*, \underline{\zeta}]$, and $(\overline{\zeta}, \infty)$. By the relation $\frac{\partial W_T^b}{\partial y^b} < 0$ in these regions, this adjustment effectively amounts to changing $y^b$. Thereby, indirect effects of financing arise, as expressed by the sign of $y_b'(\chi)$.[3]

**Step 3** results in the induced *provisioning of equity capital* at time $t = 0$. Capital provisioning is defined as how much equity capital of (constant) total equity $W_0^b$ has to be 'put aside', in order to pay for the modifications of the portfolio strategy necessary to generate the direct (Step 1) and indirect (Step 2) changes. $W_0^b - W_0^i(y^b)$ measures exactly the amount of capital that has to be put aside.

The possibility of this separation is due to the new model proposed and solved above.

**Remark.** *The following arguments are with respect to equity capital. As the total assets $V$ are a monotonous transformation of $W^b$, the same results are valid for the total assets. Debt $D$ also inherits the same characteristics in the distress and crisis regions $\xi_T > \zeta^*$.*

For further reference, Table 3.3 at the end of this section lists derivatives with respect to the VaR probability $\alpha$, the Cooke ratio $n$, costs $\lambda$, heterogeneity parameter $\beta$, and nominal debt level $F$. In Step 1, the sign of the derivatives needed for the direct effect, namely the ones of the distress level $W_*$, crisis level $\underline{W}$, VaR boundary $\overline{\zeta}$, and the costs sensitivity $h$ are displayed. Step 2 shows the impact on the Lagrange multiplier $y^b$. Lastly, the change in capital provision for Step 3 is given.

Finally we have to address of how to measure financial (in)stability. Going further, the literature (and regulatory authorities) does not even provide a generally accepted

[3]In order to keep notation simple we define $y_b'(\chi) = \frac{dy(\chi)}{dy^b}$.

definition of what constitutes a stable (fragile) financial system, what the systemic risks are, and how to define a crisis in the financial system, see de Bandt and Hartmann (2000), Davis (2003) or Goodhart (2006) for a discussion. In our analysis we follow the micro-economically founded models of Tsomocos (2003), Catarineu-Rabell et al. (2005), Goodhart et al. (2006), and Goodhart and Tsomocos (2007) and use not only the probability of distress as a measure, but also the wealth of the economy in crisis and the sensitivity of the economy to the driving economic factor in distress and/or crisis. These authors note that an increasing probability of distress might just represent increased risk taking, but is not harmful to the economy if it is not accompanied by serious loss to profitability (in our case equity wealth) in the economy.

Let $H$ be an endogenous variable representing an aspect of financial stability and $\chi$ one of the exogenous parameters, then the (total) derivative can be decomposed into

$$\frac{d}{d\chi}H(y^b(\chi),\chi) = \underbrace{\frac{\partial H}{\partial y_b} y_b'(\chi)}_{\text{indirect effect}} + \underbrace{\frac{\partial H}{\partial \chi}}_{\text{direct}} \tag{3.21}$$

Table 3.3, Panel 2 displays the sign of all three relevant derivatives. $H$ are variables that measure aspects of financial stability, which are the probability of distress, $\mathbf{P}[D_T < F]$, the probability of crisis, $\mathbf{P}[W_T^b < nV_T]$, capital provision (needed for Step 3), $W_0^b - W_0^i(y^b)$, the jump size $J$, and the equity value in crisis, $\mathcal{I}(hy^b\xi_T)|_{\xi_T>\bar{\zeta}}$.

### Regulation by the VaR probability

As an example, we discuss the case where the banking system faces stricter regulation by the VaR probability, that is, $\alpha$ *decreases*. The decomposition of the effects on the optimal solution are displayed in Figure 3.4. The

**direct effect of Step 1** is displayed in Figure 3.4, Panel 1. The VaR boundary $\bar{\zeta}$, by virtue of its definition increases, see Change 1. The optimal solution (3.12) is not affected in its principal structure ($W_*$ and $\underline{W}$ are constant), only the region, where wealth is kept at level $\underline{W}$ becomes wider, see Region A. Thus,

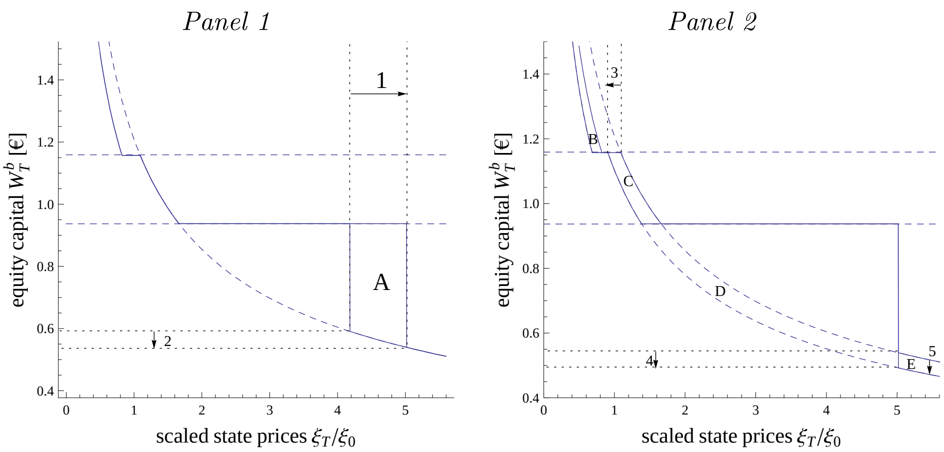


Figure 3.4: **Impact of a Change in the Regulation Parameter** $\alpha$
*This figure decomposes the impact of stricter regulation on terminal wealth into its direct effect in Panel 1 and into the indirect effect in Panel 2, as in Equation (3.21).*

there are some states where wealth has to be raised from the lower unrestricted wealth $\mathcal{I}(hy^b\xi_T)$ to the level $\underline{W}$. Furthermore, the jump size $J$ is enlarged, see Change 2. The

**wealth transfers of Step 2** are displayed in Figure 3.4, Panel 2. Obviously, the budget equation does not hold any more; this profile needs more initial capital due to region A in Panel 1. As the resistance levels $W_*$ and $\underline{W}$ do not change, the only way to 'refinance' these positive wealth differences is to reduce the wealth in all non-restricted states, i.e. the Regions B, C, and E in Panel 2, i.e. an increase in the shadow price of equity capital $y^b$, exactly up to the point where the budget equation will hold again. The present value of regions B, C, and E is identical to the one of region A in order to fulfil the budget equation. The

**provisioning of equity capital in Step 3** can be deduced, when comparing the region that is enclosed by the solution of the unrestricted financial intermediary $i$ (dashed line in Panel 1) and the solution of the banking system $b$ before a change of $\alpha$ (solid line, before changes), with the same region in Panel 2 (after changes); the difference is the combined Regions A and D. Thereby we obtain a raised risk provision that can be directly attributed to the VaR constraint.

However, this form of regulation impairs financial stability by its side effects: reducing the probability of crisis $\mathbf{P}[W_T^b < nV_T] = \mathbf{P}[\xi_T > \overline{\zeta}]$ by means of stricter regulatory policy comprises indirectly a higher probability of distress, $\mathbf{P}[D_T < F] = \mathbf{P}[\xi_T > \zeta^*]$, as $\zeta^*$ decreases through the indirect refinancing effect, see Panel 2, Change 3. Moreover, the crisis becomes more pronounced as the jump size $J$ heightens, both directly as a result of the regulatory impact of $\alpha$ on the VaR boundary $\overline{\zeta}$ (Panel 1, Change 2) as well as indirectly through the effect of wealth transfers, as expressed by $y^b$, on $\zeta^*$, see Panel 2, Change 4. The jump size is of interest with regard to financial stability, as it is related to the question to what extent the banking system reacts to a small change in the underlying economic situation; it thus can be seen as a way to measure the escalation potential of less favourable economic situations. In addition, the severity of crises, as measured by aggregate wealth in case of a crisis $\mathcal{I}(hy^b\xi_T)|_{\xi_T>\overline{\zeta}}$, will be lower than in a less regulated economy, see Panel 2, Change 5.

This discussion of the change in the VaR probability $\alpha$ on the solution vividly illustrates that (stricter) regulation, i.e. Change 1 in Panel 1, leads to complex modifications in the behaviour of the banking system. These modifications may result in less financial stability when other viable measures of financial stability are in view. In particular it is to stress that, even with the minimal assumption in place, the probability of distress in the financial system is increasing due to regulation. Tracing back the result to the static derivative portfolio of equation (3.20) shows that this side effect will also happen in world with sufficient innovations in the derivative markets.

Capital Requirement

While $\alpha$ restricts the probability of a crisis, a positive change in the *capital requirements* $n$ alleviates the severity of the distress in the banking sector by increasing the retention level $\underline{W}$.

By an analogous argument, the indirect financing effect countervails this positive influence, since a similar argument as before shows that the probability of distress, the jump size and the severity in crisis increase, see Table 3.3.

From the viewpoint of regulatory policy, both instruments, the capital adequacy $n$ and the VaR probability $\alpha$ imply adverse reactions: while stability benefits directly from regulation the way one expects, indirect (optimal) wealth transfers are to the detriment of stability in the banking system.

Relation to the Structural Parameters of the Economy

Even though, within the modelling framework, the nominal debt level $F$, costs $\lambda$, and heterogeneity $\beta$ are given constants, they may, in fact, not be constant, since they are subject to the Lucas (1978) critique. Thus a study of the effect stemming from changes in theses variables is necessary in order to assess the robustness of regulatory policies. The arguments in the following are based on Table 3.3.

The banking system e.g. might respond to a change in regulatory policy by downsizing its business activities by lowering *aggregate debt exposure* $F$. This in turn lessens both retention levels in absolute terms. Lower retention levels do not matter in this case, as on a relative scale the ratios $W_*/F$ or $\underline{W}/F$ will not change. Yet, even on a relative scale, the jump size is reduced and positive wealth transfers additionally amend the situation with respect to the probability of distress and the recovery rates of debt $D_T/F$. When combined in this way, the reaction of the banking system to a change in regulation ameliorates the secondary negative effects of regulation.

Without any VaR restriction, i.e. the *unregulated* financial intermediary $i$, *costs* $\lambda$ decrease $h$ and thus raise the terminal equity value of the financial intermediary $W_T^i$ in the distress region $\xi_T > \zeta^*$. The increase, though, is financed over both unrestricted areas, $\xi_T < \zeta_*$ and $\xi_T > \zeta^*$. Consequently, terminal wealth in distress $W_T^i|_{\xi_T > \zeta^*}$ is still increased after shifts due to financing. If there are no cost $\lambda = 0$, a Modigliani and Miller (1958) type of argument holds and there is no disturbance of the unrestricted policy with terminal profile $\mathcal{I}(y^b \xi_T)$.

When financial intermediaries already expect at time $t = 0$ to be hit by some losses due to defaults within the financial system, they behave as if they are more risk averse. If every bank indeed shows this behaviour, the unregulated financial system behaves unambiguously as being more risk averse in order to avoid these exogenous costs. This is in contrast to standard results in literature, as in Jensen et al. (1976), where there are risk-increasing incentives due to limited liability. In this model framework, financial intermediaries do know that they will be hit by some losses from other institutions, which in turn lessen their willingness to take larger risks.

The *regulated banking sector* $b$ differs with respect to *costs* by the additional retention level to crisis $\underline{W}$; for $\lambda = 0$ it is zero, i.e. VaR is non-binding. As $\lambda$ increases, $\underline{W}$ increases and may become binding; when leverage is too low, it will never be binding. Relative to the unrestricted case, the retention level $\underline{W}$ must be financed, resulting in a faster *relative* increase of the budget tightness $y^b$. On the other hand, if $\lambda$ is very high, the VaR restriction is effectively not binding, as the costs themselves are already a big enough incentive. Thus, regulation is non-monotonous in systemic costs.
Costs reduce the riskiness of the banking system. However, the effectiveness of a VaR-based regulation depends on the degree of systemic costs; it only works well for moderate costs. If the costs are 'prohibitively' high, there is simply no need for regulation, as agents voluntarily reduce their exposure. Two properties of costs in our model framework are essential: if costs are not 'expected' to appear by market participants, they will not change their optimal decision; regulation becomes ineffective as well. Moreover, the costs in our model framework are borne by the banking system. If the 'total' costs (consisting of direct, indirect, systemic, real feedback) do not generate enough costs, which are borne by equity holders of banks, regulation is effectively impaired as well. Note that one of the main determinants of the costs fraction $\lambda$ consist of the degree of connectivity within the banking system. Hence, a well-functioning interbank-market has a disciplining effect.

**Remark.** *The impact of heterogeneity $\beta$ is indistinctive, as even the direct effects of the unregulated financial intermediary are ambivalent. Both retention levels $W_*$ and $\underline{W}$ are (progressively) increasing inducing a tighter budget equation. At the same time, the increase in $h$ lessens the restrictiveness of the budget. Without further assumption, it is unclear which effect prevails.*

| | | | | | $\alpha$ | $n$ | $\lambda$ | $\beta$ | $F$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $\frac{\partial\cdot}{\partial y}$ | $\frac{d\cdot}{d\alpha}$ | $\frac{d\cdot}{dn}$ | $\frac{d\cdot}{d\lambda}$ | $\frac{d\cdot}{d\beta}$ | $\frac{d\cdot}{dF}$ |
| STEP 1 | distress level | $i, b$ | $W_* = \frac{\beta}{1-\beta}F$ | . | . | . | . | + | + |
| | cum/ex sensitivity | $i, b$ | $h = \frac{\beta}{\beta+\lambda(1-\beta)}$ | . | . | . | − | + | . |
| | crisis level | $b$ | $\underline{W} = \delta\frac{\beta}{1-\beta}F$ | . | . | + | + | − | + |
| | crisis boundary | $b$ | $\overline{\zeta}$ | . | − | . | . | . | . |
| STEP 2 | budget | $(i/b)$ | $y$ | | (−/.) | (+/.) | (+/+) | (?/?) | (+, +) |
| | | | | $\frac{\partial\cdot}{\partial y}$ | $\left(\frac{\partial\cdot}{\partial\alpha}, \frac{d\cdot}{d\alpha}\right)$ | $\left(\frac{\partial\cdot}{\partial n}, \frac{d\cdot}{dn}\right)$ | $\left(\frac{\partial\cdot}{\partial\lambda}, \frac{d\cdot}{d\lambda}\right)$ | $\left(\frac{\partial\cdot}{\partial\beta}, \frac{d\cdot}{d\beta}\right)$ | $\left(\frac{\partial\cdot}{\partial F}, \frac{d\cdot}{dF}\right)$ |
| STEP 3 | risk provision | $b$ | $W_0^b - W_0^i(y^b)$ | + | (−, −−) | (+, ++) | (?, ?) | (−, ?) | (+, ++) |
| | probability of distress | $i, b$ | $\mathbf{P}[D_T < F]$ | + | (., −) | (., +) | (−, ?) | (+, ?) | (+, ++) |
| | probability of crisis | $b$ | $\mathbf{P}[W_T^b < nV_T]$ | . | (+, +) | (., .) | (., .) | (., .) | (., .) |
| | jump size | $b$ | $J = \underline{W} - \mathcal{I}(hy^b\overline{\zeta})$ | + | (−, −−) | (+, ++) | (?, ?) | (?, ?) | (+, ++) |
| | wealth in crisis (distress) | $b, (i)$ | $\mathcal{I}(hy^{\cdot}\xi_T)\vert_{\xi_T>\bar{\zeta}}$ | − | (., −) | (., −) | (+, ?) | (−, ?) | (., −) |

Table 3.3: **List of the Optimal Solution's Derivatives**

*This table reports the derivatives of the variables in the first column with respect to the exogenous VaR probability $\alpha$, the capital requirement $n$, the cost $\lambda$, the heterogeneity in the banking system $\beta$, and the nominal debt level $F$. '.' denotes no dependence at all, '?' undetermined dependence, '+' positive, and '−' negative dependence. A doubling of '+' or '−' marks cases, where direct (*STEP 1*) and indirect effects (*STEP 2*) have the same sign. For further reference see the Appendix.*

## 3.3 The Banking Sector in Partial Equilibrium

In order to obtain *quantitative* statements or to explore the characteristics of the solution at some *arbitrary* point in time $t \in (0, T)$, we need to impose further structure on the investment opportunity set.

In the previous section, we presented the solution where all the agents take the investment opportunity $(r, \mu, \sigma)$ set as a given. It was required to be continuous and adapted to the filtration $\mathcal{F}$. In order to be consistent with the pure exchange equilibrium in the following chapter, we assume a constant interest rate

$$r_t = r \geq 0 \ , \quad \forall t \ .$$

Moreover, the adapted processes $(\mu, \sigma)$ are connected in such a way as to guarantee a constant market price of risk

$$\kappa_t = \frac{\mu_t - r}{\sigma_t} = \kappa > 0 \ , \quad \forall t \ .$$

With these additional assumptions, the state price process $\xi$ in equation (2.5) on p. 16 follows a geometric Brownian motion. This enables us to evaluate the expectation operators (semi-)analytically.

Yet, there is no need to specify the drift and the volatility of the risky asset more explicitly. In contrast to the *following* chapter, prices are still exogenous by the choice of $r$ and $\kappa$.

**Remark.** *On the impact of this structural assumption, see Remark 1 on page 28.*

In this section, we analyze the restrictiveness of the VaR. We do this by characterizing $(i)$ the point, where the VaR restriction becomes binding and $(ii)$ the capacity of the banking system to load on nominal debt.

We then quantify the equity capital provisions of the VaR constraint and the impact on the banks' debt market.

Finally, we discuss the evolvement of the portfolio decision of the aggregate banking sector over time and states and link it to an instantaneously adjusted risk aversion, caused by the incentives given through the VaR restriction.

**Remark.** *Analogous to the discussion in Remark 2 on p. 36, this framework can be seen as a Cox et al. (1985a)-type production economy. Then, prices have to be interpreted as quantities.*

### 3.3.1 Value of the Optimal Solution

The wealth of the unrestricted investor is, as shown in (2.19) on p. 28,

$$W_t^u(y^u) = \mathcal{I}(y^u \xi_t)\left(a(T-t) + A(T-t)\right) .$$

Using the static budget equation of the unrestricted investor $W_t^u(y^u)|_{t=0} = W_0^u$, we can solve for the Lagrange multiplier analytically

$$y^u = \left(\frac{a(T) + A(T)}{W_0^u}\right)^{\gamma} \frac{1}{\xi_0} .$$

By the same procedure, the aggregate equity position of the banking system $b$ is, as sketched in the Appendix,

$$\begin{aligned} W_t^b(y^b) \;=\; & (A(T-t) + a(T-t))\,\mathcal{I}(y^b\xi_t) \\ & + a(T-t)\,\mathcal{I}(y^b\xi_t) \Big( -\mathcal{I}(h)(\Phi(-d_1(\zeta_*)) - h\Phi(-d_1(\zeta^*))) \\ & \qquad -\mathcal{I}(h)(h\Phi(-d_1(\underline{\zeta})) - h\Phi(-d_1(\overline{\zeta}))\Big) \\ & + e^{-r(T-t)}\, W_* \Big(\Phi(-d_2(\zeta_*)) - h\Phi(-d_2(\zeta^*))\Big) \\ & + e^{-r(T-t)}\, \underline{W} \Big(h\Phi(-d_2(\underline{\zeta})) - h\Phi(-d_2(\overline{\zeta}))\Big) \end{aligned} , \tag{3.22}$$

where $\Phi$ is the standard normal distribution and

$$\begin{aligned} d_1(\zeta) &= \frac{\left(r - \frac{\kappa^2}{2}\right)(T-t) + \log\left(\frac{\zeta}{\xi_t}\right) + \frac{\kappa^2}{\gamma}(T-t)}{\sqrt{T-t}\,\kappa} \\ d_2(\zeta) &= \frac{\left(r - \frac{\kappa^2}{2}\right)(T-t) + \log\left(\frac{\zeta}{\xi_t}\right)}{\sqrt{T-t}\,\kappa} \end{aligned}$$

are as in equation (2.31) on p. 35.

The budget equation $W_t^b(y^b)|_{t=0} = W_0^b$ cannot be solved in closed form. The function $W_t^b(y^b)|_{t=0}$ is strictly (and progressively) decreasing in $y^b$, see equation (3.45); therefore, a unique Lagrange multiplier $y^b$, satisfying the budget equation, exists, given that the feasibility condition is fulfilled.

Equipped with the numerical solution to the Lagrange multiplier, one can recover other variables of interest. Even though we have to rely on a numeric procedure for $y^b$ itself, derivatives with respect to some parameter $\chi$ of the Lagrange multiplier, $y_b'(\chi)$, can be analytically derived, by applying the implicit function theorem on the static budget equation, $W_t^b(y^b(\chi), \chi)|_{t=0} = W_0^b$.

**Remark.** *When directly comparing the regulated banking sector $b$ with the unregulated financial intermediaries $i$, we use the superscript $b$ or $i$ also on total asset $V$, debt $D$, or other variables of interest. If no superscript is used, the regulated banking sector $b$ (alone) is discussed.*

Figure 3.3 shows the balance sheet of the banking sector at time $t = 4$ as a function of the scaled state price $\xi_t/\xi_0$. The aggregate assets $V_t^n$ are shown in Panel 1, equity $W_t^n$ in Panel 2, and debt $D_t^n$ in Panel 3, where $n \in \{b, i\}$. The corresponding formulae are given in the Appendix.

The decomposition of terminal wealth presented in equation (3.20) illustrates that aggregate wealth has a formula of the Black and Scholes (1973) type. The principal structure of $W_t^b$ as a function of the state price $\xi_t$ is similar to the one at time $T$; however, time to maturity has a smoothing effect. Since aggregate equity capital $W_T^b$ is monotonous in the state prices $\xi_T$, the continuous solution in $t$ is strictly monotonous as well. The non-convexities of $W_T^b$ in $\xi_T$ are also transmitted. It is easy to imagine how the curve shapes out over time into the retention levels $W_*$, $\underline{W}$, and the jump $J$ at the VaR boundary $\overline{\zeta}$. The additive first term captures the value of future consumption.

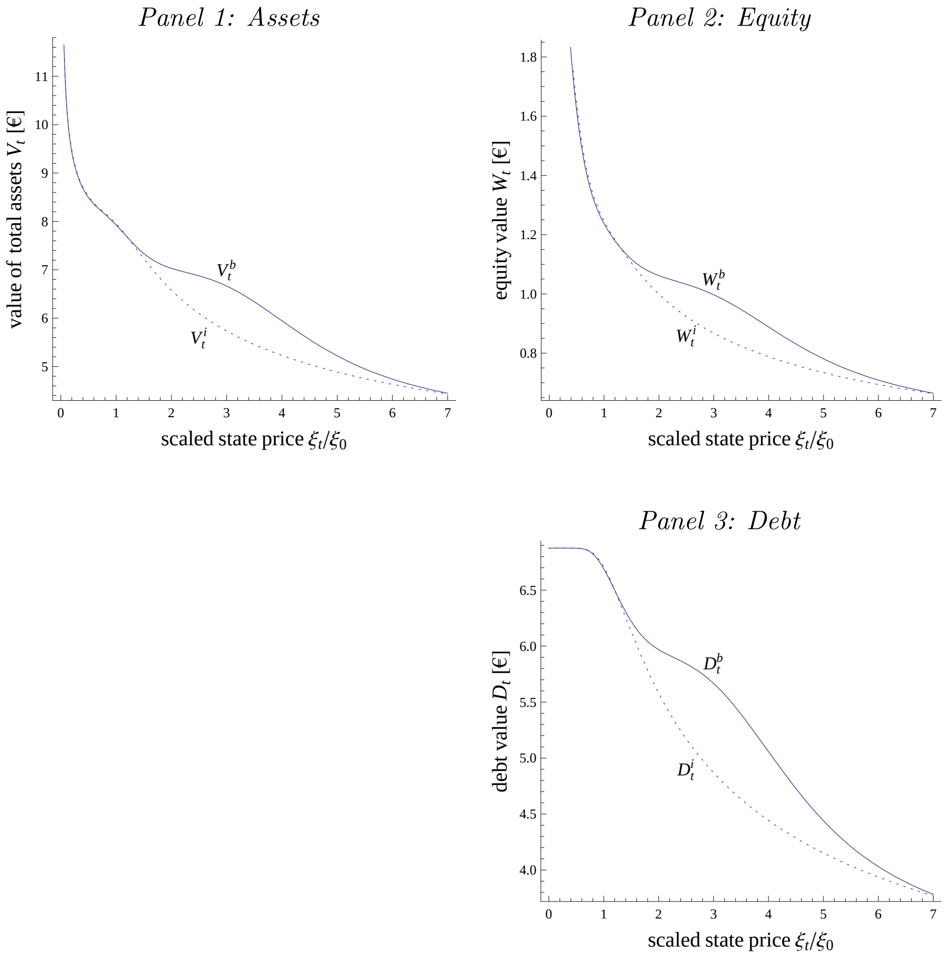


Figure 3.5: **Value of Total Assets, Equity and Debt**

*This figure shows the balance sheet of the aggregate banking sector in present value terms at time $t = 4$, for the restricted banking sector as solid lines and for the unrestricted financial intermediary as dotted lines. Each panel is plotted over the scaled state-price-density $\xi_t/\xi_0$, where low values represent 'good' times and vice versa. Panel 1 shows the total value $V_t$, whereas Panel 2 the equity $W_t$, and Panel 3 the debt value $D_t$. The parameters are as in the standard parameter set in Table 3.2.*

The mean and volatility of the aggregate equity value $W^b$ are stochastic, even if the underlying investment opportunity set $(r, \mu, \sigma)$ is set constant. This is to be expected, as the returns are transformed by the structure of the aggregate balance sheet. However, as the total assets $V_T$ are a piecewise linear function of the aggregated wealth $W_T^b$, the total assets also have stochastic mean and volatility, even though the model framework is alike a 'classical' Black and Scholes (1973) and Merton (1973b) type of economy. This result turns out to be robust, as it only needs a fixed settlement procedure at time $T$, which results in some disturbance of the unrestricted optimal profile $\mathcal{I}(\cdot)$.

**Remark 4.** *This stochastic behaviour of $\mu$ and $\sigma$ translates into total assets as well. This contrasts many models assuming that the total assets (or cash flows) follow a geometric Brownian motion, see Merton (1974), Black and Cox (1976), Leland (1994), Leland and Toft (1996), Goldstein et al. (2001), and Dangl and Zechner (2004) regarding corporate debt contracts, and Merton (1977), Merton (1978), Fries et al. (1997), Bhattacharya et al. (2002), and Dangl and Lehar (2004) on a regulatory framework.*

### 3.3.2 Restrictiveness of the VaR Restriction

Given a regulation policy $(\alpha, n)$, the restrictiveness of the VaR restriction is characterized by two specific nominal debt levels: $(i)$ by the transition point $\vec{F}$ where the restricted solution first becomes effectively restrictive and $(ii)$ by the (maximum) debt capacity $\hat{F}$, the highest level of nominal $F$ that the banking system is able to maintain, given fixed initial value of equity.

**Remark.** *There are two laterally reversed variables of interest, $W_0^b$ and $F$, see also the feasibility constraint (3.17). The question of the maximal feasible nominal debt level $F$ is equivalent to the minimal initial equity capital $W_0^b$ or the maximal nominal leverage $F/W_0^b$. For expositional reasons, we vary the nominal debt level $F$. All conclusions are analogous if one picks one of the other variables.*

When $F = 0$, the VaR restriction is obviously non-restrictive and $y^b = y^i = y^u$. Furthermore, comparative statics in the previous section shows that $y_b'(F) \geq y_i'(F) > 0$, where the equality holds if the VaR restriction is not binding. Consequently, once VaR is first binding, it is binding from than on, when increasing the nominal debt level $F$.

The transition point $\vec{y}$, where the restricted solution changes from being effectively unrestricted to being restricted, can be characterized by

$$\vec{y} = \frac{u'(\underline{W})}{h\bar{\zeta}} \ .$$

By inserting into the budget equation $W_t^i(\vec{y})|_{t=0} = W_0^b$ and solving for $F$, on obtains the analytical solution to the transition nominal $\vec{F}$.

The maximum debt capacity $\hat{F}$ can be characterized by evaluating the feasibility condition for the solution and solving for $F$,

$$\hat{F} = e^{rT} W_0^b \frac{\beta + \lambda(1-\beta)}{\psi(n)\Phi(d_2(\bar{\zeta}(\alpha))) \cdot \beta + \lambda(1-\beta)}, \tag{3.23}$$

where $\psi(n)$ is defined in equation (3.15). Note that $\vec{F} \leq \hat{F}$, where equality holds in the non-generic cases $\alpha = 1$, $n = 0$, or $\lambda = 0$. It is in these cases that the restricted problem is in fact equivalent to the unrestricted for all $F$.

Figure 3.6 shows the type of solutions: Region $A$, where the VaR restriction is not binding, corresponds to $0 \leq F \leq \vec{F}$, and Region $B$, where the effectively restricted solution is obtained, to $\vec{F} < F < \hat{F}$.

Regulation

Panel 1 of Figure 3.6 depicts the type of solution in terms of the VaR probability $\alpha$. Debt capacity in the unrestricted banking sector is $\hat{F}(\alpha = 1, n = 0) = e^{rT} W_0^b \frac{1}{h\lambda} = 63.75$ and with maximal regulation $\hat{F}(\alpha = 0, n = \beta) = e^{rT} W_0^b \frac{1-\beta}{\beta} = 15.65$.

Since region $B$ is generically non-empty, i.e. $0 < \vec{F} < \hat{F}$, the banking system will actively manage its portfolios in order to comply with the VaR restriction. Whether

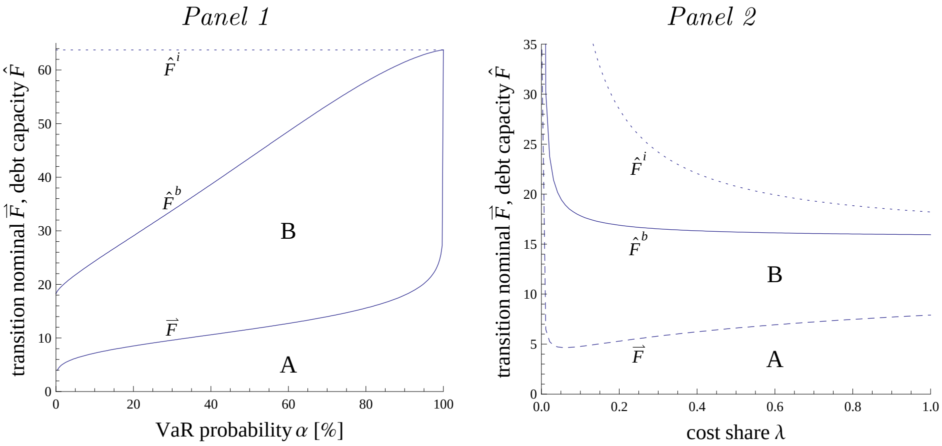


Figure 3.6: **Restrictiveness of the VaR Restriction**

*This figure shows the type of solutions if varying the regulation instrument $\alpha$ (Panel 1) or the costs fraction $\lambda$ (Panel 2) together with the nominal debt level $F$. In region A, the VaR restriction is not active, whereas in region B, the VaR restriction is binding. The transition nominal $\vec{F}$ separates Region A (non active / effectively unrestricted) from Region B (active / effectively restricted). $\hat{F}^b$ and $\hat{F}^i$ are the debt capacities of the restricted banking sector b and of the unrestricted financial intermediary i, respectively. All other parameters are as defined in the standard parameter set in Table 3.2.*

the banking sector reacts to the introduction or a change in regulation depends on the nominal debt level $F$. Furthermore, stricter regulatory policy $(n, \alpha)$ reduces the transition nominal, $\vec{F}'(\alpha) > 0, \vec{F}'(n) < 0$; the banking system reacts earlier.

If financial stability is measured by the maximum debt capacity $\hat{F}$ of the banking system, regulation is effective in reducing the maximum exposure, in our numerical example from $(i) : 63.75 \rightarrow (b) : 18.26$.

**Remark.** *The results with respect to the capital adequacy restriction $n$ is similar, see the Appendix.*

### The Underlying Economic Structure

Panel 2 of Figure 3.6 depicts the type of solution in terms of the costs $\lambda$.

The cost fraction $\lambda$ weakens the debt capacity $\hat{F}$, irrespective of being restricted, $\hat{F}^b$, or not, $\hat{F}^i$, as the aggregate banking sector expects more losses to come. Recall that the fraction of 'total' costs due to a financial system in distress, which are transmitted to the equity position of the banking system itself, depends crucially on the degree of connectivity within the banking sector. The result states that (rational)

agents are willing to extend less credit in aggregate to a financial system with a large interbank market. This contrasts with an argument, where a financial system is able to maintain *more* debt, because losses will be (effectively) more diversified across banks within the financial system.
Regulation further abates the ability of the economy to leverage up, $\hat{F}^b < \hat{F}^i$ as it entails ancillary capital provisions.

The effect of costs on the transition nominal $\vec{F}$ is ambiguous. When increasing costs at a low level of $\lambda$, the transition point $\vec{F}$ (usually) declines, as the VaR retention level $\underline{W}$ is positively related to the costs fraction $\lambda$ and thus the VaR restriction is binding earlier.
On the other side, costs increase wealth in distress, $\mathcal{I}(hy^b\xi_T)|_{\{\xi_T>\bar{\zeta}\}}$, and thereby VaR is less restraining. Since $\vec{F}$ grows at high costs, the second effect dominates the first effects.
The fraction of costs, where the derivative $d\vec{F}/d\lambda$ changes sign, also relies on risk aversion. If risk aversion is very high, in our numerical example $\gamma \geq 97.14$, the second wealth effect always dominates, $d\vec{F}/d\lambda < 0$ for all feasible parameters of regulation $(n, \alpha)$.

**Remark.** *With regard to the parameter of heterogeneity $\beta$, the arguments concerning $\vec{F}$ are similar.*
*The debt capacity $\hat{F}^i$ of the unregulated financial intermediary $i$ is lowered. However, in the regulated economy $b$ the sign of $\hat{F}_b'(\beta)$ is determined by a complex relation between the parameters $\alpha, n, \lambda$ and $\beta$ itself; hence neither $\vec{F}$ nor $\hat{F}^b$ is monotonous in the degree of heterogeneity $\beta$.*

With these results in view, the 'effectiveness' of regulation, when measured by the ability to change the decision of the banking system, depends non-monotonously on the underlying economic structure $(\lambda, \beta)$. As we directly model the aggregate banking sector, our model is subject to the Lucas (1976) Critique. If regulation *in fact* affects also the underlying structure, like the size of the interbank markets or the business model of banks, the previous analysis shows that the (net) impact of

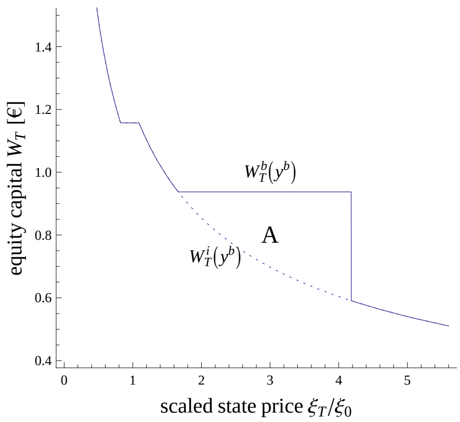


Figure 3.7: **VaR Induced Difference in Optimal Solution**

*This figure shows the VaR induced changes in equity capital in Region A by displaying equity capital with VaR, $W_T^b(y^b)$ (solid line), and equity capital of the financial intermediary using the same tightness of the budget constraint as the regulated banking sector, $W_T^i(y^b)$ (dotted line). The parameters are as defined in the standard set in Table 3.2.*

regulation is hard to predict, even within this simple economy.

### 3.3.3 VaR-induced Capital Provision

This section quantifies how much equity capital the banking system has to provide purely due to the VaR regulation. Figure 3.7 plots the terminal equity wealth $W_T^b(y^b)$ over the scaled state prices $\xi_T/\xi_0$ as a solid line. The dotted line corresponds to $W_T^i(y^b)$. It differs from the regulated banking sector $W_T^b(y^b)$ only in those states that are affected by VaR, namely $\xi_T \in (\underline{\zeta}, \overline{\zeta})$.
Accordingly, $W_0^i(y^b)$ is the initial equity capital needed to implement an unregulated portfolio strategy with a terminal equity position that coincides in all states but those directly attributable to the VaR restriction, see Region A.
Our measure of capital provision due to VaR is

$$CP = \frac{W_0^b - W_0^i(y^b)}{W_0^b} \; . \tag{3.24}$$

The normalization by the initial wealth $W_0^b$ to a proportion of equity capital makes $CP$ comparable in size to a capital requirement, based however on equity capital.

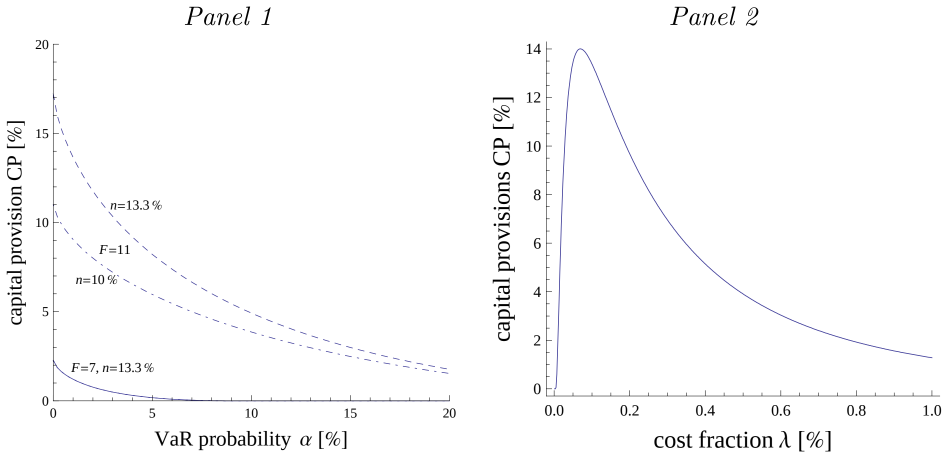


Figure 3.8: **VaR Induced Capital Provisions**

*This figure shows the (relative) capital provision $CP$ of the banking sector. Panel 1 graphs provisioning with respect to the regulatory parameter $\alpha$. The solid line represents the standard case ($F = 7, n = 13.3\%$) , whereas the dashed line the case of ($F = 11, n = 13.3\%$) and the dotted-dashes the case of ($F = 11, n = 10\%$). Panel 1 displays the capital provisioning with respect to the costs share $\lambda$. All other parameters are as defined in the standard set in Table 3.2.*

Because $W_T^i(y^b)$ captures all the equity capital that a rational agent provides due to debt financing, $CP$ can also be interpreted as (relative) excess capital to cover 'unexpected losses'.

Figure 3.8 shows (relative) capital provisions $CP$ for two nominal debt levels $F$ and two Cooke ratios $n$, depending on the regulatory control $\alpha$ in Panel 1 and depending on the cost fraction $\lambda$ in Panel 2.

Regulation

With a stricter regulation policy, the aggregate banking system's capital provisions (progressively) rise due to regulation. Even though it seems tautological, this result is not directly obvious, as regulation restricts equity capital to be less than $nV_0$ and less than $nV_T$ with probability $\alpha$. However, $CP$ is different from the capital requirement $n$, as it measure the restrictiveness of the VaR restriction in terms of initial equity. Conditioning on some regulatory policy, capital provision $CP$ may be higher or lower than the capital requirement $n$, depending on nominal debt $F$, as can be seen in Figure 3.8, Panel 1.

There is an upper bound $\widehat{CP}$ in risk provision, when ($\alpha = 0\%, n = \beta$), where debt is risk-free. In our numerical example, maximal risk provision will be ($\widehat{CP}(F = 7) =$ $5.62\%$, $\widehat{CP}(F = 11) = 39.11\%$).

Furthermore, given an arbitrary regulation, $CP$ is increasing in the nominal debt level $F$. Governing excess capital provisions of the banking system by means of a capital adequacy requirement without having control over nominal debt levels $F$ turns out to be futile within this model framework.

STRUCTURAL PARAMETERS

Panel 2 of Figure 3.8 plots the capital provisions in dependence of the costs $\lambda$. It further underlines the result from the previous subsection that the 'effectiveness' of regulation non-monotonously relies on the underlying economic structure.

For low costs $\lambda$, the VaR-regulated banking system increases capital provisions in order to finance an increased retention level $\underline{W}$. For high costs, the impact of $\lambda$ on $\underline{W}$ is almost negligible, whereas raised costs increase the wealth in distress of the unrestricted intermediary (as a direct effect, not through the budget constraint), which lessens the capital provisions.

### 3.3.4 The Impact of Regulation on Debt Markets

In this section, three standard variables of interest in debt markets - $(i)$ the probability of distress, *(ii)* recovery rates, and *(iii)* spreads - are discussed.

However, as recovery rates and spreads are calculated with aggregated quantities of the banking system, they should be interpreted rather as indicators than as true market variables.

PROBABILITY OF DISTRESS

The (conditional) probability of distress $PD_t$ is defined as

$$PD_t = \mathbf{P}_t[D_T < F] = \Phi\left( -d_0\left(\zeta^*\right) \right) \quad \text{where } \zeta^* = \frac{u'\left(\frac{\beta}{1-\beta}F\right)}{hy^b} < 0 \ . \tag{3.25}$$

The definition of $\zeta^*$ is repeated to show that it neither depends on $\alpha$ nor on $n$. The impact of regulation is only indirectly through the repercussion of wealth transfers at

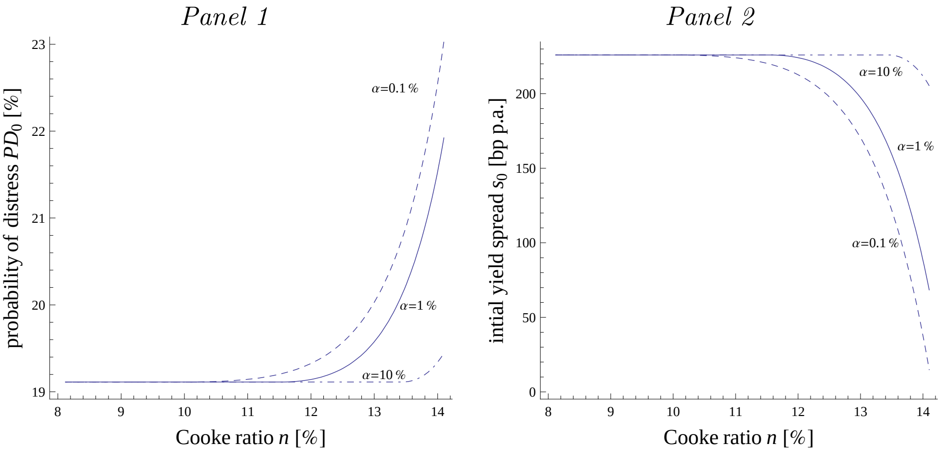


Figure 3.9: **Initial Probability of Default and Yield Spread**

*This figure shows the probability of distress in the banking system $PD_0$ (Panel 1) and the yield spread $s_0$ (Panel 2) as function of the Cooke ratio $n$. Both are illustrated for three different levels of VaR probabilities, $\alpha = 10\%$ (dotted dashed line), $\alpha = 1\%$ (solid line), and $\alpha = 0.1\%$ (dashed line). The probability of distress of the unrestricted financial intermediary is 19.1% and the yield spread 226 bp p.a. All other parameters are as defined in the standard set in Table 3.2.*

time $T$ wherby the probability of distress enlarges. Figure 3.9, Panel 1, quantifies the results regarding the initial probability of distress $PD_0$ using the standard parameter set. Clearly both regulatory instruments $(\alpha, n)$ increase the probability of distress.

Figure 3.10 shows the difference between the conditional probability of distress under the risk-neutral measure $\mathbf{Q}$ and the real one $\mathbf{P}$. The dotted line displays the unregulated financial intermediary $i$, and the solid line the regulated banking system $b$.

When the economic situation turns out to be very 'good', $\xi_t \to 0$, the chance of distress approaches zero. Using a sloppy formulation, there is no need for a change of measure. The analogous argument holds for the opposite case, where distress is almost sure when $\xi_t \to \infty$.

In between, the risk neutral $PD_t^{\mathbf{Q}}$ is, consistent with risk aversion, strictly greater than the real probability measure $PD_t^{\mathbf{P}}$, with a peak at $\xi_t = e^{-r(T-t)}\zeta^*$; empirically, Bliss and Panigirtzoglou (2004) document similar results using an options data set. When the economic situation is worsening from 'boom', prices reflect a fast-increasing risk aversion with respect to defaults, as the difference in probability of

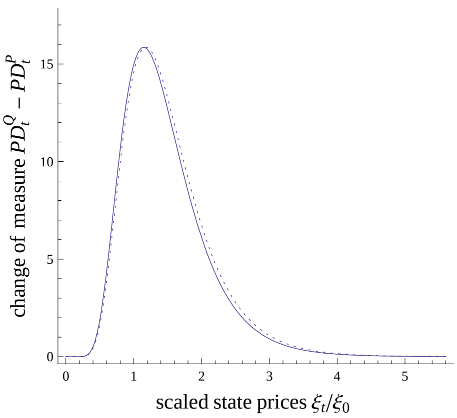


Figure 3.10: **Probability of Distress: Change of Measure**

*This figure shows the difference between the probability of distress under the risk-neutral measure and the real one, $PD_t^{\mathbf{Q}} - PD_t^{\mathbf{P}}$, dependent on the scaled state prices $\xi_t/\xi_0$ at time $t = 4$. The solid line displays the regulated banking system b, whereas the dotted line the unregulated financial intermediary i. The parameters are as defined in the standard set in Table 3.2.*

distress exhibits a large slope with respect to the state prices. In contrast, when the economy recovers from 'bust', the implicit risk aversion is more inert than otherwise. Coudert and Gex (2008) empirically find a similar result. The graph also illustrates that regulation shifts the change of measure to the right.

### Recovery Rates

The recovery rates in this model are

$$R_T = \frac{D_T}{F} \ .$$

As $D_T$ is a function of state prices $\xi_T$, the recovery rate itself is stochastic. It ranges between 1 in case of no distress and the minimum recovery rate, given by $\check{R} = \frac{h(1-\beta)\lambda}{\beta}$.

Furthermore, the expected recovery rate is

$$R_t = \mathbf{E}_t^{\mathbf{P}}[R_T] \ , \tag{3.26}$$

where the expectation is under the real-world measure $\mathbf{P}$.

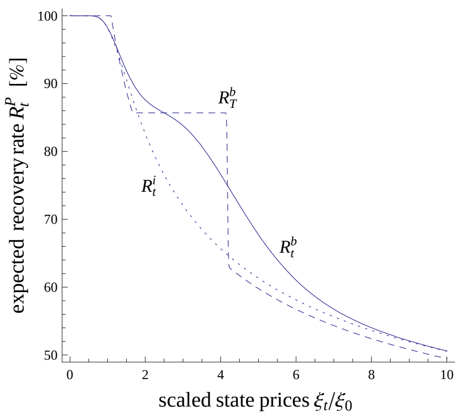


Figure 3.11: **Recovery Rates**

*This figure shows the expected recovery rates, $R_t$, dependent on the scaled state prices $\xi_t/\xi_0$ at time $t = 4$. The solid line displays the regulated banking system b, whereas the dotted line the unregulated financial intermediary i. For reference, the dashed line represents the state depended recovery rates of the baking system $R_T^b$. The parameters are as defined in the standard set in Table 3.2.*

Figure 3.11 shows the recovery rate $R_T$ (dashed line) and the expected recovery, $R_t$, depending on the scaled state prices, $\xi_T/\xi_0$, respectively $\xi_t/\xi_0$.

The model endogenously determines the state-dependent recovery rates, irrespective of using a regulated or unregulated financial system. State-dependent recovery rates are opposed to the 'standard' assumption in the credit risk literature, where a constant recovery rate is usually a modeling assumption, see Jokivuolle et al. (2003) and Schuermann (2004) for a discussion of this point with regard to advanced IRB approach in Basel II. Our model predicts that the recovery rates are in fact lower in economically worse times; Altman et al. (2004) and Acharya et al. (2003) empirically underline the same result.

Regarding regulation, the effects are twofold: on the one hand, there is a higher recovery rate in those states that are now better insured, i.e. in the crisis retention region $\underline{\zeta} < \xi_T \leq \overline{\zeta}$; on the other hand, the recovery rate is lowered by the indirect wealth transfers at time $T$, as represented by the higher Lagrange multiplier $y$, i.e. lower recovery rates in the regions $\zeta^* < \xi_T < \underline{\zeta}$ and $\xi_T > \overline{\zeta}$.

Time to maturity quickly smooths out the kinks and the jump of the terminal recovery rates $R_T$. The expected recovery rates are strictly decreasing in state prices, starting at 1 and also approximating $\check{R}$ in economically 'bad' times.

YIELD SPREAD

The (conditional) yield spread

$$s_t = \frac{\log\left(\frac{F}{D_t}\right)}{T-t} - r = \frac{\log\left(\frac{F}{\mathbf{E}_t^Q[e^{-r(T-t)}D_T]}\right)}{T-t} - r = \frac{\log R_t^{\mathbf{Q}}}{T-t} \tag{3.27}$$

is a transformation of the expected recovery rate under $\mathbf{Q}$.

The yield spread falls with stricter regulation, as displayed in the Panel 2 of Figure 3.9. Since, at regulation level ($\alpha = 0\%, n = \beta$), the debt is risk-free, the yield spread will be $s_0 = 0$.

Summarizing the impact on a hypothetical debt market, one can state that regulation puts debt holders of the banking system at time $t = 0$ in a better position, when measured by the spread, however worse off when the probability of distress is in view.

At maturity, $t = T$, the question of whether debt holders are better off or worse, depends on the economic development: they are negatively affected by regulation in crisis and at the beginning of distress, $\zeta^* < \xi_T < \underline{\zeta}$, however, they are negatively affected in between.

### 3.3.5 Implied Risk Aversion of the Banking System

The aim of this subsection is to show the impact of regulation on the portfolio decision of the banking system $b$, relative to an unregulated financial system $i$.

Since the portfolio decision is in the general case not univariate, an indirect approach is chosen. We first define an univariate measure, the implied risk aversion of an agent, by relating its portfolio decision with a hypothetical risk aversion, which results in the same portfolio decision. The ratio of the implied risk aversion of the banking sector to the one of the unregulated financial system maps the incentives given by regulation into a relative change of risk aversion derived from the portfolio.

### The Definition of Implied Risk Aversion

The optimal portfolio decision of the unrestricted investor $u$, which is identical to the portfolio of the mutual fund $G$ in the decomposition (3.20), is

$$\theta_t^u = \frac{1}{\gamma}\frac{\mu_t - r}{\sigma_t^2} = \frac{1}{\gamma\sigma_t}\kappa.$$

Imagine that an investor $n$ chooses a possibly state- and time-dependent multiple of the mutual fund $G$, namely $q^n\theta^u$; his instantaneous portfolio choice would be the same as if he were an unrestricted investor with an instantaneous relative risk aversion coefficient at time $t$ of $\gamma/q_t^n$. This inspires the definition of an *implied risk aversion* $q^n$. If $q^n < 1$ the portfolio choice is like the one of a more risk-averse agent, whereas $q^n > 1$ relates to less risk aversion, than the unrestricted investor. Note that relative risk aversion is univariate even in a case with multiple assets.

### The Definition of Implied Relative Risk Aversion

The implied relative risk aversion is defined as,

$$IRRA_t = \frac{q_t^b}{q_t^i} \tag{3.28}$$

where $q_t^n, n \in \{i, b\}$, represents the time $t$ load of the unrestricted investor $i$ and the banking system $b$ to the mutual fund portfolio of the unrestricted investor $\theta^u$. Here, the riskiness of the portfolio of the banking system is normalized to the one of the unrestricted banking system $i$. Thus, $IRRA_t$ measures the proportional change in implied risk aversion *relative* to the equivalent unrestricted banking system, when the regulated banking system behaves less riskily than the unregulated one, $IRRA_t < 1$, and vice versa.

### The Portfolio Choice of the Unregulated and Regulated Financial Sector

The portfolio of any wealth process can be derived by applying Ito's lemma to the respective time $t$ fair value $V^n$ and equating diffusionary parts with the budget dynamics in equation (3.2),

$$\begin{aligned}
\frac{\partial V_t^n}{\partial \xi}(-\kappa \xi_t) &= V_t^n \theta_t^n \sigma_t \\
&= V_t^n q_t^n \theta_t^u \sigma_t = V_t^n q_t^n \frac{1}{\gamma}\kappa \\
q_t^n &= -\gamma \left.\frac{\frac{\partial V^n}{V^n}}{\frac{\partial \xi}{\xi}}\right|_{(\xi_t,t)},
\end{aligned}$$

where, in the second line, first the definition of implied risk aversion is substituted and afterwards the optimal portfolio decision of the unrestricted investor $\theta^u$ as in equation (3.28). Hence, implied risk aversion $q^n$ is proportional to the sensitivity of the portfolio value $V^n$ with respect to the state price $\xi$.

When applied to the total assets of the regulated banking system $V^b$ (resp. to $V^i$), the implied risk aversion amounts to

$$\begin{aligned}
q_t^b &= \underbrace{1}_{(i)} \qquad\qquad (3.29)\\
&\underbrace{-e^{-r(T-t)}\frac{W_*}{V_t^b}\left(\frac{(1-\beta)}{\beta} + \big(\Phi\left(-d_2(\zeta_*)\right) - h\Phi\left(-d_2(\zeta^*)\right)\big)\right)}_{(ii)} \\
&\underbrace{-e^{-r(T-t)}\frac{h}{\beta}\frac{\underline{W}}{V_t^b}\left(\Phi\left(-d_2(\underline{\zeta})\right)\right) - \Phi\left(-d_2(\overline{\zeta})\right)\Big)}_{(iii)} \\
&\underbrace{+e^{-r(T-t)}\frac{h}{\beta}\frac{J}{V_t^b}\,\frac{\gamma}{\kappa\sqrt{T-t}}\phi\left(-d_2(\overline{\zeta})\right)}_{(iv)}.
\end{aligned}$$

It can be decomposed into four parts:

($i$) the unrestricted portfolio part,

($ii$) the replication strategy, which insures the retention level to distress $W_*$,

($iii$) the replication strategy, which that generates the minimum wealth required to comply with VaR level $\underline{W}$, and

($iv$) the replication strategy, which results in the jump $J$ at the VaR boundary $\overline{\zeta}$, which similar to the dynamic duplication of a binary option.

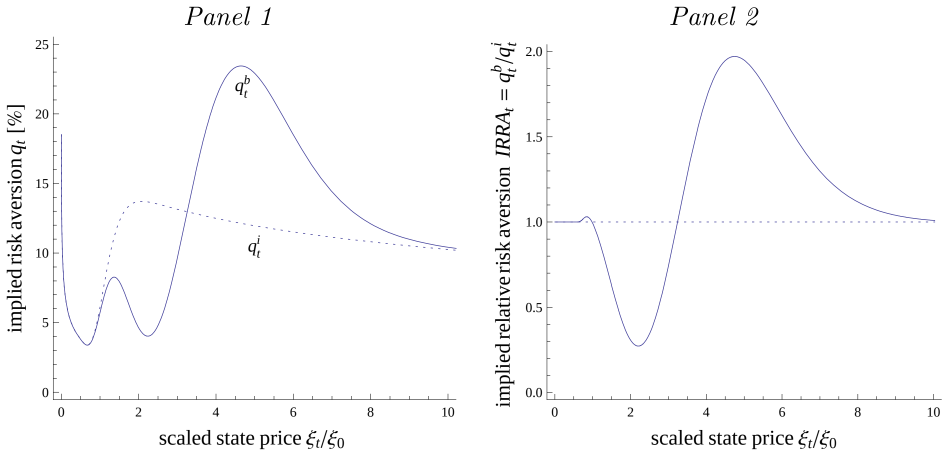


Figure 3.12: **Implied Risk Aversion of the Financial Sector**

*Panel 1 of this figure shows the implied risk aversion of the restricted banking system, $q_t^b$, as a solid line, and of the unrestricted financial intermediary, $q_t^i$, as a dotted line. Panel 2 displays the implied relative risk aversion $q_t^b/q_t^i$. The parameters are as defined in the standard parameter set in Table 3.2 and $t = 4.5$.*

The portfolio multiple $q^i$ of the unrestricted banking system comprises only the parts $(i)$ + $(ii)$. Accordingly, $(iii)$ + $(iv)$ are attributable to the impact of VaR regulation.

Implied Risk Aversion of the Financial Sector

Figure 3.12, Panel 1 shows the principal structure $q^i$ (dotted line) and $q^b$ (solid line) as a function of the scaled state prices $\xi_t/\xi_0$. The implied risk aversion of the unrestricted financial intermediary $q^i$ can be shown to lie within the unit interval, i.e. total assets $V^i$ of the intermediated financial system are less risky than the wealth of the unrestricted agent $W^u$, when the respective portfolio strategies are compared.

The structural form can be explained by using the identity $\sigma_t^{V,n} V_t^n = \sigma_t^{W,n} W_t^n + \sigma_t^{D,n} D_t^n$, $n \in \{i, b\}$. First, the behaviour at the boundaries $\xi_t \to 0$ and $\xi_t \to \infty$ is described for the case of a financial intermediary with no regulation. Following this, the intermediate behaviour is motivated. Afterwards, the previous arguments are carried over to the case of the regulated banking sector.

When times are '*really good*', i.e. $\xi_t \to 0$, the total assets $V_t^i$ of the unregulated financial intermediary $i$ are very high, hence debt is almost riskless, $\sigma_t^{D,i} \simeq 0$, and equity participates almost one-to-one in changes in the mean variance portfolio, i.e.

$\sigma_t^{W,i} \simeq \sigma_t^u$. Consequently, $q_t^i \simeq W_t^i / V_t^i$ holds.

On the other hand, if times are *'really bad'*, i.e. $\xi_t \to \infty$, $V_t^i$ slowly approaches $\check{V}$ in equation (3.5) from above in the style of a portfolio insurance strategy, thus $\sigma_t^{V,i} \simeq 0$, or $q_t^i \to 0$.

In between, the properties of terminal asset value pertain. Total assets of the unregulated financial intermediary $V_T^i$ are continuously decreasing in $\xi_T$, have two kinks at $\zeta_*$ and $\zeta^*$, and are linear in $\mathcal{I}(\cdot)$ otherwise. Accordingly, at any time $t$ the sensitivity is in $(0, 1)$ by the smoothing property of this type of dynamic problems. The kinks cause the single downward hump.

In the case of the VaR regulated banking sector $b$, the limits with respect to 'good' or 'bad' times of the portfolio multiple of the restricted banking system $q^b$ do not change by the same arguments as presented above, namely $q_t^b \to W_t^b / V_t^b$ as $\xi_t \to 0$ and $q_t^b \to 0$ as $\xi_t \to \infty$.

In the states between those two extremes, the optimal time $T$ profile $V_T^b$ has two retention levels, $\frac{h}{\beta} W_*$ and $\frac{h}{\beta} \underline{W}$, and a jump of size $\frac{h}{\beta} J$ at the VaR boundary $\overline{\zeta}$. The second retention level results in another downward hump, whereas the jump effectuates the upward hump at approximately $\overline{\zeta}$; this is due to the part (*iv*) in equation (3.29) and is more pronounced depending on the jump size $J$ and time to maturity $T - t$.

In the aggregate case, the costs due to an unregulated financial system $i$ under distress, $\xi_t > \zeta^*$, spread via the interbank market through the system and are in part absorbed by equity capital, not only by debt holders. As managers rationally incorporate this, there is a considerable risk reduction relative to an un-intermediated economy, i.e. relative to the unrestricted investor $u$ with portfolio $\theta_t^u = \frac{1}{\gamma \sigma_t} \kappa$, as $q_t^i < 1$.

When additionally introducing regulation of the financial system, the same argument still holds; however, the VaR restriction substantially increases the implied risk aversion in the proximity of the VaR boundary $\overline{\zeta}$, i.e. banks follow riskier policies under regulation. González (2005) documents in a study covering 36 countries that stricter regulation in fact increases the risk taking of banks.

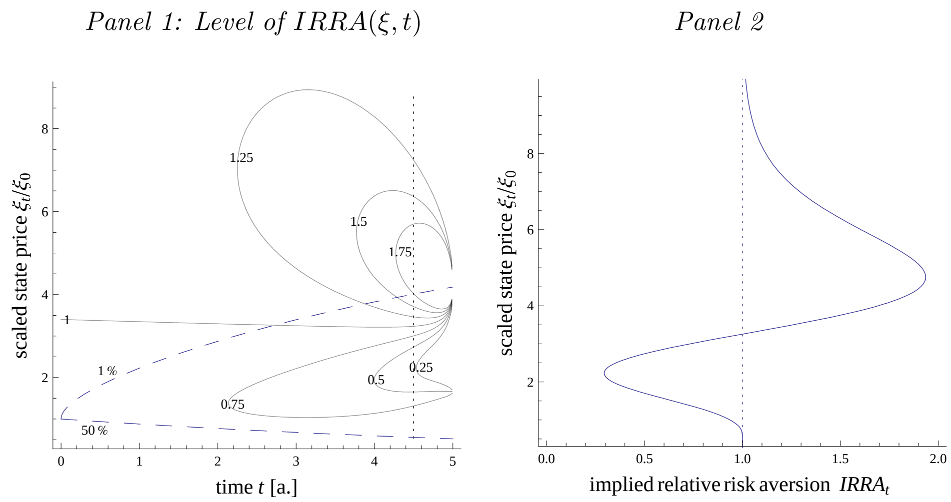


Figure 3.13: **Level Curves of Implied Relative Risk Aversion**

*This figure shows in Panel 1 the level curves of the $IRRA$ coefficient over the state space $(\xi, t)$. The dashed lines localize the 1% and 50% quantiles. Panel 2 is identical to Panel 2 in Figure 3.12 with interchanged coordinates; it is the cut through the state space at time $t = 4.5$, represented by the dotted line in Panel 1. Parameters are as defined in the standard parameter set in Table 3.2.*

## Changes in the Risk Attitude due to Regulation

The resulting implied relative risk aversion as a function of the state price, $IRRA(\xi_t)$ is displayed in Figure 3.12, Panel 2. It has the following structure which is qualitatively stable over time: it starts at 1, has a $\mathcal{S}$-shaped curve, first downwards, then upwards, and eventually converges to unity again. As a reference point, Panel 2 in Figure 3.13 as well as Panel 2 in Figure 3.14 are the same graph, albeit with interchanged coordinates.

Panel 1 of Figure 3.13 is the contour plot of the $IRRA$ coefficients. It shows for each time and state $(t, \xi_t)$ the level of the $IRRA$ coefficient. Panel 2 is the cut at time $t = 4.5$.

Both regions $IRRA \gg 1$ and $IRRA \ll 1$ disperse in time and states; time smooths the impact of the VaR restriction and the humps in the $\xi_t$ cut of the $IRRA$ level plot become less incisive, as time to maturity is further away.

As the $IRRA$ coefficient can be considerably greater than one, the banking system takes much more risk relative to being unrestricted under some specific circumstances. This behaviour can be attributed to the characteristic of the VaR measure that it limits losses by their probability, yet, the level of losses beyond the limit is not relevant for the measure. When VaR is not only used as a passive risk measure for risk, but used actively to manage the risk of the banking sector, there is an incentive to transfer losses beyond the 1% quantile, that is, into the crisis states. Within the model framework, two sources ameliorate this incentive, increasing marginal utility for decreasing wealth and the additional proportional costs to equity.

The circumstances, in which the banking system behaves in such a way are fairly special. The upper dashed line presents the (running) 1% quantile of the economy starting at $\xi_0$, whereas the lower dashed line shows the median state. In most states of the world, there is almost no effect, as $IRRA \simeq 1$ below the median, or a risk reduction, as almost all $IRRA$ coefficients below the 1% quantile are less than 1.

Even though risk taking is improbable, it happens just at the moment when the economy is already in trouble, that is, where the marginal utility of the representative agent $\xi$ is already very high. Furthermore, the level plot in Figure 3.13 shows, especially for short VaR horizons that there is almost no 'separating' area $IRRA \simeq 1$ in between the two regions $IRRA \gg 1$ and $IRRA \ll 1$. The behaviour of risk taking, both in absolute and relative terms, will be very sensitive to changes in the underlying economic variables in these cases.

Changes in Regulation

Figure 3.14 illustrates that, if the banking sector is regulated more strictly, two effects appear: first, the point where the banking sector becomes riskier than the unregulated intermediary slightly decreases to 'better' states. Second, the risk exposure is increased. Both effects are not desirable from the viewpoint of regulation. Especially when crisis is already ahead, regulation is implicitly an incentive for increased risk taking. Even worse, the more regulation, the increasing more risk loading, relative to an unrestricted financial intermediary.

**Remark.** *If not stated otherwise, the results in this subsection are not proven in a rigorous way, as the formulae are too complex to determine signs of derivatives. However, well known results from hedging binary options suggest that the stated*

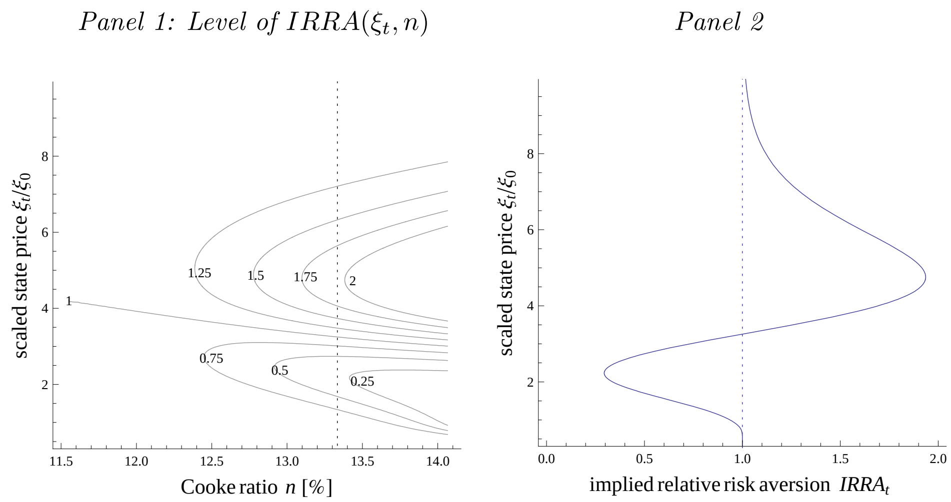


Figure 3.14: **Impact of Regulation on Implied Relative Risk Aversion**

*This figure shows in Panel 1 the level of the IRRA coefficient for various $(\xi_t, n)$ combinations at time $t = 4.5$. Panel 2 is identical to Panel 2 in Figure 3.12 with interchanged coordinates; it is the cut at $n = 13\ 1/3\%$, represented by the dotted line in Panel 1. All other parameters are as defined in the standard parameter set in Table 3.2.*

*arguments are not subject to the specific parameter set used as an example.*

**Remark 5.** *Relating to Remark 2 on p. 36 an alternative interpretation can be used. In an economy of the Cox et al. (1985a) type, equilibrium is not in the form of prices, but in the form of the quantity attributed to the real production economy with a risk-less and a risky investment opportunity. Then, Figure 3.13 implies that a stricter regulation may increase volatility of aggregate production relative to an unregulated economy and thereby generates an additional potential for amplifying economic cycles.*

### Changes in the Underlying Structure

Figure 3.15 shows the impact of costs $\lambda$ (Panel 1) and heterogeneity $\beta$ (Panel 2) on the implied risk aversion $q$.

In the case of an unrestricted financial intermediary $i$, the costs fraction $\lambda$ decreases the overall level of implied risk aversion $q^i$, as the agent is more sensitive to tail risk, namely $\frac{dh}{d\lambda} < 0$. Obviously, the same is also valid for the restricted banking system $b$. The implied relative risk aversion $IRRA$ depends heavily on the jump size $J$,

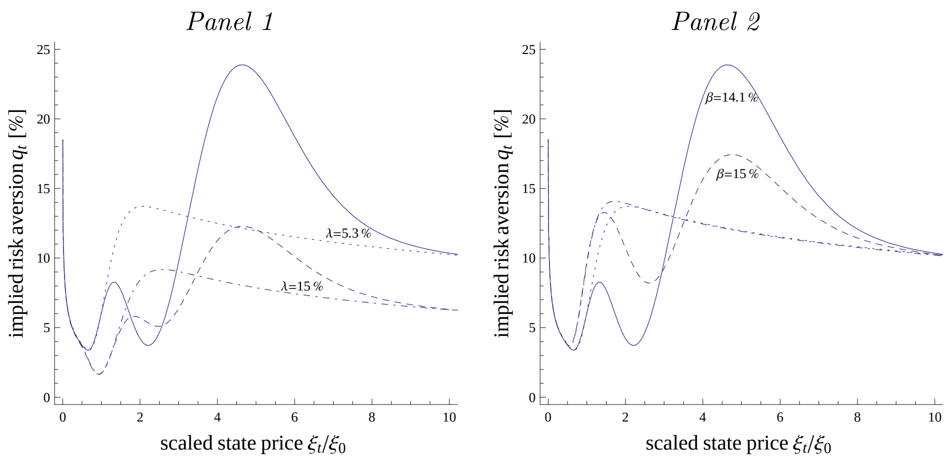


Figure 3.15: **Impact of Costs and Heterogeneity on Implied Risk Aversion**

*This figure shows the impact of the costs to the banking sector and the heterogeneity on the implied risk aversion; Panel 1 illustrates the impact of a change in the costs fraction from $\lambda = 5.3\%$ to $\lambda = 15\%$, Panel 2 of the heterogeneity parameter $\beta = 14.1\%$ to $\beta = 15\%$. Both panels contain four lines: (i) for the unrestricted financial intermediary i (dotted line, $\mathcal{S}$ shaped) and (ii) for the restricted banking sector b (solid line, double $\mathcal{S}$ shaped) with the standard parameterizations. (iii) for the unrestricted financial intermediary i (dotted-dashed line, $\mathcal{S}$ shaped) and (iv) for the restricted banking sector b (dashed line, double $\mathcal{S}$ shaped) with the alternative parametrization. All other parameters are as defined in the standard parameter set in Table 3.2.*

which is raised on the one hand due to the increase in $\underline{W}$, whereas the jump size is lowered by the impact on $h$. In relative terms, it is not clear which effect dominates.

Regarding heterogeneity $\beta$, it has no level effect in contrast to the costs $\lambda$. It shifts the 'hump' of the unrestricted financial intermediary into lower state prices, that is, into better economic situations. In the numerical example, it flattens out the second hump from the VaR restriction of the banking system. Nevertheless, one can construct (extreme) parameter constellations, where there exists the opposite effect of increasing $IRRA$ coefficients.

# 3.4 Appendix to Chapter 3

## 3.4.1 Proof of the Optimal Solution

The proof is structured into two parts. In *Part 1*, a similar problem is solved, where the VaR restriction (3.8) is replaced by

$$\mathbf{P}[W_T \geq \underline{W}] \geq (1-\alpha).$$

*Part 1* is subdivided into three sections: *Part 1.0* discusses some trivial subcases, *Part 1.1* is the main proof, *Part 1.2* outlines the boundary cases, which are formally not included in the main proof.

Finally, *Part 2* reformulates the former problem back to the original one.

Since only the banking sector $b$ is of relevance, we abstract from using the index $b$ in the following.

*Part 1.0:*
If $F = 0$, the solution of the problem is known to be $W_T = W_T^v$ as in equation (9) of Basak and Shapiro (2001). Similarly, if $\alpha = 1$, the VaR restriction is never binding and we recover the unrestricted solution with debt $W_T = W_T^i$ as in equation (5) of Basak and Shapiro (2005). Therefore, we assume in the following parts ($F > 0, \alpha \in [0,1)$).

*Part 1.1:* Assume $\underline{W} < \frac{\beta}{1-\beta}F$ and $\alpha \in (0,1)$.
Define $W_T$ as in equation (3.12) on p. 71. If $\mathbf{P}[W_T \geq \underline{W}] < \alpha$, then, by definition, $\overline{\zeta} < \underline{\zeta}$ and, hence, the VaR restriction is not binding; the solution is $W_T = W_T^i$, which is the optimal unrestricted solution, following the arguments presented in Basak and Shapiro (2005).
Otherwise, $\mathbf{P}[W_T \geq \underline{W}] = \alpha$ and thus $\overline{\zeta} \geq \underline{\zeta}$ holds. If $\underline{\zeta} = \overline{\zeta}$, the VaR restriction is effectively not binding and the unrestricted solution $W_T^i$ is obtained. Since $h \in (0,1)$, $\zeta_* = h\zeta^* < \zeta^*$, and since $\underline{W} < W_*$, $\underline{\zeta} = \frac{u'(\underline{W})}{hy} > \frac{u'(W_*)}{hy} = \zeta^*$ holds. This case corresponds a (mesh) structure $\zeta_* < \zeta^* < \underline{\zeta} < \overline{\zeta}$ and is the one, where the VaR

restriction is effectively binding.

**Lemma 1:** Pointwise, for all $\xi_T$, the optimal solution is

$$
\begin{aligned}
V_T^* &= (\mathcal{I}(y\xi_T)+F)1_{\{\xi_T<\zeta_*\}})+\frac{h}{\beta}\left(\mathcal{I}(hy\xi_T)+\lambda F\right)(1_{\{\zeta^*\leq\xi_T<\underline{\zeta}\}}+1_{\{\xi_T\geq\overline{\zeta}\}}) \\
&\quad \frac{1}{1-\beta}F1_{\{\zeta_*\leq\xi_T<\zeta^*\}}+\frac{h}{\beta}\left(\underline{W}+\lambda F\right)1_{\{\underline{\zeta}\leq\xi_T<\overline{\zeta}\}} \\
&= \arg\max_V u\big(V-D(V)-C(V)\big)-y\xi_T\big(V-D(V)\big)+y_2 1_{\{W(V)\geq\underline{W}\}} \\
&= \arg\max_V \mathcal{L}(V)\ , \qquad (3.30)
\end{aligned}
$$

where

$$
\begin{aligned}
D(V) &= F\,1_{\{(1-\beta)V\geq F\}}+1_{\{(1-\beta)V<F\}}\big((1-\beta)V\big) \\
C(V) &= 0\,1_{\{(1-\beta)V\geq F\}}+1_{\{(1-\beta)V<F\}}\big(\lambda(F-D(V)\big) \\
W(V) &= (V-F)\,1_{\{(1-\beta)V\geq F\}}+1_{\{(1-\beta)V<F\}}\big(\beta V+\lambda((1-\beta)V-F)\big) \\
y_2 &= \Big(u\big(\frac{h}{\beta}\left(\mathcal{I}(hy\overline{\zeta})+\lambda F\right)\big)-y\overline{\zeta}\big(\frac{h}{\beta}\left(\mathcal{I}(hy\overline{\zeta})+\lambda F\right)\big)\Big) \\
&\quad -\Big(u\big(\frac{\underline{W}}{1-\beta}\big)-y\overline{\zeta}\frac{\underline{W}}{1-\beta}\Big)
\end{aligned}
$$

and all other variables as in equation (3.12).

**Proof:** The function $\mathcal{L}(V)$ is not concave in $V$, but can exhibit two 'inner' maxima $\{i1, i2\}$ and two 'boundary' maxima $\{b1, b2\}$ only at

$$
\begin{cases}
V_{i1}=\mathcal{I}(y\xi_T)+F, & \text{if } (1-\beta)V_1\geq F\ ; \\
V_{i2}=\frac{h}{\beta}\left(\mathcal{I}(hy\xi_T)+\lambda F\right), & \text{if } (1-\beta)V_2<F\ ; \\
V_{b1}=\frac{1}{1-\beta}F, & \\
V_{b2}=\frac{h}{\beta}\left(\underline{W}+\lambda F\right)\ . &
\end{cases}
\qquad (3.31)
$$

Let us define the Lagrange functions

$$
\begin{aligned}
\mathcal{L}_1(V,\xi) &= u\left((V-F)\right)-y\xi(V-F)+y_2 1_{\{(V-F)\geq\underline{W}\}} \text{ if } (1-\beta)V\geq F \\
\mathcal{L}_2(V,\xi) &= u\left(\frac{\beta}{h}V-\lambda F\right)-y\xi\beta V+y_2 1_{\{\frac{\beta}{h}V-\lambda F\geq\underline{W}\}} \text{ if } (1-\beta)V<F\ .
\end{aligned}
$$

Then the global maximimizer is

$$V^*(\xi_T) = \arg\max_V \{\mathcal{L}_1(V_{i1}, \xi_T), \mathcal{L}_2(V_{i2}, \xi_T), \mathcal{L}_1(V_{b1}, \xi_T), \mathcal{L}_2(V_{b2}, \xi_T)\} \tag{3.32}$$

over the respective state space mesh $\zeta_* < \zeta^* < \underline{\zeta} < \overline{\zeta}$, corresponding to the different regions. By assumption it is in any region true that $V_{i2} > V_{i1}$ and $V_{b1} > V_{b2}$.

***Region 1,*** $\xi_T < \zeta_*$: In this region we have $V_{i1} > V_{b1} > V_{b2}$ and $V_{b1} - F > \underline{W}$. By the arguments presented in Basak and Shapiro (2005) equation (A.3), $\mathcal{L}_1(V_{i1}, \xi_T) > \mathcal{L}_1(V_{b1}, \xi_T) = \sup_{\{V > F/(1-\beta)\}} \mathcal{L}_2(V, \xi_T)$. Finally, we compare $\mathcal{L}_1(V_{i1}, \xi_T)$ and $\mathcal{L}_2(V_{b2}, \xi_T)$. Let us define

$$\begin{aligned}
a &= (u(\mathcal{I}(y\xi_T)) - y\xi_T \mathcal{I}(y\xi_T)) - (u(\underline{W}) - y\xi_T \underline{W}) \\
&> 0 \text{ if } \underline{W} < \mathcal{I}(y\xi_T) \\
b &= (\mathcal{L}_1(V_{i1}, \xi_T) - \mathcal{L}_2(V_{b2}, \xi_T)) - a \\
&= y\xi_T((h-1)\underline{W} + Fh\lambda) \\
&\geq 0 \text{ if } \underline{W} \leq \frac{\beta}{1-\beta} F.
\end{aligned}$$

Hence,

$$0 < a + b = (\mathcal{L}_1(V_{i1}, \xi_T) - \mathcal{L}_2(V_{b2}, \xi_T))$$

holds. Therefore, $V^*(\xi_T) = \mathcal{I}(y\xi_T) + F$, respectively $W^*(\xi_T) = \mathcal{I}(y\xi_T)$, is the optimal solution in this region.

***Region 2,*** $\zeta_* \leq \zeta < \zeta^*$: Following Basak and Shapiro (2005), equation (A.4), $\mathcal{L}_1(V_{b1}, \xi_T) > \mathcal{L}_2(V_{i2}, \xi_T)$ and $\mathcal{L}_1(V_{b1}, \xi_T) \geq \mathcal{L}_1(V_{i1}, \xi_T)$ is true. By the properties of the convex conjugate, $\mathcal{L}_2(V_{i2}, \xi_T) > \mathcal{L}_2(V_{b2}, \xi_T)$ holds. Thus, $V^*(\xi_T) = \frac{1}{(1-\beta)} F$, respectively $W^*(\xi_T) = \frac{\beta}{1-\beta} F$, is the optimal solution in this region.

***Region 3*:** When $\zeta^* \leq \xi_T < \underline{\zeta}$, then $V_{b1} \geq V_{i2} > V_{b2}$ holds and the unrestricted order is $\mathcal{L}_2(V_{i2}, \xi_T) \geq \mathcal{L}_2(V_{b1}, \xi_T)$. Since $\frac{\beta}{h}V_{i2} - \lambda F > \underline{W}$, we still have $\mathcal{L}_2(V_{i2}, \xi_T) > \mathcal{L}_2(V_{b2}, \xi_T)$. The optimal solution is $V^*(\xi_T) = \frac{h}{\beta}\left(\mathcal{I}(hy\xi_T) + \lambda F\right)$, respectively $W^*(\xi_T) = \mathcal{I}(hy\xi_T)$.

***Region 4*:** $V_{b1} > V_{b2} \geq V_{i2}$ holds, since $\underline{\zeta} \leq \xi_T < \overline{\zeta}$. Also it is true that

$$\begin{aligned}
\mathcal{L}_2(V_{b2}, \xi_T) &= u(\mathcal{I}(hy\overline{\zeta})) - hy\overline{\zeta}\mathcal{I}(hy\overline{\zeta}) + hy\underline{W}(\overline{\zeta} - \xi_T) - hy\xi_T\lambda F \\
&> u(\mathcal{I}(hy\xi_T)) - hy\xi_T\mathcal{I}(hy\xi_T) - hy\xi_T\lambda F \\
&= \mathcal{L}_2(V_{i2}, \xi_T) \ ,
\end{aligned}$$

where the inequality follows from $\xi_T < \overline{\zeta}$, and, for all $\xi > \underline{\zeta}$,

$$\frac{\partial}{\partial \xi}\Big(u(\mathcal{I}(hy\xi) - hy\xi\mathcal{I}(hy\xi) + hy\underline{W}\xi\Big) = hy(\underline{W} - \mathcal{I}(hy\xi) \geq 0. \qquad (3.33)$$

Therefore, $V^*(\xi_T) = \frac{h}{\beta}\left(\underline{W} + \lambda F\right)$, respectively $W^*(\xi_T) = \underline{W}$, is the optimal solution in this region.

***Region 5*:** Finally, in region $\overline{\zeta} \leq \xi_T$, the same arguments as in the previous region apply, however, the inequality in equation (3.33) is reversed; thereby we obtain the optimal solution $V^*(\xi_T) = \frac{h}{\beta}\left(\mathcal{I}(hy\xi_T) + \lambda F\right)$, respectively $W^*(\xi_T) = \mathcal{I}(hy\xi_T)$.

$y_2 \geq 0$**:** Lastly,

$$y_2 = \Big(u(\mathcal{I}(hy\overline{\zeta})) - hy\overline{\zeta}\mathcal{I}(hy\overline{\zeta}) + hy\underline{W}\overline{\zeta}\Big) - \Big(u(\mathcal{I}(hy\underline{\zeta})) - hy\underline{\zeta}\mathcal{I}(hy\underline{\zeta}) + hy\underline{W}\underline{\zeta}\Big) \geq 0 \ ,$$

by using equation (3.33).

□

Let $V(T)$ be any candidate optimal solution satisfying the static budget constraint (3.16) and the VaR restriction (3.8). Then we obtain

$$\begin{aligned}
&\mathbf{E}[u(V_T^* - D(V_T^*) - C(V_T^*))] - \mathbf{E}[u(V_T - D(V_T) - C(V_T))] \\
&= \mathbf{E}[u(V_T^* - D(V_T^*) - C(V_T^*))] - \mathbf{E}[u(V_T - D(V_T) - C(V_T))] \\
&\quad -y\xi_0 W_0 + y\xi_0 W_0 - y_2(1-\alpha) + y_2(1-\alpha) \\
&\geq (\mathbf{E}[u(V_T^* - D(V_T^*) - C(V_T^*))] - \mathbf{E}[y\xi_T V_T^* - D(V_T^*)] + \mathbf{E}[y_2 1_{\{W(V_T^*) > \underline{W}\}}]]) \\
&\quad -(\mathbf{E}[u(V_T - D(V_T) - C(V_T))] - \mathbf{E}[y\xi_T V_T - D(V_T)] + \mathbf{E}[y_2 1_{\{W(V_T) > \underline{W}\}}]]) \\
&\geq 0 \ ,
\end{aligned}$$

where the former inequality follows from the static budget constraint and the VaR restriction holding with equality for $V_T^*$ , while holding with inequality for $V_T$. The latter inequality follows from Lemma 1. Thus, $V_T^*$ is optimal and equivalently $W_T^*$. Finally, as the VaR constraint must hold with equality, the definition of $\overline{\zeta}$ follows.

From the solution (3.12) it is clear that, except at the default retention level $W_*$ and the VaR retention level $\underline{W}$, $\frac{\partial W_T^*}{\partial y} < 0$ holds.
Since $W_T^i(y) = W^b(y; \alpha = 1) \leq W_T^*(y)$ and $\frac{\partial W_T^*}{\partial \alpha} < 0$, in order to allow the budget constraint to hold with equality, we must have $y \geq y^i \geq y^u$. The last inequality is from Basak and Shapiro (2005).
Analogously, since $W^v(y) = W^b(y; F = 0) \leq W_T^*(y)$ and $\frac{\partial W_T^*}{\partial F} > 0$, in order to allow the budget constraint to hold with equality, we must have $y \geq y^v \geq y^u$. The last inequality is from Basak and Shapiro (2001). $\square$

*Part 1.2:* Boundary Cases

*Case* $\underline{W} < W_*$ *and* $\alpha = 0$: In this case $\overline{\zeta} \underset{\alpha \to 0}{\to} \infty$ and the last part of solution (3.12) vanishes. It corresponds a modified portfolio insurance solution, as wealth $W_T \geq \underline{W}$. Also note, that this solution is continuous. The proof is analogous to the previous one, however, by definition there is no *Region 5*.

*Case* $\underline{W} = W_*$ *and* $\alpha \in (0, 1)$: In this case $\zeta^* = \underline{\zeta}$ holds, hence, the unrestricted part of the solution (3.12) between $\zeta^*$ and $\underline{\zeta}$ disappears. The default boundary is $\overline{\zeta}$, thus, the probability of default equals the VaR probability $\alpha$. The proof is analogous to the previous one, however, by definition, there exists no *Region 3*.

*Case* $\underline{W} = W_*$ *and* $\alpha = 0$: This is the combined case of the former ones, $\overline{\zeta} \underset{\alpha \to 0}{\to} \infty$ and $\zeta^* = \underline{\zeta}$. In this case, there is no default, i.e. $D_T = F$ and the solution in terms of wealth $W_T$ equals the portfolio insurance solution of Basak (1995). The proof is analogous to the previous one, however, by definition there exits no *Region 3* and no *Region 5*.

There exist further cases as well. However, these are of no interest due to economic reasons. If $\underline{W} > W_*$, there is more insurance to equity holders than necessary to protect debt; however, a part of the associated costs will be paid by debt holders. As they do not benefit from this insurance, this solution cannot be sustained in a rational equilibrium.

*Part 2:*

Next we reformulate the VaR restriction of the former proof in such a way, that it corresponds to the original problem.

$$
\begin{aligned}
\mathbf{P}\left[\frac{W_T}{V_T} \le n\right] &= \mathbf{P}\left[\frac{W_T}{\frac{h}{\beta}(W_T+\lambda F)} \le n\right] \quad \text{if } \; n<\beta \\
&= \mathbf{P}\left[W_T \ge \left(\frac{hn(\beta-1)\lambda}{(hn-\beta)\beta}\right)\frac{\beta}{1-\beta}F\right] \quad \text{if } \; n<\frac{\beta}{h} \\
&= \mathbf{P}\left[W_T \ge \psi(n)\frac{\beta}{1-\beta}F\right] \\
&= \mathbf{P}\left[W_T \ge \underline{W}\right]
\end{aligned}
$$

The condition $n < \frac{\beta}{n}$ is always fulfilled, since $n < \beta$ in order to satisfy $\underline{W} < W_*$. Moreover, $\psi(n) \in [0,1]$, with $\psi(0)=0$ and $\psi(\beta)=1$.
The feasibility constraint can be derived by

$$
\frac{D_0}{W_0} \le \frac{1-\beta}{\beta},
$$

since otherwise it is optimal to close down business as of today. Therefore, the constraint reads, by letting $y \to \infty$,

$$
\begin{aligned}
W_0 &\ge \frac{\beta}{1-\beta}\mathbf{E}\left[\frac{\xi_T}{\xi_0}\left(D_T(W=\underline{W})1_{\{\xi_T\le\bar{\zeta}\}} + D_T(W=0)1_{\{\xi_T>\bar{\zeta}\}}\right)\right] \qquad (3.34)\\
&= \frac{\psi(n)\beta+\lambda(1-\beta)}{\beta+\lambda(1-\beta)}\frac{\beta}{1-\beta}F\mathbf{E}\left[\frac{\xi_T}{\xi_0}1_{\{\xi_T\le\bar{\zeta}\}}\right] + \frac{\beta\lambda}{\beta+\lambda(1-\beta)}F\mathbf{E}\left[\frac{\xi_T}{\xi_0}1_{\{\xi_T>\bar{\zeta}\}}\right]
\end{aligned}
$$

□

### 3.4.2 Comparative Statics

As most variables of interest, say $H$, depend on $\chi \in \{\alpha, n, \lambda, \beta, F\}$, but the multiplier $y$ as well, one can more easily determine signs by first calculating the derivatives $y'(\chi)$, using the budget equation (3.16), $\mathcal{B}(y, \chi) = \xi_0 W_0$ (under the assumption, that we do not change the case), and afterwards the comparative static of the variable of interest

$$\frac{d}{d\chi} H(y(\chi), \chi) = \frac{\partial H}{\partial y} y'(\chi) + \frac{\partial H}{\partial \chi} \tag{3.35}$$

The standing assumptions used are:

$$\begin{cases} F > 0 \\ \beta \in (0, \frac{1}{2}) \\ \lambda \in (0, 1) \\ n < \beta \\ \alpha \in (0, 1) \end{cases} \tag{3.36}$$

as well as

$$\begin{array}{rcl} \psi(n) & \in & (0,1) \\ h & = & \frac{\beta}{\beta + \lambda(1-\beta)} \in (0,1) \ . \end{array} \tag{3.37}$$

The derivation of the sign of a variable $H$ has the following structure:

**Step 1:** Derive the impact of $\chi$ on the relevant variables $W_*, \underline{W}, \overline{\zeta}$ and $h$.

**Step 2:** If the analysis in Step 1 results in a monotone increase (monotone decrease) of the optimal profile $W_T$, i.e.

$$W_*' \geq 0 \wedge \underline{W}' \geq 0 \wedge \overline{\zeta}' \geq 0 \wedge h' \leq 0 \ , \tag{3.38}$$

(respectively all inequalities reversed), with at least one inequality holding as equality, the budget in $t = 0$ is strictly increasing (strictly decreasing ) as well, $\frac{\partial \mathcal{B}(y,\chi)}{\partial \chi} > 0$. If is is not possible fulfill one of the two condition, no general conclusion about the change of the budget can be derived without further

assumptions.
As the budget is strictly decreasing in $y$, $\frac{\partial \mathcal{B}(y,\chi)}{\partial y} < 0$, the sign of the derivative $y'(\chi)$ can be determined, as the budget equation $\mathcal{B}(y,\chi) = \xi_0 W_0$ must be fulfilled as well.

**Step 3:** By using (3.35), the sign of the variables of interest $H$ is (possibly) determined.

The results are presented Table 3.3 on page 84.
Step 1 can be easily verified by checking the sign of derivatives of the respective variables, using the stated assumptions (3.36) and (3.37). Step 2 is obtained by the previous argument. For Step 3, the following characteristics are useful:

$$\begin{aligned}
\mathbf{P}[D_T < F] &= \mathbf{P}[\xi_T > \zeta *] \text{ with } \zeta^*(y,F) = \frac{u'(\frac{\beta}{1-\beta}F)}{hy} \\
\mathbf{P}[W_T < nV_T] &= \mathbf{P}[\xi_T > \overline{\zeta}] \\
\mathcal{I}'(x) &= -\frac{\mathcal{I}(x)}{\gamma x} < 0 \quad (x > 0, \gamma > 0)
\end{aligned}$$

For further reference we need to show that $y_b'(F) > y_i'(F)$ holds, if the VaR restriction is binding:
$h$ and $\overline{\zeta}$ do not depend on $F$ and $\frac{dW_*^b}{dF} = \frac{dW_*^i}{dF} = \frac{\beta}{1-\beta} > 0$. Furthermore, $\frac{d\underline{W}}{dF} = \psi(n)\frac{\beta}{1-\beta} > \frac{d\overline{W}^i}{dF} = 0$ and $\frac{\partial \underline{\zeta}}{\partial F} = -\frac{\gamma \underline{\zeta}}{F} < 0$ hold. Hence, region A in Figure 3.7 on page 93 is increasing in $F$, while all partial effects are identical between $i$ and $b$. Consequently, $y_b'(F) > y_i'(F)$.

### 3.4.3 Deduction of Dynamic Wealth

This section shows how to derive the dynamic value of the optimal solution or one of its transformations, given a constant interest rate $r$ and a constant market price of risk $\kappa$.

All solutions consist of two types of functional forms (i) $\mathcal{W}_1 = \mathcal{I}(y\xi_T)1_{\{\xi_T \leq \zeta\}}$ (respectively $\mathcal{I}(y\xi_T)1_{\{\xi_T > \zeta\}}$) or (ii) $\mathcal{W}_2 = K1_{\{\xi_T > \zeta\}}$ (respectively $K1_{\{\xi_T \leq \zeta\}}$). By knowing these, one can easily reproduce the given solutions. Note, that if $\mathcal{W}_1 + \mathcal{W}_2$ is continuous, it is equivalent to a portfolio insurance on the unrestricted wealth with pay-off profile $\max\{W^u, K\}$.

Solutions for constant parts $\mathcal{W} = K\,1_{\{\xi_T > \zeta\}}$:
By arbitrage free pricing we obtain

$$
\begin{aligned}
\xi_t \mathcal{W}_t &= \mathbf{E}_t[\xi_T \mathcal{W}_T] \\
&= \mathbf{E}_t[\xi_T K 1_{\{\xi_T > \zeta\}}] \\
&= K\mathbf{E}_t[\xi_T (1 - 1_{\{\xi_T \leq \zeta\}})] \\
&= K\mathbf{E}_t[\xi_T 1] - K\mathbf{E}_t[\xi_T 1_{\{\xi_T \leq \zeta\}})]) \\
&= K\xi_t e^{-r(T-t)} - K\xi_t e^{-r(T-t)} \Phi\big(\, d_2(\xi_t, (T-t), \zeta)\big) \\
&\Leftrightarrow \\
\mathcal{W}_t &= K e^{-r(T-t)} \Phi\big(-d_2(\xi_t, (T-t), \zeta)\big) \ , \qquad (3.39)
\end{aligned}
$$

where the $\Phi$ is the standard normal distribution and $d_2$ is given in equation (3.22) on p. 86. The derivation of the expectation $\mathbf{E}_t[\xi_T 1_{\{\xi_T \leq \zeta\}})]$ is similar to a Black and Scholes (1973) and Merton (1973b) framework, as

$$
\ln \xi_T | \ln \xi_t \sim \mathcal{N}\big(\ln \xi_t - (r + \frac{1}{2}\kappa^2)(T-t), \kappa^2(T-t)\big) \ .
$$

Solutions for unregulated-style parts $\mathcal{W} = \mathcal{I}(y\xi_T)1_{\{\xi_T \leq \zeta\}}$:

By arbitrage free pricing we obtain

$$
\begin{aligned}
\xi_t \mathcal{W}_t &= \mathbf{E}_t[\xi_T \mathcal{W}_T] \\
&= \mathbf{E}_t[\xi_T \mathcal{I}(y\xi_T)1_{\{\xi_T \leq \zeta\}}] \\
&= \mathcal{I}(y)\mathbf{E}_t[\xi_T \mathcal{I}(\xi_T)1_{\{\xi_T \leq \zeta\}}] \\
&= \mathcal{I}(y)\mathbf{E}_t[\xi_T^{(-\frac{1}{\gamma}+1)} 1_{\{\xi_T \leq \zeta\}})]) \\
&= \mathcal{I}(y)\xi_T^{(-\frac{1}{\gamma}+1)} a(T-t)\Phi\big(\, d_1(\xi_t, (T-t), \zeta)\big) \\
&\Leftrightarrow \\
\mathcal{W}_t &= \mathcal{I}(y\xi_t)a(T-t)\Phi\big(\, d_1(\xi_t, (T-t), \zeta)\big)\ . \qquad (3.40)
\end{aligned}
$$

The derivation of the conditional expectation $\mathbf{E}_t[\xi_T^{(-\frac{1}{\gamma}+1)} 1_{\{\xi_T \leq \zeta\}})]$ is similar to a modified Black and Scholes (1973) framework, as

$$
\ln \xi_T^c | \ln \xi_t^c \sim \mathcal{N}\big(c \ln \xi_t - (r + \frac{1}{2}\kappa^2)(T-t), c^2\kappa^2(T-t)\big) \quad c \in (-\infty, 1).
$$

The time-dependent factor $a$ captures the relative prudence effect on portfolios. In case of $\gamma = 1$, $a = 1$ and we recover the well known Black and Scholes (1973) formula.

The Portfolio Insurance

$$
\begin{aligned}
\mathcal{W}_1 + \mathcal{W}_2 &= \mathcal{I}(y\xi_t)a(T-t)\Phi\big(\, d_1(\xi_t, (T-t), \zeta)\big) + Ke^{-r(T-t)}\Phi\big(-d_2(\xi_t, (T-t)\zeta)\big) \\
&= \mathcal{I}(y\xi_t)a(T-t)\Big(1 - \Phi\big(-d_1(\xi_t, (T-t), \zeta)\big) \\
&\quad + Ke^{-r(T-t)}\Phi\big(-d_2(\xi_t, (T-t)\zeta)\big)\Big)
\end{aligned}
$$

Value of Debt and Total Value

Analogously, one can construct the debt value

$$
\begin{aligned}
D_t &= e^{-r(T-t)} F \left(1 - h\Phi\left(-d_2(\zeta^*)\right)\right) \\
&\quad + e^{-r(T-t)} \tfrac{h}{\beta}(1-\beta)\underline{W}\left(\Phi\left(-d_2(\underline{\zeta})\right) - \Phi\left(-d_2(\overline{\zeta})\right)\right) \\
&\quad \tfrac{h}{\beta}(1-\beta)a(T-t)\mathcal{I}(hy\xi_0)\left(\Phi\left(-d_1(\zeta^*)\right) + \Phi\left(-d_1(\overline{\zeta})\right) - \Phi\left(-d_1(\underline{\zeta})\right)\right)
\end{aligned}
$$

and the value of total assets

$$
\begin{aligned}
V_t &= A(T-t)\mathcal{I}(y\xi_t) + a(T-t)\mathcal{I}(y\xi_t) \\
&\quad * \left(\frac{(\Phi(-d_1(\zeta^*)) + \Phi(-d_1(\overline{\zeta})) - \Phi(-d_1(\underline{\zeta})))h^{\frac{\gamma-1}{\gamma}}}{\beta} - \Phi(-d_1(\zeta_*)) + 1\right) \\
&\quad + \frac{e^{r(t-T)}F(-\Phi(-d_2(\zeta_*))\beta + \beta + h\Phi(-d_2(\zeta^*)) - 1)}{\beta - 1} \\
&\quad + \frac{e^{r(t-T)}h\underline{W}(\Phi(-d_2(\underline{\zeta})) - \Phi(-d_2(\overline{\zeta})))}{\beta}
\end{aligned} \tag{3.41}
$$

Useful Facts for Applications:

The most important partial derivatives used are

$$
\begin{aligned}
\zeta^{*(1,0)}(y,F) &\rightarrow -\frac{\zeta^*}{y} \\
\zeta^{*(0,1)}(y,F) &\rightarrow -\frac{\gamma\zeta^*}{F} \\
\underline{\zeta}^{(1,0)}(y,F) &\rightarrow -\frac{\underline{\zeta}}{y} \\
\underline{\zeta}^{(0,1)}(y,F) &\rightarrow -\frac{\gamma\underline{\zeta}}{F} \\
\bar{\zeta}'(\alpha) &\rightarrow -\frac{e^{rT}\sqrt{T}\kappa\bar{\zeta}^2}{\xi_0\phi\left(-d_2\left(\xi_0, T, \bar{\zeta}\right)\right)}\ .
\end{aligned} \tag{3.42}
$$

**Lemma 2:** For the function

$$H_1(y,F) = \frac{e^{-rT} F\beta}{1-\beta} - C_0(y,F),$$

where $C_0$ is the price of a call option on the underlying $\mathcal{I}(y\xi_T)$ with strike $\underline{W}$,

$$C_0(y,F) = \mathcal{I}(y\xi_0)a(T)\left(1 - \Phi\left(-d_1(\zeta_*)\right)\right) + \frac{e^{-rT} F\beta\Phi\left(-d_2(\zeta_*)\right)}{1-\beta} \ ,$$

the following relation holds under the budget restriction

$$H_1 < 0 \ , \quad \forall (F, y(F)) \ . \tag{3.43}$$

**Proof:** The function $H_1$ is strictly increasing in $F$

$$\begin{aligned} H_1^{(1,0)}(y,F)y'(F) + H_1^{(0,1)}(y,F) &= (\mathcal{I}(y\xi_0)a(T)(1-\Phi(-d_1(\zeta_*))))\frac{y'(F)}{\gamma y} \\ &\quad + e^{-rT}\frac{\beta}{1-\beta}(1-\Phi(-d_2(\zeta_*))) \\ &> 0 \end{aligned}$$

and has the limits, as $F \to 0$, $H_1(y(0),0) = -a(T)\mathcal{I}(y^u\xi_0) < 0$ and, as $F \to \hat{F}$, $H_1(y(\hat{F}),\hat{F}) = 0$. Thus, $H_1 < 0$ holds.

□

**Lemma 3:** For the function

$$H_2(y,F) = h\beta\Phi(-d_2(\zeta^*)) - \beta\Phi(-d_2(\zeta_*)) + h\beta\delta(\Phi(-d_2(\overline{\zeta})) - \Phi(-d_2(\underline{\zeta})))$$

the following relation holds under the budget constraint

$$-e^{rT}(1-\beta)\frac{W_0}{F} < H_2 < 0 \ \forall (F, y(F)) \ . \tag{3.44}$$

**Proof:** From the previous analysis, the property $\frac{\partial W_0(y,F)}{\partial F} > 0$ is known. Inserting $H_2$ we obtain

$$\frac{\partial W_0(y,F)}{\partial F} = -H_2 \frac{e^{-rT}}{1-\beta} > 0$$

hence, $H_2 < 0$.

Furthermore, since $y'(F) = -\frac{\partial W_0(y,F)}{\partial F} / \frac{\partial W_0(y,F)}{\partial y} > 0$, we obtain

$$y'(F) = -\frac{H_2 y\gamma}{H_2 F - e^{rT} W_0(\beta - 1)} > 0 \; .$$

Hence, $H_2 > -\frac{e^{rT} W_0(1-\beta)}{F}$.

□

With these substitutions the partial derivatives of equity and debt value are

$$\begin{aligned}
{W_0}^{(0,1)}(y,F) &= \frac{e^{-rT} H_2}{\beta - 1} > 0 \\
{W_0}^{(1,0)}(y,F) &= \frac{e^{-rT} F H_2}{y(\beta-1)\gamma} - \frac{W_0}{y\gamma} < 0 && (3.45) \\
{W_0}^{(0,1)}(y,\alpha) &= \frac{F h \beta \bar{\zeta}\,(\psi - \psi_2)}{(\beta-1)\xi_0} < 0 \\
{D_0}^{(0,1)}(y,F) &= \frac{e^{-rT}\left(H_2 + \beta\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right)\right)}{\beta} > 0 \\
{D_0}^{(1,0)}(y,F) &= \frac{(\beta-1)H_1}{y\beta\gamma} - \frac{e^{-rT} F\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right)}{y\gamma} + \frac{(1-\beta){W_0}^{(1,0)}(y,F)}{\beta} < 0 \\
{D_0}^{(0,1)}(y,\alpha) &= -\frac{F h \bar{\zeta}\,(\psi - \psi_2)}{\xi_0} < 0 \\
y'(F) &= -\frac{{W_0}^{(0,1)}(y,F)}{{W_0}^{(1,0)}(y,F)} > 0 \; . && (3.46)
\end{aligned}$$

In case ${D_0}^{(1,0)}(y,F)$, the sign cannot be directly seen from the formula itself; it is shown in the the derivation in equation (4.7) on p. 160. However, it can be also easily deducted from the time $T$ solution of the problem.

### 3.4.4 Proofs of the Results in Partial Equilibrium

RESTRICTIVENESS

By evaluating the feasibility constraint (3.17), under the restriction $W_0^i = W_0^b = W_0$, we obtain the *debt capacity*

$$\hat{F}^i \stackrel{(r,\kappa)}{=} \frac{e^{rT} W_0(\beta(1-\lambda)+\lambda)}{\beta\lambda}$$

and

$$\begin{aligned} \hat{F}^b &= \frac{W_0(\beta-1)(\lambda\beta-\beta-\lambda)}{\beta(\mathbf{E}[\frac{\xi_T}{\xi_0}\mathbf{1}_{\{\xi_T\leq\overline{\zeta}\}}]\beta\psi(n)-\beta\lambda+\lambda)} \\ &\stackrel{(r,\kappa)}{=} e^{rT} W_0 \frac{(1-\beta)(\beta(\lambda-1)-\lambda)}{\beta((\beta-1)\lambda-\beta\psi(n)\Phi(d_2(\overline{\zeta})))} \end{aligned}$$

Further properties of the debt capacity are

$$\hat{F}^b \in [\frac{e^{rT}W_0(1-\beta)}{\beta}, \hat{F}^i] \ ,$$

and

$$\begin{aligned} \hat{F}^i &= \hat{F}^b(n=0,\alpha=1) \\ &= \hat{F}^b(n=0) = \hat{F}^b(\alpha=1) = \hat{F}^b(\lambda=0) \\ \hat{F}_b'(n) &< 0 \\ \hat{F}_b'(\alpha) &> 0 \\ \hat{F}^b(n=\beta,\alpha=0) &= \tfrac{e^{rT}W_0(1-\beta)}{\beta} \ , \end{aligned}$$

in particular, $\hat{F}^b < \hat{F}^i$ holds, unless $n=0$ or $\alpha=1$, i.e. under no regulation.

The *transition point* $(\vec{F}, \vec{y})$ is defined as the tuple, where the VaR starts to bind and the budget equation is fulfilled, i.e. the solution of the system

$$\left\{ \begin{array}{rcr} \mathcal{I}(hy\overline{\zeta}) &=& \psi(n)\frac{\beta}{1-\beta}F \\ \mathcal{B}(y,F) &=& W_0 \end{array} \right\}$$

The system can be successively solved for the solution.

In order to show $\vec{F} < \hat{F}^b$, we address the problem more generally. Let us define $y_1 = u'(\underline{W})/(hy)$; because *(i)* $\lim_{F\to 0} y_1 = \infty$, $y_1'(F) < 0$, and $\lim_{F\to\infty} y_1 = 0$, and $(ii)$ - with the previous results from the budget equation $\mathcal{B}(y,F) = \xi_0 W_0^b$ - $y(0) > 0, y'(F) > 0$, and $\lim_{F\to\hat{F}} y(F) = \infty$, there exists a unique solution to the system $(\vec{y}, \vec{F})$. In particular, $\vec{F} < \hat{F}^b$ holds.

Capital Provisions

Because $y^b(F) > y^i(F)$ for $F > \vec{F}$, $\frac{\partial W_0^i}{\partial y^i} < 0$, and $W_0^i > \check{W}$, we obtain

$$CP \in (1,0) \ .$$

Define stricter regulation by $\chi \in \{-\alpha, n\}$. Then,

$$\frac{dCP}{d\chi} = -\underbrace{\frac{(W_0^i)'(y^b)}{W_0^i}}_{<0} y_b'(\chi) > 0.$$

holds.

Using a simplified notation with $\frac{\partial W_0^i}{\partial x} = W_x^i$, we obtain the result for increasing nominal debt

$$\begin{aligned} \frac{dCP}{dF} &= -\frac{1}{W^i}\left(W_y^i y_b'(F) + W_F^i\right) \\ &= \frac{W_F^i}{W^i}\left(\left(-\frac{W_y^i}{W_F^i}\right) y_b'(F) - 1\right) \\ &= \underbrace{\frac{W_F^i}{W^i}}_{>0} \underbrace{\left(\frac{y_b'(F)}{y_i'(F)} - 1\right)}_{>0} > 0 \quad (F > \vec{F}) \ . \end{aligned}$$

### Debt Markets

The comparative statics of the *Probability of Distress* is

$$
\begin{aligned}
\frac{dPD_t}{d\alpha} &= \frac{d\mathbf{P}_t[D_T < F]}{d\alpha} = \frac{d\Phi\left(d_0\left(\zeta^*(y(\alpha))\right)\right)}{d\alpha} = \frac{\phi(-d_0(\zeta^*))}{\sqrt{T-t}y\kappa}\, y'(\alpha) < 0 \\
\frac{dPD_t}{dn} &= \frac{d\Phi\left(d_0\left(\zeta^*(y(n))\right)\right)}{d\alpha} = \frac{\phi(-d_0(\zeta^*))}{\sqrt{T-t}y\kappa}\, y'(n) > 0 \\
\frac{dPD_t}{dF} &= \frac{d\Phi\left(d_0\left(\zeta^*(y(F),F)\right)\right)}{d\alpha} = \frac{\phi(-d_0(\zeta^*))}{\sqrt{T-t}y\kappa}\, (y\gamma + Fy'(F)) > 0 \\
\frac{dPD_t}{d\lambda} &= \frac{d\Phi\left(d_0\left(\zeta^*(y(\lambda))\right)\right)}{d\alpha} = \frac{\phi(-d_0(\zeta^*))}{\sqrt{T-t}y\kappa}\, y'(\lambda) < 0\ .
\end{aligned}
$$

For the *spread* $s_0$, see equation (3.27), we obtain the analogous results.

$$
\begin{aligned}
\frac{ds_0}{d\alpha} &= \frac{e^{(r+s_0)T}\beta(\psi-\psi_2)\overline{\zeta}\left(-H_1 e^{rT}(\beta-1) - F\beta + F\beta\Phi(-d_2(\zeta_*))\right)}{T\left(H_2F - e^{rT}W_0(\beta-1)\right)\left(\beta(\lambda-1)-\lambda\right)\xi_0} < 0 \\
\frac{ds_0}{dF} &= \frac{1}{F}\Bigg(1 - \frac{F\beta(\psi-\psi_2)\overline{\zeta}\left(-H_1 e^{rT}(\beta-1) - F\beta + F\beta\Phi(-d_2(\zeta_*))\right)}{\left(H_2F - e^{rT}W_0(\beta-1)\right)\left(\beta(\lambda-1)-\lambda\right)\xi_0}\Bigg) > 0\ .
\end{aligned}
$$

Define the *Change of Measure* for the probability of distress by

$$
\Delta PD_t = PD_t^{\mathbf{Q}} - PD_t^{\mathbf{P}} = \Phi\left(d_2\left(\zeta^*(y)\right)\right) - \Phi\left(d_0\left(\zeta^*(y)\right)\right)\ .
$$

The first order condition $\frac{d\Delta PD_t}{d\xi} = 0$ shows, the maximum is attained at $\hat{\xi}(t) = e^{-r(T-t)}\zeta^*$. Furthermore, $\frac{d\Delta PD_t|_{\hat{\xi}(t)}}{dy} = 0$ holds, i.e. the size of the (maximum) change of measure is not changed by regulation. However, because the structural form is the same and $\hat{\xi}^i(t) > \hat{\xi}^b(t)$, the change of measure is shifted to the left due to regulation. By calculating

$$
\frac{\frac{d\Delta PD_t|_{\xi=\hat{\xi}\varepsilon}}{d\varepsilon}}{\frac{d\Delta PD_t|_{\xi=\hat{\xi}/\varepsilon}}{d\varepsilon}} = -1
$$

we additionally obtain the property that the change of measure is increasing faster when $\xi_t$ is raising from a low level, than increasing when $\xi_t$ is falling from a high level.

### 3.4.5 Deduction of the Portfolio Decision

In this section we derive the portfolio decision along the lines presented in Chapter 2. As a building block, the Portfolio Insurance example of Section 3.4.3 is revisited. The diffusion part of any wealth dynamics follows $\mathcal{W}_t\theta_t\sigma_t dw_t$. On the other hand, by using Ito's lemma, the same diffusion part is $\frac{\partial\mathcal{W}(\xi_t,t)}{\partial\xi}(-\kappa\xi_t)dw_t$. Thus one obtains

$$\begin{aligned}\theta_t &= \frac{1}{\sigma_t}\cdot\frac{\frac{\partial\mathcal{W}(\xi_t,t)}{\partial\xi}(-\kappa\xi_t)}{\mathcal{W}(\xi_t,t)}\\ &= \frac{1}{\gamma\sigma_t}\kappa\cdot\big(-\gamma\frac{\partial\mathcal{W}}{\partial\xi}/\frac{\mathcal{W}}{\xi_t}\big)=\theta^u\cdot q_t^{\mathcal{W}}\end{aligned}$$

By this decomposition, one can easily see that the deduced portfolio policy is to invest into a stochastic multiple of the myopic mean-variance portfolio.

The partial derivatives with respect to the state price $\xi$ of these two functional types are

$$\begin{aligned}\frac{\partial\mathcal{W}_2}{\partial\xi} &= e^{-r(T-t)}F\frac{\phi(-d_2(\xi_t,T-t,\zeta))}{\sqrt{T-t}\kappa\xi_t}\\ \frac{\partial\mathcal{W}_1}{\partial\xi} &= -\mathcal{I}(y\xi_t)a(T-t)\Phi(d_1(\xi_t,T-t,\zeta))\frac{1}{\gamma\xi_t}-e^{-r(T-t)}\mathcal{I}(y\zeta)\frac{\phi(-d_2(\xi_t,T-t,\zeta))}{\sqrt{T-t}\kappa\xi_t}\ .\end{aligned}$$

Adding both parts we obtain as the multiplier

$$\begin{aligned}q_t^{\mathcal{W}} &= 1-e^{-r(T-t)}\frac{F}{\mathcal{W}_t}\Phi(d_2(\xi_t,T-t,\zeta)))\\ &\quad +(e^{-r(T-t)}\frac{\mathcal{I}(y\zeta)}{\mathcal{W}_t}\left(1-\frac{F}{\mathcal{I}(y\zeta)}\right)\frac{\gamma\phi(-d_2(\xi_t,T-t,\zeta))}{\sqrt{T-t}\kappa}\ .\end{aligned}$$

If the solution is continuous, then, by definition, $\mathcal{I}(y\zeta)=K$ and thus

$$q_t^{\mathcal{W}}=1-e^{-r(T-t)}\frac{F}{\mathcal{W}_t}\Phi(d_2(\xi_t,T-t,\zeta))\in(0,1)$$

holds. Otherwise, if there is an downward discontinuity, there exist cases depending on the state $(\xi,t)$, where the volatility multiplier $q$ is even greater than 1. An analogous derivation is applicable for more complex solutions.

# Chapter 4

# The Equilibrium Impact of a Regulated Banking Sector on Financial Markets

In equilibrium, the aggregated decisions of agents influence market prices and thus indirectly affect other market participants as well. In particular, the market value of aggregate debt to the real sector is endogenous, but also equity and banks' debt prices and volatilities.

When all agents are fully rational, they are able to anticipate all of these effects. However, if this endogeneity of prices cannot be properly 'predicted' by market participants, their decisions can be heavily biased. Prediction is not meant with respect to the underlying economic uncertainty as represented by the Brownian motion, but with respect to the indirect impact of the restriction on market prices. When, as an example, a company of the real sector plans its financial decision purely based on market prices, but is not able to anticipate the effect of the VaR restriction correctly, its 'wrong' state-contingent decision may lead to procyclicality in the economy.

## 4.1 Construction of the Pure Exchange Equilibrium

The influence of the aggregate banking sector is not of a strategic nature, as all banking institutes within the banking sector will not take the impact of their own decision on the decision of other banks into account. Therefore, we discuss the market in a competitive pure exchange equilibrium in the style of Lucas (1978).
First, we state the equilibrium condition and then solve for the resulting equilibrium. We characterize the investment opportunity set in equilibrium and discuss prices and volatility.

### 4.1.1 Definition and Existence

Produced by the real sector of the economy, there is an exogenously given cash flow from the coupons, $\delta$, which follows a geometric Brownian motion

$$\frac{d\delta_t}{\delta_t} = \mu_\delta dt + \sigma_\delta dw_t \ . \tag{4.1}$$

The constant $\mu_\delta$ is the instantaneous growth rate of cash-flows, and $\sigma_\delta$ its volatility. The process starts at a known $\delta_0 > 0$.
As in the previous chapter, this cash-flow represents aggregate coupon payments of the aggregate debt to the real economy, the risky asset in the economy. It has fair value $P$, paying coupon stream $\delta$. Additionally, there exists a riskless investment opportunity, the money market account, which is in zero net supply.
The unrestricted agent $u$ is endowed with a fraction $\omega \in [0, 1]$ of aggregate debt.

**Remark.** *Specifying the cash-flows exogenously implies that there are no feed back effects between the evolvements on the financial side of the economy, such as asset prices, and the real one, as expressed by the amount of cash-flows. If this attitude is supported, the following model constitutes a general equilibrium.*
*This independency might not be a realistic setting, especially when considering times*

*of economic distress. With this argumentative viewpoint, the model is still a partial equilibrium, since a feedback of a possibly occurring credit crunch in the real economy will not affect asset prices on the financial market. Yet, this modeling technique permits the analysis in equilibrium where, at least, the direct effect of regulation on the financial markets can be studied.*

### Definition of Equilibrium

The definition of equilibrium in this pure exchange economy is a collection of optimal consumption, optimal terminal wealth, and optimal total assets of the banking sector $(c^u, c^b, W^u, V)$ and the investment opportunity set $(r, \mu, \sigma)$, such that the markets for consumption, for the risky asset, and for the money market account clear any time $t$, i.e. recalling the market clearing conditions (2.33) on p. 38,

$$\begin{aligned} c_t^u + c_t^b &= \delta_t \\ \theta_t^u W_t^u + \theta_t^b V_t &= P_t \\ (1-\theta_t^u) W_t^u + (1-\theta_t^b) V_t &= 0 \quad , \end{aligned}$$

where $\theta^b$ is the optimal portfolio decision of the banking sector $b$ and where $P$ is the total (financial) value of the cash-flow-producing real economy, see (2.10) on p. 18,

$$\xi_t P_t = \mathbf{E}_t \left[ \int_t^T \xi_s \delta_s ds + \xi_T P_T \right] .$$

Analogously, equilibrium in an unregulated economy is defined by replacing the banking sector $b$ with the unregulated financial system $i$.

### Existence

By the arguments presented in Chapter 2, if there exists a state price process $\xi$, satisfying

$$\delta_t \quad = \quad \mathcal{I}(y^u \xi_t) + \mathcal{I}(y^b \xi_t), \quad \forall t \ ,$$

where $\{y^b > 0, y^u > 0\}$ solve the static budget equations with the optimal solutions $(c^u, W^u)$ (equation (3.10) on p. 71) and $(c^b, W^b)$ (equation (3.12) on p. 71) substituted,

$$\begin{aligned}\mathbf{E}\left[\int_0^T \xi_s c_s^u(y^u)ds + \xi_T W_T^u(y^u)\right] &= \omega\xi_0\mathbf{E}_0[\int_0^T \xi_s\delta_s ds + \xi_T P_T] \\ \mathbf{E}\left[\int_0^T \xi_s c_s^b(y^b)ds + \xi_T(W_T^b(y^b) + C(W_T^b(y^b))\right] &= (1-\omega)\xi_0\mathbf{E}_0[\int_0^T \xi_s\delta_s ds + \xi_T P_T]\ ,\end{aligned} \tag{4.2}$$

then all equilibrium conditions (2.33) on p. 38 are satisfied.

Inverting the unique alternative equilibrium representation, the state prices are given by

$$\xi_t = \frac{u'(\delta_t)}{u'(\mathcal{I}(y^u) + \mathcal{I}(y^b))}\ . \tag{4.3}$$

Normalizing $u'(\mathcal{I}(y^u) + \mathcal{I}(y^b)) = 1$, substituting into optimal consumption and terminal wealth, the expectations in the static budget equations (4.2) are well defined for all $(y^b > 0, y^u > 0)$.

There exists a solution $(y^b, y^u)$ to the static budget equations (4.2), if the endogenous equity value of the aggregate banking sector $W_0^b$ is sufficient to support the nominal debt level $F$; formulated as a restriction on the exogenous parameters, there exists an equilibrium $\omega \in [0, \hat{\omega})$, where

$$\frac{1}{\hat{\omega}} = 1 + \frac{e^{-rT}hF\Big(-\beta\lambda + \lambda + \beta\psi(n)\Phi\left(d_2\left(\bar{\zeta}\right)\right)\Big)}{(a(T) + A(T))\delta_0\big((1-\beta)\beta\big)}\ .$$

The proof can be found in the Appendix.

**Remark.** *We introduce some notation to facilitate comparisons. The letter $B$ or $\mathcal{B}$ in super/subscripts refers to a regulated economy. It comprises the regulated banking system $b$ and the unrestricted representative investor $u$. $I$ or $\mathcal{I}$ denotes an economy with an unrestricted investor $u$ and the unregulated financial intermediary $i$. Pure exchange equilibria are calligraphic $\mathcal{B}$ or $\mathcal{I}$. If one takes the equity value $W_0^u$ and $W_0^b$, resp. $W_0^i$, in the static budget equations (3.11) and (3.16) as exogenously fixed, then we refer to this case as a partial equilibrium $B$, resp. $I$, in standard typeface.*

### 4.1.2 The Investment Opportunity Set in Equilibrium

The endogenous interest rate $r_t$ and the market price of risk $\kappa_t = \frac{\mu_t - r_t}{\sigma_t}$ can be recovered by applying Ito's lemma to the inverted state price in equation (4.3) and comparing coefficients with the dynamics in (2.6) on p. 16, to obtain

$$\begin{aligned} r &= \gamma\mu_\delta - \tfrac{1}{2}\gamma(1+\gamma)\sigma_\delta^2 \\ \kappa &= \gamma|\sigma_\delta| \end{aligned} \tag{4.4}$$

which shows that both the interest rate and market price of risk are constants. The stochastic discount factor $\xi$ is independent of the aggregate behaviour of agents and initial endowments. It only depends on risk aversion as well as the growth and volatility of the aggregate coupon payments in the economy.

Consequently, the same endogenous riskless interest rate and market price of risk will prevail, if there is only an unrestricted investor, or some unregulated financial intermediaries, or a regulated banking system. Neither the introduction nor a change in the weight of the banking system hence induces a 'valuation risk' in the sense that there is a change in the state price process. All the observed effects can be attributed to a 'cash-flow risk'. This separation is on the one hand convenient and enables us, to solve in closed form; on the other hand, there will be no additional ambiguous effects resulting from distortion due to valuation.

This valuation invariance is the implication of the specific exogenously given aggregate cash flows, of the aggregation property of the utility functions, and of complete markets.

By evaluating $P_t$ explicitly and applying Ito's lemma, one can derive from the dynamics

$$dP_t + \delta_t dt = \mu_t P_t dt + \sigma_t P_t dw_t$$

the instantaneous expected return $\mu_t$ of the market in equilibrium

$$\mu_t = r + \sigma_t \cdot \kappa$$

and the volatility $\sigma_t$ of the market in equilibrium

$$
\begin{aligned}
\sigma_t \quad = \quad & \underbrace{\sigma_\delta - \sigma_\delta \, e^{-r(T-t)} \frac{h}{\beta} \frac{W_*}{P_t} \left( \frac{(1-\beta)}{\beta} + \Phi\left(-d_2(\delta_*)\right) - h \Phi\left(-d_2(\delta^*)\right) \right)}_{(i)} \\
& \underbrace{+ \sigma_\delta \, \frac{h}{\beta} e^{-r(T-t)} \frac{\overline{W}}{P_t} \left( \Phi\left(-d_2(\underline{\delta})\right) - \Phi\left(-d_2(\overline{\delta})\right) \right)}_{(ii)} \qquad (4.5) \\
& \underbrace{+ e^{-r(T-t)} \frac{h}{\beta} \frac{J}{P_t \sqrt{T-t}} \;\; \phi(-d_2(\underline{\delta}))}_{(iii)} \; .
\end{aligned}
$$

The transformation of the boundaries on the level of state prices

$$\{\zeta_*, \zeta^*, \overline{\zeta}, \underline{\zeta}\}$$

into the corresponding boundaries on the level of coupon payments

$$\{\delta^*, \delta_*, \underline{\delta}, \overline{\delta}\}$$

are given in the Appendix.

The solution to the unregulated economy $\mathcal{I}$ is identical to the first term $(i)$ of equation (4.5). It can be shown to be strictly less than $\sigma_\delta$. In a world without any financial intermediary, regulated or not, the volatility of the market is $\sigma_t = \sigma_\delta$.

**Remark.** *As the banking systems needs enough equity capital to support the debt, see equation ((3.23) on p. 90), the case $\omega = 1$ is in general not nested. However, the solution of an economy without any financial intermediation can be found in equation (2.50) on p. 47.*

The market volatility of the regulated economy $\sigma_t$ in equation (4.5) retains the same structure as the implied risk aversion $q_t^b$ in (3.29) on p. 101 of the regulated banking system. Hence, if the VaR restriction is binding, volatility inherits the same properties. Therefore the arguments made in partial equilibrium are qualitatively still valid; VaR regulation induces in some states of the economy higher volatility relative to an unregulated banking sector.The effects, however, are smoothed by the presence of the unrestricted investor. The main difference to the formula of the implied risk aversion is that $V_t$ is replaced by $P_t = W_t^u + V_t > V_t$. Thus, the impact of the banking sector on volatility is diluted due to the additional unrestricted capital $W^u$.

**Remark.** *Given that the underlying economy is not influenced by the financial market, i.e. $\sigma_\delta$ remains constant, this derivation formalizes the 'market wisdom' that volatility is an indicator of how 'nervous' the market is. Buraschi and Jiltsov (2006) derive a similar result from a differences in beliefs aproach and empirically finds that volatility is increasing in heterogeneity. In our model framework, volatility is mainly driven by implied risk aversion of the banking sector $q^b$, which can be, loosely speaking, seen as another measure of how 'nervous' the banking sector is.*

**Remark.** *Relating to equation (2.55) on p. 51, the equilibrium investment opportunity set is stochastic. When discussing the effects of regulation, some authors assume a constant opportunity set for total assets. This assumptions only sustains a discussion in a partial equilibrium, see Dangl and Lehar (2004), Kupiec (2007) or Bodie et al. (2007) as examples.*

### 4.1.3 Asset Prices and Volatility in Equilibrium

After deriving the equilibrium prices of aggregate debt to the real sector $P$, market prices for total assets, equity, and debt as well as their volatilities can be analyzed as well.

The Banking Sector's Balance Sheet in Equilibrium: Value

Figure 4.1 shows the balance sheet of the banking sector, where dotted lines show the comparison with the case of an unregulated economy. Note that the scaled cash-flows $\delta_t/\delta_0$ are proportional to the value of the underlying 'mutual' fund of the decomposition (3.20) on p. 76.

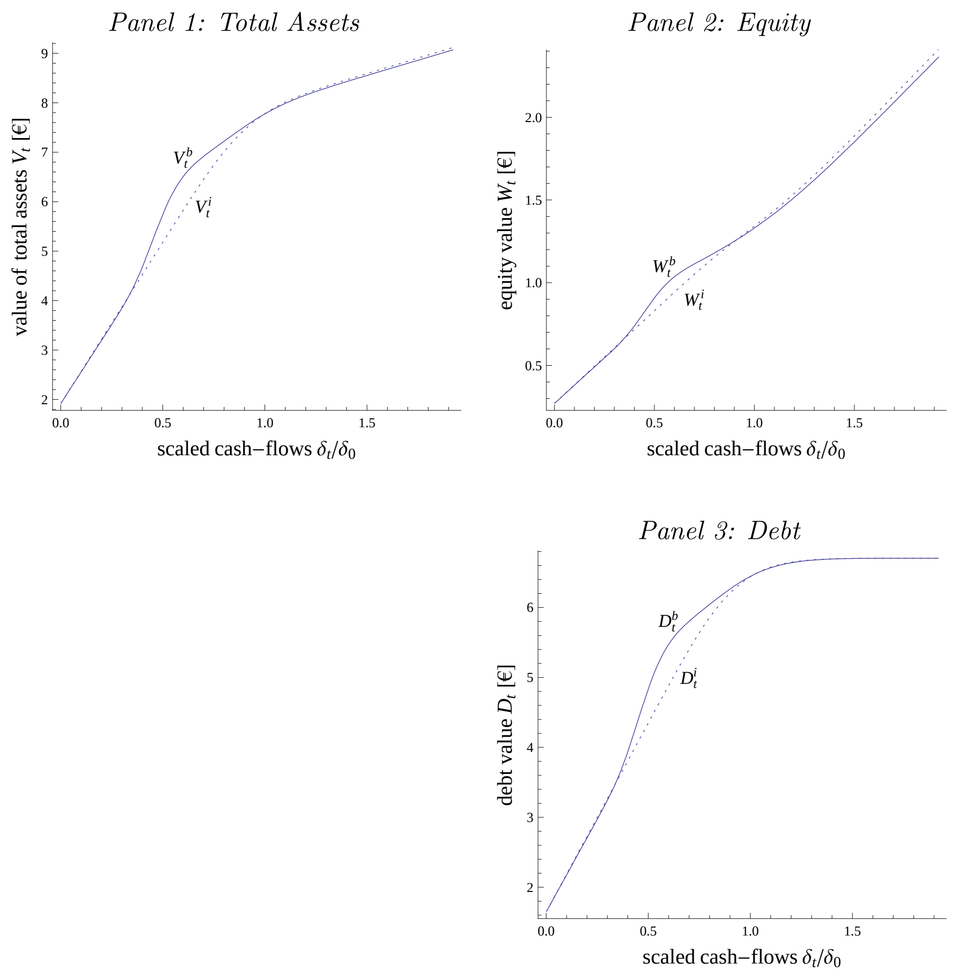


Figure 4.1: **Equilibrium Balance Sheet of the Banking Sector: Value**

*This figure shows the banking sector's balance sheet in value at time $t = 4$ in equilibrium. The dotted lines represents the unregulated financial intermediary $i$, whereas the solid lines the one of the regulated banking system $b$. The parameters are as defined in the standard parameter set in Table 3.2.*

In both economies, the regulated and the unregulated one, there is a minimum value if the economy almost breaks down, $\delta_t \to 0$. If the economic situation turns out to be very favorable, equity profits one-to-one from increases in the total asset, whereas debt is almost riskless.

In the unregulated economy $\mathcal{I}$, the equity value is almost linear in between and the debt title behaves as in a Merton (1974) type of economy. In contrast to the unregulated balance sheets, the VaR restriction introduces a dent in the total assets, which distributes to equity and debts as well.

In consequence, the equity position (or equivalently total assets) is higher under regulation in economically worse times, whereas it is less in good times, as can be seen by the small difference in the slopes of the total assets or equity in very good economic situations. This transfer of wealth between states of the economy in equilibrium can be seen as evidence that the regulation of the banking sector renders the economy in fact less prone to a credit crunch relative to the corresponding unregulated economy.

### The Banking Sector's Balance Sheet in Equilibrium: Volatilities

Figure 4.2 shows the balance sheet from the perspective of volatilities; the 'budget' equation in volatility terms is $\sigma_t^V V_t = \sigma_t^W W_t + \sigma_t^D D_t$.

The reasoning regarding the structural form of volatility again follows a similar argument as in the discussion of the implied risk aversion. In very adverse economic situations, the portfolio decision resembles that of a portfolio insurance. This behaviour transmits through the balance sheet, as all volatilities approach zero. In favorable economic devolvements, equity volatility is approximately $\sigma_\delta$, whereas debt, as it is now almost risk-free, has no volatility any more. The volatility of the total assets in this part is approximately $\sigma_t^V \simeq \frac{W_t}{V_t}\sigma_\delta$.

In the unregulated economy $\mathcal{I}$, there is a hump in volatility in between those extremes which transmits mainly into the debt title; this also applies to the volatility effect of the VaR restriction.

This results in a situation, where - in the numerical example - debt holders face

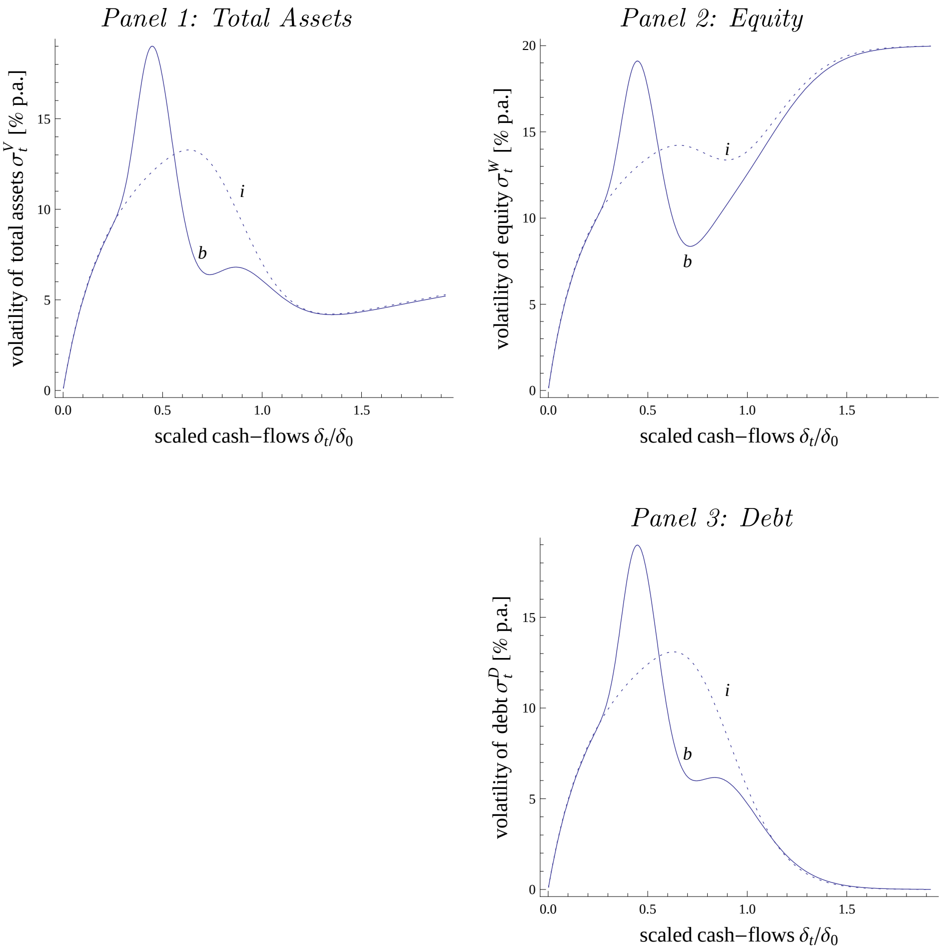


Figure 4.2: **Equilibrium Balance Sheet of the Banking Sector: Volatility**

*This figure shows the banking sector's balance sheet in volatility terms at time $t = 4$ in equilibrium. The dotted lines represent the unregulated financial intermediary $i$, whereas the solid lines the one of the regulated banking system $b$. The parameters are as defined in the standard parameter set in Table 3.2.*

in some states of the economy, namely around the VaR boundary $\underline{\delta}$, a volatility of almost 20%, which is not only substantially higher than in an unregulated economy $\mathcal{I}$, but is in fact comparable to the volatility in a totally un-intermediated economy or to the volatility of equity in a regulated economy. If regulation is also intended to protect debt holders, regulation fails at this point.

**Remark.** *Note that the volatility of equity (or any other part of the balance sheet) seen as a function of equity value is non-linear and also non-invertible, as there are multiple equity values, which result in the same volatility. Both, volatility and the volatility of volatility, are stochastic.*
*Another notable feature of the model is that decreasing prices correspond to increasing volatilities, when the financial system reaches a region where distress becomes probable, i.e. where leverage is quickly increasing. Thus, the model reproduces the well-documented empirical fact that deteriorating economic development leads to higher volatilities as expressed for example by the VIX. Moreover, volatility of volatility is countercyclical as well, see Jones (2003) and Corsi et al. (2008).*
*When crisis is actually almost inevitable, the banking system substantially reduces risky investments in order to stabilize the asset side of its balance sheet. Not even in this simple economy is a decreasing volatility after some time of distress with high volatility an unmistakable sign of a recovering economy.*

**Remark.** *This remark briefly discusses the impact of the fraction of intermediated capital $\omega$ on volatility.*
*When, as an example, the banking system is the dominant factor, $\omega \to 0$, the risky asset only comprises the total assets of the banking system, $P = V$, and the implied risk aversion of the banking system translates one-to-one into market volatility. However, the specific form of volatility is dependent on whether the VaR restriction is binding or not. When increasing the initial endowment $\omega$ of the unrestricted investor u, the endogenously determined equity capital of the banking sector $W^b$ decreases; thus, the typical VaR behaviour becomes more pronounced when seen relative to the unregulated economy $\mathcal{I}$. The bigger the share of the banking sector, the smaller the*

*effect of VaR. This possibly counterintuitive fact is generated by keeping nominal debt $F$ constant, while at the same time increasing equity capital that supports this debt, thereby effectively reducing the leverage. A higher share of intermediated capital, regulated or not, benefits financial stability, if not combined at the same time with a higher leverage of the economy.*

## 4.2 Comparing Partial and Pure Exchange Equilibria

In this section, the impact of regulation on prices and their implicit impact through wealth transfers is discussed. For comparison, there are two equilibria of interest: the pure exchange equilibrium $\mathcal{B}$ with a regulated banking system and the pure exchange equilibrium $\mathcal{I}$ with an unregulated financial intermediary. When comparing the benchmark case $\mathcal{I}$ with $\mathcal{B}$, differences resulting from regulation are of interest, taking all endogenous price impacts into account.
In addition, a specific partial equilibrium is of special interest: $B$ is defined by keeping equity prices fixed to the benchmark case $\mathcal{I}$, i.e.

$$\begin{aligned} W_0^u|_B &= W_0^u|_{\mathcal{I}} \\ W_0^b|_B &= W_0^i|_{\mathcal{I}} \, . \end{aligned}$$

Using this equilibrium, the differences in equilibria $\mathcal{I}$ and $\mathcal{B}$ can be decomposed into two parts: when comparing $\mathcal{I}$ with $B$, the introduction of regulation is of interest, while keeping equity prices fixed. The comparison of economy $B$ with economy $\mathcal{B}$ isolates the effects of regulation on prices since, in both economies, regulation does not change.

First we discuss in which way equilibrium effects the demand for the risky asset. Afterwards, it is argued that the endogeneity of prices effectively results in a transfer of wealth between the participating members of the economy.
Finally, reflecting the discussion under partial equilibrium, differences attributable

to the endogenous price reaction with respect to the restrictiveness of regulation, capital provisions and the banks' debt markets are briefly discussed.

Aggregate Demand

Equilibrium does not *directly* change the agents' decisions. This is an implication of $(i)$ competitive markets, because agents take prices in their decision as given, as well as $(ii)$ the valuation invariance characteristic, because the VaR boundary itself is independent of prices. Hence, the demand for the risky asset is only affected *indirectly* through the endogeneity of prices. Equilibrium modifies financing costs, as expressed by the Lagrange multiplier $y^b$.

**Remark.** *The normalization $\mathcal{I}(y^b) + \mathcal{I}(y^u) = 1$ allows us to reduce the system of equations (4.2) to a single equation. Only the Lagrange multiplier of the banking system $y^b$ needs to be solved for; the one of the unrestricted agent follows from the normalization. In the following, we simplify notation by using $y$ instead of $y^b$.*

It is shown in the Appendix that $y|_{\mathcal{B}} \geq y|_B \geq y^i|_{\mathcal{I}}$, where equality holds only for an effectively unregulated economy. Consequently, the risky asset is in equilibrium less valuable than in the corresponding equilibrium with fixed prices, $P_0|_{\mathcal{B}} \leq P_0|_B$, as $\frac{dP}{dy} < 0$. This result demonstrates that there is an increased aggregate demand due to regulation as well as due to the endogeneity of prices. Furthermore, as $\frac{dW_0^b}{dy} < 0$, the equity value of the banking sector is less in equilibrium, while at the same time the wealth of the unrestricted agent increases $\frac{dW_0^u}{dy} > 0$. Hence, equilibrium shifts wealth between agents.

| | $\mathcal{I} \to B$ | $\mathcal{I} \to \mathcal{B}$ |
|---|---|---|
| $\Delta W_0^u / \Delta P_0$ | 0 | $\omega > 0$ |
| $\Delta W_0^b / \Delta P_0$ | 0 | $-\epsilon < 0$ |
| $\Delta D_0 / \Delta P_0$ | 1 | $0 < 1 - \omega + \epsilon < 1$ |
| $\sum$ | 1 | 1 |

Table 4.1: **Wealth Transfers due to Endogenous Prices**

*This table reports the wealth transfers' effect that arise if regulation becomes stricter, i.e. $\Delta P_0 > 0$, assuming VaR is effectively restricting . $\mathcal{I} \to B$ denotes the case where regulation is introduced, yet, prices are fixed, whereas $\mathcal{I} \to \mathcal{B}$ denotes the case, where regulation is introduced and prices adapt. Furthermore, $\epsilon > 0$.*

### Wealth Transfers

Table 4.1 reports who profits and who looses from stricter regulation, in the case with fixed prices in economy, i.e. $\mathcal{I} \to B$, and in the case where prices are endogenous in economy, i.e. $\mathcal{I} \to \mathcal{B}$.

With a change from the unregulated economy in equilibrium $\mathcal{I}$ to the partial equilibrium $B$, the total surplus generated by a stricter regulation policy $\Delta P_0 > 0$ is completely attributed to debt $D_0$; as prices are fixed, $W_0^u$ and $W_0^b$ do not react to a change in regulatory environment.

The endogeneity of prices in equilibrium $\mathcal{B}$ results in an indirect wealth transfer. The equilibrium condition $W_0^u = \omega P$ implies $\Delta W_0^u|_{\mathcal{B}} = \omega \Delta P_0 > 0$, meaning that the unrestricted investor shares $\omega$ of the surplus. The additional value is $\Delta V_0|_{\mathcal{B}} = (1 - \omega)\Delta P_0 > 0$.

Prices move against the interest of the banking sector $\Delta W_0^b|_{\mathcal{B}} < 0$, while at the same time debt still profits from the additional value, $0 < \Delta D_0 / \Delta P_0|_{\mathcal{B}} < 1$. Accordingly, debt is only relatively worse off in equilibrium $\mathcal{B}$, as it does not profit from the total regulatory surplus. The banking system is losing even on an absolute basis, since it indirectly subsidizes the unrestricted investor $u$. This distributional impact of regulation on other market participants has not been discussed in the literature.

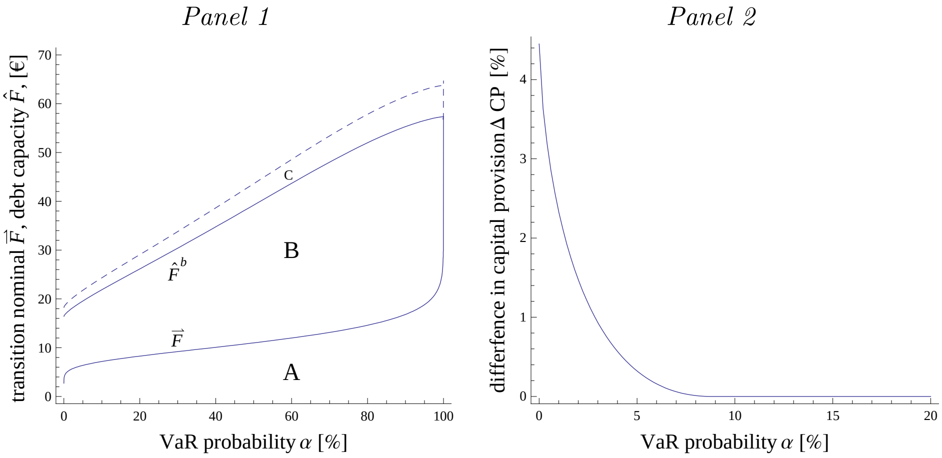


Figure 4.3: **Comparing Equilibria: Debt Capacity and Capital Provisions**

*This figure shows the differences in the restrictiveness of regulation by comparing partial and pure exchange equilibrium. In region A in Panel 1 the banking sector is effectively unregulated, whereas in region B the VaR restriction is binding. The additional region C, encompassed by the dashed line, illustrates the reduction in debt capacity due to endogenous prices. Panel 2 plots the differences in capital provisions $CP$ due to the endogeneity of prices. All other parameters are defined as in the standard parameter set in Table 3.2.*

## Restrictiveness of Regulation and Capital Provision

The fact that endogenous prices increase financing costs can also be seen by comparing the debt capacity $\hat{F}$ under fixed prices $B$ and with endogenous prices $\mathcal{B}$. It can be shown that that the debt capacity is less, $\hat{F}|_{\mathcal{B}} \leq \hat{F}|_{B}$, as can be seen in Panel 1 of Figure 4.3, Region C.

Aggregate excess demand due to regulation reduces prices, not only as of today, but there will be also a price impact in the future as the economy evolves. In this light, the difference in capital provisions $\Delta CP = CP|_{\mathcal{B}} - CP|_{B}$ can be interpreted as (net) premium, which is necessary to cover the future price impact of regulation. This premium constitutes an essential part of the total capital provisions due to VaR in equilibrium. As it is intuitive, this premium increases with higher leverage or stricter regulation.

Let us flip the side: if regulatory policy is not aware of its own effect on prices, capital provisions may turn out to be too low. If the banking sector uses the wrong $CP|_{B}$ instead of $CP|_{\mathcal{B}}$, they will not be able to keep the necessary positions as to

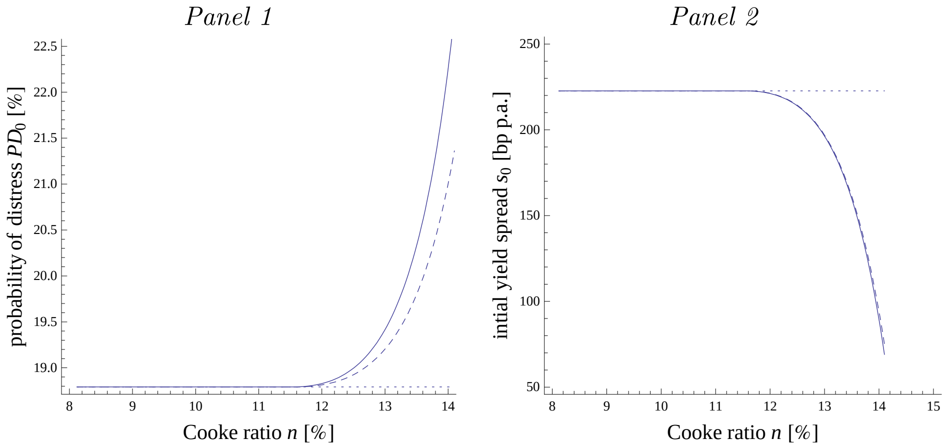


Figure 4.4: **Comparing Equilibria: Probability of Distress and Spreads**

*This figure shows the effect of endogenous prices on the probability of distress $PD_0$ and the yield spread $s_0$. In Panel 1, $PD_0$ is graphed over the Cooke ratio $n$, whereas in Panel 2, the yield spread $s_0$ is displayed. All other parameters are defined as in the standard parameter set in Table 3.2.*

comply with the VaR when times turn 'bad' and prices move against them. Even worse, in this case, regulation generates a problem that wasn't there before.

The Banking Sector's Debt Market

Relative to the case without an endogenous price effect, the probability of distress in the banking system is increasing in equilibrium, while the spread $s_0$ is also higher in equilibrium. Notwithstanding, the overall effect on the spread is still lowered by regulation. The qualitatively same effects arise by an analysis of the impact of regulation via the Cooke ratio $n$.

**Remark.** *Changing the initial value $\delta_0$ is equivalent to an economy where the initial distribution is adequately adapted. It is therefore not discussed.*

## 4.3 Beyond Full Rationality

The question we ask in this section is what happens on the *individual level*, if not all the agents within the economy have the necessary information (informational asymmetries) and/or are able (the problem is too complex) or willing (bounded rationality, sparse time) to base their decision on full information and/or full rationality. We do not to discuss the impact of such agents on equilibrium for technical reasons. Hence, two further assumption are needed in order to derive *consistently* the results in this section.

1. Some agents neglect parts of the impact of endogenous prices for some exogenous reason not modeled explicitly. This assumption generates the results.

2. These agents have a measure of zero. This assumption in fact removes the price impact of these agents; it is needed in order to avoid an inconsistency when equilibrium is calculated the same way as before.
   When abstracting from this assumption, equilibrium prices are affected in a complex way, as there arise feedback effects. Sircar and Papanicolaou (1998) for example analyze an equilibrium where some participants apply for exogenous reasons a $\Delta$-hedging strategy.

Alternatively, one can also interpret the results as an 'as if' study to give first insights into the first-order effects in an economy with a 'true' general equilibrium.

**Remark.** *In contrast to most other results presented throughout this thesis, the conclusions in this section are only based on numerical evidence and are not proved in a formal way.*

### 4.3.1 Realized vs. Expected Shocks

The question under consideration in this subsection is whether the equity capital of the banking system evolves better under the same economic development ($i$) when unexpected or ($ii$) when expected. We demonstrate the differences by using the same underlying development $\delta_0 \to \delta_T$, but under two extreme assumptions, namely

**case (i) with only realized shocks,** where the whole movement is solely due to the realization of the Brownian motion $w_T - w_0$ and thus not expected, i.e.

$\mu_\delta = 0$. The contrary extreme is

**case (ii) with only expected shocks,** where there is no unexpected movement, that is, $w_T = 0$, but $\delta_T$ is only explained by $\mu_\delta$. Note that in between $t = 0$ and $t = T$ the economy evolves along the same path as before, as expressed by aggregate coupon payments $\delta_t$.

**Remark.** *The source of 'incomplete' rationality is that agents do not take the assumed dynamics into account in their decisions in advance. Otherwise, there will be arbitrage, since $\delta$ then follows a Brownian Bridge; see e.g. the discussion in Loewenstein and Willard (2000) or Liu and Longstaff (2004)*

Figure 4.5 shows terminal wealth $W_T$ over a range of possible $\frac{\delta_T}{\delta_0}$, in Panel 1 for the unregulated economy $\mathcal{I}$ and in Panel 2 for the regulated economy $\mathcal{B}$. The ordinates are in

**case (i) with only realized shocks** (dotted lines in Figure 4.5; constant zero expectation $\mu_\delta = 0$)

$$\begin{aligned}\text{terminal wealth} \quad & W_T\Big(y(\mu_\delta = 0), \mu_\delta = 0, \delta_T(\mu_\delta = 0, w_T)\Big)\\ \text{over} \quad & \frac{\delta_T}{\delta_0}\Big(\underset{\substack{\uparrow\\ \text{no expectations}}}{\mu_\delta = 0}, w_T\Big) = e^{\left(0 - \frac{\sigma_\delta^2}{2}\right)T + \sigma_\delta\sqrt{T}w_T},\end{aligned}$$

whereas in

**case (ii) with only expected shocks** (solid lines; no unexpected shock $w_T = 0$)

$$\begin{aligned}\text{terminal wealth} \quad & W_T\Big(y(\mu_\delta), \mu_\delta, \delta_T(\mu_\delta, w_T = 0)\Big)\\ \text{over} \quad & \frac{\delta_T}{\delta_0}\Big(\mu_\delta, \underset{\substack{\uparrow\\ \text{pure expectations}}}{w_T = 0}\Big) = e^{\left(\mu_\delta - \frac{\sigma_\delta^2}{2}\right)T + \sigma_\delta\sqrt{T}0}.\end{aligned}$$

$\frac{\delta_T}{\delta_0}(\mu_\delta = 0, w_T = 0)$ is approximately 0.90.

**Remark 6.** *The analysis is evaluated at time $t = T$ for simplicity. Although similar, 'smoother' results can be obtained when using some arbitrarily time in between $(0, T)$.*

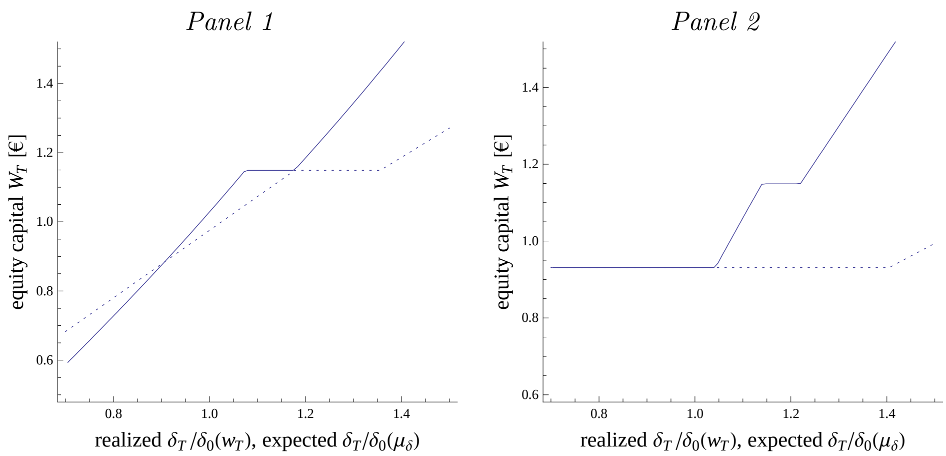


Figure 4.5: **Expected vs. Realized Shocks**

*This figure shows terminal equity wealth $W_T$ for a range of paths $\delta_T/\delta_0$. Solid lines represent the case where the path was perfectly expected, i.e. $\delta_T/\delta_0(\mu_\delta)$, whereas dotted lines, if the realization was purely unexpected, $\delta_T/\delta_0(w_T)$. Panel 1 plots the results for the unregulated economy $\mathcal{I}$ and Panel 2 for the regulated banking system $\mathcal{B}$. All other parameters are as defined in the standard parameter set in Table 3.2.*

In the economy without regulation, terminal equity wealth $W_T^i$ is not identical for the same economic development, that is for the same path $\delta_0 \rightarrow \delta_T$. With a successful development of the economy, e.g. $(\delta_T/\delta_0) \geq (\delta_T/\delta_0)(\mu_\delta = 0, w_T = 0)$, the equity value is larger, when the boom was expected, $w_T = 0$ (solid line in Panel 1 in Figure 4.5) than when it just happened, $\mu_\delta = 0$ (dotted line in Panel 1), where, ex post, the banking system behaved in too precautionary a manner. The analogous result turns out to be true on the opposite side: equity value is larger when recession is a surprise, rather than when it was expected.

When one imagines an additional period, where the model is restarted again, the equity wealth at time $t = T$ constrains the ability to give credit to the real sector in the next period. Extrapolating the above-mentioned result, where equity wealth was higher in 'good' times and lower in 'bad' times, procyclicality in $t = T$ due to (common) expectations of the unregulated banking sector at time $t = 0$ can arise.

Within a regulated economy, the same argument for equity wealth $W_T^b$ as before holds, at least in booms. Yet, regulation eliminates the analogous effect in recessionary times. Expectations do not affect equity wealth any more.

With this result at hand, regulation eliminates the procyclical effect of expectations, at least in the numerical example, for both cases, realized recessions ($\mu_\delta = 0, w_T \leq 0$; solid line in Panel 2) and expected ones, ($\mu_\delta \leq 0, w_T = 0$; dotted line in Panel 2). It comes at the cost of lower equity levels in 'good' times.
This is true for moderate parameterization. When using extreme parameterization, one will still find the procyclical effect mentioned in the unregulated case. Nevertheless, procyclicality occurs only for much more extreme economic developments $\delta_0 \rightarrow \delta_T$ and terminal wealth is still larger under regulation than without.

Therefore, we cannot conclude that regulation feeds additional procyclicality into the real sector, neither when recession is just occurring nor when it is expected; on the contrary, it might have a dampening effect.

### 4.3.2 Endogeneity of Volatility and VaR Estimations

VaR regulation superimposes additional effects on market volatility, as argued e.g. by Danielsson et al. (2001) and Danielsson (2002). To illustrate the impact of endogenous market volatility, we study the impact of regulation with VaR on the VaR estimates of the market itself. One agents is able to incorporate fully all the endogenous effects into his VaR calculation; the other one uses only market data as of today for his VaR estimate.

If all the effects of endogeneity of prices are incorporated, that is, considering all the agent's aggregate influence on prices by market clearing, the true probability distribution of any price in the future $P_{t+\tau},\ \tau \in [T-t, 0)$ is known. The endogenous VaR, E-VaR, of an investment in the market portfolio measured over a period $\tau$ is therefore

$$\begin{aligned}
\text{E-VaR}_{[t;\tau]} &:= \inf\{x \geq 0 : \mathbf{P}[P_t - P_{t+\tau} \geq x | \mathcal{F}_t] < 1\%\} \\
\Rightarrow\quad \text{E-VaR}_{[t,\tau]} &= P(t, \delta_t) - P(t+\tau, \overline{\delta}_{[t;\tau]}) \\
\overline{\delta}_{[t;\tau]} &= e^{\left(\mu_\delta - 1/2\sigma_\delta^2\right)\tau + \sigma_\delta \Phi^{(-1)}[0.01]\sqrt{\tau}} \delta_t.
\end{aligned}$$

However, since some market participants on the aggregate market may not anticipate the behaviour of other market participants and the resulting effects on prices and volatility, they will report a VaR estimate that can only depend on past market data and actual market data. Since our economy is Markovian, including past data will not render any systematical enhancement of the VaR. By using only market data as of today,$(\mu_t, r, \sigma_t)$, the estimate of the VaR based on market data, M-VaR, is

$$\begin{aligned} \text{M-VaR}_{[t;\tau]} &:= \inf\{x \geq 0 : \mathbf{P}[P_t - \mathcal{P}_{t+\tau} \geq x | \mathcal{F}_t] < 1\%\} \\ \mathcal{P}_{t+\tau} &= P_t + \int_{s=t}^{t+\tau} \mu_t \mathcal{P}_s ds + \int_{s=t}^{t+\tau} \sigma_t \mathcal{P}_s dw_s \\ \Rightarrow \quad \text{M-VaR}_{[t;\tau]} &= P(t, \delta_t) - P(t, \delta_t) e^{\left(\mu_t - 1/2\sigma_t^2\right)\tau + \sigma_t \Phi^{(-1)}[0.01]\sqrt{\tau}} \end{aligned}$$

Table 4.2 on p. 153 reports the market value $P_t$ and the two VaR numbers as negative annualized percentage returns

$$\begin{aligned} \text{E-VaR\%} &= -\frac{\ln\left(\frac{P(t+\tau,\bar{\delta})}{P(t,\delta_t)}\right)}{\tau} \\ \text{M-VaR\%} &= -\frac{(\mu_t - 1/2\sigma_t^2)\tau + \sigma_t \Phi^{(-1)}[0.01]\sqrt{\tau})}{\tau}, \end{aligned}$$

at which the endogenous VaR and, analogously, the market VaR are situated in the distribution. The fourth number shows the difference between the two different VaR calculations. The states of the world are chosen in such a way as to represent 'bad' states in economic development; the values of the aggregate payments $\delta_t$ correspond to the $\{0.1\%, 1\%, 2.5\%, 10\%\}$ quantiles of the $\delta$-distribution.

The maximum underestimation in percentage returns is 32.35%, whereas the maximum overestimation is 12.77%, both in the same case ($F = 11, \alpha = 1\%$). With higher nominal debt, estimates seem to be more biased, both in mean and maxima. Increasing the nominal debt results in a market behaviour that is more determined by the financial intermediary than by the unrestricted agent. It seems to be plausible that increased debt worsens the approximation quality.

The relation between states of the world $\delta_t$ and the endogenous VaR turns out to

be non-monotonous in the states of the world; neither does market data in 'bad' times imply higher VaR estimations. They may go even in adverse directions, as in the case of ($F = 7, \alpha = 1\%; \delta_t = 0.27$ vs. $0.21$) the endogenous E-VaR decreases by 7.8%, whereas the market M-VaR estimate increases by 3.4%.
Given the data in Table 4.2, regulation does not systematically reduce underestimation. Moreover, it seems more likely that it worsens the problem at hand, as the mean over the four states increases with regulation in both cases $F = 7$ and $F = 11$.

The results in this model of financial intermediation shows that the endogeneity of volatility due to VaR regulation poses a substantial problem in the use of VaR as a regulatory tool itself. Not only there seems to be a tendency to underestimate VaR in 'bad' times, but also there is no clear systematic effect that could be captured by a 'simple' multiplier to the VaR calculations. In 'good' times, regulation has almost no influence on the VaR estimates (not shown). It underpins the arguments put forward by Danielsson (2002) and references therein that VaR approaches are misleading or even break down in crisis, as markets do not behave 'regularly' any more and exhibit very different statistical properties. Lehar (2005) (indirectly) illustrates the argument in his empirical study.

### 4.3.3 Procyclicality and Credit Crunch

In this section, we discuss whether regulation introduces procyclicality effects on the real side of the economy and whether regulation contributes to a credit crunch.
In order to illustrate this topic we imagine a single (atomic) corporation, whose strategy is to invest only in the market portfolio $P$. The corporation is equipped with equity capital $\mathfrak{W}$. In addition, the corporation leverages the investment with nominal debt $\mathfrak{F}$ and maturity of one year from the banking system. For an exogenous reason, the corporation is willing to pay at most a (constant) credit spread of $\mathfrak{s} = 55bp$.
There are no frictions due to bankruptcy, so banks use a Merton (1974) option-style framework for valuing the debt claim $\mathfrak{D}$ and the equity claim of this company $\mathfrak{W}$. In

contrast to the standard Merton case, the task of valuing the options has to be done numerically, as the market volatility $\sigma^P(\delta_t, t)$ is stochastic. Note that in an economy without any financial intermediation, the volatility of the market is constant and the standard Merton case emerges. Consequently, given that the equity position is equal over time, the nominal debt level $\mathfrak{F}$ will also be constant.

**Remark.** *The notation for the corporate is analogous to the banking sector, yet with gothic letters. Note further that the corporation has no impact on the equilibrium. The underlying economy is not changing due to the existence of this additional market participant.*

### Dynamic View

First, we take a picture at time $t = 4$; nevertheless, the qualitative structure remains stable over time. Figure 4.6 shows in Panel 1 the maximum leverage ratio $\mathfrak{D}_t/\mathfrak{W}_t$ the corporation is able to maintain for adverse states of the economy, given the previous assumptions. Panel 2 shows the corresponding conditional volatility $\sigma^P$ for this time instant.

When comparing Panel 1 and Panel 2 in Figure 4.6 one immediately notes the (inverse) relation between the volatility and the leverage over the different states $\delta_t$. Since we used the Merton model for valuing the claim to the company, the debt claim is the combination of a riskless asset and a put short. At a low volatility, debt is worth more. As we keep the equity wealth $\mathfrak{W}$ and the spread $\mathfrak{s}$ constant, the debt volume $\mathfrak{D}$ consequently is higher with lower volatility.

The mapping is not as 'perfect' as described above, because volatility is, in contrast to the Merton model, time- and state-dependent and thus evolving over time.

When trying to measure the procyclicality effects of the banking system in our dynamic economy, there are two components of interest. The

**level impact** should prevent a possible shortage of the availability of credit to good creditors. It is desirable to have a higher nominal debt *level* in economically hard times,

$$\mathfrak{F}^{\mathcal{B}} > \mathfrak{F}^{\mathcal{I}} .$$

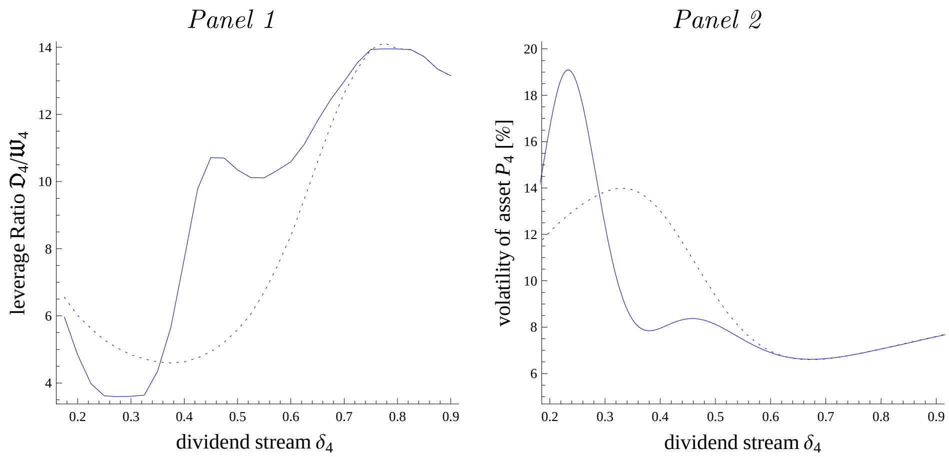


Figure 4.6: **Procyclicality**

*This figure shows in Panel 1 the leverage ratio $\mathfrak{D}_t/\mathfrak{W}_t$ and in Panel 2 the volatility $\sigma^P$ over adverse states $\delta$ at time $t = 4$.*

However, if the economic situation is doing well, we may accept the case, where debt is lower than in an unregulated economy. The

**relative impact** is more in the spirit of a cyclical development in an economy. Here, the question is, whether the impact of regulation dampen both ups and downs in the economy by reducing leverage in good times and increasing leverage in (temporarily) bad situations,

$$\frac{\frac{d\mathfrak{F}^{\mathcal{B}}}{d\delta}}{\frac{d\mathfrak{F}^{\mathcal{I}}}{d\delta}} \leq 1 \ .$$

By this definition of procyclicality, nominal debt levels increase relative to an unregulated economy, if the economy is in recession. However, when the economy recovers, leverage is less and the company will profit less from the upturn.

The common pattern is as follows:
When the state of the economy is very good, volatilities almost coincide and thus there is no significant impact on the availability of credit due to regulation, neither on the debt *level* nor on the *relative* effect.
When the economic state worsens, under regulation the corporation is able to obtain

a higher credit *volume* than in an unregulated economy; regulation is also first not procyclical in *relative* terms. As there is a second, downward hump in volatility in the proximity of $\delta_4 = 0.4$, it even translates into an anticyclical (relative) behavior, as the company can borrow more since even sign of the relative measure changes. However, as the recession worsens, regulation has a large procyclical effect in relative terms.
In the third part, beyond some threshold, regulation affects credit *volume* in such a way that the corporation is given less by banks than in an unregulated economy. However, on a *relative* basis, procyclicality vanishes again. If the economy is well in crisis, the impact of regulation is almost non-existent, as both economies behave identically.

With these results at hand, there is no clear-cut answer to the question of procyclicality, neither on a relative nor on an absolute level. As a recession emerges, debt capacity is less scarce and even may increase slightly; nevertheless, when recession deepens further, the possibility for the real economy to obtain credit from the regulated banking system is reduced even more than in an unregulated economy. The discussion above relies on a one point in time perspective and is heavily reliant on the volatility.

### Dynamic and Path-Dependent View

Up to now, the analysis has been dynamic, but not path dependent, since the value of equity was reset to a constant each time. When the economy evolves, not only does the availability of credit change, but also the equity position itself is influenced.

In Table 4.3 the combined effects are exemplified in the style of a binomial tree. For each time-state-path combination there is a panel of the following structure: the time-state pair $(t, \delta_t)$ is given in the upper left corner of each panel, and the respective paths are encoded underneath by the possible up ($u$) and down ($d$) combinations. In the first two rows, the value of the equity position $\mathfrak{W}_t$ and the debt level $\mathfrak{F}_t = \mathfrak{F}_t(\mathfrak{W}_t)$ at the constant spread of $\mathfrak{s} = 55bp$ of the company is displayed, for both the economy with a regulated banking sector $\mathcal{B}$ as well as with unregulated financial intermediaries $\mathcal{I}$. The small numbers set underneath the numbers in the unregulated case $\mathcal{I}$ represent the difference from the regulated case $\mathcal{B}$ in percentage terms, $100\frac{\mathfrak{W}_t|_{\mathcal{B}}-\mathfrak{W}_t|_{\mathcal{I}}}{\mathfrak{W}_t|_{\mathcal{I}}}$ for the two underlying economies $\mathcal{I}$ and $\mathcal{B}$.

The third row is the availability of debt, when the shock to the equity position in the last period is neglected, $\mathfrak{F}_{-1} = \mathfrak{F}_t(\mathfrak{W}_{t-1})$, that is, without the path dependency in the last period. Underset are the percentage differences to the case with a shock to the equity capital, $100\frac{\mathfrak{F}_t(\mathfrak{W}_t)-\mathfrak{F}_t(\mathfrak{W}_{t-1})}{\mathfrak{F}_t(\mathfrak{W}_{t-1})}$ for the two underlying economies $\mathcal{I}$ and $\mathcal{B}$.

The corporation starts with the same initial equity $\mathfrak{W}_0 = 1.35$; yet, under regulation, the nominal debt level $\mathfrak{F}$ is less by 1.8%.

When comparing the extreme paths $uuu$ and $ddd$, the equity position $\mathfrak{W}_t$ is less extreme, i.e. higher in bad times and lower in good ones, which is also true for the debt level $\mathfrak{F}_t$. One might argue that this is simply the result of the reduced initial debt level in case of regulation. Although, when comparing the debt capacity without the equity increase or decrease in the last period $\mathfrak{F}_{-1}$ with the respective debt level $\mathfrak{F}_t$, the absolute difference in percentage terms (i.e. comparing the absolute percentage terms beneath $\mathfrak{F}_{-1}$) is higher throughout the whole table. This is evidence that the dampening effect of regulation cannot be attributed to the initial conditions alone.

The path dependency of this dynamic approach can be best seen in $t = 3$. The results with two ups $\delta_3 = 0.57$ or two downs $\delta_3 = 0.48$ are very dispersed; however, when the boom was first ($u$ at front position) and the bust late ($d$ on last position), the equity position is more depressed. At the same time, regulation again ameliorated the situation.

This dynamic and path dependent study is evidence that regulation does not introduce additional procyclicality; it is even more reasonable to assume that regulation in fact has anticyclical effects in dynamic and path-dependent framework. Hence, the anticyclical property of the valuation as shown in Section 2.1 of this chapter dominates the partially procyclical effects of volatility as illustrated in the above discussed perspective.

| $\alpha$ | $\delta_t$ | F=7 | | | | F=11 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $P_t$ | E-VaR% | M-VaR% | $\Delta$ | $P_t$ | E-VaR% | M-VaR% | $\Delta$ |
| 10% | 0.40 | 7.73 | 28.55 | 19.89 | -8.66 | 9.29 | 40.32 | 33.83 | -6.49 |
| | 0.31 | 6.52 | 28.07 | 22.55 | -5.52 | 7.25 | 28.23 | 30.99 | 2.76 |
| | 0.27 | 5.94 | 26.94 | 22.10 | -4.85 | 6.50 | 23.33 | 23.44 | 0.11 |
| | 0.21 | 4.93 | 24.11 | 19.68 | -4.43 | 5.51 | 18.16 | 14.02 | -4.14 |
| 1% | 0.40 | 8.03 | 21.51 | 12.05 | -9.46 | 11.49 | 23.51 | 6.01 | -17.50 |
| | 0.31 | 7.09 | 35.15 | 18.40 | -16.75 | 10.12 | 59.30 | 26.96 | -32.35 |
| | 0.27 | 6.47 | 36.18 | 24.91 | -11.27 | 8.84 | 58.30 | 42.44 | -15.86 |
| | 0.21 | 5.10 | 28.39 | 28.33 | -0.06 | 6.09 | 32.40 | 45.17 | 12.77 |
| 0.10% | 0.40 | 8.09 | 12.79 | 8.61 | -4.17 | 11.71 | 6.37 | 0.01 | -6.36 |
| | 0.31 | 7.51 | 13.31 | 4.94 | -8.38 | 11.29 | 13.24 | 2.25 | -10.99 |
| | 0.27 | 7.24 | 25.27 | 7.52 | -17.76 | 10.91 | 33.33 | 8.63 | -24.70 |
| | 0.21 | 6.20 | 46.85 | 28.01 | -18.83 | 8.93 | 68.83 | 41.00 | -27.83 |

Table 4.2: **Endogenous VaR and VaR Estimate Based on Market Data**

*This table reports in each case the quadruplet of numbers representing the market value $P_t$, the endogenous VaR (E-VaR%), and the market-based VaR estimation (E-VaR%), both as a percentage term of $P_t$, as well as their difference $\Delta$, as of time $t = 3.5$ over $\tau = 1$ years. The E-VaR is the VaR of an investment in the market portfolio with endogenous time-varying volatility, whereas the M-VaR is calculated with constant volatility. The cases include two different levels of debt $F \in \{7, 11\}$, three regulation regimes, $\alpha \in \{10\%, 1\%, 0.1\%\}$, and four possible states of the world, $\delta_t$, corresponding to the lower quantiles $\{0.1\%, 1\%, 2.5\%, 10\%\}$.*

| (0.52) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| – | $\mathcal{B}$ | 1.35 | 7.4 | |
| | $\mathcal{I}$ | 1.35 | 7.53 (−1.8) | |

| (0.57) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $u$ | $\mathcal{B}$ | 1.69 | 10.96 | 8.77 (24.9) |
| | $\mathcal{I}$ | 1.73 (−2.3) | 11.48 (−4.6) | 8.96 (28.2) |

| (0.48) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $d$ | $\mathcal{B}$ | 1.13 | 6.46 | 7.7 (−16.2) |
| | $\mathcal{I}$ | 1.07 (5.6) | 6.25 (3.2) | 7.89 (−20.7) |

| (0.62) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $uu$ | $\mathcal{B}$ | 1.96 | 15.24 | 13.13 (16.1) |
| | $\mathcal{I}$ | 2.06 (−4.9) | 16.54 (−7.8) | 13.86 (19.3) |

| (0.52) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $ud$ | $\mathcal{B}$ | 1.14 | 7.21 | 10.66 (−32.3) |
| | $\mathcal{I}$ | 1.08 (5.6) | 7.09 (1.6) | 11.32 (−37.3) |
| $du$ | $\mathcal{B}$ | 1.47 | 9.3 | 7.15 (30.1) |
| | $\mathcal{I}$ | 1.51 (−2.6) | 9.87 (−5.7) | 7 (41) |

| (0.43) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $dd$ | $\mathcal{B}$ | 0.99 | 5.41 | 6.19 (−12.6) |
| | $\mathcal{I}$ | 0.89 (11.2) | 4.93 (9.9) | 5.9 (−16.5) |

| (0.68) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $uuu$ | $\mathcal{B}$ | 2.05 | 23.04 | 21.94 (5) |
| | $\mathcal{I}$ | 2.22 (−7.7) | 24.87 (−7.4) | 23.13 (7.5) |

| (0.57) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $uud$ | $\mathcal{B}$ | 0.96 | 7.31 | 14.89 (−50.9) |
| | $\mathcal{I}$ | 0.95 (1.1) | 7.54 (−3.1) | 16.41 (−54) |
| $udu$ | $\mathcal{B}$ | 1.45 | 11.01 | 8.68 (26.8) |
| | $\mathcal{I}$ | 1.47 (−1.4) | 11.72 (−6) | 8.62 (35.9) |
| $duu$ | $\mathcal{B}$ | 1.74 | 13.27 | 11.2 (18.5) |
| | $\mathcal{I}$ | 1.81 (−3.9) | 14.38 (−7.7) | 11.99 (20) |

| (0.48) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $udd$ | $\mathcal{B}$ | 0.92 | 5.33 | 6.58 (−19.1) |
| | $\mathcal{I}$ | 0.86 (7) | 5.04 (5.7) | 6.37 (−20.9) |
| $dud$ | $\mathcal{B}$ | 1.05 | 6.03 | 8.49 (−28.9) |
| | $\mathcal{I}$ | 0.83 (26.5) | 4.86 (24.1) | 8.86 (−45.2) |
| $ddu$ | $\mathcal{B}$ | 1.33 | 7.67 | 5.71 (34.4) |
| | $\mathcal{I}$ | 1.3 (2.3) | 7.66 (0.1) | 5.25 (46) |

| (0.40) | | $\mathfrak{W}_t$ | $\mathfrak{F}_t$ | $\mathfrak{F}_{-1}$ |
|---|---|---|---|---|
| $ddd$ | $\mathcal{B}$ | 0.88 | 4.47 | 5.01 (−10.6) |
| | $\mathcal{I}$ | 0.77 (14.3) | 3.91 (14.5) | 4.56 (−14.3) |

Table 4.3: **Credit Crunch**

## 4.4 Appendix to Chapter 4

### 4.4.1 Proofs for the General Equilibrium Section

Proof of Existence and Uniqueness of Equilibrium

In order to show the existence and uniqueness of the equilibrium, all we need to show is, that there exists a unique solution $(y^b, y^u)$ to the system of equations (2.33) under the state price density (4.3).

First, the normalization (1 degree of freedom) is used

$$u'\left(\mathcal{I}(y^u) + \mathcal{I}(y^b)\right) = 1 \ . \tag{4.6}$$

Thus $y_u'(y^b) < 0$ holds.

In the following the notation is shortened, namely $y^b \rightarrow y$, $W_0^u \rightarrow W_u, W_0^b \rightarrow W_b, V_0 \rightarrow V$, and $P_0 \rightarrow P$. Furthermore, $W_b, W_u, V$, and $P$ are seen as functions of $y$ alone. Because

$$\begin{aligned} \frac{d}{dy}\left(\frac{W_u}{P}\right) &= \frac{W_n'P - W_nP'}{P^2} = \frac{W_n'P - W_n(W_n' + V')}{P^2} \\ &= \frac{W_n'(P - W_n) - W_nV'}{P^2} = \frac{W_n'V - W_nV'}{P^2} \\ &> 0 \ , \end{aligned}$$

i.e. $\frac{W_u}{P}$ is strictly increasing and

$$\begin{aligned} \frac{W_u}{P} &\underset{(y^u\to 1)}{\overset{y\to\infty}{\longrightarrow}} \frac{W_u(1)}{e^{-rT}(V_T(W = \underline{W})\Phi(d_2(\overline{\zeta})) + V_T(W = 0)\Phi(-d_2(\overline{\zeta}))) + W_u(1)} \\ &=: \hat{\omega} \\ \frac{W_u}{P} &\underset{(y^u\to\infty)}{\overset{y\to 1}{\longrightarrow}} 0, \end{aligned}$$

we obtain that there exists a unique equilibrium, if the unrestricted investor holds a share of the market $\omega \in [0, \hat{\omega}$. Results from the partial analysis, namely $V' < 0$ and $W_u' = W_u'(y^u)y_u'(y) > 0$, were used.

Transformations: Using the Lagrange multiplier of the equilibrium condition and the following list of transformations

$$
\begin{aligned}
\xi_t &\rightarrow u'(\delta_t) \\
\mathcal{I}(\xi_t) &\rightarrow \delta_t \\
r &\rightarrow \gamma\mu_\delta - \frac{1}{2}\gamma(1+\gamma)\sigma_\delta^2 \\
\kappa &\rightarrow \gamma\sigma_\delta \\
\zeta_* &\rightarrow \delta^* = \frac{W_*}{\mathcal{I}(y^b)} \\
\zeta^* &\rightarrow \delta_* = \frac{W_*}{\mathcal{I}(hy^b)} \\
\underline{\zeta} &\rightarrow \overline{\delta} = \frac{\underline{W}}{\mathcal{I}(hy^b)} \\
\overline{\zeta} &\rightarrow \underline{\delta} = e^{\left(\mu_\delta - 1/2\sigma_\delta^2\right)\tau + \sigma_\delta \Phi^{(-1)}[0.01]\sqrt{\tau}}\delta_0
\end{aligned}
$$

on the formulas in the partial equilibrium section, one can recover all relevant equilibrium values $W_t^u$ ((2.19) on p. 28), $W_t^b$ ((3.22) on p. 86), $D_t$ ((3.41) on p. 119), $V_t$ ((3.41) on p. 119), and $P_t = V_t + W_t^u$. Applying the same techniques as presented in Section 2.3.2 and Section 3.4.5 of the Appendix to Chapter 3 the corresponding volatilities $\sigma_t = \sigma_t^P$ as well as $\sigma_t^W, \sigma_t^D, \sigma_t^V$ can be recovered.

### 4.4.2 Comparison of Equilibria

The goal is to show the relations $y^b|_{\mathcal{B}} \geq y^b|_B \geq y^i_{\mathcal{I}}$, which hold as strict inequalities, if the VaR restriction is binding, $F > \vec{F}$. This is approached by a constructive proof with the following steps:

**Step 1:** Set $\alpha = 1$ and all other parameters to the identical ones for $\mathcal{B}$ and $\mathcal{I}$; match the exogenous $(r, \kappa; W_0^b = W_0^i, W_0^u)$ for $B$, such that they match the endogenous $(r, \kappa; W_0^b = W_0^i, W_0^u)$ of the economies $\mathcal{B}$ and $\mathcal{I}$.
By construction

$$\begin{aligned} L_0^b(\alpha = 1)|_{\mathcal{B}} &= L_0^b(\alpha = 1)|_B &= L_0^i|_{\mathcal{I}} \\ y^b(\alpha = 1)|_{\mathcal{B}} &= y^b(\alpha = 1)|_B &= y^i|_{\mathcal{I}} \end{aligned}$$

where $L_0^b = \frac{D_0^b}{W_0^b}$ and analogously for $i$.

**Step 2** Decrease $\alpha$ up to the transition point $\vec{\alpha} < 1$:
$(i)$ $y^i(\alpha)|_{\mathcal{I}} = y^i|_{\mathcal{I}}$, as the economy with agent $i$ is not regulated, thus nothing changes, in particular, $(W_0^u|_{\mathcal{I}}, W_0^i|_{\mathcal{I}})$ as well as $L_0^i|_{\mathcal{I}}$ is constant.
$(ii)$ Up to the point, where the VaR restriction becomes binding, neither $(W_0^u|_{\mathcal{B}}, W_0^b|_{\mathcal{B}})$ nor $(W_0^u|_B, W_0^b|_B)$ (and the corresponding leverage ratios) react. The VaR restriction starts to bind at the same point $\vec{\alpha}|_{\mathcal{B}} = \vec{\alpha}|_B = \vec{\alpha}$.

**Step 3** Decrease $\alpha$ further beyond $\vec{\alpha} < 1$:
$(i)$ It is irrelevant to the economy $\mathcal{I}$.
$(ii)$ In economy $B$ the prices are exogenous, but $y_b'(\alpha)|_B < 0$ still applies, by the results from the previous sections. Hence

$$\begin{cases} L_0^b(\alpha)|_B > L_0^i|_{\mathcal{I}} \\ y^b(\alpha)|_B > y^i|_{\mathcal{I}} \ . \end{cases}$$

**Step 4:** Construct a general equilibrium $\mathcal{B}(F(\alpha))$ relative to $B$:

As the VaR restriction starts to be binding at the same point $\vec{\alpha}$, we will fulfill

$$\left.\begin{array}{rcl} L_b'(\alpha)|_{\mathcal{B}(F(\alpha))} & = & L_b'(\alpha)|_B \\ y_b'(\alpha)|_{\mathcal{B}(F(\alpha))} & = & y_b'(\alpha)|_B \end{array}\right\}$$

in order to obtain $y^b(\alpha)|_{\mathcal{B}(F(\alpha))} = y^b(\alpha)|_B$ and $L_0^b(\alpha)|_{\mathcal{B}(F(\alpha))} = L_0^b(\alpha)|_B$.

The result is, that $F'(\alpha) < 0$, $\forall \alpha < \vec{\alpha}$, and, hence, $F(\alpha) < F$, if the VaR restriction is binding.

**Step 5:** Finally, we obtain the original equilibrium $\mathcal{B}$ by adjusting $F(\alpha)$ back to $F$ within the equilibrium economy. Because $y_b'(F)|_{\mathcal{B}} > 0$, the final result follows, $y^b|_{\mathcal{B}} > y^b|_B > y^i_{\mathcal{I}}$, if the VaR restriction is binding, and $y^b|_{\mathcal{B}} = y^b|_B = y^i_{\mathcal{I}}$ otherwise.

Proof of Step 2: The VaR restriction starts to bind at the same point

Inverting the transition point $F = \vec{F}(W_0^b, \overline{\zeta}(\alpha))$ on obtains (a unique) $\vec{\alpha}(W_0^b, F)$. The only variable, which might be different in equilibrium $\mathcal{B}$, relative to economy $B$, where $W_0^b$ is exogenously given, is $W_0^b$ itself, since it is an endogenous variable. However, the endogenous variable $W_0^b$ will only react to regulation, if the VaR restriction first becomes binding. Consequently, the VaR restriction starts to bind at $\vec{\alpha}$, irrespective of the type of economy.

□

Proof of Step 3: $y^b(\alpha)|_B > y^i|_{\mathcal{I}}$

As economy $\mathcal{I}$ is equivalent to an economy $I$, when varying regulation, it follows from the previous result that $(i)$ $y^b(\alpha)|_B \geq y^i|_I$, and $(ii)$ $\frac{dD_0}{d\alpha} > 0$, where a strict inequality holds, if the VaR restriction is binding. Furthermore, $\frac{dW_0}{d\alpha} = 0$ holds in any partial equilibrium.

□

PROOF STEP 4: $F'(\alpha) < 0 \ \forall \alpha < \vec{\alpha}$

In the following the notation is shortened, namely $y^b \to y$, $W_0^u \to W_u$, and $D_0, \to D$. $f^{(0,1,0)}$ denotes the partial derivative with respect to the second argument, other derivatives are defined analogously.

Totally differentiating the equilibrium condition

$$W_u(y) = \omega \left(D(y, F, \alpha) + W_b(y, F, \alpha) + W_u(y)\right)$$

and inserting

$$y'(\alpha)|_{\mathcal{B}(\alpha, F(\alpha))} = -\frac{W_b{}^{(0,0,1)}(y, F, \alpha)}{W_b{}^{(1,0,0)}(y, F, \alpha)}$$

one can solve for

$$\frac{dF}{d\alpha} = \frac{-(1-\omega)W_u' W_b{}^{(0,0,1)} + \omega \left(W_b{}^{(0,0,1)} D^{(1,0,0)} - D_b{}^{(0,0,1)} W_b{}^{(1,0,0)}\right)}{\omega \left(D^{(0,1,0)} + W_b{}^{(0,1,0)}\right) W_b{}^{(1,0,0)}}$$

where the arguments were omitted.

The sign of partial derivatives can be found in Table 3.3.

The denominator is

$$\omega \left(D^{(0,1,0)} + W_b{}^{(0,1,0)}\right) W_b{}^{(1,0,0)} < 0$$

since $\omega > 0 \wedge W_b{}^{(0,1,0)} > 0 \wedge D^{(0,1,0)} > 0 \wedge W_b{}^{(1,0,0)} < 0$.

The first part of the numerator is

$$-(1-\omega)W_u' W_b{}^{(0,0,1)} > 0$$

since $0 < \omega < 1 \wedge W_u' > 0 \wedge W_b{}^{(0,0,1)} < 0$.

The second part of the numerator is

$$\omega \left(W_b{}^{(0,0,1)} D^{(1,0,0)} - D^{(0,0,1)} W_b{}^{(1,0,0)}\right) > 0$$

since $D^{(0,0,1)} < 0 \wedge W_b^{(0,0,1)} < 0 \wedge D^{(1,0,0)}$ and because

$$\frac{W_b^{(1,0,0)}(y,F,\alpha)}{W_b^{(0,0,1)}(y,F,\alpha)} - \frac{D^{(1,0,0)}(y,F,\alpha)}{D^{(0,0,1)}(y,F,\alpha)} = \frac{e^{-rT}\xi_0\left(e^{rT}(\beta-1)H_1 - F\beta\left(\Phi\left(-d_2\left(\zeta_*\right)\right)-1\right)\right)}{Fhy\beta\gamma\bar{\zeta}\left(\psi-\psi_2\right)} > 0, \tag{4.7}$$

where Lemma 2, equation (3.43) on p. 120, was used. Hence, $F'(\alpha) < 0$ must hold.

□

Proof Step 5: $y'(F)|_{\mathcal{B}} > 0$

The equilibrium condition is

$$W_u(y) = \omega P(y,F) = \omega(W_u(y) + V(y,F)) \ .$$

Applying the implicit function theorem, we obtain the sign of the derivative

$$y'(F) = -\frac{-V^{(0,1)}}{W_u'(1-\omega) - V^{(1,0)}} > 0 \ ,$$

together with the assumption/results from the previous sections, namely $V^{(0,1)} > 0, W_u' > 0, V^{(1,0)} < 0$, and $0 < \omega < 1$.

□

# Chapter 5

# A Two-Sided Equilibrium Model

In all the previous chapters and in most parts of the relevant literature, the liability side, i.e. the nominal debt level $F$ of the economy, is fixed. In the following, we argue that the overall debt level of the economy is, at least partly, under the control of the supervising authorities; for ease of reference, we call this part of the supervising authorities the central bank. In principle, the central bank is willing to disburse any nominal debt level $F$ at the fair price of the modeled economy. To demonstrate the implications, we propose two different goals of the central bank, which constrain their willingness to disburse debt, namely $(i)$ keeping the leverage ratio in present value terms under control; or $(ii)$ keeping the probability of distress under control.

With this approach, a first step into a combined asset-liability management of regulated banks is undertaken. The intention of this chapter is to give a first insight to the question of how banks may use this additional degree of freedom in their decision. However, it is only a first step, as it incorporates only a market for debt once, namely at time $t = 0$. However, when advancing this road further to a completely dynamic and double-sided model, technical complexity substantially increases.

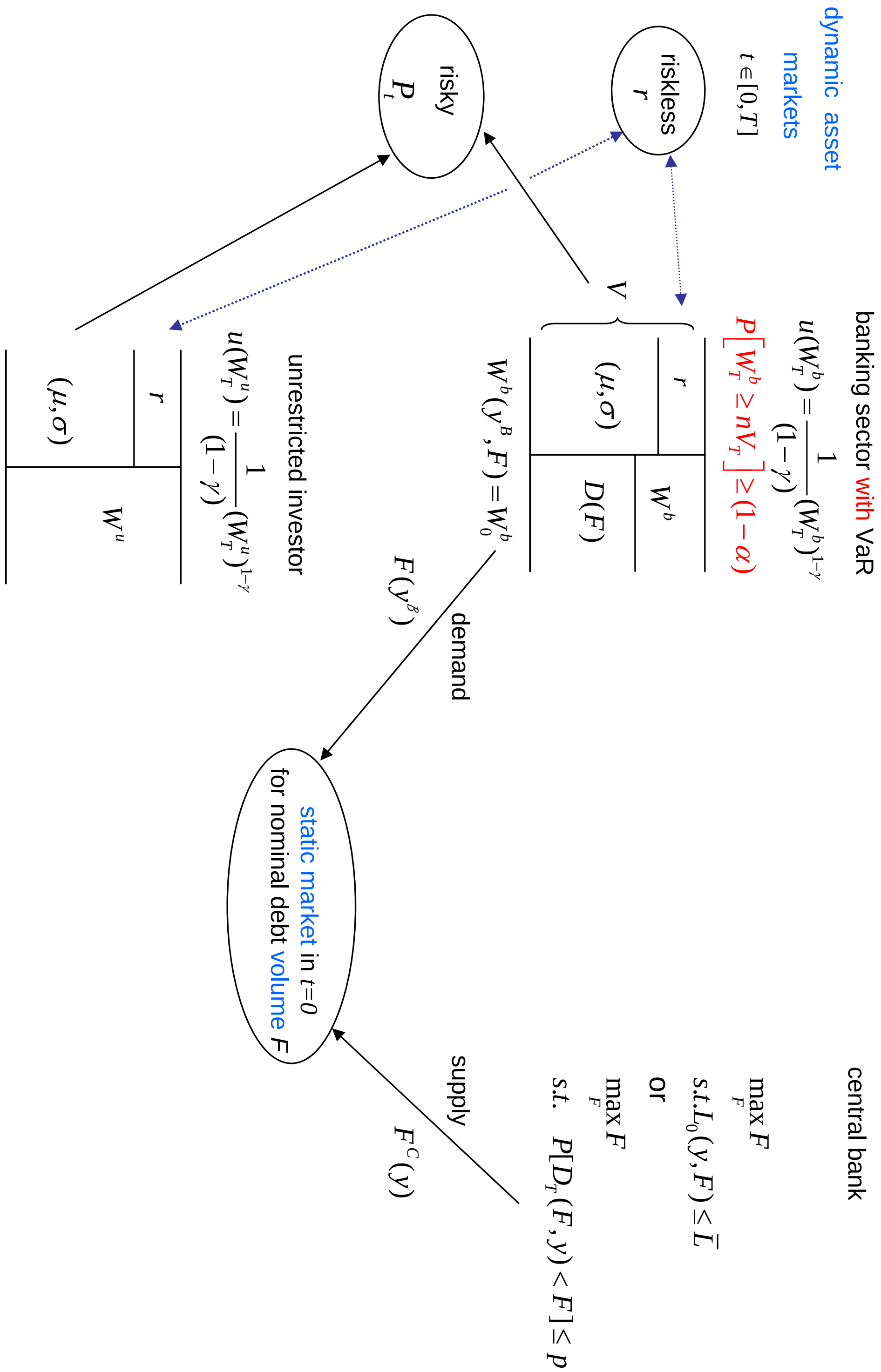


Figure 5.1: **Structure of the Two-Sided Economy**

## 5.1 The Structure of the Two-Sided Equilibrium

We propose a two sided market which endogenizes the previously exogenous nominal debt level $F$. The structure of the economy is illustrated in Figure 5.1.

Dynamic Market for Financial Assets

On the left-hand side in Figure 5.1, there is the dynamic economy as described in the previous chapters with an aggregate banking sector $b$ (or alternatively as a benchmark the unregulated investors $i$) and the unrestricted investor $u$. Together they form a continuous competitive market for loans $P$, which is a claim to the coupon payment stream $\delta$.

The banking sector takes nominal debt $F$ as a given its optimal decision. The Lagrange multiplier $y^{\mathcal{B}}(F)$ is the shadow price of the static budget equation in utility terms. Thus, if the price for a marginal unit of nominal debt $F$ is low, i.e. $y^{\mathcal{B}}$ is small, the banking sector has a large incentive to take on more debt, et vice versa. By inverting $y^{\mathcal{B}}(F)$, we therefore deduct a demand curve for nominal debt, given the willingness to add debt, as expressed by the (shadow) price of debt $y^{\mathcal{B}}$, $F(y^{\mathcal{B}})$.

Static Market for Nominal Debt

The 'central bank' uses its instrument, namely setting the 'price' for the supply of nominal debt by the choice of interest rate it charges to banks, which refinance their capital needs in order to control the supply. Within our model, the central bank uses its market power to set the shadow price $y$ given the optimal answer of the banking system.

We do not consider other possibilities of central banks to control the money supply, like e.g. minimum reserve requirement, open market operations, or buying covered bonds.

Note that the central bank is *not a price-taking* economic agent and is *leading* the market. We define the residual *supply*/demand $-F^C$, which the central bank supplies, by the *demand*/supply of the total market $F^M$ less the *demand* of the banking system $F^{\mathcal{B}}$,

$$-F^C = F^M - F^{\mathcal{B}} \, .$$

For simplicity, we assume that the central bank has discretionary power over the debt volume and there exists no additional demand/supply, i.e. a credit multiplier of 1, then $F^M = 0$, and in equilibrium

$$F^C = F^{\mathcal{B}} \, .$$

Hence, the optimization problem resulting in the nominal debt volume in equilibrium $\bar{F}$ is

$$\bar{F} = \underset{F \geq 0}{\arg\max} \quad F \tag{5.1}$$

$$s.t. \quad \begin{cases} \frac{D_0(y,F)}{W_0^b(y,F)} \leq L_0 \text{ or } \mathbf{P}[D_T(y,F) < F] \leq p & \text{CB GOALS} \\ W^u(y,F) = \omega P(y,F) & \text{FM EQUILIBRIUM} \end{cases}$$

The optimal solution to the optimization problem reduces to solving two equations, the demand function of the banking system $F(y^{\mathcal{B}})$ from the Financial Market equilibrium, and the supply function of the central bank $F^C(y)$ from the Central Bank's goal, to obtain the equilibrium $(\bar{y}, \bar{F})$.
Since the budget equations of the unrestricted investor $u$ and the banking system $b$ hold with equality, all markets clear any time in the two-sided equilibrium.

**Remark.** *Since, for simplicity, the central bank holds all debt, it is naturally exposed to all the credit risk. Alternatively one could model $F^M > 0$ as capital market debt, whereas the central bank only holds with its share $F^C$ senior debt of the banking system. The resulting spread will be substantially reduced.*
*Nevertheless, the central bank (better supervisory authorities) charges a yield spread relative to the riskless instrument, which can also be interpreted as a fair deposit insurance premium.*

**Remark.** *In the following, we denote both $F(y^{\mathcal{B}})$ and the inverted function $y^{\mathcal{B}}(F)$ a supply function and $F^C(y)$ and the inverted function $y(F^C)$ a demand function.*

## 5.2 Constant Leverage Ratio

Let $\eta_0 = W_0^b/W_0^u$ measure the capital distribution (of equity wealth) and $L_0 = D_0/W_0^b$ the leverage ratio in present value terms. Then initial endowment can be rewritten as

$$\eta_0(1 + L_0) = \frac{1-\omega}{\omega}.$$

In partial equilibrium $B$, where prices are fixed, the behavior of the banking sector, as expressed in $L_0$, is independent of the capital distribution $\eta_0$. In the pure exchange equilibrium prices adjust in such a way as to ensure that this functional relation between leverage and capital distribution holds.

If there is a change in an economic fundamental, say $\chi$, and if the central bank reacts to this change by choosing a new money supply $F(\chi)$, such that the leverage ratio $L_0$ does not change, i.e. $dL_0 = 0$ by intervention, then the new pure exchange equilibrium $\mathcal{B}$ with endogenous prices is identical to the respective partial equilibrium $B$ with fixed prices, because the capital distribution $\eta_0$ will not change in a pure exchange equilibrium, i.e. $d\eta_0 = 0$ by equilibrium mechanism. With this assumed reaction of the central bank, prices of equity $W_0^b$, debt $D_0$ and the wealth of the investor $W_0^u$ do not change in the pure exchange equilibrium, only the nominal debt level $\bar{F}(\chi)$.

We illustrate consequences by using the probability of distress $PD_0$, see Panel 1 of Figure 5.2. From comparative static analysis of Chapter 3 stricter regulation affected probability of distress negatively, i.e. $dPD_0/d\alpha < 0$. However, when banks are forced to react (or react themselves) in the assumed way, regulation does in fact generate more stability when it is measured through distress probability, $dPD_0/d\alpha|_{L_0} > 0$. The difference is as follows: the debt title $D_0$ is more valuable due to the regulation and banking system is thereby more levered. Thus, regulation forces banks to maintain a constant leverage, $\bar{F}'(\alpha) < 0$. This reduction implies a decline in the probability of distress, which turns out to outweigh the first effect. At the same time, debt gains twofold, as both stricter regulation and the reduction

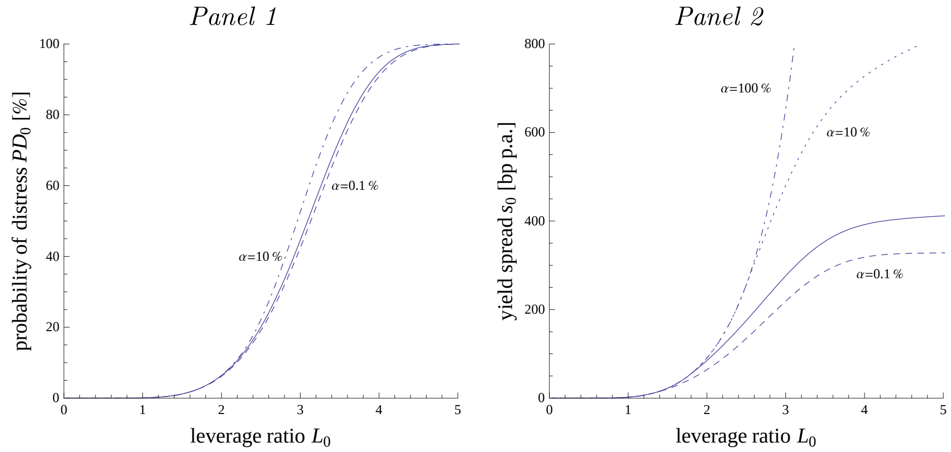


Figure 5.2: **Probability of Distress and Spread: Constant Leverage Ratio**

*This figure shows probability of distress in the financial system $PD_0$ and yield spread $s_0$, dependent on the leverage ratio in present value terms $L = D_0/W_0$. Panel 1 plots the probability of distress $PD_0$ for different levels of VaR probabilities $\alpha \in \{10\%, 1\%, 0.1\%\}$. Panel 2 graphs the yield spread $s_0$ alike. All other parameters are as defined in the standard parameter set in Table 3.2.*

of the nominal debt level reduce the risk, as can be seen inspecting the spreads in Panel 2 of Figure 5.2. In our numerical example, there is a substantial reduction due to regulation in this case.

The key difference is the reaction of the nominal debt level in equilibrium $\bar{F}$. We assumed that the central bank keeps the leverage ratio in present value terms constant. This simple assumption reversed the destabilizing impact of regulation into a stabilizing one.

**Remark.** *The experience of the years between the introduction of banking regulation and the financial crisis seems to suggest that the reaction to regulation was in fact to increase nominal leverage and risk exposures, as is empirically documented in Adrian and Shin (2008b). This behaviour may be attributed to a sense of better control over risk which in turn entrapped banking institutions into enhancing their return on equity by leveraging up with new financial products that had low regulatory capital requirements; see Peltzman (1975) for a similar case. With this behavioural assumption in mind, regulation is doubly adverse to financial stability in our framework.*

*Hellwig (1995) already mentions the concern that financial innovations - partly driven due to regulatory arbitrage - undermine the effectiveness of regulation; thereby, regulation may in fact induce an increased systemic risk.*
*Additionally, regulation will indirectly support a higher degree of homogeneity in the banking portfolios. For a detailed analysis of the homogenization impact of regulation see Freixas et al. (2007) and Wagner (2008).*

## 5.3 Controlling the Probability of Distress

In this part, we consider as the target of the central bank to control the probability of distress

$$p = \mathbf{P}[D_T < F] = \Phi\left(-d_0(\zeta^*(y, F))\right). \tag{5.2}$$

The central bank (supervising authorities) uses the price the banking sector is willing to pay to derives a supply of debt. If the price for debt $y$ is low, banks are more aggressive in taking on higher debt levels. This comes at the cost of a higher probability of distress. Banks increase the debt level up to the point where it matches the fixed probability $p$ of the supervising authorities.

Panel 1 of Figure 5.3 shows the two (inverted) supply and demand functions. The downward-sloping (inverted) supply function $y(F^C)$ can be derived analytically and is identical in both cases of an unregulated or restricted economy. The second dashed line represents the supply function at the lower distress probability $p/2$. It is shown in the Appendix that a decline in the distress probability results in a downward shift of $y(F^C)$.
The upward-sloping solid line shows the (inverted) demand $y^{\mathcal{B}}(F)$ for the regulated banking system, whereas the dotted-dashed line represents the unregulated demand $y^{\mathcal{I}}(F)$. Due to the properties of these supply and demand curves, there exists a unique equilibrium $(\bar{y}, \bar{F})$, which is shown in the Appendix. As the demand does not change with respect to central bank's policy $p$ in both cases $\mathcal{B}$ or $\mathcal{I}$, any equilibrium $(\bar{y}(p), \bar{F}(p))$ is along the demand curve $y^{\mathcal{B}}(F)$.

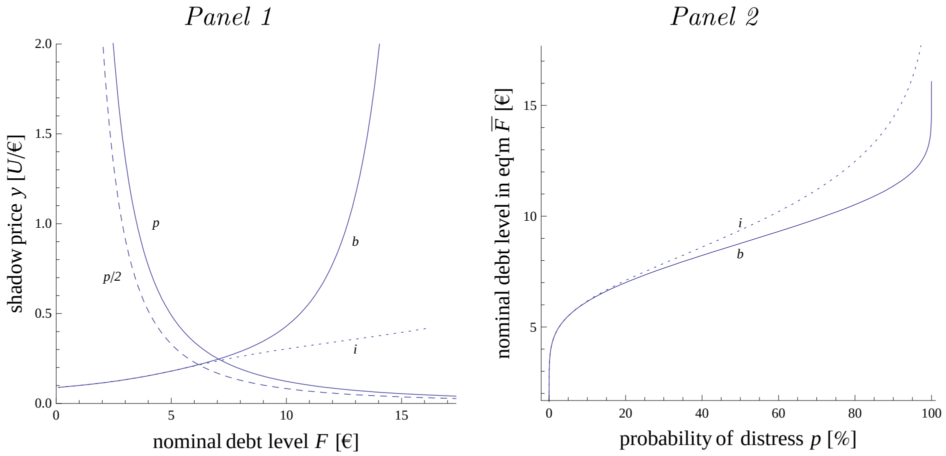


Figure 5.3: **Varying the Probability of Distress**

*This figure shows the equilibrium $(\bar{y}, \bar{F})$ and its dependence on the control variable $p$ of the central bank. Panel 1 reflects demand (downward-sloping lines) and supply (upward-sloping lines). The dashed line graphs the supply for $p/2$, whereas the dotted line the demand in the unrestricted economy. Panel 2 plots the equilibrium debt volume $\bar{F}$ as a function of the underlying control $p$. All other parameters are as defined in the standard parameter set in Table 3.2.*

Panel 2 of Figure 5.3 shows the impact of the policy $p$ of the central bank on the nominal debt volume in equilibrium $\bar{F}(p)$ for both cases, the regulated economy $\mathcal{B}$ (solid line) and the unregulated economy $\mathcal{I}$ (dotted-dashed line).

If the probability of distress is effectively unrestricted, $p = 100\%$, the banks chooses the maximum debt capacity $\hat{F}^{\mathcal{B}}$ and $\hat{F}^{\mathcal{I}}$, respectively. On the other hand, if $p = 0$ is required, the banking sector is unable to take on any debt. At the transition point $\bar{y} = \vec{y}$, in our numerical example, at $p \leq 3.2\%$, the restricted economy $\mathcal{B}$ changes to the unrestricted solution, i.e. behaves as if being effectively unrestricted. As the equilibrium $(\bar{y}(p), \bar{F}(p))$ is along $y^{\mathcal{B}}(F)$ and increasing $p$ shifts $y(F^C)$ upwards, the equilibrium inherits the properties $(\frac{d\bar{y}}{dp} > 0, \frac{d\bar{F}}{dp} > 0)$ as well.

Consequently, we recover in equilibrium the standard result that a declining leverage results in less probability of distress (or the probability of default of the first individual bank).

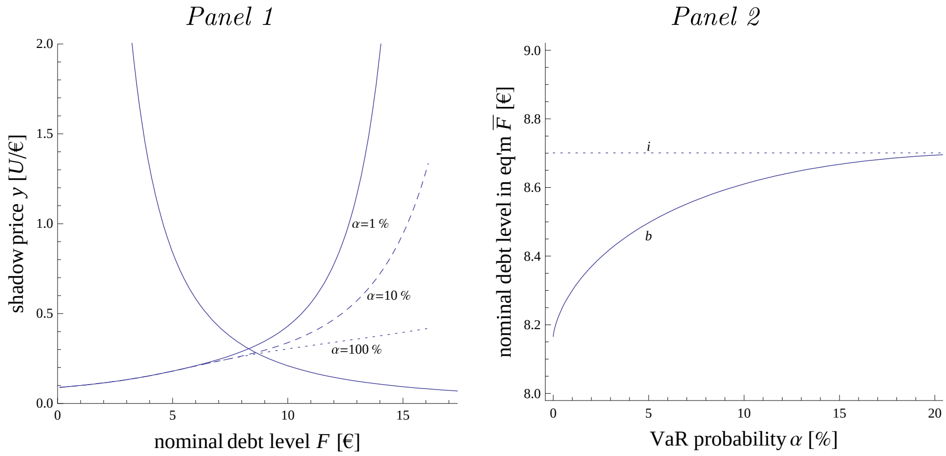


Figure 5.4: **Varying the Probability of Crisis**

*This figure shows the equilibrium $(\bar{y}, \bar{F})$ and its dependence on the control variable $\alpha$ of the regulator. Panel 1 reflects demand (downward-sloping line) and supply (upward-sloping lines). The dotted line represents the demand of the unrestricted financial intermediary ($\alpha = 100\%$), whereas the dot-dashed line the the demand of a restricted banking system with a VaR probability of 10%. Panel 2 plots the equilibrium debt volume $\bar{F}$ as a function of the regulatory control $\alpha$. All other parameters are as defined in the standard parameter set in Table 3.2.*

When discussing the effects of regulation on the equilibrium, first note that the downward-sloping supply $y(F^C)$ does not rely on the regulation parameters $(\alpha, n)$. On the demand side, we know from the previous analysis that $y^{\mathcal{B}}(F)$ will (increasingly) shift upwards with stricter regulation. The equilibrium $(\bar{y}, \bar{F})$ therefore evolves upwards along the supply curve $y(F^C)$ with stricter regulation, that is, $(\frac{d\bar{y}}{d\alpha} < 0, \frac{d\bar{F}}{d\alpha} > 0)$ and $(\frac{d\bar{y}}{dn} > 0, \frac{d\bar{F}}{dn} < 0)$. Panel 1 of Figure 5.4 shows the unrestricted case $\alpha = 100\%$ as a dotted-dashed line, a case with $\alpha = 10\%$ as a dashed line, and the standard case with $\alpha = 1\%$ as a solid line. Panel 2 of Figure 5.4 shows the nominal debt level in equilibrium $\bar{F}$ over a range of VaR probabilities.

The notable result of the analysis is that under regulation lower debt levels are needed than in an equivalent unregulated economy, if the same probability of distress in the banking system should be maintained. Even keeping the nominal debt level constant increases the probability of distress in fact.

The economic intuition behind the result is as follows: if there is regulation, debt is better secured and thus more valuable. This wealth increase of debt investors is

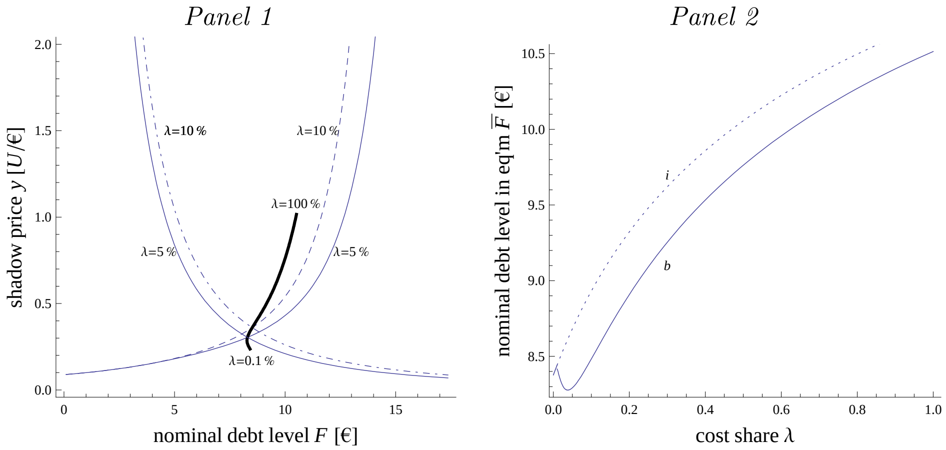


Figure 5.5: **Varying Costs**

*This figure shows the equilibrium $(\bar{y}, \bar{F})$ and its dependence on the cost share $\lambda$. Panel 1 reflects demand (downward-sloping lines) and supply (upward-sloping linea). The dotted-dashed line displays the demand with increased costs of $\lambda = 10\%$, whereas the dashed line graphs the change of supply due to the the same change in costs. Panel 2 plots the resulting equilibrium debt volume $\bar{F}$ as a function of the costs fraction $\lambda$, the solid line illustrates the solution for the restricted banking sector b, whereas the dotted line for the unrestricted financial intermediary i. All other parameters are as defined in the standard parameter set in Table 3.2.*

financed by adding some risk to the asset side, when risk is measured by the distress probability. To keep the probability of distress constant, the nominal debt level must be declining to compensate the former effect.

In an economy without any regulation, $\mathcal{I}$ or $I$, a high fraction of costs prevent agents from using excessive tail risk in their portfolio choices, as $\frac{dW_T}{d\lambda}|_{W_T<\underline{W}} > 0$, see Table 3.3. From the viewpoint of the supervising authorities, $h$ is a measure of punishment of tail risks. Consequently, with rising costs, the central bank is willing to provide more credit, $F_C'(\lambda)|_{\mathcal{I}} > 0$, while still keeping the probability of distress constant. Regulation needs financing of the retention level $\underline{W}$, in addition. Panel 1 of Figure 5.6 measures these costs by the necessary capital provisions. However, this financing effect increases the probability of distress. Thus, within a regulated economy, the equilibrium outcome may result in a case where there is a decrease in the nominal volume, i.e. $F_C'(\lambda)|_{\mathcal{B}} < 0$, as one can see in Panel 2 of Figure 5.5. This side effect is even more pronounced when regulation itself is very efficient, that is, when

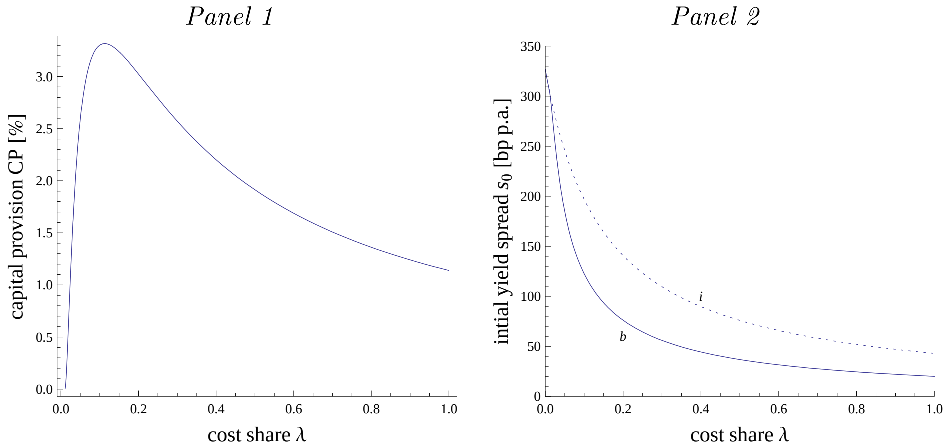


Figure 5.6: **Constant Probability of Distress: The Impact of Cost**

*This figure shows the dependence of capital provisions $CP$ (Panel 1) and the yield spread $s_0$ (Panel 2) on the cost share $\lambda$. The dotted line in Panel displays the corresponding yield spread in an unregulated economy. All other parameters are as defined in the standard parameter set in Table 3.2.*

there are large capital provisions due to the VaR restriction with moderate systemic costs. This can be seen by comparing $\bar{F}(\lambda)$ in Figure 5.5, Panel 2, with $CP(\lambda)$ in Figure 5.6, Panel 1 in the region $\lambda \simeq 5\%$. Even though the difference in capital provisions is small, the impact an the yield spread is substantial when compared to the corresponding unregulated financial system, see Figure 5.6, Panel 2.

When the combined approach is relevant for systemic stability, namely keeping the probability of crisis and the one of distress constant, regulation introduces interferences that need a different behaviour of the central bank, depending on the (expected) level of systemic costs. Furthermore, internalizing more costs into the banking sector, has the secondary impact that the jump size $J$ (as well as the relative jump size $J/F$) increases.

## 5.4 Appendix to Chapter 5

### 5.4.1 Constant Leverage

From the equilibrium condition we obtain

$$\eta_0 \frac{dL_0}{dF} + (L_0 + 1)\frac{d\eta_0}{dF} = 0 \ .$$

Consequently, if $dL_0 = 0$, then $d\eta_0 = 0$ holds in equilibrium. Therefore, when neither $L_0$ nor $\eta_0$ change in equilibrium, it is equivalent, but more convenient, to use the budget equation together with the restriction on the leverage than the equilibrium condition itself.

In order to evaluate the impact of the VaR restriction under a constant leverage ratio $L_0$, the system of equations

$$\left\{ \begin{array}{l} D_0(y^b, F, \alpha) = L_0 W_0^b(y^b, F, \alpha) \\ W_0^b(y^b, F, \alpha) = W_0^b \end{array} \right\}$$

is totally differentiated with respect to the VaR probability $\alpha$.

In the following the notation is shortened, namely $y^b \to y$, $W_0^b \to W_b$, and $D_0, \to D$.

The resulting reaction functions are

$$\begin{aligned} \frac{dF}{d\alpha} &= -\frac{D^{(0,0,1)}(y,F,\alpha)W_b^{(1,0,0)}(y,F,\alpha) - W_b^{(0,0,1)}(y,F,\alpha)D^{(1,0,0)}(y,F,\alpha)}{W_b^{(0,1,0)}(y,F,\alpha)D^{(1,0,0)}(y,F,\alpha) - D^{(0,1,0)}(y,F,\alpha)W_b^{(1,0,0)}(y,F,\alpha)} \\ &= -\frac{Fh\beta\bar{\zeta}\left(\psi - \psi_2\right)\left(F\beta\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right) - e^{rT}(\beta - 1)H_1\right)}{(\beta - 1)\xi_0\left(\beta W_b\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right) - H_1 H_2\right)} \\ &> 0 \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\alpha} &= \frac{D^{(0,0,1)}(y,F,\alpha)W_b^{(0,1,0)}(y,F,\alpha) - W_b^{(0,0,1)}(y,F,\alpha)D^{(0,1,0)}(y,F,\alpha)}{D^{(0,1,0)}(y,F,\alpha)W_b^{(1,0,0)}(y,F,\alpha) - W_b^{(0,1,0)}(y,F,\alpha)D^{(1,0,0)}(y,F,\alpha)} \\ &= \frac{Fhy\beta^2\gamma\bar{\zeta}\left(\psi - \psi_2\right)\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right)}{(\beta - 1)\xi_0\left(\beta W_b\left(\Phi\left(-d_2\left(\zeta_*\right)\right) - 1\right) - H_1 H_2\right)} \\ &< 0 \end{aligned}$$

where $\psi_2$ is defined by the identity

$$\psi_2 \frac{\beta}{1-\beta} F = \mathcal{I}(hy\bar{\zeta}) \ ,$$

hence, $\psi_2 < \psi$ for $F > \vec{F}$.

To determine signs, Lemma 2, equation (3.43) on p. 120, and Lemma 3, equation (3.44) on p. 120, are used.

Finally, the change of the probability of distress under constant leverage is

$$\left.\frac{d}{d\alpha} PD_0(F, y, \alpha)\right|_{L_0} = \frac{\phi(-d_0(\zeta^*))\,(y\gamma F'(\alpha) + Fy'(\alpha))}{Fy\kappa\sqrt{T}} \ .$$

One can easily determine the sign of

$$\begin{aligned} y\gamma F'(\alpha) + Fy'(\alpha) &= -\frac{e^{rT} Fhy\beta\gamma\bar{\zeta}H_1\,(\psi - \psi_2)}{\xi_0\,(H_1H_2 + \beta W_0 - \beta W_0 \Phi\,(-d_2\,(\zeta_*)))} \\ &> 0 \quad (F > \vec{F}) \end{aligned}$$

by the use of Lemma 2, equation (3.43) on p. 120, and Lemma 3, equation (3.44) on p. 120. The result follows, namely that $\frac{dPD_0|_{L_0}}{d\alpha} > 0$ holds, i.e. stricter regulation decreases the probability of default.

### 5.4.2 Constant Probability of Distress

From the previous analysis we know that the Lagrange multiplier, $y^{\mathcal{B}}(F) \in [1, \infty)$ for $F \in [0, \hat{F})$. Furthermore, $y'_{\mathcal{B}}(F) > 0$, $y''_{\mathcal{B}}(F) < 0$, and $y^{\mathcal{B}}(F \to \hat{F}) \to +\infty$.
As

$$\Phi\left(-d_0(\zeta^*(y, F^C))\right) = p$$

is solvable to a continuous supply function $F^C(y)$, or directly to the inverted supply function

$$y(F^C) = \frac{e^{(r+\frac{\kappa^2}{2})T+\kappa\Phi^{(-1)}[p]\sqrt{T}}u'(\frac{\beta}{1-\beta}F^C)}{h\xi_0} \tag{5.3}$$

with the properties

$$\begin{aligned} y'(F^C) &= -\frac{y\gamma}{F^C} < 0 \\ y''(F^C) &= \frac{y\gamma(\gamma+1)}{(F^C)^2} > 0 \\ y(F^C \to 0) &\to +\infty \ , \end{aligned}$$

there exists a unique fixpoint $(\bar{y}, \bar{F})$ to the system of equations in partial equilibrium

$$\{W_0^b(y, F) = W^b \ , \Phi\left(-d_0(\zeta^*(y, F))\right) = p\}$$

and in pure exchange equilibrium

$$\{W_0^b(y, F) = \omega P_0(y, F) \ , \ \Phi\left(-d_0(\zeta^*(y, F))\right) = p\} \ .$$

Moreover, the system is also solvable in closed form, for both types of economies, by first inserting the supply $y(F^C)$ in (5.3) into the budget equation (partial equilibrium) or the initial endowment equation (pure exchange equilibrium), then solving for $\bar{F}$, and finally substituting back into the supply function (5.3).
By totally differentiating the system with respect to some parameter, say $\chi$, one can also recover the derivatives $(\bar{y}'(\chi), \bar{F}'(\chi))$.

Central Banks Policy $p$:

Since the control for the probability of distress $p$ shifts only the supply function upwards,

$$\frac{dy(F^C,p)}{dp} = \frac{\sqrt{T}y\kappa}{\phi(-d_0(\zeta^*))} > 0 \ , \quad \forall F^C$$

the result in equilibrium is $(\bar{y}'(\theta) > 0, \bar{F}'(\theta) > 0)$ (together with the remaining properties of the demand and supply function). As the supply function of the unrestricted investor is less increasing, $y^{\mathcal{I}}(F) \leq y^{\mathcal{B}}(F)$, the nominal debt level will be higher (equal if not binding) in economy $\mathcal{I}$ than in economy $\mathcal{B}$.

Regulation:

$y(F^C)$ is independent of regulation $\chi \in \{-\alpha, n\}$; thus the supply does not shift. On the other hand, from the previous analysis we know, that the demand shift upwards with tighter regulation, $y'_{\mathcal{B}}(\chi) > 0$. Consequently, $(\bar{y}'(\theta) > 0, \bar{F}'(\theta) < 0)$ holds. Analogously to the previous argument, nominal debt level will be higher (equal if not binding) in economy $\mathcal{I}$ than in economy $\mathcal{B}$.

Cost Share $\lambda$:

Costs shift the *supply* upwards,

$$\frac{dy(F^C,\lambda)}{d\lambda} = y\frac{1-\beta}{\beta+\lambda(1-\beta)} > 0 \ , \quad \forall F^C > 0 \ , \tag{5.4}$$

as well as the demand side, $y'_{\mathcal{B}}(\lambda) > 0$. Analogously to the previous argument, nominal debt level will be higher (equal if not binding) in economy $\mathcal{I}$ than in economy $\mathcal{B}$, since $y^{\mathcal{B}}(\lambda) \geq y^{\mathcal{I}}(\lambda)$.

# Chapter 6

# Summary and Conclusion

## 6.1 Summary of Results

What are the consequences of regulation, which can be deduced from the previous model framework? We summarize the results by answering a set of questions.

*Does the banking system (in general, rationally) respond to regulation? Does regulation impair the ability of the aggregate banking sector to load on debt?*
If the banking system has a low leverage, the capital requirements are automatically fulfilled and banks do not respond to regulation in their risk management. Otherwise, both regulation instruments, the VaR probability and the (average) capital requirement, effectively reduce debt capacity.

*How much equity capital does the banking system in fact provide in order to sustain the regulation requirement (not the capital requirement)?*
Both regulatory instruments, the VaR probability and the (average) capital requirement, effectively increase capital provisions. Furthermore, capital provisioning increases in the overall riskiness of the banking system as expressed by the leverage. Our model framework enables us to separate different parts of capital provisioning. As banks already rationally provide some equity capital for debt without any regulation, capital provisioning that is solely attributable to VaR regulation is different from the (total) capital requirements demanded by regulation.

It can be greater or less than the Cooke ratio $n$, depending on the parameter set. Without control over leverage, managing capital provisions is ineffective.

*What are the distortions of regulation on the portfolio decision?*

In this model framework, the (risky) portfolio decision does not change in its relative proportions: holding the growth optimal/myopic portfolio is still optimal. However, regulation alters the amount held in the myopic portfolio. This property enables us to relate the portfolio decision to an implied risk aversion coefficient, which captures the incentives induced by the VaR regulation.

In most economic circumstances, the banking system behaves in a more risk-averse manner than an unregulated one. Nevertheless, as the economic situation worsens, the banking system may (rapidly) convert to a much more risky strategy.

*Does regulation accentuate the sensitivity of the aggregate banking sector and/or the market to an economic change?*

Especially when the VaR horizon is short and the economy development turns out to be in the proximity of the VaR quantile, the optimal portfolio decision is very sensitive to the fundamental economic dynamics.

*Does regulation reduce the probability of the banking system being in distress? If so, how does regulation affect aggregate losses?*

Regulation raises the probability of distress; this implication is not directly caused by regulation, but through the increased risk provisions. The effect on losses in distress is ambiguous: while regulation reduces losses in some sates, refinancing the charge for regulation increases losses in some other states.

*What is the impact of regulation on the value and volatility of the (total asset) market?*

Regulation has a positive effect on the total asset value. In the pure exchange equilibrium, price adjustments further improve the (initial) value of the market.

Relative to an unregulated economy, the properties of the implied risk aversion of the aggregate banking system are transformed in equilibrium into similar characteristics in the volatility. Namely, as the economic situation worsens, the volatility may increase significantly. The situation further escalates due to the high sensitivity of portfolio decisions.

*Are debt holders impaired by regulation?*

At maturity, debt holders profit on the one hand from regulation, as they gain a higher recovery rate in the states that are affected by VaR; on the other hand, the losses are higher in the non-affected states. In terms of the initial spread on (aggregate) banks' debt, regulation increases the value of debt.

With prices fixed, debt holders gain the whole share of the increased asset value of the economy. In equilibrium, the debt value still increases due to regulation, but there is an indirect wealth transfer to the unrestricted investors due to adjusting prices. Furthermore, by virtue of the banking sector's balance sheet, the specific properties of market volatility are also transmitted into debt.

*How is banks' equity affected?*

By definition, regulation does not affect the initial equity value in partial equilibrium. In equilibrium, price adjustments result in a loss of value. Effectively, there is an indirect transfer via prices to the un-intermediated sector.

*Does the estimation of VaR with market data pose a significant problem if the estimation neglects the endogeneity of volatility?*

Standard VaR calculations depend on past and current data. When comparing these calculations with the VaR which fully incorporates the endogenous nature of volatility the calculated VaR underestimates, in most cases substantially, the true VaR by neglecting the endogenous behaviour of market participants.

*What happens in the real sector in response to regulation?*

If the aggregate dividend is below a threshold, the real sector will be affected in a procyclical way due to regulation, meaning that a given corporation will gain less credit volume at the same contractual specifications relative to an unregulated economy. Above this threshold, the credit volume will be anticyclical with respect to an unregulated economy; in some instances, the credit volume will be expanded in equilibrium, hence being anticyclical in absolute terms.

The presented dynamic and path-dependent example is evidence that regulation can be even anticyclical.

## 6.2 Robustness of the Results

This section qualitatively discusses how the results will change when the essential assumptions for the derivation of the results are altered.

**Linear Approximations for Redemption Payments at the Terminal Date:** The solution (3.12) on p. 71, as displayed in Figure 3.3, may look artificial at first sight, but it is robust to changes in the underlying economic modelling. The retention level at the distress boundary is clearly attributable to the simple $\beta$-linearization in equation (3.3) on p. 59. If one instead imagines a more general piecewise linear function, the sequence of kinks in the $(V_T, W_T)$ space (Figure 3.2 on p. 58) induces a sequence of resistance levels in the $(W_T, \xi_T)$ space (Figure 3.3 on p. 74), each of them continuously connected by parts of the type $\mathcal{I}(\cdot\xi_T)$. Thus, a more general, progressively increasing function $W_T(V_T)$, as displayed in Figure 3.2 as dotted line, still generates a region where wealth is much slower decaying in state prices $\xi_T$ than the unrestricted profile.

An analogous argument can be constructed with regard to the cost structure $C_T$.

With the first approach (piecewise defined approximation) one obtains a more complex solution without rendering substantial further insights, whereas in the second approach (continuous function) analytical solvability is in general lost and a numerical procedure is needed. However, the main arguments and conclusions put forward in this paper will not be affected, as this smoother model only lessens the quantitative impacts, but not their sign.

**VaR boundary:**

The VaR boundary itself is independent of an individual bank's characteristics; it is the same state of the world $\overline{\zeta}$, irrespective of any individual bank that comprises the banking system. Consequently, there is a jump in aggregate terms as well, if there is at least one effectively regulated bank of positive measure. This argument is valid under the assumption that regulation and risk management is homogeneous in the sense that every single bank uses the same probability $\alpha$. Multiple regulatory probabilities will soften the single jump event by dispersing it to multiple jumps over a small part of state space. Consequently, a regulation regime that brings different risk management systems of banks more in line with each other inherently increases systemic risk in the sense that a small change in the economic underlying may cause a rapid movement in endogenous variables such as total assets. Even worse, this happens exactly when the world turned 'bad', i.e. for high state prices $\xi_T$.
Generalizing the model framework with respect to some of these features does not render new insights. Their main consequence lies in smoothing out the 'extreme' behaviour due to the replication of a binary option needed to generate the jump in the optimal profile. As the main interest is in the regulated banking sector with the unregulated banking system as a reference, the main implications remain true as long as there is only smoothing but no adverse effects due to a modification in the model framework.

**Other Measures of Risk:**

In this study the Value-at-Risk is used as a way to regulate risk in the banking sector. There are other measures of risk proposed in literature with 'better' properties to determine risk-sensitive weights. When comparing the portfolio decisions of Basak and Shapiro (2001) (expected loss under $\mathbf{Q}$), Gabih et al. (2005) (expected loss under $\mathbf{P}$, utility-weighted loss), or Gundel and Weber (2008) (utility-based shortfall risk), one obtains that the 'tail' effects from the VaR measure (jump) are not existent. Still, there are ways to 'game' the risk measure which result in a more risky portfolio choice of the regulated banking sector relative to an unregulated financial system.

When expected shortfall is used, as example, the terminal wealth profile is continuous (see Basak and Shapiro (2001), Equation (14) on p. 386 or Figure 6 on p. 387), i.e. the resulting portfolio is less risky than the corresponding unregulated one, however, there is still a 'hump' (see Basak and Shapiro (2001), Proposition 5 on p. 387 or Figure 8(b) on p. 390) due to the kink in the optimal wealth profile. Consequently, the substitution of the VaR based capital requirement with a one based on expected loss will still render a more risky portfolio behaviour than in the analogous unregulated economy.

The results can be viewed as an example of the Corollary 1 to Goodhart's Law (Goodhart (1974))

> 'A risk model breaks down when used for regulatory purposes.' See Danielsson (2002) p. 1276.

**Consumption:**

In our model framework, the banking system is a consuming one. This assumption is essential in deriving analytical results. When using instead only the terminal wealth objective, the interest rate and the market price of risk are stochastic. Optimal portfolio decision are altered in order to form hedge portfolios to account for the changes in the stochastic investment set. Some results can still be deduced, when recovering the optimal portfolio decision by the use of Malliavin calculus. For examples of how to recover a portfolio decision in a framework with a complete market see Detemple et al. (2003) or Stefanova (2008); for a technique to derive the equilibrium see e.g. Serrat (2001).

However, the optimal terminal wealth solution is unaffected, since the markets are still complete.

**Incomplete Markets:**

The single most important assumption we need is a complete (financial) market; without it, the derivation of the feasible optimal profile(s) is drastically aggravated. The assumption of continuous processes substantially reduces the complexity of the problem, but is not essential as long as the financial market remains complete; see e.g. Liu and Pan (2003) for the question of how to recover the portfolio profile in such a case. However, when discussing systemic risk, *non-diversifiable* jump events (macroeconomic shocks) should, in particular, not be excluded from a further analysis, since these systemic jumps will have a significant influence on the portfolio decisions, see e.g. Das and Uppal (2004); in the banking context; Koziol and Lawrenz (2009) highlight the importance of considering jump events with regard to banking regulation.

**Reaction of Banks:**

Besides Chapter 5, the standing assumption was that banks do not respond to changes in the economic environment by adapting their nominal debt level. However, we also showed in Chapter 5 within a two-sided equilibrium that this point turns out to be crucial if one wants to assess the problem at hand from a holistic viewpoint, as 'pure' comparative static results may reverse.

Furthermore, the strand of literature analyzing the liability side of banks, e.g. Blum (1999) or Koziol and Lawrenz (2009), suggests that a deeper understanding will be reached if a continuous adjustment of the passive side is possible. For a first approach of how to *technically* tackle the problem of continuous changes in the deposit volume, see Kaniel and Hugonnier (2008) and Breton et al. (2008).

## 6.3 Conclusion

We conclude that the debt capacity is reduced under regulation, capital provisions are increased, and spreads are lowered. These 'static' indictors show that the banking system benefits from regulation. However, the detriment of regulation is intrinsic to dynamics. As the VaR measure is indifferent to the extent of losses within the VaR quantile, there is an incentive to enter into contracts or portfolio strategies

that contain tail risks. When (part of) these risks realize due to an unfavourable economic development, the banking sector rapidly changes between two extreme positions: first, the banking sector *excessively adds new business* by buying in new risks, and then, in a *flight-to-quality* type of reaction, it sells the risky portfolio and invests (almost) all in the risk-less security. Neither investment positions are in the interest of financial stability. Adding risk in 'bad' times is obviously not to the benefit of financial stability; but a massive flight to quality depresses prices of the risky investments.
The arguments made above hinge on the notion of a 'bad' economic development. The recent financial crisis did not happen in economically 'bad' times, when seen from the viewpoint of the real sector. From the position of standard commercial banks, the last years before the crisis were, in fact, 'bad' times, as credit spreads were, historically seen, very low and, in addition, there existed little 'standard' banking business to expand into. At the same time, however, there was a boom in other banking activities shortly before the crisis, where banks loaded on tail risk while keeping capital requirements almost stable. Especially these markets are now illiquid in crisis, while at the same time risk-less investments soar.

Banking regulation should not be considered as a matter of risk-sensitive ex ante capital requirements alone: the combination of the Value-at-Risk - not as a measure of risk, but as a tool to manage risks actively - and limited liability - possibly enhanced by implicitly too-big-to-fail guarantees - creates externality effects, since tail risks become attractive under VaR. Other measures such as expected shortfall may reduce the problem of tail risks; still, gambling the risk measure on limited liability remains attractive (regulatory arbitrage). Finally, even if a bank does not use tail risk, systemic risk can be only in parts accommodated by individual risk management systems.

An equally important regulatory aspect is the distribution of costs ex post, arising if the banking system is in distress. Letting the costs of distress that are shared by the banking sector be zero, capital regulation with VaR is effectively useless. On the other hand, if banks are burdened with all costs due to a distress in the

financial system, no regulation is needed, as there are no defaults. This extreme case is obviously not a realistic one, but illustrates the impact.
Thus, moderate parts of total costs due to a distress in the banking sector should be existent, if they are seen as a purely exogenous parameter, or needed, if they are under the control of a regulatory authority. Only under moderate costs, capital requirements are working well.

Finally, even if those two effects interact well, a leverage restriction is still necessary, since the usual risk-increasing incentive of limited liability may still work.
There are three reasons to consider:

1. Banks circumvent regulation by directly or indirectly (financial innovation) increasing leverage.

2. The regulating authority is unable to detect risky portfolios (transparency) or is unable to charge fines (political pressure).

3. Bank mangers expect in advance that limited liability is not strictly adhered to in times of crisis, as individual banks might be too-big-to-fail.

Hence, a cap on the leverage ratio increases the equity capital in crisis (at the horizon), thereby enabling an efficient regulation.

Theoretically Bichsel and Blum (2005) and Blum (2008) propose a similar approach, although, founded on a model with informational asymmetries and externality effects. The empirical studies from Avery and Berger (1990), Estrella et al. (2000), and Hovakimian and Kane (2000) suggest that the combination of risk-sensitive capital requirements and an additional leverage constraint indeed serves financial stability better than either one alone.

# Bibliography


Acharya, V. (2009). A theory of systemic risk and design of prudential bank regulation. *Journal of Financial Stability*, In Press, Corrected Proof:–.

Acharya, V., Bharath, S., and Srinivasan, A. (2003). Understanding the recovery rates on defaulted securities. *SSRN eLibrary*.

Acharya, V., Shin, H., and Yorulmazer, T. (2009). Fire-sale FDI. *SSRN eLibrary*.

Adrian, T. and Shin, H. (2008a). Financial intermediary leverage and Value-at-Risk. *SSRN eLibrary*.

Adrian, T. and Shin, H. (2008b). Liquidity and leverage. *Journal of Financial Intermediation*, In Press, Corrected Proof:–.

Ahn, D., Boudoukh, J., Richardson, M., and Whitelaw, R. (1999). Optimal risk management using options. *Journal of Finance*, 54(1):359–375.

Alexander, G. and Baptista, A. (2004). A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. *Management Science*, 50(9):1261–1273.

Alexander, G. and Baptista, A. (2006). Does the Basle capital accord reduce bank fragility? An assessment of the Value-at-Risk approach. *Journal of Monetary Economics*, 53(7):1631–1660.

Allen, F. and Carletti, E. (2006). Credit risk transfer and contagion. *Journal of Monetary Economics*, 53(1):89–111.

Allen, F. and Gale, D. (2000). Financial contagion. *Journal of Political Economy*, 108(1):1–33.

Altman, E., Resti, A., Sironi, A., Sarfatti, V., and Isonzo, V. (2004). Default recovery rates in credit risk modelling: A review of the literature and empirical evidence. *Economic Notes*, 33(2):183–208.

Arrow, K. J. (1970). *Essays in the theory of risk-bearing*. North-Holland, Amsterdam and London.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Avery, R. and Berger, A. (1990). *Risk-based capital and deposit insurance reform*. Federal Reserve Bank of Cleveland, Research Department.

Baele, L., De Jonghe, O., and Vander Vennet, R. (2007). Does the stock market value bank diversification? *Journal of Banking and Finance*, 31(7):1999–2023.

Basak, S. (1995). A general equilibrium model of portfolio insurance. *Review of Financial Studies*, 8(4):1059–1090.

Basak, S. (2002). A comparative study of portfolio insurance. *Journal of Economic Dynamics and Control*, 26(7-8):1217–1241.

Basak, S. and Chabakauri, G. (2009). Dynamic mean-variance asset allocation. *Review of Financial Studies*, Advance Access:–.

Basak, S. and Shapiro, A. (2001). Value-at-Risk based risk management: Optimal policies and asset prices. *Review of Financial Studies*, 14(2):371–405.

Basak, S. and Shapiro, A. (2005). A model of credit risk, optimal policies, and asset prices. *Journal of Business*, 78(4):1215–1265.

Berkelaar, A., Cumperayot, P., and Kouwenberg, R. (2002). The effect of VaR based risk management on asset prices and the volatility smile. *European Financial Management*, 8(2):139–164.

Bhattacharya, S., Plank, M., Strobl, G., and Zechner, J. (2002). Bank capital regulation with random audits. *Journal of Economic Dynamics and Control*, 26(7-8):1301–1321.

Bichsel, R. and Blum, J. (2005). Capital regulation of banks: Where do we stand and where are we going? *Quarterly Bulletin, Swiss National Bank*, 4:42–51.

Bjoerk, T. (2009). *Arbitrage theory in continuous time.* Oxford University Press.

Black, F. and Cox, J. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31(2):351–367.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–659.

Bliss, R. and Panigirtzoglou, N. (2004). Option-implied risk aversion estimates. *Journal of Finance*, 59(1):407–446.

Blum, J. (1999). Do capital adequacy requirements reduce risks in banking? *Journal of Banking and Finance*, 23(5):755–771.

Blum, J. (2008). Why 'Basel II' may need a leverage ratio restriction. *Journal of Banking and Finance*, 32(8):1699–1707.

Bodie, Z., Gray, D., and Merton, R. (2007). New framework for measuring and managing macrofinancial risk and financial stability. *SSRN eLibrary.*

Breeden, D. (1979). An intertemporal asset pricing model with stochastic investment and consumption opportunities. *Journal of Financial Economics*, 7(3):265–296.

Breton, M., Hugonnier, J. N., and Masmoudi, T. (2008). Mutual fund competition in the presence of dynamic flows. *SSRN eLibrary.*

Buraschi, A. and Jiltsov, A. (2006). Model uncertainty and option markets with heterogeneous beliefs. *Journal of Finance*, 61(6):2841–2897.

Calem, P. and Rob, R. (1999). The impact of capital-based regulation on bank risk-taking. *Journal of Financial Intermediation*, 8(4):317–352.

Caprio, G. and Klingebiel, D. (2002). Episodes of systemic and borderline banking crises. *Managing the real and fiscal effects of banking crises*, page 31.

Cass, D. and Stiglitz, J. (1970). The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. *Journal of Economic Theory*, 2(2):122–160.

Catarineu-Rabell, E., Jackson, P., and Tsomocos, D. (2005). Procyclicality and the new Basel accord-banks' choice of loan rating system. *Economic Theory*, 26(3):537–557.

Chamberlain, G. (1988). Asset pricing in multiperiod securities markets. *Econometrica*, 56(6):1283–1300.

Committee on Banking, Finance, and Urban Affairs of the United States Congress (1984). Inquiry into Continental Illinois Corp. and Continental Illinois National Bank: Hearings before the Subcommittee on Financial Institutions Supervision, Regulation and Insurance of the Committee on Banking, Finance, and Urban Affairs. 98th Congress, 2nd Session.

Constantinides, G. (1990). Habit formation: A resolution of the equity premium puzzle. *Journal of Political Economy*, 98(3):519–543.

Corsi, F., Mittnik, S., Pigorsch, C., and Pigorsch, U. (2008). The volatility of realized volatility. *Econometric Reviews*, 27(1):46–78.

Coudert, V. and Gex, M. (2008). Does risk aversion drive financial crises? Testing the predictive power of empirical indicators. *Journal of Empirical Finance*, 15(2):167–184.

Cox, J. and Huang, C. (1991). A variational problem arising in financial economics. *Journal of Mathematical Economics*, 20(5):465–487.

Cox, J. and Huang, C.-F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49(1):33–83.

Cox, J., Ingersoll, J., and Ross, S. (1985a). An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384.

Cox, J., Ingersoll, J., and Ross, S. (1985b). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407.

Cox, J., Ross, S., and Rubinstein, M. (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7:229–263.

Cuoco, D. and Liu, H. (2006). An analysis of var-based capital requirements. *Journal of Financial Intermediation*, 15(3):362–394.

Dangl, T. and Lehar, A. (2004). Value-at-Risk vs. building block regulation in banking. *Journal of Financial Intermediation*, 13(2):96–131.

Dangl, T. and Zechner, J. (2004). Credit risk and dynamic capital structure choice. *Journal of Financial Intermediation*, 13(2):183–204.

Danielsson, J. (2002). The emperor has no clothes: Limits to risk modelling. *Journal of Banking and Finance*, 26(7):1273–1296.

Danielsson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O., and Shin, H. (2001). *An academic response to Basel II.* School of Economics, Financial Markets Group.

Danielsson, J., Jorgensen, B., Mandira, S., Samorodnitsky, G., and de Vries, C. (2005). Subadditivity re-examined: The case for Value-at-Risk. *Working Paper.*

Danielsson, J., Shin, H., and Zigrand, J.-P. (2004). The impact of risk regulation on price dynamics. *Journal of Banking and Finance*, 28(5):1069–1087.

Das, S. and Uppal, R. (2004). Systemic risk and international portfolio choice. *Journal of Finance*, 59(6):2809–2834.

Davis, E. (2003). Towards a typology for systemic financial instability. *Economics.*

de Bandt, O. and Hartmann, P. (2000). Systemic risk: A survey. *European Central Bank Working Paper Series*, No. 35.

Decamps, J., Rochet, J., and Roger, B. (2004). The three pillars of Basel II: Optimizing the mix. *Journal of Financial Intermediation*, 13(2):132–155.

Detemple, J., Garcia, R., and Rindisbacher, M. (2003). A Monte Carlo method for optimal portfolios. *Journal of Finance*, 58(1):401–446.

Dewatripont, M. and Tirole, J. (1994). *The prudential regulation of banks.* MIT press Cambridge, MA.

Duffie, D. (2001). *Dynamic asset pricing theory.* Princeton University Press, Princeton, NJ.

Duffie, J. D. (2008). Innovations in credit risk transfer: Implications for financial stability. *SSRN eLibrary.*

Eisenberg, L. and Noe, T. (2001). Systemic risk in financial systems. *Management Science*, 47(2):236–249.

Elsinger, H., Lehar, A., and Summer, M. (2006). Risk assessment for banking systems. *Management Science*, 52(9):1301–1314.

Estrella, A. (2004). Bank capital and risk: Is voluntary disclosure enough? *Journal of Financial Services Research*, 26(2):145–160.

Estrella, A., Park, S., and Peristiani, S. (2000). Capital ratios as predictors of bank failure. *Economic Policy Review*, 6(2):33–52.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, 6(4):429–447.

Freixas, X., Lóránth, G., and Morrison, A. (2007). Regulating financial conglomerates. *Journal of Financial Intermediation*, 16(4):479–514.

Freixas, X., Parigi, B., and Rochet, J. (2000). Systemic risk, interbank relations, and liquidity provision by the central bank. *Journal of Money, Credit and Banking*, 32(3):611–638.

Freixas, X. and Rochet, J. (1999). *Microeconomics of banking*. MIT press.

Fries, S., Mella-Barral, P., and Perraudin, W. (1997). Optimal bank reorganization and the fair pricing of deposit guarantees. *Journal of Banking and Finance*, 21(4):441–468.

Gabih, A., Grecksch, W., and Wunderlich, R. (2005). Dynamic portfolio optimization with bounded shortfall risks. *Stochastic Analysis and Applications*, 23(3):579–594.

Goldstein, R., Ju, N., and Leland, H. (2001). An EBIT-based model of optimal capital structure. *Journal of Business*, 74(4):483–512.

González, F. (2005). Bank regulation and risk-taking incentives: An international comparison of bank risk. *Journal of Banking and Finance*, 29(5):1153 – 1184.

Goodhart, C. (1974). Public lecture at the reserve bank of australia.

Goodhart, C. (2006). A framework for assessing financial stability? *Journal of Banking and Finance*, 30(12):3415–3422.

Goodhart, C., Sunirand, P., and Tsomocos, D. (2006). A model to analyse financial fragility. *Economic Theory*, 27(1):107–142.

Goodhart, C. and Tsomocos, D. (2007). Analysis of financial stability. *Working Paper*.

Gordy, M. (2003). A risk-factor model foundation for ratings-based bank capital rules. *Journal of Financial Intermediation*, 12(3):199–232.

Gray, D. and Malone, S. (2008). *Macrofinancial risk analysis*. John Wiley & Sons Inc., Hoboken, NJ.

Gundel, A. and Weber, S. (2008). Utility maximization under a shortfall risk constraint. *Journal of Mathematical Economics*, 44(11):1126–1151.

Hakansson, N. H. (1971). Capital growth and the mean-variance approach to portfolio selection. *The Journal of Financial and Quantitative Analysis*, 6(1):517–557.

Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408.

Harrison, J. M. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.*, 11(3):215–260.

Hart, O. and Jaffee, D. (1974). On the application of portfolio theory to depository financial intermediaries. *Review of Economic Studies*, 41(125):129–147.

Hellmann, T., Murdock, K., and Stiglitz, J. (2000). Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough? *American Economic Review*, 90(1):147–165.

Hellwig, M. (1995). Systemic aspects of risk management in banking and finance. *Swiss Journal of Economics and Statistics*, 131(4/2):723–738.

Hombert, J. (2007). Fire sales: Revisiting the market equilibrium approach. *SSRN eLibrary*.

Honohan, P. and Klingebiel, D. (2000). Controlling the fiscal costs of banking crises. *World Bank Policy Research Working Paper*.

Hovakimian, A. and Kane, E. (2000). Effectiveness of capital regulation at US commercial banks, 1985 to 1994. *Journal of Finance*, 55(1):451–468.

Huang, C. (1987). An intertemporal general equilibrium asset pricing model: The case of diffusion information. *Econometrica*, 55(1):117–142.

Inada, K.-I. (1963). On a two-sector model of economic growth: Comments and a generalization. *Review of Economic Studies*, 30(2):119–127.

James, C. (1991). The losses realized in bank failures. *Journal of Finance*, 46(4):1223–1242.

Jeitschko, T. and Jeung, S. (2005). Incentives for risk-taking in banking–A unified approach. *Journal of Banking and Finance*, 29(3):759–777.

Jensen, M., Meckling, W., Field, S., and Park, T. (1976). Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics*, 3(4):305–360.

Jokivuolle, E., Peura, S., and Bank, S. (2003). Incorporating collateral value uncertainty in loss given default estimates and loan-to-value ratios. *European Financial Management*, 9(3):299–314.

Jones, C. (2003). The dynamics of stochastic volatility: Evidence from underlying and options markets. *Journal of Econometrics*, 116(1-2):181–224.

Kahane, Y. (1977). Capital adequacy and the regulation of financial intermediaries. *Journal of Banking and Finance*, 1(2):207–218.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291.

Kaniel, R. and Hugonnier, J. N. (2008). Mutual fund portfolio choice in the presence of dynamic flows. *SSRN eLibrary*.

Kaplanski, G. and Levy, H. (2007). Basel's Value-at-Risk capital requirement regulation: An efficiency analysis. *Journal of Banking and Finance*, 31(6):1887–1906.

Karatzas, I., Lehoczky, J., and Shreve, S. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM Journal on Control and Optimization*, 25(6):1557–1586.

Karatzas, I., Lehoczky, J., and Shreve, S. (1990). Existence and uniqueness of mulit-agent equilibrium in a stochastic, dynamic consumption/investment model. *Mathematics of Operations Research*, 15(1):80–128.

Karatzas, I. and Shreve, S. (1998). *Methods of mathematical finance*. Springer Verlag, New York.

Kim, D. and Santomero, A. (1988). Risk in banking and capital regulation. *Journal of Finance*, 43(5):1219–1233.

Koehn, M. and Santomero, A. (1980). Regulation of bank capital and portfolio risk. *Journal of Finance*, 35(5):1235–1244.

Korn, R. (1997). *Optimal portfolios*. World Scientific Publ.

Koziol, C. and Lawrenz, J. (2009). What makes a bank risky? Insights from the optimal capital structure of banks. *Journal of Banking and Finance*, 33(5):861–873.

Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950.

Kupiec, P. (2007). Financial stability and Basel II. *Annals of Finance*, 3(1):107–130.

Latane, H. (1959). Criteria for choice among risky ventures. *The Journal of Political Economy*, 67(2):144.

Lehar, A. (2005). Measuring systemic risk: A risk management approach. *Journal of Banking and Finance*, 29(10):2577–2603.

Leippold, M., Trojani, F., and Vanini, P. (2006). Equilibrium impact of Value-at-Risk regulation. *Journal of Economic Dynamics and Control*, 30(8):1277–1313.

Leland, H. and Pyle, D. (1977). Informational asymmetries, financial structure, and financial intermediation. *Journal of Finance*, 32(2):371–387.

Leland, H. and Toft, K. (1996). Optimal capital structure, enogeneous bankruptcy, and the term-structure of capital spreads. *Journal of Finance*, 51(3):987–1019.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49(4):1213–1252.

Liu, J. and Longstaff, F. (2004). Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities. *Review of Financial Studies*, 17(3):611–641.

Liu, J. and Pan, J. (2003). Dynamic derivative strategies. *Journal of Financial Economics*, 69(3):401–430.

Loewenstein, M. and Willard, G. (2000). Local martingales, arbitrage, and viability: Free snacks and cheap thrills. *Economic Theory*, 16(1):135–161.

Lucas, R. (1976). Econometric policy evaluation: A critique. u: Brunner karl i meltzer allan h.(ur.) the phillips curve and labor markets: Carnbegie-rochester conference series on public policy. *North-Holland*, 1:19–46.

Lucas, R. (1978). Asset pricing in an exchange economy. *Econometrica*, 46(6):1429–1445.

Merton, R. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, pages 247–257.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413.

Merton, R. (1973a). An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887.

Merton, R. (1973b). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4(1):141–183.

Merton, R. (1974). On the pricing of corporate debt. *Journal of Finance*, 29(2):449–470.

Merton, R. (1978). On the cost of deposit insurance when there are surveillance costs. *Journal of Business*, 51(3):439–52.

Merton, R. (1990). *Continuous-time Finance*. Blackwell Publishers, Cambridge, MA.

Merton, R. C. (1977). An analytic derivation of the cost of deposit insurance and loan guarantees an application of modern option pricing theory. *Journal of Banking and Finance*, 1(1):3–11.

Modigliani, F. and Miller, M. (1958). The cost of capital, corporation finance, and the theory of investment. *American Economic Review*, 48(3):261–297.

Nielsen, L. and Vassalou, M. (2006). The instantaneous capital market line. *Economic Theory*, 28(3):651–664.

Nier, E., Yang, J., Yorulmazer, T., and Alentorn, A. (2007). Network models and financial stability. *Journal of Economic Dynamics and Control*, 31(6):2033–2060.

Obstfeld, M. (1994). Risk-taking, global diversification, and growth. *American Economic Review*, 84(5):1310–1329.

Peltzman, S. (1975). The effects of automobile safety regulation. *Jounal of Political Economy*, 83(4):677–725.

Pliska, S. (1986). A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operations Research*, 11(2):371–382.

Pliska, S. (1997). *Introduction to mathematical finance: discrete time models*. Blackwell Publishers.

Pratt, J. (1964). Risk aversion in the small and in the large. *Econometrica*, 32(1-2):122–136.

Pyle, D. (1971). On the theory of financial intermediation. *Journal of Finance*, 26(3):737–747.

Repullo, R. and Suárez, J. (2004). Loan pricing under Basel capital requirements. *Journal of Financial Intermediation*, 13(4):496–521.

Rochet, J. (1992). Capital requirements and the behaviour of commercial banks. *European Economic Review*, 36(5):1137–1170.

Ross, S. (1978). Mutual fund separation in financial theory—the separating distributions. *Journal of Economic Theory*, 17(2):254–286.

Rubinstein, M. (1974). An aggregation theorem for securities markets. *Journal of Financial Economics*, 1(3):225–44.

Schachermayer, W., Sîrbu, M., and Taflin, E. (2009). In which financial markets do mutual fund theorems hold true? *Finance and Stochastics*, 13(1):49–77.

Schuermann, T. (2004). What do we know about loss given default? *SSRN eLibrary*.

Serrat, A. (2001). A dynamic equilibrium model of international portfolio holdings. *Econometrica*, 69(6):1467–1489.

Shin, H. (2008). Risk and liquidity in a system context. *Journal of Financial Intermediation*, 17(3):315–329.

Sircar, K. and Papanicolaou, G. (1998). General Black-Scholes models accounting for increased market volatility from hedging strategies. *Applied Mathematical Finance*, 5(1):45–82.

Stefanova, D. (2008). Dependence modeling of joint extremes via copulas: A dynamic portfolio allocation perspective. *Working Paper*.

Strotz, R. (1955). Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies*, 23(3):165–180.

Tsomocos, D. (2003). Equilibrium analysis, banking, and financial instability. *Journal of Mathematical Economics*, 39(5-6):619–655.

von Neumann, J. and Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton University Press, Princeton, NJ.

Wagner, W. (2006). Diversification at financial institutions and systemic crises. *Working Paper*.

Wagner, W. (2008). The homogenization of the financial system and financial crises. *Journal of Financial Intermediation*, 17(3):330–356.

# Lebenslauf

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|---|---|
| Seit Aug. 2005 | **Universität Mannheim**<br>Lehrstuhl für ABWL und Finanzierung<br>Wissenschaftlicher Mitarbeiter |
| Okt. 2000 - Juni 2005 | **Universität Karlsruhe (TH)**<br>Studium des Wirtschaftsingenieurwesens<br>Abschluss: Diplom-Wirtschaftsingenieur |
| Juni 1999 | Abitur am Ludwig-Wilhelm-Gymnasium, Rastatt |
| Okt. 1979 | Geboren in Malsch |