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Moritz Kuhn

Contents

List of Figures

List of Tables

Introduction

In industrialized countries, the single most important source of income for most people is income earned in the labor market. It is used to finance, among other things, all kinds of consumption ranging from everyday expenditures for food, clothes, and services like hair-cuts to long-lasting consumption goods like cars or TVs. Uncertainty about future individual labor income raises worries that a current standard of living has to be adjusted in the future to respond to new income situations. It is common to most people that such risky expectations harm their well-being. Consequently, if they had the choice they would prefer to live in a world where they face no income risk but earn just average income. This *risk aversion* leads people to seek insurance to insulate consumption, i.e. their standard of living, from income risk. But in general, insurance markets for individual income risk are missing or at least not perfect in the sense that not every contingency can be insured. If insurance markets are in this way *incomplete*, then agents seek for other ways to achieve *consumption smoothing*.

The questions of how households adjust their consumption-saving behavior in the presence of income risk and of how much insurance they can still achieve in a world with incomplete insurance markets have been a fruitful field of economic research for decades. In my dissertation, I build on existing results and extend a workhorse model to empirically relevant, yet unexplored, economic environments to learn more about these important questions.

Obviously, the effects of income risk on individual decisions depend on the properties and characteristics of income risk. Transitory fluctuations in income might be easily dealt with by agents without much harm to their welfare. Agents can put aside some money in good times to be prepared for a rainy day. To do so, a simple risk-free asset suffices. This way households are able to smooth their consumption and insulate consumption from income risk on their own, without sophisticated financial instruments or insurance contracts. In economics, this *self-insurance* behavior is well-understood and generally known as *buffer stock saving*. But what happens if income risk is not transitory but permanent? This question has attracted increasing attention over the last years for two reasons: (*i*) The empirical literature on labor income risk has provided broad support for permanent components in individual income risk, and (*ii*) the intuitively appealying idea of *buffer stock saving* does no longer apply in cases with permanent income risk.

In chapters 1 and 2 of my dissertation, I deal with the optimal consumption-saving behavior in the presence of permanent income risk. I do this, first from a theoretical perspective and then by studying the quantitative consequences of the theory. In these chapters, income risk is represented by exogenous fluctuations in income. Chapter 3 departs from the consumption-saving problem and constitutes a starting point to understand better the sources and characteristics of income risk. In this chapter, which is joint work with Philip Jung, we study job losses and creation of new jobs in the labor market. Although the analysis remains at this stage at an aggregate level, we hope to develop, starting from there, a better understanding of the sources and characteristics of individual income risk. The next paragraphs outline the chapters of my dissertation and describe the main results.

In chapter 1, I provide a new model to study the consumption-saving decision in the presence of permanent income risk. It is an extension of the classical Aiyagari model. Aiyagari style economies are a workhorse model of quantitative research. For this economy, I prove the existence of a recursive competitive equilibrium and show that there exist equilibria where borrowing constraints are never binding. This allows me to establish a non-trivial lower bound on the equilibrium interest rate. To solve the individual consumption-saving problem, I present a new approach that uses lattices of consumption functions to deal with the non-compact state space and the unbounded utility function of the problem. The approach uses only the first order conditions of the problem (*Euler equations*). The proof is constructive and it serves as a theoretical foundation for the convergence of a policy function iteration procedure.

Chapter 2 uses the model presented in chapter 1 for a quantitative analysis. The paper builds on the existing results that in models with transitory income shocks and trade in a riskless asset the welfare loss of missing insurance markets is small, and that with permanent income shocks and no trade in assets the welfare losses can become substantial. I consider an empirically relevant case in between the two extreme cases with permanent income shocks but trade in a riskless asset. I show that welfare losses from missing insurance markets in this model are still substantial. Furthermore, I show that one can closely align the welfare effects in a model without asset trade to the welfare effects in a model with asset trade by scaling the volatility of income uncertainty. I derive a scaling factor that coincides with the labor income share of total income and provide a closed form approximation formula to describe the welfare costs of permanent income shocks in the presence of market incompleteness and self-insurance. From these findings, I conclude that asset trade provides an effective channel for self-insurance also in models with permanent income shocks.

In chapter 3, that is joint work with Philip Jung, we document that the firing rate volatility in Germany is 2.5 times as high as in the U.S. and contributes $60-70\%$ to the aggregate unemployment volatility, the opposite of what is found for the US. We show that wage rigidities are not at the root of these large differences. To explain the cross-country differences, we develop a labor market search model with endogenous firings, quits on the job, and match heterogeneity. We calibrate the model for Germany and the US jointly to study the institutional differences that generate the differences in business cycle behavior. We show that the model predicts the observed time-series pattern of important labor market variables for both countries well. We show that institutional differences generating lower average hiring and firing rates amplify the response of the economy to aggregate shocks. At the same time, they are responsible for substantial differences in the persistence of the unemployment rate, explaining the sluggish response to shocks in Germany compared to the US.

The three chapters of the thesis are self-contained and can be read separately. All chapters are followed by an appendix with additional material.

Chapter 1

Recursive equilibria in an Aiyagari style economy with permanent income shocks

1.1 Introduction

Over the last two decades, a large literature has studied the effects of income uncertainty on individual behavior in heterogeneous agents incomplete markets economies, a model class that is widely known as Aiyagari style models.[1] While applied researchers have extensively studied this class of models numerically, theoretical results on the existence, characterization, and computation of equilibria are rare. This paper makes three theoretical contributions with important economic implications. We prove the existence of recursive competitive equilibria (RCE) for an Aiyagari style model where income shocks are permanent. The proof is constructive and contains a convergence proof for a popular computational algorithm based on the first-order conditions of the agent's problem (*policy function iteration*). Regarding the characterization, we prove the existence of equilibria with non-binding borrowing constraints and with a non-trivial lower bound on the equilibrium interest rate. The characterization of the equilibrium allocation allows us to derive further important implications for the optimal consumption-saving decision of agents in equilibrium.

Applied researchers studying Aiyagari style economies have focused on finding RCE numerically trusting on their existence. In line with these studies, Duffie et al. (1994) and Miao (2006) have provided existence proofs for RCE where the state space is a compact set. The elements of the equilibrium description, like the optimal policy function or the distribution over individuals on the state space, are then functions (distributions) on a compact domain (support). Although the assumption of a compact state space seems to be a rather technical issue, it imposes important

[1] See for example Aiyagari (1994), Huggett (1993), Telmer (1993) or the textbook by Ljungqvist and Sargent (2000)

economic restrictions on individual income processes. For example, it rules out the possibility that the income process contains a unit root. However, the unit-root specification has become quite popular in the empirical literature on income risk because various empirical studies have provided evidence that individual income risk contains transitory and permanent (unit root) components.[2] Therefore, the analysis of a model with a non-compact state space does not only address a theoretical gap but it also provides the foundation to study the implications of permanent income shocks on the consumption-saving decision in Aiyagari style economies.

The equilibrium existence proof comprises three steps. The first step is to show the existence of an optimal solution to the agents' problem. The seminal textbook by Stokey and Lucas (1989) establishes the value function approach, the contraction property of the Bellman equation, and the principle of optimality as the standard tools to prove the existence of a solution for this kind of problem. In this paper, we depart from this approach by relying only on first order conditions of the agents' problem (*Euler equations*) to prove the existence of an optimal policy function.[3] Similar approaches have been taken in Deaton and Laroque (1992), Coleman (1991), and Rabault (2002). All three papers deal with functions on a metric space and in the case of Deaton and Laroque (1992) and Coleman (1991) apply only to problems with a compact state space and bounded utility.[4] Instead of dealing with functions in a metric space, we use a lattice of consumption functions and apply Tarski's fixed point theorem to prove the existence of a recursive policy function. This allows us to deal with the non-compactness of the state space and unboundedness of the utility function. Since the proof is constructive it establishes the convergence of the *policy function iteration* algorithm for consumption-saving problems, and thereby provides a theoretical justification for its widespread use. To our knowledge, this proof has been missing from the literature.[5]

In the second step of the existence proof, we show that a unique stationary distribution exists, and in step three we derive the existence of a market clearing interest rate. As it turns out, the presence of prudence, i.e. strictly convex marginal utility, is crucial in order to get precautionary savings in an equilibrium with permanent income shocks. The reason is that borrowing constraints are potentially non-binding. This complements findings in Huggett and Ospina (2001), who have shown that in models with mean-reverting shocks, prudence of agents is not needed to get precautionary savings because borrowing constraints are always binding

[2]For example Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Blundell, Preston, and Pistaferri (2008).

[3]Although the present paper focuses on the case of permanent income shocks, this step of the proof is presented for a general class of consumption-saving problems with Markovian income processes.

[4]Coleman (1991) analyzes a representative agent model. This changes the operator on the Euler equation.

[5]The approach in Deaton and Laroque (1992) and Coleman (1991) covers only the case of a compact state space. Furthermore, the operator in Coleman applies only to a representative agent economy. The approach by Rendahl (2007) assumes bounded utility and still relies on the convergence of the value function iteration.

for some agents.

In fact, the existence of equilibria with non-binding borrowing constraints follows as a corollary to the existence proof. This result is of particular interest because it opposes the finding in standard incomplete markets models, where there is an intimate link between the existence of equilibria and binding borrowing constraints. Hence, it shows that the non-existence result for RCE with non-binding borrowing constraints on a compact state space (Krebs (2004)) does not extend to the case of a non-compact state space. The two sources of market incompleteness, namely missing insurance markets for idiosyncratic risk and borrowing constraints, can now be disentangled. This suggests that the existence of precautionary savings in Huggett and Ospina (2001) is indeed driven by the market imperfection induced by the borrowing constraint rather than by incomplete insurance markets, although the two sources of market incompleteness are intimately linked in models with mean-reverting shocks.

The present paper is not the first to study the implications of permanent income shocks. Constantinides and Duffie (1996) and Krebs (2007) are two examples that do this in a general equilibrium setup. The prediction for the consumption-saving decision from these papers is, however, highly stylized. The structure of the endowment process in these models allows the construction of no trade equilibria where all agents consume their endowment of the current period.[6] In contrast to these models, we consider a production economy. The consumption-investment good is produced using capital and labor as inputs to a neoclassical production function. Consequently, in equilibrium some agents have to hold positive assets, for which they receive a deterministic income in return. This rules out autarkic equilibria as they are constructed in the earlier papers.

Turning to our last result, we show that non-binding borrowing constraints imply a non-trivial lower bound on the equilibrium interest rate. This lower bound coincides with the equilibrium interest rate in no trade economies as in Krebs (2007). The reason for the higher interest rate in our model stems from the fact that in a production economy agents must hold on average assets in positive net supply.[7] The lower bound allows us to relate our results to existing partial equilibrium studies that examine consumption-saving decisions with permanent income shocks, like Deaton (1991) and Carroll (2004). In these studies, the authors restrict the interest rates to values that are below the lower bound that we establish. This provides an explanation for why they find borrowing constraints to be always binding.[8] These models predict, therefore,

[6]Heathcote et al. (2009) build on this model setup to sustain analytic tractability in a model with permanent shocks but they allow for insurance of a certain fraction of income shocks.

[7]In Krebs (2007), the bond is in zero net supply.

[8]Carroll (2004) allows for zero income shocks and for transitory shocks. These additional shocks induce savings in his model. If we drop these additional shocks, the model reduces to the Deaton (1991) case, and we will find again that borrowing constraints are always binding.

long-run consumption dynamics that are similar to those of models with autarkic equilibria like in Constantinides and Duffie (1996) and Krebs (2007), where consumption tracks income one-to-one.[9] In contrast, the model in this paper features asset trade in equilibrium, so that income shocks will not affect consumption one-to-one.

The rest of the paper is structured as follows: Section 1.2 presents the model. The existence of an optimal solution to the individual's problem is established in section 1.3. This section is more general and applies to a large class of Markovian income processes. In section 1.4, we prove the existence of a stationary distribution, and in section 1.5, we prove that a RCE exists. The discussion on borrowing constraints and the implications for the consumption-saving decision follows in section 1.6. Section 1.7 concludes. All proofs can be found in the appendix.

1.2 The model

We take time to be discrete and the periods are labeled by an index $t \in \mathbb{N}$. The economy is populated by a continuum of mass 1 of ex ante identical agents.[10] Every agent has an infinite planning horizon, but faces a constant probability of death in every period. An agent who dies is replaced by a newborn agent. The initial endowment in assets and labor productivity $\{a_0, z_0\}$ is drawn from a possibly degenerate distribution $\lambda(a, z, r)$. At the beginning of her life every agent chooses a recursive policy function that determines her behavior over time. We normalize the time endowment of every agent in every period to unity and assume an inelasitc labor supply of this unit of time. The only choice the agent has to make in the model is a consumption-saving decision. We assume that the preferences of agents over recursively generated consumption plans can be represented by the expected discounted sum of constant relative risk aversion (CRRA) utility functions.

Assumption 1. *The period utility function is of the CRRA type*

$$u(c) = \begin{cases} \log(c) & \gamma = 1 \\ \frac{c^{1-\gamma}}{1-\gamma} & otherwise \end{cases} \tag{1.1}$$

We denote the productivity state in period t by z_t.[11] The shocks to labor productivity are

[9] However, the model by Carroll (2004) generates a reaction that is less than one-to-one if all sources of income risk (transitory and zero income shock) as specified in the model are employed.

[10] We are aware of the technical issues regarding the measurability problem for models with a continuum of agents and i.i.d. income shocks. But we refer the interested reader to Green (1994) for detailed discussion of the appropriate construction of the set of agents to preserve measurability for all subset of agents. From now on we apply the law of large numbers in this paper without further discussion.

[11] Throughout, we do not use subscripts for individuals because they only increase the notational burden and are not necessary for the proofs.

permanent, and we allow for a wide range of distributions for the innovation term. To capture the fact that an agent who died is replaced by a newborn agent, we use the following augmented labor productivity process

$$z_{t+1} = \begin{cases} z_t \varepsilon_{t+1} & \eta_{t+1} = 1 \\ z_0 & otherwise \end{cases} \tag{1.2}$$

ε_{t+1} denotes the shock to labor productivity that is realized at the beginning of period $t+1$, and η_{t+1} denotes a survival shock. For simplicity we assume that η_{t+1} has a binomial distribution. A realization $\eta_{t+1} = 1$ means that an agent survives the transition from period t to $t+1$. We also allow for transitory i.i.d. income shocks. We denote the transitory income shock in period t by ζ_t. We make the following assumptions on the random variables

Assumption 2. *The distributions of ε, ζ and η satisfy*

(i)	$\nexists e \in supp(\varepsilon) : Prob(e) = 1$	*(vi)*	$\mathbb{E}[\zeta] = 1$	
(ii)	$Prob(\varepsilon > 0) = 1$	*(vii)*	$Prob(\zeta > 0) = 1$	
(iii)	$Prob(\eta = 0) = \theta > 0$	*(viii)*	$\mathbb{E}[\zeta_t \varepsilon_s] = \mathbb{E}[\zeta_t]\,\mathbb{E}[\varepsilon_s]$	$\forall s, t \geq 0$
(iv)	$\mathbb{E}[\varepsilon] = 1$	*(ix)*	$\mathbb{E}[\zeta^{1-\gamma}] = M < \infty$	
(v)	$\beta \mathbb{E}[\varepsilon^{1-\gamma}] < 1$			

1.2.1 Agent's problem

We assume that the objective of the agent is to maximize her expected discounted lifetime utility from consumption. The objective function is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} ((1-\theta)\tilde{\beta})^t \, u(c_t) \middle| \mathcal{F}_0\right] \tag{1.3}$$

where $\tilde{\beta}$ is the time discount factor and $(1-\theta)$ is the probability of surviving from period t to $t+1$. Hence, expectations are only taken with respect to the realization of the stochastic productivity process $\{\varepsilon_{t+1}\}_{t=0}^{\infty}$ and the sequence of transitory income shocks $\{\zeta_{t+1}\}_{t=0}^{\infty}$. By $\mathcal{F}_t$ we denote the information set of the agent in period t. The set of admissible consumption choices is restricted by the fact that every plan must satisfy the intertemporal budget constraint

$$c_t + a_{t+1} = (1+r)a_t + w_t z_t \zeta_t \tag{1.4}$$

together with a no Ponzi condition. The condition we impose to rule out Ponzi schemes is an ad hoc debt constraint $a_{t+1} \geq 0$ for all periods $t > 0$. We discuss the impact of this borrowing constraint in section 1.6.

The state space S for this problem is the Cartesian product of possible asset holdings and productivity states. The information set $\mathcal{F}_t$ for every period contains the current state of the agent $\{a_t, z_t\}$ and all prices.
When we collect all ingredients to the agent's decision problem, we can write it as an optimal control problem under uncertainty

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} ((1-\theta)\tilde{\beta})^t \, u(c_t) \middle| \mathcal{F}_0\right] \\
s.t. \quad & c_t + a_{t+1} = (1+r)a_t + w_t z_t \zeta_t \qquad \forall t \\
& z_{t+1} = \; z_t \varepsilon_{t+1} \qquad \forall t \\
& \{a_{t+1}, c_t\} \in [0, \infty) \times \mathbb{R}_+ \;\; \forall t \\
& \{a_0, z_0\} \subset \mathcal{F}_0
\end{aligned} \tag{1.5}$$

To simplify notation, we replace $(1-\theta)\tilde{\beta}$ by an implicit discount rate β

$$\beta := (1-\theta)\tilde{\beta}$$

Assumption 3. *θ and $\tilde{\beta}$ are such that $\beta < 1$.*

1.2.2 Firm's problem

Production in the model takes place in a perfectly competetive production sector. We model the production side of the economy as a representative firm producing at marginal costs. We assume that production takes place using a standard neoclassical production function.

Assumption 4.

$$Y_t = F(K_t, L_t) = L_t f(k_t) \tag{1.6}$$

$$F(0, L_t) = F(K_t, 0) = 0$$

and $f'(k_t) > 0$, $f''(k_t) < 0$.

where L_t denotes labor in productivity units, i.e. labor supply times productivity aggregated over all individuals. We construct the productivity process below such that aggregate effective labor supply is $L_t \equiv 1$ in all periods. From the first order conditions there exists a one-to-one mapping from wages to interest rates

$$w = f(f'^{-1}(r+\delta)) - (r+\delta) f'^{-1}(r+\delta) \tag{1.7}$$

We make the following assumption for the depreciation rate and the discount factor.

Assumption 5. *At $\bar{k}$ defined by*

$$\delta\bar{k} = f(\bar{k})$$

it holds that

$$\left(\beta(1 + f'(\bar{k}) - \delta)^{1-\gamma}\right)^{\frac{1}{\gamma}} < 1$$

The assumption imposes joint restrictions on the preferences of individuals and the production technology. This technical assumption is only needed to make sure that for every possible aggregate capital stock there exists a strictly positive lower bound to the consumption function. It can be easily verified that for a risk aversion parameter $\gamma \leq 1$, which includes the important case of log utility, the assumption does not impose any additional restrictions on the choice for model parameters.

1.2.3 Bequests and the probability of death

The reason to assume a constant probability of death is to guarantee the existence of a stationary distribution. To make the bequest scheme resource feasible, we require that in equilibrium bequests must be equal to asset holdings of agents who die.

Assumption 6. *The initial endowments $\{a_0, z_0\}$ of agents are drawn from some distribution $\lambda(a, z, r)$ that is continuous in r and satisfies*

$$\begin{aligned} \int z\lambda(da, dz, r) &= 1 \\ \int a\lambda(da, dz, r) &= f'^{-1}(r + \delta) \end{aligned}$$

The assumptions on the means ensure that the average labor productivity in the population is always one and that the assets allocated to the newborn generation equal on average the bequests of the old generation in equilibrium.

1.2.4 Equilibrium

We define a *recursive competitive equilibrium* (RCE) for this economy as a set of recursively generated asset choices $\{a^*_{t+1}\}$ and consumption choices $\{c^*_t\}$, a capital and labor demand K^d and L^d of the production sector together with equilibrium prices r^* and w^* and a stationary equilibrium distribution $\mu(a, z)$ over asset and productivity levels of agents such that

1. For every agent there is the sequence of recursively generated asset choices $\{a^*_{t+1}\}_{t=0}^{\infty}$ and consumption choices $\{c^*_t\}_{t=0}^{\infty}$ that solve the agent's optimization problem in (1.5) given equilibrium prices w^* and r^*.

2. The firm's demand for capital K^d and labor L^d maximizes firm's profits given equilibrium prices w^* and r^*.

3. Equilibrium prices are such that

$$\begin{aligned} \int a_t^* \mu(da, dz) = \quad & K^* \quad = K^d \qquad \forall t \\ \int z_t \mu(da, dz) = \quad & L^* \quad = L^d \qquad \forall t \end{aligned}$$

1.3 Individual problem

In this section, we consider a more general consumption-saving problem where we allow for a larger class of Markovian labor productivity processes and looser ad hoc debt constraints. However, we still require that

$$Prob(wz_t\zeta_t - rD > 0) = 1$$

The generalized consumption-saving problem is

$$\begin{aligned} \max_{c_t, a_{t+1}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_0 \right] \\ s.t. \quad & c_t + a_{t+1} = (1+r)a_t + wz_t\zeta_t \\ & z_{t+1} = f(z_t, \varepsilon_{t+1}) \\ & a_{t+1} \geq -D \\ & c_t \geq 0 \\ & \{a_0, z_0\} \subset \mathcal{F}_0 \end{aligned} \tag{1.8}$$

where $f(z_t, \varepsilon_{t+1})$ is the (Markovian) law of motion for $\{z_t\}_{t=0}^{\infty}$. We reformulate the problem using cash-at-hand. We define

$$x_t := (1+r)a_t + wz_t\zeta_t + D$$

and get

$$
\begin{aligned}
\max_{c_t} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\middle|\, \mathcal{F}_0\right] \\
s.t. \quad & x_{t+1} = (1+r)(x_t - c_t) + w z_{t+1}\zeta_{t+1} - rD \\
& z_{t+1} = f(z_t, \varepsilon_{t+1}) \\
& x_t \geq c_t \\
& c_t \geq 0 \\
& \{x_0, z_0\} \subset \mathcal{F}_0
\end{aligned}
\tag{1.9}
$$

1.3.1 Characterization of the optimal solution

We know that every optimal solution to (1.9) must satisfy the first order conditions.

$$
c_t^{-\gamma} + \kappa_t = \beta(1+r)\mathbb{E}\left[c_{t+1}^{-\gamma}\middle|\, \mathcal{F}_t\right] \qquad \forall t \tag{1.10}
$$

$$
\kappa_t(x_t - c_t) = 0 \qquad \forall t \tag{1.11}
$$

where κ_t denotes the Lagrange multiplier on the debt constraint. In a RCE the optimal consumption plan must obey a recursive structure. Therefore, we restrict attention to optimal solutions that have a recursive structure of the form

$$
c_t = c(x_t, z_t)
$$

where the dependence on z_t is necessary if the conditional distribution of income next period depends on the current state[12].

Once we have restricted the optimal solution to obey a recursive structure, the problem of finding a solution to the first order conditions can be formulated as finding a fixed point to the following equation

$$
c(x,z) = \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[(c(x',z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \tag{1.12}
$$

where the min-operator captures the complementary slackness condition in (1.11). This approach has been proposed by Deaton and Laroque (1992) and has been applied to consumption-

[12] It has been shown for example in Deaton (1991) that this dependence can be removed in the case of permanent income shocks.

saving problems in Deaton (1991) and Rabault (2002)[13]. In the following, we establish the existence of a fixed point $c(x, z)$ to the modified Euler equation in (1.12). To establish the existence of a fixed point, we restrict the interest rate to a set $\left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$. As we show below, this is sufficient to establish the existence of a RCE.

1.3.2 Existence of an optimal solution

We have formulated the search for an optimal solution to the agents' problem as a fixed point problem of the modified Euler equation. To prove the existence of a fixed point to this equation, we construct a lattice of consumption functions and an operator that is a selfmap on this set of functions. We then apply a version of Tarski's fixed point theorem to establish the existence of a fixed point to this operator in a constructive way. All definitions can be found in the appendix.

In the first step, we construct a set of candidate consumption functions for the optimal solution to the consumption-saving problem. We restrict attention to the following set of consumption functions

$$
\begin{aligned}
C_0 := \{ & c : X \times Z \to \mathbb{R}_+ | \\
& \forall x_1, x_2 \in X : x_1 > x_2 \Rightarrow c(x_1, z) \geq c(x_2, z) \wedge x_1 - x_2 \geq c(x_1, z) - c(x_2, z)\}
\end{aligned}
$$

Hence, we only consider consumption functions that are increasing and Lipschitz continuous (with Lipschitz constant $L = 1$) in their first argument. For this class of functions, we apply the usual pointwise ordering

$$c_1(x, z) \geq c_2(x, z) \qquad \forall (x, z) \in X \times Z \Rightarrow c_1 \geq c_2$$

In the appendix, we show (lemma 10) that we can restrict the set of candidate solutions further by imposing an upper and a lower bound (c^u and c^l) on the set of consumption functions. The reason is that the operator that we will construct below is inward pointing[14] at the bounds. The restricted set of candidate solutions in which we are looking for a solution is the set C

$$C := \{c \in C_0 : c^l \leq c \leq c^u\}$$

[13] Both authors iterate on the optimal marginal utility function whereas we iterate on the optimal consumption policy directly.

[14] We call the operator T inward pointing if for the upper bound $\bar{x}$ it holds that $T\bar{x} \leq \bar{x}$ and respectively for the lower bound $\underline{x}$ it holds that $T\underline{x} \geq \underline{x}$.

The next step is to show that this set C together with the ordering just defined forms a complete lattice. To this end, we need to show that the supremum and the infimum for arbitrary sets always exist. In the appendix, we prove that we get the supremum (infimum) of two consumption functions as the upper (lower) envelope. Hence, we obtain the supremum $\bar{c}$ (infimum $\underline{c}$) by taking the pointwise maximum (minimum).

$$\begin{aligned}\bar{c}(x,z) &= \max\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z \\ \underline{c}(x,z) &= \min\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z\end{aligned}$$

Equivalently, we get the supremum $\bar{c}^{\infty}$ (infimum $\underline{c}^{\infty}$) of a possibly infinite subset of consumption functions $C' \subset C$ as the upper (lower) envelope.

$$\begin{aligned}\bar{c}^{\infty}(x,z) &= \sup_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \\ \underline{c}^{\infty}(x,z) &= \inf_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z\end{aligned}$$

Since the set C has an upper bound c^u and a lower bound c^l the supremum and the infimum always exist, and it holds that $\bar{c}^{\infty} \leq c^u$ and $\underline{c}^{\infty} \geq c^l$. It follows that $(C, \leq)$ is a complete lattice.

In the next step, we construct an operator on this set of functions. The operator T maps an element $c_i \in C$ to an element c_{i+1}

$$c_{i+1} = Tc_i$$

by the following operation

$$\begin{aligned}&\forall (x,z) : \quad c_{i+1}(x,z) = \lambda \text{ where } \lambda \text{ solves} \\ &\lambda = \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}\end{aligned}$$

and we define the following function

$$G_i(x,z,\lambda) := \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} - \lambda \quad (1.13)$$

such that we can represent the operator as $c_{i+1} = Tc_i$ with $c_{i+1}(x,z) = \lambda$ iff $G(x,z,\lambda) = 0$ for all (x,z).

In the appendix, we prove that the function $G(x,z,\lambda)$ is (i) increasing and continuous in x, (ii) strictly decreasing and continuous in λ, and (iii) for fixed (x,z) there is a unique solution λ^* that solves $G(x,z,\lambda^*) = 0$. It follows, that the operator maps every element $c_i \in C$ to a unique element c_{i+1}. We prove that the operator has the properties of being (i) monotone increasing and (ii) a selfmap, i.e. $T : C \to C$. Furthermore, we prove that imposing an upper bound and a lower bound on the possible set of consumption functions is valid because the operator is inward pointing at these bounds. Thus, we have constructed a monotone increasing operator that is a selfmap on a complete lattice. This is already sufficient to prove the existence of a fixed point to the modifed Euler equation in (1.12) using the fixed point theorem by Tarski (1955).

Tarski 1. *Every monotone increasing mapping $T : X \to X$ on a complete lattice X has a smallest and a greatest fixed point.*

As the theorem does not require a contraction property of the operator it also lacks the uniqueness result of a contracting operator. The proof is not constructive and establishes only the existence of a fixed point. However, constructiveness is certainly a desirable property. A constructive version of Tarski's theorem exists for continuous operators. The continuity of the operator T can be proven by exploiting the properties of the lattice of consumption fucntions. This fact allows us to apply the constructive version of Tarski's fixed point theorem[15].

Tarski 2. *For $x^u := \sup(X)$, $x^l := \inf(X)$ and a continuous increasing mapping $T : X \to X$ on a complete lattice X we get that $\lim_{n\to\infty} T^n x^u$ and $\lim_{n\to\infty} T^n x^l$ converge to the largest resp. lowest fixed point $\bar{x}$ resp. $\underline{x}$ of $T : X \to X$.*

This constructive version of the iteration procedure proves the convergence of the standard numerical approach of policy function iteration. The policy function iteration algorithm starts with an initial guess for the policy function and applies the operator T repeatedly to this guess. If c^u is taken as initial guess, then iterating on the operator T will attain a fixed point to the modified Euler equation.

Since the first order conditions are only necessary for an optimal solution, we still have to check if the transversality condition is satisfied at our candidate solution. In the appendix, we show that under the maintained assumptions the transversality condition for the case of permanent income shocks is satisfied. We also state additional conditions for the case of general Markovian income processes and borrowing constraints with $D > 0$. We can summarize the results of this section in the following proposition.

Proposition 1. *Under the maintained assumptions there exists for every interest rate $r \in \left[f'(\bar{k}) - \delta, \beta^{-1} - 1\right]$ an optimal recursive policy function to the agents' problem. It can be found as $\lim_{n\to\infty} T^n c^u$.*

[15] The constructive version of the theorem results from Kleene's (1952) first recursion theorem. See Cousot and Cousot (1979) for discussion and further references.

1.4 Stationary distribution

For the existence of a stationary distribution, we again restrict attention to the case of permanent income shocks with a constant probability of death.[16]
The joint stochastic process for asset holdings and productivity is

$$\begin{bmatrix} a_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} \eta_{t+1}((1+r)a_t + wz_t - c^*(x_t, z_t)) + (1-\eta_{t+1})a_0 \\ \eta_{t+1} z_t \varepsilon_{t+1} + (1-\eta_{t+1}) z_0 \end{bmatrix}$$

where $c^*(x_t, z_t)$ denotes the optimal policy given r and w, and a_0 and z_0 are draws from $\lambda(a, z, r)$. In the appendix, we prove that a unique stationary probability distribution for the process always exists. The idea of the proof is to exploit the renewal structure induced by the constant probability of death. With a positive probability of death the expected life-time of an agent is finite. Every time an agent dies there is a draw from a fixed distribution λ and the process starts from the support of λ. This implies that all sets with positive λ-mass must also have positive μ-mass. These two features of the stochastic process imply that the process is *recurrent* and *irreducible* such that a unique stationary distribution exists.[17]
We also establish the continuity in the interest rate of the stationary distribution on the interval $[f'(\bar{k}) - \delta, \beta^{-1} - 1]$. The proof relies on a result by Le Van and Stachurski (2007).
We summarize the results of the current section in the follwoing proposition

Proposition 2. *Under the maintained assumptions there exists for every interest rate* $r \in [f'(\bar{k}) - \delta, \beta^{-1} - 1]$ *a unique stationary distribution* μ_r *that is continuous in* r *on* $[f'(\bar{k}) - \delta, \beta^{-1} - 1]$.

Indeed, the stationary distribution in this model is a mixture over distributions of agents of different *'age cohorts'*, where an *age cohort* at time T contains all agents that have survived for t periods from $T - t$ to T. If we introduce an operator P that maps the distribution of agents' asset holdings and productivity levels of one cohort to their next period's distribution conditional on survival, then the stationary distribution can be shown to be an infinite mixture over initial distributions

$$\mu_r = \sum_{t=0}^{\infty} (1-\theta)^t P^t \lambda(a, z, r)$$

Remark 1. *The operator P maps asset holdings and productivity from the current period's distribution to next periods distribution conditional on survival, it depends therefore on the optimal consumption policy because the consumption policy affects the transition of assets.*

[16] All proofs also apply to the more general case of a Markovian process, if there is a positive probability of death, and an optimal recursive consumption policy exists.

[17] Further details and an extensive study of stability of Markovian processes can be found in the textbook by Meyn and Tweedie (1993).

1.5 Equilibrium

In the previous sections, we have established the existence of an optimal recursive solution to the agents' problem and the existence of a stationary distribution for a wide range of interest rates. To statisfy the equilibrium conditions of a RCE in section 1.2.4, we have to find a stationary distribution μ_{r^*} such that all markets clear. The labor market is cleared by construction, and in the appendix, we show that the goods market clears for at least one interest rate in the set of interest rates for which an optimal solution to the agents' problem and a stationary distribution exist. The idea of the proof is to show that there is an interest rate low enough such that asset demand exceeds asset supply and an interest rate high enough such that the converse is true. Since asset demand and asset supply are continuous in the interest rate, there must be at least one interest rate in between where asset markets clear. This proves the existence of a RCE for this model.[18]

We summarize the results of this section again in a proposition.

Proposition 3. *Under the maintained assumptions a recursive competitive equilibrium always exists.*

When we establish the existence of an interest rate for which there is aggregate excess supply of capital, we find that for sufficiently high interest rates and only permanent income shocks borrowing constraints are not binding. For this case, we need that consumers are prudent, i.e. have a positive third derivative of the utility function, to rule out equilibria without positive precautionary savings. This case provides an example where the argument by Huggett and Ospina (2001) for the existence of precautionary savings does not apply. Their result of the irrelevance of prudence relies on the fact that borrowing constraints must be binding in equilibrium. However, as we show below, there are equilibria with incomplete markets and idiosyncratic income risk where borrowing constraints are non-binding and precautionary savings arise only due to prudence of consumers.[19]

1.6 Borrowing constraints

We have established the existence of a RCE in a model with permanent and transitory income shocks. In this section, we remove transitory income risk. This allows us to prove some

[18] Since a proof for the monotonicity of asset supply in the interest rate is lacking, we can not establish uniqueness of the equilibrium.

[19] The same bound for the interest rate at which borrowing constraints would be non-binding has been established in Rabault (2002) who studies the consumption-saving decision in a partial equilibrium framework. However, he puts it as an open question whether non-binding borrowing constraints can be sustained indefinitely if marginal utility at the optimal solution is bounded.

interesting properties of the equilibrium in this model. Especially, we prove that borrowing constraints *must* be non-binding. The following proposition summarizes this result

Proposition 4. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then borrowing constraints must be non-binding.*

To establish this result, it is important to recognize that the state space can be reduced to a single ratio variable[20]: *cash-at-hand to permanent labor income.* This variable is defined as

$$\tilde{x}_t := \frac{x_t}{wz_t} = (1+r)\frac{a_t}{wz_t} + 1$$

The reduction of the state space implies that the decision whether to save or not becomes independent of the current income level. However, the amount saved will still depend on the current level. This characteristic property[21] allows us to develop an intuitive understanding why borrowing constraints are non-binding.

Consider the case where asset holdings are zero ($\tilde{x}_t = 1$). At this point, the decision whether to save or not is the same for *all* agents. Suppose now that agents with no asset holdings decided not to save, to sustain a positive aggregate capital stock in equilibrium, some agents with already higher *cash-at-hand to permanent labor income* ratios must then decide to save. However, as we prove in the appendix, this behavior is not optimal in equilibrium. Hence, an optimal policy that is compatible with an equilibrium must be a policy where agents with zero assets do save, and borrowing constraints are non-binding. This intuitive explanation leads us to associate the result of non-binding borrowing constraints rather with the existence of permanent income shocks than with the non-compactness of the state space although the two properties are inherently related.

Exploiting the same property also provides a good starting point to develop an intuitive understanding for the optimal consumption-saving decision. First recall the case of mean-reverting shocks. In this situation, agents save income when they expect a future decline in income, and they spend additional funds —if available— when they expect a future growth in income. Hence, in situations with low income and low assets the borrowing constraint will be binding, and the savings decision depends crucially on the level of the current income state relative to the long-run mean of income. Intuitively, in a situation with mean-reverting shocks agents smooth income around the long-run mean by accumulating and decumulating assets. This behavior is generally known as *buffer-stock saving.* Optimal behavior with permanent income shocks must differ from this case because a long-run mean no longer exists. The current income is now

[20] This result is well-known and can be found in Deaton (1991). We establish the result in the appendix (lemma 16).

[21] The state space reduction requires both permanent income shocks and CRRA utility.

the best predictor for future income, and a policy that aims at smoothing income around this income level results either in a Ponzi scheme or in accumulating an infinite amount of assets. The optimal policy does therefore not smooth around an income level but aims at balancing the risk exposure of total income. If agents have a low *cash-at-hand to permanent labor income*, they face relatively much income risk because labor income constitutes a large fraction of total income, and as a result, they want to accumulate additional assets to reduce overall income risk. If agents have high *cash-at-hand to permanent labor income*, they face relatively little overall income risk and given the return on assets and their impatience, they are willing to reduce their asset holdings. This intuition implies that there is one *target insurance ratio* in the state space where agents do not want to rebalance their overall risk exposure further. Indeed, we prove the existence of a unique *target insurance ratio.*[22]

Corollary 1. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then there is a unique* $\bar{\bar{x}}$ *(target insurance ratio) such that the optimal policy is* $a_t = a_{t+1}$.

The corollary formally defines the *target insurance ratio* as the state in the reduced state space where the optimal decision of the agent is to keep assets constant between periods.[23] The uniqueness of the *target insurance ratio* implies that the dynamics induced by the optimal consumption saving decision drive —apart from stochastic fluctuations— the agents' cash-at-hand ratio towards the *target insurance ratio.* This aligns nicely with the intuition provided above that agents aim at balancing their risk exposure rather than sustaining a constant income level.

As a further corollary to the result of non-binding borrowing constraints, we can establish a non-trivial interval for the equilibrium interest rate[24].

Corollary 2. *If a RCE with non-binding borrowing constraints exists, then the equilibrium interest rate* r *lies in the interval* $[\underline{r}, \bar{r}] := \Big((\beta \mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1; \beta^{-1} - 1 \Big)$

The lower bound interest rate $\underline{r}$ separates three ranges for the interest rate that have all been independently studied in different strands of the literature with quite different implications for the consumption-saving decision.

[22] The proof can be found in the appendix.

[23] It is important to notice, that this does not coincide with the target insurance rate as defined in Carroll (2004) which is

$$\mathbb{E}[\tilde{x}_{t+1} | \mathcal{F}_t] = \tilde{x}_t$$

To see this, plug $\tilde{c}_t = \frac{r}{1+r}\tilde{x}_t + \frac{1}{1+r}$ in the law of motion for the ratio variable, this yields

$$\mathbb{E}[\tilde{x}_{t+1} | \mathcal{F}_t] = \mathbb{E}[\varepsilon^{-1}](\tilde{x}_t - 1) + 1 \neq \tilde{x}_t$$

[24] The proof can be found in the appendix.

One strand of the literature has studied economies where the interest rate is exactly at the lower bound $\underline{r}$. These are the endowment economies as studied for example in Krebs (2007). In this model, assets are in zero net supply and the interest rate is chosen to balance the desire to accumulate and decumulate assets for all agents and there will be no trade in equilibrium. In this situation, the *target insurance ratio* is exactly at one ($\bar{\tilde{x}} = 1$). This situation is not compatible with an equilibrium in a production economy where capital is an essential input in the production technology. Intuitively, the higher interest rate in the production economy can then be explained by the fact that agents need an additional incentive to accumulate assets.
The interest rates below the lower bound, i.e. $r < \underline{r}$, have been extensively studied in partial equilibrium models developed by Deaton (1991) and Carroll (1997, 2004). Deaton (1991) conjectures that agents always run down assets to zero, become borrowing constrained, and stay borrowing constrained forever. We prove that his interest rate is never an equilibrium interest rate, once we impose equilibrium restrictions on prices. The bound on the interest rate in the models by Deaton and Carroll naturally arises in the proof for the existence of an optimal policy function. It can, however, be shown that this condition can be slightly relaxed without loosing existence of the optimal solution if the lower bound on the optimal consumption function $\underline{c}$ (lemma 10) is taken into account. We exploit this property to prove that the transversality condition is always satisfied.

1.7 Conclusions

In this paper, we prove the existence of a recursive competitive equilibrium (RCE) for an Aiyagari style economy with permanent income shocks and a perpetual youth structure. The proofs presented in the literature for the existence of an equilibrium do not apply to this economy because they require a compact state space. To prove that there exists an optimal recursive solution to the agent's problem in our economy, we present an approach based only on first order conditions (*Euler equation*) and use lattices of consumption functions together with Tarski's fixed point theorem. This allows us to deal with the non-compact state space and an unbounded utility function. We present the approach for a general setting of Markovian income processes and show that it can be applied for a large class of consumption-saving problems. The fact that the proof is constructive serves as a theoretical foundation for the convergence of a *policy function iteration* algorithm that is popular in the quantitative literature.
In the second part of the paper, we prove that if there exists an equilibrium where only permanent income shocks are present, then borrowing constraints must always be non-binding. This shows that the non-existence result of equilibria with non-binding borrowing constraints on compact state spaces by Krebs (2004) does not extend to the case of a non-compact state

space. Importantly, this result is driven by the fact that income shocks are permanent rather than by the fact that the state space is non-compact.

From this result, we can establish the existence of a unique target insurance ratio and a non-trivial lower bound on the equilibrium interest rate. If we compare this lower bound to the interest rates in existing studies, we find that the interest rates in these studies are not compatible with the equilibrium interest rates in our model.

Appendix

A.1 Proofs and definitions for the existence of an optimal solution

A.1.1 Mathematical preliminaries

The definitions are taken mostly from Zeidler (1986).

Definition 1. *1. A set M is called ordered iff M is nonempty and for certain pairs $(x, y) \in M \times M$ there is a relation $x \leq y$ which satisfies*

(a) $x \leq x$ for all $x \in M$

(b) if $x \leq y$ and $y \leq x$ then $x = y$

(c) if $x \leq y$ and $y \leq z$ then $x \leq z$

The notation $x < y$ means that $x \leq y$ and $x \neq y$

2. Let $N \subseteq M$ and let M be ordered. The set N is called a chain (of M) iff N is nonempty and for all $x, y \in N$, one of the two conditions $x \leq y$ and $y \leq x$ holds.

3. Let $N \subseteq M$ again. The element $x \in N$ is called greatest or smallest in N iff $y \leq x$ or $x \leq y$, respectively, for all $y \in N$. The element $x \in N$ is called a maximal element of N iff there is no $y \in N$ such that $x < y$.

4. The ordered set M is called well ordered iff every nonempty subset of M has a smallest element.

Definition 2. *Let $y \in M$ and $N \subseteq M$. Then y is called the supremum (smallest upper bound) of N iff y is an upper bound of N, i.e. $x \leq y$ for all $x \in N$, and $y \leq u$ for all upper bounds u of N. We write $y = \sup(N)$. Similarly, $\inf(N)$ is defined to be the greatest lower bound.*

Definition 3. *By a lattice we mean an ordered set M with the property that* $\inf(\{x,y\})$ *and* $\sup(\{x,y\})$ *exist for all $x, y \in M$. A lattice is called complete iff* $\inf(N)$ *and* $\sup(N)$ *exist for all nonempty subsets N of M.*

Definition 4. *An operator T is called continuous iff for every chain S*

$$\sup T(S) = T(\sup(S))$$

and

$$\inf T(S) = T(\inf(S))$$

Definition 5. *An operator T is called monotone increasing if for $x \geq y$ it holds that $Tx \geq Ty$.*

A.1.2 Set of consumption functions as complete lattice

Define

$$\begin{aligned} \bar{c}(x,z) &:= \max\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z \\ \underline{c}(x,z) &:= \min\{c_1(x,z), c_2(x,z)\} \qquad \forall (x,z) \in X \times Z \end{aligned}$$

Lemma 1. *For every two consumption functions $c_1, c_2 \in C$, it holds that $\underline{c} = \inf\{c_1, c_2\}$ and $\bar{c} = \sup\{c_1, c_2\}$. Furthermore, it holds that $\underline{c}, \bar{c} \in C$.*

Proof. Suppose not. Suppose there is a $\hat{c}$ such that $\hat{c} \geq c_1$ and $\hat{c} \geq c_2$ but $\hat{c} < \bar{c}$. This yields immediately a contradiction because $\bar{c}(x,z) = \max\{c_1(x,z), c_2(x,z)\}$ and it holds that either $\hat{c} \ngeq c_1$ or $\hat{c} \ngeq c_2$ or $\hat{c} \leq c_1$ or $\hat{c} \leq c_2$. The argument for $\underline{c}$ is equivalent.
We have $c_1, c_2 \in C$, and therefore, it holds that $\bar{c} \in C$ because $\bar{c}$ is the piecewise continuous composition of parts of c_1 and c_2. □

Define

$$\begin{aligned} \bar{c}^{\infty}(x,z) &:= \sup_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \\ \underline{c}^{\infty}(x,z) &:= \inf_{c \in C'}\{c(x,z)\} \qquad \forall (x,z) \in X \times Z \end{aligned}$$

Lemma 2. *For every subset of consumption functions $C' \subset C$, it holds that $\underline{c}^{\infty} = \inf(C')$ and $\bar{c}^{\infty} = \sup(C')$. Furthermore, it holds that $\underline{c}^{\infty}, \bar{c}^{\infty} \in C$.*

Proof. Suppose not. Suppose there exists a $\hat{c} < \bar{c}^{\infty}$ such that $c \leq \hat{c}$ for all $c \in C'$. This implies that there exist (x,s) such that $\hat{c}(x,z) < \bar{c}^{\infty}(x,z)$. By definition, it holds that

$\bar{c}^{\infty}(x,z) = \sup_{c\in C'}\{c(x,z)\}$, hence, $\hat{c}(x,z) \geq c(x,s)$ implies that $\hat{c}(x,z) \geq \sup_{c\in C'}\{c(x,z)\}$ which yields a contradiction because

$$\sup_{c\in C'}\{c(x,z)\} = \bar{c}^{\infty}(x,z) > \hat{c}(x,z) \geq \sup_{c\in C'}\{c(x,z)\}$$

It follows immediately from the fact that all $c \in C'$ are Lipschitz continuous that $\bar{c}^{\infty}(x,z)$ is also Lipschitz continuous such that $\bar{c}^{\infty} \in C$ holds. An equivalent argument applies for the infimum. □

Remark 2. *The fact that $\bar{c}^{\infty} \in C$ holds follows directly from the Lipschitz property because for all (x_1, z) and (x_2, z) with $x_1 \leq x_2$ it holds that*

$$\begin{aligned}
\bar{c}^{\infty}(x_2,z) &= \sup_{c\in C'}\{c(x_2,z)\} \\
&\leq \sup_{c\in C'}\{c(x_1,z) + x_2 - x_1\} \\
&= \sup_{c\in C'}\{c(x_1,z)\} + x_2 - x_1 \\
&= \bar{c}^{\infty}(x_1,z) + x_2 - x_1
\end{aligned}$$

and the same argument applies to the infimum.

Lemma 3. $(C, \geq)$ *is a complete lattice.*

Proof. From lemma 1 it follows that $(C, \geq)$ is a lattice, and from lemma 2 follows that it is complete. □

A.1.3 Properties of $G(x, z, \lambda)$

Lemma 4. $G_i(x, z, \lambda)$ *is*

(a) increasing and continuous in x

(b) strictly decreasing and continuous in λ

Proof. We consider the two arguments of the min-operator first separately

1. Suppose $G_i(x, z, \lambda) = x - \lambda$, (a) and (b) are obviously satisfied.

2. Suppose

$$G_i(x,z,\lambda) = \left(\beta(1+r)\mathbb{E}\left[\left(c_i\left((1+r)(x-\lambda) + wz'\zeta' - rD, s'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} - \lambda \qquad \text{(A.14)}$$

Since $u'(\cdot)$ is a strictly decreasing function, its inverse is strictly decreasing as well. By assumption, $c_i(\cdot, z)$ is increasing and continuous in x. It follows that (A.14) must be increasing in x. The continuity of $c_i(\cdot, z)$ together with the continuity of $u'(\cdot)$ and its inverse imply that (A.14) satisfies (a) because $c_i \geq c^l > 0$. We apply the same arguments for (b) and $\lambda \leq x$, and we get that (A.14) satisfies (b).

Finally, we have to show that the min-operator preserves the properties of $G_i(\cdot, z, \cdot)$. The min-operator forms the lower envelope of two continuous and increasing respectively strictly decreasing functions in x and λ. It preserves, therefore, the monotonicity and continuity of these functions. Hence, $G_i(\cdot, z, \cdot)$ satisfies (a) and (b). □

Lemma 5. *For every (x, z), $G(x, z, \lambda) = 0$ has a unique solution λ.*

Proof. It follows from the properties of $u'(\cdot)$ that for $\lambda = 0$, $G(x, z, \lambda) \geq 0$ and for $\lambda \to x$, it follows from lemma 4 that $G(x, z, \lambda)$ is strictly decreasing with $G(x, z, \lambda) \leq 0$ if $\lambda = x$. Hence, the solution $G(x, z, \lambda) = 0$ must be unique. □

A.1.4 Properties of T

Lemma 6. *The operator T is monotone increasing.*

Proof. Take $c_i^1 > c_i^2$. It follows from the fact that $u'(\cdot)$ and its inverse are strictly decreasing functions that

$$\min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^1((1+r)(x-\lambda) + wz'\zeta' - rD, z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \geq$$
$$\min\left\{x, \left(\beta(1+r)\mathbb{E}\left[\left(c_i^2((1+r)(x-\lambda) + wz'\zeta' - rD, z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}$$

From lemma 4, we know that $G_i(x, z, \cdot)$ is decreasing in λ. Since it holds that $G_i^1(x, z, \cdot) \geq G_i^2(x, z, \cdot)$, it follows that for all (x, z) we get that $\lambda^1 \geq \lambda^2$. □

Lemma 7. *The operator T maps elements of C to continuous and increasing functions.*

Proof. Again, we proceed in two steps. First, we show that if $c_i(\cdot, z)$ is continuous and increasing, then $c_{i+1}(\cdot, z)$ will be increasing, and in a second step, we show that it is also continuous.

1. (*increasing*)

   (a) If $\lambda = x$, this is obvious.

   (b) If $\lambda = \left(\beta(1+r)\mathbb{E}\left[\left(c_i((1+r)(x-\lambda) + wz'\zeta' - rD, z')\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}$ pick $x_1 > x_2$. Lemma 4 implies that $G_i(x_1, z, \lambda) \geq G_i(x_2, z, \lambda)$ and it follows that $\lambda_1 \geq \lambda_2$ because $G_i(x_1, z, \cdot)$ is strictly decreasing.

From steps (1a) and (1b) it follows that $c_{i+1}(\cdot, z)$ must be an increasing function.

2. (*continuous*) The continuity of the optimal solution follows directly from the implicit function theorem (Kumagai (1980))[25]. To see this, note that $G_i(\cdot, z, \cdot)$ is a continuous map $G_i : X \subset \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. From lemma 5, we know that for all (x_0, z) there exists a unique solution $G_i(x_0, z, \lambda_0) = 0$, and from Kumagai (1980), it follows that $c_{i+1}(\cdot, z)$ is continuous in a neighborhood of x_0 if and only if there are open neighborhoods $B \subset X$ and $A \subset \mathbb{R}_+$ of x_0 and λ_0, respectively, and

$$\forall x_0 \in B : G_i(x_0, z, \cdot) : A \to \mathbb{R}$$

is locally one-to-one (injective). From lemma 4, we know that $G(x, z, \cdot)$ is strictly decreasing, and therefore, it is locally one-to-one. Hence, $c_{i+1}(x, z)$ will be continuous in x.

□

Lemma 8. *If $x_1 > x_2$ and $G(x_2, z, \lambda_2) = 0$ with $x_2 > \lambda_2$, then for $G(x_1, z, \lambda_1) = 0$ it holds that $x_1 > \lambda_1$.*

Proof. Suppose not. It follows from lemma 4 that

$$\begin{aligned}
\lambda_1 &= x_1 \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i(wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&\leq \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_2-\lambda_2) + wz'\zeta' - rD, z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
&= \lambda_2 \\
&< x_2
\end{aligned}$$

This yields a contradiction, and hence, it holds that if $x_1 > x_2$ and $x_2 > \lambda_2$, then also $x_1 > \lambda_1$. □

Lemma 9. *The operator T is a self-map. It maps Lipschitz continuous, increasing functions $c_i(\cdot, z)$ to Lipschitz continuous, increasing functions $c_{i+1}(\cdot, z)$ with Lipschitz constant $L = 1$, i.e.*

$$c_i(x_1, z) - c_i(x_2, z) \leq x_1 - x_2 \qquad \forall x_1, x_2 \in X$$

Proof. From lemma 7, we know that T maps continuous and increasing functions to continuous and increasing functions. Consider the case where $x_1 > x_2$. We know from lemma 7 that $\lambda_1 \geq \lambda_2$. We consider now all possible combinations

[25]Kumagai proves a theorem for the case of non-differentiable function.

I. $\lambda_1 = x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 = \lambda_1 - \lambda_2$.

II. $\lambda_1 < x_1$ and $\lambda_2 = x_2 \quad \Rightarrow \quad x_1 - x_2 > \lambda_1 - \lambda_2$.

III. $\lambda_1 = x_1$ and $\lambda_2 < x_2$. Not possible, see lemma 8.

IV. $\lambda_1 < x_1$ and $\lambda_2 < x_2$.

(a) $\lambda_1 = \lambda_2 \Rightarrow x_1 - x_2 > \lambda_1 - \lambda_2$

(b) $\lambda_1 > \lambda_2$: (Proof by contradiction) Suppose that $x_1 - x_2 < \lambda_1 - \lambda_2$. This implies $x_1 - \lambda_1 < x_2 - \lambda_2$.

$$\begin{aligned}\lambda_1 &= \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_1-\lambda_1)+wz'\zeta'-rD,z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\ &\leq \left(\beta(1+r)\mathbb{E}\left[(c_i((1+r)(x_2-\lambda_2)+wz'\zeta'-rD,z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\ &= \lambda_2\end{aligned}$$

but $\lambda_1 \leq \lambda_2$ yields a contradiction, because we started with the assumption that $\lambda_1 > \lambda_2$.

Hence, it must be true that

$$x_1 - \lambda_1 \geq x_2 - \lambda_2 \quad \Longleftrightarrow \quad x_1 - x_2 \geq \lambda_1 - \lambda_2$$

and the proof is complete. □

Lemma 10. *For every r such that $\beta(1+r) \leq 1$ and $1 - (\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}} > 0$ there exists a supersolution c^u and a subsolution c^l to the operator T.*

1. *For $c^u(x,s) = x$, it holds that $Tc^u \leq c^u$.*

2. *For $c^l(x,s) = \iota x$ with $\iota := 1 - (\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}}$, it holds that $Tc^l > c^l$.*

Proof. 1. By construction, we get that $c_1 = Tc^u \leq x$. Since $c_1(x,s) = \lambda \leq x$ where λ solves

$$\lambda = \min\left\{x, \left(\beta(1+r)\mathbb{E}\left[(c^u((1+r)(x-\lambda)+wz'\zeta',z'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}$$

and it follows that $Tc^u \leq c^u$

2. Take $c^l(x,z) = \iota x$ and suppose that $G^l(x,z,\lambda) = 0$ for $\lambda \leq \iota x$ for some x. This implies that

$$\begin{aligned}
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}} \\
&\quad \left(\mathbb{E}\left[\left(c^l\left((1+r)(x-\iota x) + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
\iota x &\geq (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left(\iota\left((1+r)(1-\iota)x + wz'\zeta' - rD, z'\right)\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
x &> (\beta(1+r))^{-\frac{1}{\gamma}} \left(\mathbb{E}\left[\left((1+r)(1-\iota)x\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \\
1 &> (\beta(1+r))^{-\frac{1}{\gamma}} (1+r)(1-\iota) \\
(1-\iota) &> (1-\iota)
\end{aligned}$$

which yields a contradiction. Hence, it must be true that $\lambda > \iota x$ for all (x,z), and therefore, it holds that $Tc^l > c^l$.

□

Lemma 11. *The operator $T : C \to C$ is continuous.*

Proof. For finite chains the proof is obvious. For infinite chains, take a chain $C^S \subset C$. Define $\bar{c}^\infty = \sup(C^S)$. Denote the image set of C^S by $C^{S'} = \left\{c' \in C : c' = Tc \, \forall c \in C^S\right\}$ and $\bar{c}' = \sup(C^{S'})$. For all $(x,z) \in X \times Z$, we have $c_i'(x,z) = \lambda_i^*$ where λ_i^* solves $G_i(x,z,\lambda) = 0$. Again, $\bar{c}'$ is defined pointwise as $\bar{c}'(x,z) = \sup \lambda^* =: \bar{\lambda}^*$. Since T is monotone increasing and C^S is a chain, it holds that $\lambda_i^* \geq \lambda_j^*$ if $c_i \geq c_j$. It follows from the definition of a chain that for all $c_i, c_j \in C^S$ we either have $c_i \geq c_j$ or $c_i \leq c_j$. Now fix $(x,z,\bar{\lambda}^\infty)$ where $\bar{\lambda}^\infty = T\bar{c}^\infty(x,z)$. Put $c_i \in C^S$ in increasing order and define $\Delta_i := G_i(x,z,\bar{\lambda}^\infty)$. The $\{\Delta_i\}$ sequence is increasing and bounded because $\bar{\lambda}^\infty$ solve $G(x,z,\bar{\lambda}^\infty) = 0$ for $\bar{c}^\infty$. Since we have $\bar{c}^\infty = \sup(C^S)$, it follows from the proof of lemma 2 that for every c_i there exists a $c_{i+1} \in C^S$ such that $\bar{c}^\infty \geq c_{i+1} \geq c_i$ because otherwise $\bar{c}^\infty$ can not be the supremum of C^S. It follows that $\sup(\Delta_i) = 0$. Hence, $G_i(x,z,\bar{\lambda}^\infty) \to 0$ holds, and this implies that $\lambda_i^* \to \bar{\lambda}^\infty$ because λ_i^* solves $G_i(x,z,\lambda) = 0$ and $G_i(x,z,\cdot)$ is continuous in λ. Hence, we get $\bar{\lambda}^* = \bar{\lambda}^\infty$ for all (x,z) such that $T\bar{c}^\infty = \sup(Tc)$ holds. The equivalent argument applies to the infimum and the elements of the chain put in decreasing order. It follows that according to definition 4, $T : C \to C$ is a continuous operator. □

A.1.5 Transversality condition

The transversality condition reads

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] = 0 \tag{A.15}$$

In the following, we need the definition for cash-at-hand $x_t = (1+r)a_t + wz_t\zeta_t + D$ and the result from lemma 10 that $c^*(x_t, z_t) > \iota x_t$ for all (x_t, z_t).

$$\begin{aligned}
\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] &= \lim_{t\to\infty} \beta^t \mathbb{E}\left[\left(\frac{c_t}{x_t}x_t\right)^{-\gamma}((1+r)a_t + wz_t\zeta_t + D - wz_t\zeta_t - D)\right] \\
&= \lim_{t\to\infty} \beta^t \mathbb{E}\left[\left(\frac{c_t}{x_t}\right)^{-\gamma} x_t^{-\gamma}(x_t - wz_t\zeta_t - D)\right] \\
&\leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma} - x_t^{-\gamma}wz_t\zeta_t - x_t^{-\gamma}D)\right] \\
&\leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma})\right]
\end{aligned}$$

Consider first the case of log utility ($\gamma = 1$)

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-1}x_t^0\right] = \lim_{t\to\infty} \beta^t \iota^{-1} = 0$$

For the $\gamma > 1$ case, we get

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}x_t^{1-\gamma}\right] \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(wz_t\zeta_t - rD)^{1-\gamma}\right]$$

We make the following additional assumption for the general case

Assumption 7. *If $\gamma \geq 1$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(wz_t\zeta_t - rD)^{1-\gamma}\right] = 0$$

From assumption 7, it follows that

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0$$

For the case $D = 0$, assumption 7 condition simplifies to

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(wz_t\zeta_t)^{1-\gamma}\right] = 0$$

and we get for the case of permanent income shocks the sufficient condition

$$\beta\mathbb{E}\left[\varepsilon^{1-\gamma}\right] < 1$$

This condition is satisfied by assumption 2.

Finally, consider the $\gamma < 1$ case

$$\begin{aligned}
\lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(x_t^{1-\gamma})\right] &\leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[\iota^{-\gamma}(1+(1-\gamma)(x_t-1))\right] \\
&\leq \lim_{t\to\infty} \left(\beta^t(\iota^{-\gamma}-(1-\gamma)) + \beta^t\mathbb{E}\left[\iota^{-\gamma}x_t\right]\right)
\end{aligned}$$

We can determine an upper bound on $\mathbb{E}[x_t]$

$$\begin{aligned}
\mathbb{E}[x_t] &= \mathbb{E}\left[(1+r)a_t + wz_t\zeta_t + D\right] \\
&= \mathbb{E}\left[(1+r)a_t\right] + \mathbb{E}\left[wz_t\zeta_t\right] + D \\
&\leq \mathbb{E}\left[(1+r)\bar{a}_t\right] + \mathbb{E}\left[wz_t\zeta_t\right] + D
\end{aligned}$$

where $\bar{a}_t$ is defined as follows

$$\begin{aligned}
\bar{a}_1 &= (1+r)a_0 + wz_0\zeta_0 - \iota((1+r)a_0 + wz_0\zeta_0) \\
\bar{a}_1 &= (1-\iota)((1+r)a_0 + wz_0\zeta_0) \\
\bar{a}_2 &= ((1-\iota)(1+r))^2 a_0 + (1-\iota)^2(1+r)wz_0\zeta_0 + (1-\iota)wz_1\zeta_1 \\
\bar{a}_3 &= ((1-\iota)(1+r))^3 a_0 + (1-\iota)^3(1+r)^2 wz_0\zeta_0 + (1-\iota)^2(1+r)wz_1\zeta_1 + (1-\iota)wz_2\zeta_2 \\
&\vdots \\
\bar{a}_t &= ((1-\iota)(1+r))^t a_0 + (1-\iota)\sum_{s=0}^{t-1}((1-\iota)(1+r))^s wz_{t-1-s}\zeta_{t-1-s}
\end{aligned}$$

We have $\beta(1+r) \leq 1$, and therefore, we get

$$\bar{a}_t \leq a_0 + \frac{1}{1+r}\sum_{s=0}^{t-1} wz_{t-1-s}\zeta_{t-1-s}$$

and

$$\begin{aligned}
\mathbb{E}[x_t] &\leq \mathbb{E}\left[\sum_{s=0}^{t} wz_{t-s}\zeta_{t-s}\right] + D + a_0(1+r) \\
&= x_0 + \mathbb{E}\left[\sum_{s=0}^{t-1} wz_{t-s}\zeta_{t-s}\right] \\
&= x_0 + \mathbb{E}\left[\sum_{s=0}^{t-1} wz_{t-s}\right]
\end{aligned}$$

where the last equality holds because of assumption 2.

For the general case we have to make an additional assumption

Assumption 8. *If $\gamma < 1$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\sum_{s=0}^{t-1} wz_{t-s}\right] = 0$$

For the case of permanent income shocks, the expression simplifies to

$$\lim_{t\to\infty} \beta^t wz_0 \sum_{s=0}^{t-1} \left(\mathbb{E}[\varepsilon]\right)^{t-s} = 0$$

and is satisfied because of assumption 2.

Hence, if for the general case 7 resp. 8 holds, then there exists an upper bound for the transversality condition

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0$$

For the case of permanent shocks assumption 2 is sufficient for the existence of the upper bound. To establish a lower bound, note that if $D = 0$, then the lower bound is trivially at zero. For the general case of $D > 0$ we need an additional assumption.

Assumption 9. *If $D > 0$, then it holds that*

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(wz_t\zeta_t - rD)^{-\gamma}\right] = 0$$

We have established an upper bound and an lower bound for the transversality condition

$$0 \leq \lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] \leq 0 \qquad \Longrightarrow \qquad \lim_{t\to\infty} \beta^t \mathbb{E}\left[c_t^{-\gamma}(1+r)a_t\right] = 0$$

and we can conclude that the transversality condition is satisfied. Hence, the fixed point to the modified Euler equation is an optimal solution to the agents' problem in (1.9).

A.2 Proofs and definitions for the existence of a stationary distribution

A.2.1 Mathematical preliminaries

The definitions are taken mostly from Meyn and Tweedie (1993). Let the state space for the stochastic process of labor productivity and asset holdings be S and the Borel σ-algebra on S be $\mathcal{B}(S)$. The stochastic process $\{a_t, z_t\}_{t=0}^{\infty}$ is denoted by Φ and the state in period t by $\Phi_t = \{a_t, z_t\}$.

Definition 6. *The return time probability from state Φ_0 to a set $A \in \mathcal{B}(S)$ is defined as*

$$L(\{a_0, z_0\}, A) := Prob(\Phi_t \text{ ever enters } A | \{a_0, z_0\})$$

Definition 7. *We call a Markov chain φ-irreducible if there exists a measure φ on $\mathcal{B}(S)$ such that, whenever $\varphi(A) > 0$, we have $L(\{a, z\}, A) > 0$ for all $\{a, z\} \in S$*

Definition 8. *The Markov chain is called ψ-irreducible if it is φ-irreducible for some φ and the measure ψ is a maximal irreducibility measure ($\psi \succ \varphi$).*

From the definitions and proposition 4.2.2 in Meyn and Tweedie (1993) we get immediately that if the Markov chain is φ-irreducible, it is also ψ-irreducible. Next, we introduce the concepts of *recurrence* and *transience*.

Definition 9. *The set A is called recurrent if $\mathbb{E}\left[\mathbb{1}(\Phi_t \in A)|(a, z)\right] = \infty$ for all $(a, z) \in A$. The set A is called uniformly transient if there exists a $M < \infty$ such that $\mathbb{E}\left[\mathbb{1}(\Phi_t \in A)|(a, z)\right] \leq M$ for all $(a, z) \in A$.*

These concepts can be extended to chains in the following way

Definition 10. *If every state is recurrent, the chain is recurrent, and if every state is transient, the chain is transient.*

Theorem 1. *Under the maintained assumptions there exists for every r with $\beta(1 + r) \leq 1$ a unique stationary probability distribution μ_r.*

Proof. By construction, Φ is λ-irreducible, and every set in the support of λ is recurrent, hence, Φ is a recurrent chain (cf. theorem 8.1.2 Meyn and Tweedie (1993)). It follows from theorem 10.0.1 in Meyn and Tweedie (1993) that Φ has a unique stationary measure. It holds furthermore that the expected hitting time for every set in the support of λ is finite, and therefore, the stationary measure can be normalized to be a probability measure. □

It is important to notice that the intial endowments of agents are only resource feasible in equilibrium. If goods markets do not clear, then also the mean over assets of the exogenously fixed distribution does not coincide with the mean asset holdings of the agents' that died.

Remark 3. *The proof for the existence and uniqueness of a stationary distribution does not require that initial endowments $\{a_0, z_0\}$ are uncorrelated with $\{a_t, z_t\}$. It only requires that the conditional distribution for $\{a_0, z_0\}$ has the same support as $\lambda(a, z, r)$ and that the unconditional distribution over $\{a_0, z_0\}$ is $\lambda(a, z, r)$. Hence, we can allow for correlation in assets and productivity levels of agents that leave and their successors.*

Lemma 12. *The stationary distribution is continuous in the interest rate on the interval $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$.*

Proof. See proof of theorem 1 in Le Van and Stachurski (2007). The assumptions can be easily verified. Assumption 1 holds because the optimal consumption choice is continuous in the interest rate, the individual choice is independent from the cross-sectional distribution, and the initial distribution is continuous in the interest rate. Assumption 2 is satisfied[26] because we have for every r in $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$ a unique stationary distribution (theorem 1) such that we can directly evaluate at the limit. The bound for the stationary moments follow immediately from the positive probability of death (our assumption 2) and the lower bound on consumption (lemma 10). Finally, assumption 3 follows by a similar argument using that a highest sustainable capital stock exists (our assumption 5) and that the variance of productivity is bounded. We have already shown that the stationary distribution is unique (theorem 1), and hence, the stationary distribution is continuous in the interest rate (see remark 1 in Le Van and Stachurski). □

A.3 Proof for the existence of a RCE

In this section, we establish the existence of an equilibrium interest rate in the interval $(f'(\bar{k}) - \delta, \beta^{-1} - 1)$ such that all markets clear. We need the following lemmata.

Lemma 13. *If only permanent shocks are present, $D = 0$, and r is such that $\beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \geq 1$, then borrowing constraints are non-binding.*

Proof. The borrowing constraints are non-binding if for all (x, z) it holds that $G(x, z, x) < 0$. If only permanent income shocks are present, then it can be easily checked that the inequality always holds if

$$1 > \beta(1 + r)\mathbb{E}\left[\varepsilon^{-\gamma}\right]$$

[26] Using as Lyapunov function $V(a, z) = a + (z - \mathbb{E}[z])^2 = a + (z - 1)^2$.

Hence, we get that for all r that satisfy this inequality, borrowing constraints must be non-binding. □

Lemma 14. *For $\beta(1+r)=1$ aggregate asset supply is larger than aggregate asset demand.*

Proof. It follows from theorem 1 that a stationary distribution exists. Aggregate asset supply K^s is the sum of asset supply of newborn agents K^{new} and the asset holdings of agents that survived from the last period K^{old}, we get

$$K^s = \theta K^{new} + (1-\theta) K^{old}$$

The asset supply of the newborn generation K^{new} is determined by the initial distribution $\lambda(a,z,r)$. The asset supply of the surviving generation K^{old} has been determined by a sequence of optimal consumption choices. The consumption choice is characterized by the first order conditions of the agent's problem. We have to distinguish two cases.
(1) If borrowing constraints are binding for some agents, it follows from the first-order conditions (see Huggett and Ospina (2001)) that for $\beta(1+r)=1$ there is expected consumption grwoth in the cross-section conditional on survival

$$1 > \mathbb{E}_\mu \left[\left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right] \Rightarrow \mathbb{E}_\mu \left[c^*_t \right] < \mathbb{E}_\mu \left[c^*_{t+1} \right]$$

where the μ subscript denotes the fact that the expectations are taken with respect to the stationary distribution μ.
(2) If lemma 13 applies, then borrowing constraints are non-binding. The Euler equation holds as an equality, and the argument by Huggett and Ospina (2001) does not apply.

$$1 = \mathbb{E} \left[\left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right]$$

There is only one riskless asset. Hence, $c_{t+1} = c_t$ is not an optimal choice for all realizations of ε_{t+1}. Hence, Jensen's inequality for strictly convex functions[27] applies, we get

$$1 = \mathbb{E} \left[\left(\frac{c^*_{t+1}}{c^*_t} \right)^{-\gamma} \right] > \left(\mathbb{E} \left[\frac{c^*_{t+1}}{c^*_t} \right] \right)^{-\gamma} \Rightarrow 1 < \mathbb{E} \left[\frac{c^*_{t+1}}{c^*_t} \right] \Rightarrow \mathbb{E}_\mu \left[c^*_t \right] < \mathbb{E}_\mu \left[c^*_{t+1} \right]$$

and again we get conditional on survival consumption growth in the cross-section.[28]

[27] Note that marginal utility is strictly convex if and only if $\frac{\partial^3 u(x)}{\partial x^3} > 0$.

[28] The same argument applies, if borrowing constraints were binding. The argument by Huggett and Ospina (2001) could therefore be replaced by this argument but to highlight the importance of prudence in the model with permanent shocks we decided to present the proof in two steps.

Since expected labor income is constant, consumption growth can only be financed by accumulating on average higher assets. If assets grow for all surviving agents between periods, it follows that $K^{old} > K^{new}$ because the average capital of all generations at the beginning of the life has been K^{new}. As a consequence, we get $K^s > K^{new} = K^d$. ☐

Lemma 15. *There exists an interest rate low enough such that aggregate asset demand is larger than aggregate asset supply.*

Proof. Suppose not. First determine the highest sustainable capital stock given zero consumption

$$\bar{k} = (1-\delta)\bar{k} + f(\bar{k})$$

Fix the interest rate at the implied interest rate

$$\underline{r} = f'(\bar{k}) - \delta$$

and allocate $\bar{k}$ arbitrarily in the population. Draw initial productivity levels from the stationary marginal distribution of productivity levels. To sustain the capital stock, all agents must consume $c_t = 0$ but this is never optimal. Hence, aggregate consumption must be positive and capital supply must be smaller than capital demand, but this yields a contradiction. ☐

Theorem 2. *Under the maintained assumptions a recursive competitive equilibrium (RCE) exists.*

Proof. We have already shown that an optimal solution to the agents optimization problem and a stationary distribution exist. The stationary distribution is continuous in the interest rate. Lemmata 14 and 15 together with the fact that asset demand is downward sloped[29] imply that there must exist at least one interest rate such that the goods market clears. The labor market clears by construction. Hence, a recursive competitive equilibrium exists. ☐

A.4 Proof of non-binding borrowing constraints

Lemma 16. *If all income shocks are permanent or transitory and i.i.d., then the optimal policy only depends on a single variable.*

[29] This follows immediately from assumption 4.

Proof. (i) Start with $c_0(x,z) = c^u(x,z) = x$.

$$\begin{aligned}
\lambda &= \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[((1+r)(x-\lambda)+wz'\eta)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\frac{\lambda}{wz} &= \min\left\{\frac{x}{wz}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left((1+r)\frac{(x-\lambda)}{wz}+\varepsilon\eta\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\tilde{\lambda} &= \min\left\{\tilde{x}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\varepsilon^{-\gamma}\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

where we define for all variables $\tilde{x} := \frac{x}{wz}$. It follows that $\tilde{c}_0(\tilde{x}) = \tilde{x}$, because $\tilde{x}' = \frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta$ and $\tilde{c}_1(\tilde{x}) = \tilde{\lambda}$ for all $\tilde{x}$.

(ii) Suppose $c_i(x,z) = wz\tilde{c}_i(\tilde{x})$, it follows that

$$\begin{aligned}
\lambda &= \min\left\{x, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left(\tilde{c}_i\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\frac{\lambda}{wz})+\eta\right)wz\varepsilon\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\} \\
\tilde{\lambda} &= \min\left\{\tilde{x}, (\beta(1+r))^{-\frac{1}{\gamma}}\left(\mathbb{E}\left[\left(\tilde{c}_i\left(\frac{(1+r)}{\varepsilon}(\tilde{x}-\tilde{\lambda})+\eta\right)\varepsilon\right)^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right\}
\end{aligned}$$

it follows that $\tilde{c}_{i+1}(\tilde{x}) = \tilde{\lambda}$ will also only be a function of $\tilde{x}$.

□

For this policy we use the result from Carroll and Kimball (1996) that the optimal consumption function $c(\tilde{x})$ is concave[30]. Using this result, we prove that for the case where only permanent shocks are present borrowing constraints must be non-binding.

Theorem 3. *Assume only permanent income shocks are present. If a stationary recursive equilibrium exists, then borrowing constraints must be non-binding.*

Proof. The optimal recursive policy function of a RCE satisfies $c^* > c^l$ (Lemma 10). From Carroll and Kimball (1996) and Carroll (2004) it follows that $\tilde{c}(\tilde{x})$ is concave. This implies that ι as defined in Lemma 10 is also a lower bound to the slope of the optimal policy function in ratio form $\tilde{c}(\tilde{x})$. If an equilibrium exists, there must exist states where agents spend less than there current income, and states where they spend more than their current income. Current

[30] The result can also be used on the reduced state space as it is shown in Carroll (2004). The argument by Carroll and Kimball (1996) involves iteration on the Bellman equation but applies here as well because the sequences of consumption functions of the two approaches are equivalent. This can be easily verified because $G_i(x,z,\lambda) = 0$ is the necessary condition for updating the value function using the Bellman equation.

income in the reduced state space is

$$\frac{r}{1+r}\tilde{x} + \frac{1}{1+r}$$

and it can be easily shown that $\frac{r}{1+r} \leq \underline{\iota}$ in equilibrium because $\beta(1+r) \leq 1$

$$\underline{\iota} = 1 - (1+r)^{-1} (\beta(1+r))^{\frac{1}{\gamma}} \geq 1 - \frac{1}{1+r} = \frac{r}{1+r}$$

If borrowing constraints are binding, then it holds for some $\tilde{x}$ that $\tilde{c}(\tilde{x}) = \tilde{x}$ and the continuity and the slope restriction for $\tilde{c}(\tilde{x})$ imply $\tilde{c}(\tilde{x}) > \frac{r}{1+r}\tilde{x} + \frac{1}{1+r}$ for all $\tilde{x}$. However, a situation where agents always spend more than their current income is not compatible with the existence of an equilibrium. This contradiction proves that borrowing constraints must always be non-binding in a RCE of this model. □

Corollary 1. *Assume only permanent income shocks are present. If a recursive competitive equilibrium exists, then there is a unique* $\bar{\tilde{x}}$ *(*target insurance rate*) exists such that the optimal policy yields* $a_t = a_{t+1}$.

Proof. In equilibrium the optimal policy of the agent must such that optimal consumption is for some state smaller and for some states larger than current income. It follows directly from the continuity and concavity of the optimal policy function together with the lower bound c^l on the optimal policy that there must be a unique intersection of the optimal policy with current income. This intersection characterizes $\bar{\tilde{x}}$. □

Corollary 2. *Given the assumptions of theorem 3, the equilibrium interest rate* r *lies in the interval* $\left((\beta\mathbb{E}[\varepsilon^{-\gamma}])^{-1} - 1; \beta^{-1} - 1\right)$

Proof. The upper bound follows from lemma 14. The lower bound can be derived from the fact that borrowing constraints are always non-binding. The Euler equation for the reduced state space variables and zero assets implies that if borrowing constraints are non-binding, then

$$\begin{aligned} 1 &< \beta(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}\right] \\ \Longleftrightarrow \quad r &> \left(\beta\mathbb{E}[\varepsilon^{-\gamma}]\right)^{-1} - 1 \end{aligned}$$

□

Chapter 2

Welfare analysis with permanent income shocks

2.1 Introduction

An important finding from the large literature on the individual consumption-saving problem is that a small *buffer stock* in a riskless asset already suffices to achieve almost perfect consumption insurance when income risk is transitory and insurance markets are missing. As a consequence, welfare losses due to missing insurance markets are small.[1] On the other hand, it has been shown that welfare losses of market incompleteness can be substantial once income shocks are permanent.[2] However, results for permanent income shocks are obtained in models where the structure of the economy delivers a highly stylized optimal policy rule with no asset trade, and hence, no consumption smoothing in equilibrium. In these models, consumption tracks income one-to-one and *self-insurance* —which is highly effective for transitory risk— is shut down.

This paper contributes to the literature by examining a model with permanent income shocks and asset trade in equilibrium. This case constitutes an empirically relevant extension to existing models because it contains, on the one hand, permanent income shocks that have received broad support in the empirical literature[3], and, on the other hand, it features non-degenerate asset trade is present in reality. Furthermore, the model combines the main driver for welfare losses —namely permanent income shocks —with an active channel for self-insurance— namely equilibrium asset trade.

[1]See for example Telmer (1993), Heaton and Lucas (1996), and for a theoretical argument Levine and Zame (2001). Kubler and Schmedders (2001) provide alternative calibrations to challenge the argument by Levine and Zame (2001).

[2]This result originates from a class of models based on Constantindes and Duffie (1996) and Krebs (2007).

[3]For example Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Blundell, Preston, and Pistaferri (2008).

The framework for the analysis is one of the workhorse models in quantitative research. It is an Aiyagari style[4] economy with two modifications : (1) individual income shocks are permanent and (2) there is a perpetual youth structure. For this model, we show that asset trade in a realistically calibrated economy significantly reduces the welfare costs of market incompleteness. We argue that the smaller welfare costs result from effectively lower income risk. This risk reduction is induced by the optimal consumption-saving behavior (*self-insurance*). We propose an approximate stylized consumption rule that captures the key properties of the optimal consumption-saving decision, and show how the risk reduction can be captured by a simple scaling factor. This allows us to derive a closed form approximation of the welfare costs of market incompleteness in an otherwise analytically intractable model. The result shows that models without asset trade overstate the welfare loss of market incompleteness, but as we argue, their stylized consumption rules and closed form welfare formulas still apply in a more realistic model with equilibrium asset trade.

The quantitative analysis starts by solving for the optimal consumption policy and the equilibrium allocation. We show that the optimal policy is almost linear and propose an intuitive approximation to the policy function that highlights the basic properties of the optimal consumption-saving decision. In the welfare analysis, we account for endowment effects that arise from changes in the aggregate capital stock by taking the transition to the new steady state into account. This becomes important when we study the effectiveness of *self-insurance* because it allows us to abstract from income effects. Furthermore, borrowing constraints are non-binding in equilibrium, and we can attribute the welfare loss exclusively to changes in income risk.

We quantify the effectiveness of self-insurance by comparing the welfare effects in our model with asset trade to an endowment economy without equilibrium asset trade. In the latter economy, agents consume only their current endowment, consumption responds one-to-one to shocks, and no self-insurance takes place. This results in welfare costs of market incompleteness that will, ceteris paribus, be larger. We propose an intuitive scaling factor to the volatility of the endowment process that measures and accounts for the self-insurance effect of asset trade. The scaling factor coincides with the individual labor income share and accounts for other sources than labor income that finance consumption and are unaffected by individual labor income risk. We show that after applying this scaling factor the welfare consequences of the two economies align very closely. Since the endowment economy is analytically tractable, we can derive a simple closed form expression to approximate the welfare effects of market incompleteness and verify its good performance. This result shows that we can asses the welfare costs of

[4]We use the term *Aiyagari style* for a heterogeneous agents, incomplete markets economy with a neoclassical production sector (Aiyagari (1994)).

uninsurable permanent income risk in an economy with self-insurance using a simple analytic welfare formula.

In a final step, we check the model's quantitative predictions for other measures of the consumption response to income shocks that have been proposed in the literature.

We are not the first who study the possibilities of self-insurance against permanent income shocks within a consumption-saving model. Heathcote, Storesletten, and Violante (2008, 2009) examine self-insurance with permanent income shocks, but focus on self-insurance possibilities that arise due to a labor-leisure choice.[5] They develop an analytically tractable model framework with a labor-leisure decision, perfect risk-sharing within groups, but imperfect risk-sharing between groups. The construction of the equilibrium allocation and the analytical tractability is closely related to the work by Constantinides and Duffie (1996). The analytic tractability comes at the cost that the self-insurance channel of asset trade is again shut down in their model. Heathcote et al. find welfare effects from incomplete risk-sharing that are much larger than the welfare costs of business cycles. They argue, therefore, that the welfare gains from progressive taxation and wage compression are much larger than the effects from a policy that aims at smoothing out business cycle fluctuations.

Kaplan and Violante (2009) study a life-cycle partial equilibrium model to assess whether the empirical estimates by Blundell et al. (2008) for partial insurance can be explained by self-insurance. Their model generates too little self-insurance and a life-cycle profile that is inconsistent with the empirical findings. They note that the reason for this finding is the life-cycle motive in the consumption-saving decision that mainly drives the capital accumulation decision. The perpetual youth structure in our model allows us to focus exclusively on the interaction between income risk and the consumption-saving decision. We derive welfare implications of the partial insurance result, whereas Kaplan and Violante focus only on the consumption response.

Carroll (2009) studies the consumption response to permanent shocks in a partial equilibrium model. He finds an average marginal consumption response[6] to permanent shocks that is less than one-to-one. Jappelli et al. (2008) study the same partial equilibrium model and compare it to empirical estimates from Italian panel data. They conclude that the model can only poorly be reconciled with their empirical findings for the consumption response to permanent income shocks.

The partial equilibrium parameterization in Carroll (2009) and Jappelli et al. (2008) is com-

[5]Storesletten, Telmer, and Yaron (2007) also use a model with permanent income shocks. However, they have an overlapping generations structure with a fixed life time. Hence, a strong life-cycle savings motive arises in their model that governs most of the optimal behavior. In their model it is, therefore, hard to disentangle the effect of permanent income shocks.

[6]Carroll calculates the average marginal propensity to consume out of permanent income.

plementary to the calibration in our paper because we impose general equilibrium restrictions. As shown in chapter 1, the partial equilibrium parameterization in these studies is inconsistent with general equilibrium restrictions.
Finally, in a highly influential paper Blundell, Preston, and Pistaferri (2008) perform an empirical analysis on a merged income and consumption panel data set. They find a consumption response to permanent income shocks that is substantially smaller than one-to-one.
The rest of the paper is structured as follows. Section 2.2 presents the model, the calibration, and the equilibrium together with the optimal policy and the approximation to the optimal policy. Section 2.3 contains the analysis of the welfare costs of market incompleteness and derives the closed form approximation formula to assess the welfare effects of market incompleteness. Section 2.4 discusses the consumption response to permanent income shocks. Section 2.5 concludes. The appendix contains a model where also transitory shocks are present and an extensive sensitivity analysis of the results.

2.2 Model

We use the Aiyagari style framework proposed in chapter 1 to study the welfare costs of market incompleteness when income shocks are permanent.

2.2.1 Setup

There is a continuum of ex ante identical agents who experience permanent shocks to their labor productivity. Labor income is determined as the product of the realized labor productivity and the wage rate. We abstract from a labor-leisure choice and a participation decision of workers. We assume that shocks to labor productivity are i.i.d. over time and individuals. There is no aggregate uncertainty. Every agent chooses a recursive consumption plan at the beginning of her life-time. As agents face a constant probability θ of dying each period, life-time is stochastic. If an agent died, she is immediately replaced by a newborn agent. The intitial distribution of newborn agents is exogenously fixed. Regarding productivity, it captures factors that determine initial labor market heterogeneity and that are outside the model. Regarding assets, it redistributes the accidental bequests of the preceding generation that died. Agents take their initial endowment as given. The objective function of an agent is her discounted expected life-time utility and the consumption good serves as the unit of account for this economy.
We denote the level of labor productivity of an agent in period t by z_t.[7] We assume that labor

[7]We omit an index for individuals throughout because it only increases the notational burden.

productivity follows a random walk in logs.

$$\log(z_{t+1}) = \ log(z_t) + \log(\varepsilon_{t+1}) \qquad \log(\varepsilon_{t+1}) \overset{iid}{\sim} \mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right)$$

The innovation term is normally distributed with mean $-\frac{\sigma^2}{2}$ and variance σ^2. This construction guarantees that there is no drift in the labor income process over time, i.e. $\mathbb{E}[\varepsilon_{t+1}] = 1$. Since we are interested in the welfare costs associated with permanent income shocks, we abstract from transitory risk in the main part of the paper. In the appendix, we study an economy with permanent and transitory income risk.

The utility function is of the constant relative risk aversion (CRRA) type. The objective function for the agent is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} ((1-\theta)\tilde{\beta})^t \, \frac{c_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_0\right]$$

where $\tilde{\beta}$ is the individual time discount factor, $(1-\theta)$ denotes the constant probability of surviving from period t to $t+1$, and $\mathcal{F}_0$ denotes the information filtration of the agent in period $t = 0$.[8] The agent faces the standard intertemporal budget constraint

$$c_t + a_{t+1} = (1+r)a_t + w_t z_t$$

and to rule out Ponzi schemes, we impose an ad hoc debt constraint $a_{t+1} \geq 0$ for all t.[9]

When we collect all ingredients to the agent's decision problem, we can write it as

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}} \quad & \mathbb{E}\left[\sum_{t=0}^{\infty} ((1-\theta)\tilde{\beta})^t \, \frac{c_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_0\right] \\
s.t. \quad & c_t + a_{t+1} = (1+r)a_t + w_t z_t \\
& log(z_{t+1}) = \ log(z_t) + log(\varepsilon_{t+1}), \qquad log(\varepsilon_{t+1}) \overset{iid}{\sim} \mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right) \\
& \{a_{t+1}, c_t\} \in [0, \infty) \times \mathbb{R}_+ \ \forall t \\
& \{a_0, z_0\} \subset \mathcal{F}_0
\end{aligned} \tag{2.1}$$

[8] The expectation operator only refers to labor income uncertainty.

[9] In chapter 1, we prove that this constraint will be never binding in equilibrium. We verify this result here computationally.

To simplify notation, we replace $(1-\theta)\tilde{\beta}$ by an implicit discount rate β

$$\beta := (1-\theta)\tilde{\beta}$$

Production takes place in a perfectly competitive production sector represented by a single firm that produces at marginal costs using a Cobb-Douglas production function

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} \tag{2.2}$$

where L_t denotes labor in productivity units, i.e. labor supply times productivity aggregated over all individuals. The stationary wage rate w and the (net) interest rate r are then

$$\begin{aligned} r &= \alpha \left(\frac{L}{K}\right)^{1-\alpha} - \delta \\ w &= (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \end{aligned}$$

where δ is the depreciation rate for capital employed in production.

2.2.2 Equilibrium

We define a *recursive competitive equilibrium* (RCE) for this economy as a set of recursively generated asset choices $\{a_{t+1}^*\}$ and consumption choices $\{c_t^*\}$, a capital and labor demand K^d and L^d of the production sector together with equilibrium prices r^* and w^* and a stationary equilibrium distribution $\mu(a,z)$ over asset and productivity levels of agents such that

1. For every agent there are sequences of recursively generated asset choices $\{a_{t+1}^*\}_{t=0}^{\infty}$ and consumption choices $\{c_t^*\}_{t=0}^{\infty}$ that solve the agent's optimization problem in (2.1) given equilibrium prices w^* and r^*.

2. The firm's demand for capital K^d and labor L^d maximizes firm's profits given equilibrium prices w^* and r^*.

3. Equilibrium prices are such that

$$\begin{aligned} \int a_t^* \mu(da, dz) = \quad & K^* \quad = K^d \qquad \forall t \\ \int z_t \mu(da, dz) = \quad & L^* \quad = L^d \qquad \forall t \end{aligned}$$

2.2.3 State space reduction

Following the approach by Deaton (1991) and Carroll (2004), we solve the optimization problem on a reduced one-dimensional state space. We use this section to introduce the notation and present the Euler equation for the problem with a reduced state space.
With non-binding borrowing constraints the first order necessary condition for an optimal solution to the problem in (2.1) is

$$c_t^{-\gamma} = \beta(1+r)\mathbb{E}\left[c_{t+1}^{-\gamma}\middle|\mathcal{F}_t\right]$$

We define a *cash-at-hand* variable $x_t := (1+r)a_t + wz_t$ and a class of ratio variables that we denote by a $\sim$ on top of it. These variables denote the original variable normalized by *permanent labor income*[10], e.g. $\tilde{x}_t := \frac{x_t}{wz_t}$. Applying this definition, we can derive[11]

$$\tilde{c}_t^{-\gamma} = \beta(1+r)\mathbb{E}\left[\varepsilon_{t+1}^{-\gamma}\tilde{c}_{t+1}^{-\gamma}|\mathcal{F}_t\right]$$

and the corresponding stochastic law of motion is

$$\tilde{x}_{t+1} = \frac{(1+r)}{\varepsilon_{t+1}}(\tilde{x}_t - \tilde{c}_t) + 1$$

2.2.4 Calibration

A time period is taken to be one year. The constant probability of death θ is chosen to match the average years a worker is in the labor market. It should therefore be interpreted as the probability of leaving the labor market rather than physical death. The structure of the model can then be thought of as a labor market with several cohorts where agents drop out of cohorts randomly and a new cohort of workers enters the labor market in every period. We target a working life of 35 years, this implies $\theta = 0.028571$, and corresponds to the working life in Kaplan and Violante (2009). The coefficient of relative risk aversion is chosen to be $\gamma = 1$. This choice corresponds to log utility, and is within the range used in the literature. Krebs (2007) considers values for γ between 1 and 4, Kubler and Schmedders (2001) use values in the range from 0.5 to 2.5, Carroll (1997, 2009) chooses $\gamma = 2$, and Krusell and Smith (1997) also use *log* utility. We choose $\sigma = 0.1$ for the standard deviation of permanent income risk in the benchmark model. The same value is used in Krebs (2007), Carroll (1997, 2009) and Kaplan

[10]Current labor income is due to the property that income follows a random walk without drift the best predictor for future labor income, and therefore, we call it *permanent labor income*.
[11]This state space reduction only works because income growth is i.i.d.. It does not work with persistent but non-permament income shocks.

and Violante (2009). We provide a sensitivity analysis with respect to income risk and risk aversion in the appendix.

For the production parameters we follow Cooley and Prescott (1995).[12] We set the capital share in production to $\alpha = 0.4$ and calibrate the time discount factor and the depreciation rate to match a capital-to-output ratio of 3.32 and an investment-to-output ratio of 0.2523. This implies a depreciation rate of $\delta = 0.076$.

For the initial distribution, we follow Kaplan and Violante (2009) who set the initial dispersion of productivity to $\sigma_{z0} = 0.3873$ to match income dispersion at age 60.[13] In the benchmark economy, we choose a coefficient of variation for assets $\sigma_{a0} = 0.3873$ and impose a correlation of $\rho(a_0, z_0) = 1.0$ for the initial endowment draws[14]. The correlation of initial wealth and income implies a constant ratio of wealth-to-income for newborn agents. It is also consistent with a zero correlation of income and the wealth-to-income ratio, which corresponds to the SCF data reported by Kaplan and Violante (2009) who find a correlation of 0.02 between income and the wealth-to-income ratio. We perform a sensitivity analysis with respect to the initial distribution parameters in the appendix.[15] The baseline parametrization together with the calibrated parameters can be found in table 2.1.

Table 2.1: Parametrization and calibration

Parameter	value	target/description
α	0.4	Capital's share in output
θ	0.02857	Expected lifetime 35 years
γ	1	Risk aversion
ρ	1	Correlation of initial wealth and income
σ_{z0}	0.3873	Initial residual dispersion of income
σ_{a0}	0.3873	Initial residual dispersion of wealth
σ	0.1	Standard deviation of permanent income shock
$\tilde{\beta}$	0.9799	Capital-to-output ratio (3.32)
δ	0.076	Investment-to-output ratio (0.2523)

[12]Cooley and Prescott (1995) explicitly account in their calibration for the fact that the model does neither include government spending, nor trade, or consumer durables.

[13]Our calibration approach requires to choose the initial dispersion σ_{z0} and the probability of death θ jointly.

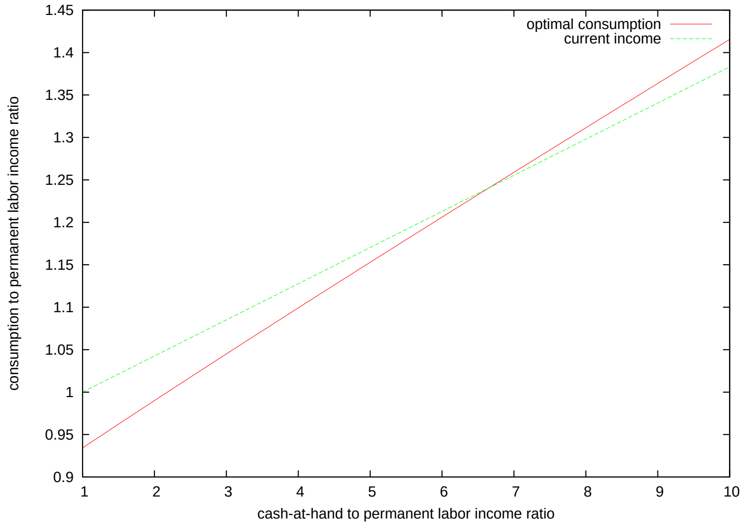
[14]This choice results in a Gini coefficient of 0.22 for assets of newborn agents.

[15]We do not target the dispersion of wealth in the economy, but it is important to emphasize that the choice of the initial distribution is not innocuous for higher order moments of the stationary distribution. This is shown in the appendix. Krusell and Smith (1997) have critized the model for this reason because it generates only a very small cross-sectional variation in the stationary asset distribution. We show that this feature is a consequence of the degenerate initial distribution and that the cross-sectional variation for the asset distribution can be significantly increased if we allow for a non-degenerate initial distribution.

2.2.5 Equilibrium allocation

The equilibrium of this model has no solution in closed form, and we have to solve for the optimal consumption policy and the implied equilibrium allocation numerically. We implement the numerical algorithm along the steps of the equilibrium existence proof of chapter 1.[16] We plot the optimal policy for the benchmark parameterization together with current income transformed to the reduced state space[17] in figure 2.1.

Figure 2.1: Optimal consumption policy



Notes: Optimal consumption is expressed relative to permanent income ($\tilde{c}(\tilde{x})$). The state variable $\tilde{x}$ is the cash at hand to permanent labor income ratio.

The first thing to recognize is that the optimal policy function is almost linear. Furthermore, optimal consumption is below current income for low cash-at-hand to permanent labor income ratios. Especially for $a_t = 0$, the agent consumes less than current income, and hence, the borrowing constraint is not binding. This verifies the theoretical result of chapter 1 that borrowing constraints must be non-binding in equilibrium.

For high cash-at-hand to permanent labor income ratios the consumption policy is above current income, and there is one state in the state space, that we label $\bar{\tilde{x}}$, where agents just consume their current income. This state is the *target insurance ratio* of agents. Chapter 1 proves the existence and uniqueness of this *target insurance ratio*. The ratio is defined as the point in the state space where assets stay constant $a_t = a_{t+1}$.[18] Again, we verify computationally the existence and uniqueness. Importantly, we see that, apart from stochastic fluctuations, the policy function implies saving dynamics that drive asset holdings towards the *target insurance*

[16] Details can be found in the computational appendix.

[17] Notice that $\tilde{x} = 1$ corresponds to the lowest point in the state space because $a = 0$ implies $\tilde{x} = 1$.

[18] Carroll (2004) introduces a slightly different definition, where he defines the target rate as the $\tilde{x}$ such that $\mathbb{E}[\tilde{x}] = \tilde{x}$ holds. As noted in chapter 1 this implies a different target ratio.

ratio, i.e. accumulate additional assets below the target ratio and decumulate assets above the target ratio.

Linear approximation of the optimal policy

The almost linearity of the policy function lends itself to a linear approximation. To do so, we use that at $\bar{\tilde{x}}$ the optimal policy satisfies

$$\tilde{c}(\bar{\tilde{x}}) = \frac{r}{1+r}\bar{\tilde{x}} + \frac{1}{1+r}$$

such that the agent only consumes her current income and assets stay constant.[19] We expand the policy linearly around this point in the following way

$$\tilde{c}(\tilde{x}) = \frac{r}{1+r}\tilde{x} + \frac{1}{1+r} + \xi(\tilde{x} - \bar{\tilde{x}})$$

For an appropriately chosen parameter ξ this yields a very accurate approximation to the optimal policy.[20] We use this linear approximation below to provide some intuition for our results.

Stationary distribution

Regarding the stationary distribution of the model, Krusell and Smith (1997) have argued that the model generates a wealth distribution that is almost degenerate to one point. We do an extensive sensitivity check with respect to the specification of the initial distribution and it seems that their result stems from their choice of the initial distribution. Following Constantinides and Duffie (1996) they set all newborn agents to mean productivity and mean asset holdings of the economy, i.e. they use a degenerate initial distribution. Since, numerically, the *target insurance ratio* coincides almost exactly with mean endowments in the reduced state space, the distribution for assets will stay quite close to the initial distribution. If we allow for initial heterogeneity in endowments, the stationary distribution features a significant cross-sectional dispersion of asset levels.[21] For the benchmark economy, we get a Gini coefficient for wealth that is 0.24, for labor income that is 0.37, and for consumption that is 0.33. If we take the log variance as measure of dispersion, we get a value for labor income that is 0.52 and for

[19] It can be easily verified that $\frac{ra+wz}{wz} = \frac{r}{1+r}\tilde{x} + \frac{1}{1+r}$ using the definitions introduced above.

[20] We tried different parametric specifications for ξ and it turned out that $\xi = \frac{\gamma+1}{\gamma}\frac{\sigma^2}{2}$ works quite well. See appendix for further details.

[21] However, the model still suffers from the same shortcomings as the model with mean-reverting shocks that it can not explain the high degree of wealth inequality observed in data. An extended model that generates among other things more cross-sectional dispersion is part of an ongoing research project.

consumption that is 0.36. Hence, the model is able to generate a consumption inequality that is smaller than labor income inequality.[22] In the appendix, we provide a sensitivity analysis for the stationary distribution with respect to risk aversion, income risk, and the initial distribution. It demonstrates that the degenerate wealth distribution is an artifact of the choice of a degenerate initial distribution.

2.3 Welfare analysis

In this section, we determine the welfare costs of market incompleteness. We perform the hypothetical experiment of shutting down income risk to determine the equivalent variation in consumption that agents would be willing to give up to live under certainty rather than facing income risk. For this experiment, we take the transition phase to the new deterministic economy explicitly into account.

2.3.1 Welfare costs of market incompleteness

It is a standard computational exercise to derive the transition dynamics from the situation with a positive precautionary savings to the steady state of the deterministic economy without precautionary savings.[23] Given the transition path, we can derive expected life-time utility of an agent living through the transition who is initially endowed with asset holdings a and labor productivity z. We denote the expected life-time utility by $v(a, z)$. The model with uncertainty is solved using standard numerical methods on a reduced state space. We denote the expected life-time utility in the stochastic economy by $\tilde{v}(a, z)$. We determine the welfare cost of market incompleteness as the equivalent variation Δ in consumption of the agent living through the transition phase.[24] Straightforward calculations yield the following expressions for Δ

$$\Delta = 1 - \exp((1-\beta)(\tilde{v}(a, z) - v(a, z)))$$

for the log-utility case and

$$\Delta = 1 - \left(\frac{\tilde{v}(a, z)}{v(a, z)}\right)^{\frac{1}{1-\gamma}}$$

for the $\gamma \neq 1$ case.

[22]The numbers here are not directly comparable to Krueger and Perri (2005) who study the relation between income and consumption inequality over time because their sample selection in the empirical analysis reduces the level of inequality significantly. Results that account for their sample selection are available upon request.

[23]All details can be found in the computational appendix.

[24]In our analysis Δ is always a share of the consumption stream in the terminal economy.

In the following discussion, we focus mainly on the average agent in the economy, i.e. someone who holds mean assets and is endowed with mean productivity, but we also derive several welfare cost measures for the cross-section of agents. As it turns out, the focus on the mean agent is justified because the results for the cross-section are very close.

When we calculate the welfare costs of market incompleteness for the mean agent, we get $\Delta = 5.90\%$. This welfare loss is substantial and it is by far larger than the numbers found for transitory (modestly persistent) income risk.[25] Hence, although equilibrium asset trade allows for consumption smoothing the welfare costs of market incompleteness are still large for the average agent. For the cross-section of agents, we calculate the *mean equivalent variation*, the *mean equivalent variation for newborn agents*, the *equivalent variation of social welfare*, and the *equivalent variation of social welfare for newborn agents*. We report these average measures together with the number for the mean agent in the benchmark economy in table 2.2. The results show that the equivalent variation for the mean agent is close to the measures in the cross-section, and that the cross-sectional measures are close to each other given the overall size of the welfare effects.

Table 2.2: Welfare costs of market incompleteness in the benchmark economy

Welfare measure	Equivalent variation
Equivalent variation for the mean agent	5.90%
Mean equivalent variation	5.83%
Mean equivalent variation for newborn agents	5.90%
Equivalent variation of social welfare	5.83%
Equivalent variation of social welfare for newborn agents	5.90%

Notes: Welfare effects are given in terms of the equivalent variation in consumption of the deterministic economy. The welfare effects take the transition path to the deterministic steady state into account.

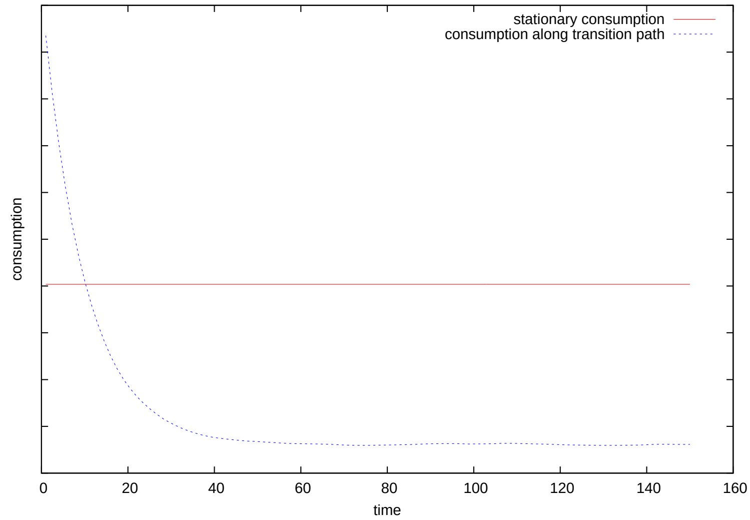
This shows that the focus on the mean agent is justified because it provides a good measure for the welfare effects in the cross-section of all agents and in particular for newborn agents. In the appendix, we provide an extensive sensitivity analysis with respect to parameters of the initial distribution, income risk, and risk aversion.

2.3.2 Endowment effect

In the stochastic economy, the agent accumulates an higher capital stock compared to a situation under certainty. This is in general referred to as *precautionary savings*. In a production economy this implies for a given labor input more output. In a situation where income risk has been removed the agent no longer wants to sustain the high capital stock and the economy

[25]As we show in the appendix, including transitory shocks with standard deviation $\sigma_\eta = 0.2$ increase the welfare costs to $\Delta = 5.95\%$. The welfare effect associated with transitory income risk is therefore negligible.

Figure 2.2: Consumption transition paths



Notes: Consumption paths are for mean income in the stationary equilibrium with idiosyncratic income risk and along the transition path where income risk has been shut down.

converges to a steady state with lower capital stock, and hence, less output. However, during the transition phase the agents can finance additional consumption by running down the capital stock. Next, we quantify the welfare effects of this *endowment effect*. To do so, we compare the mean consumption stream in the stochastic economy to mean consumption along the transition path. We denote the consumption stream in the stochastic economy by $\{\tilde{c}_t\}_{t=0}^{\infty}$ and the consumption stream that is consumed along the transition path by $\{c_t\}_{t=0}^{\infty}$. $\{c_t\}_{t=0}^{\infty}$ is obtained as the optimal policy along the transition path starting from mean assets and mean productivity. The $\{\tilde{c}_t\}_{t=0}^{\infty}$ sequence is an artificial construct. It is a series of constant consumption where consumption equals income obtained if endowed with average capital and average productivity in the equilibrium of the stochastic endowment economy

$$\tilde{c}_t := r\bar{k} + w\bar{z} \qquad \forall t$$

r and w are the interest rate and the wage rate in the stationary equilibrium of the stochastic economy and $\bar{k}$ and $\bar{z}$ are the mean capital stock and the mean effective labor endowment in the stationary equilibrium of the stochastic economy. The two paths for the benchmark economy can be seen in figure 2.2.

When we calculate the discounted utility of the two sequences, we can derive the equivalent variation in consumption that can be associated with the *endowment effect*. We solve for Δ^{endow} that satisfies

$$\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t(1 - \Delta^{endow})) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

For the benchmark economy, we get $\Delta^{endow} = 0.07\%$. This effect is negligible compared to the overall welfare effect, and therefore, we abstract from endowment effects in the welfare analysis. Furthermore, we have documented that borrowing constraints are never binding in equilibrium, so that no additional welfare costs arise due to missing borrowing possibilities. We associate, therefore, the welfare costs in the previous section entirely to missing insurance markets for idiosyncratic income risk.

2.3.3 Endowment economy without equilibrium asset trade

To derive the welfare costs in a model without asset trade, we consider an endowment economy with permanent income shocks along the lines of Krebs (2007). This economy is particular in terms of its equilibrium allocation because agents choose optimally not to trade any assets and agents consume only the current realization of their stochastic endowment stream.[26] This economy is analytically tractable and the value function can be derived in closed form.[27] For a deterministic endowment economy the expected life-time utility of an agent can also be derived easily.[28] Based on the closed form expressions for the value functions, we get a closed form expressions for the equivalent variation. For the log utility case, we get

$$\Delta = 1 - \exp\left(-\frac{\sigma^2}{2}\frac{\beta}{1-\beta}\right)$$

[26]See Constantinides and Duffie (1996) and Krebs (2007) for examples of such endowment economies. The same idea is used in Heathcote, Storesletten, and Violante (2008, 2009) to shut down asset trade between groups.

[27]For $\gamma = 1$ the expected life-time utility of a newborn agent using the distributional assumptions made above is

$$\tilde{v}(z) = \frac{\log(wz)}{1-\beta} - \frac{\beta\sigma^2}{2(1-\beta)^2}$$

and for the case of $\gamma \neq 1$, it is

$$\tilde{v}(z) = \frac{(wz)^{1-\gamma}}{(1-\gamma)\left(1-\beta exp\left(-\frac{1}{2}(1-\gamma)\sigma^2\gamma\right)\right)}$$

where w is a scaling factor of the endowment process that is only introduced to highlight the equivalence to the labor income process of the production economy. See Krebs (2007) for details.

[28]We get

$$v(z) = \frac{\log(wz)}{1-\beta}$$

and

$$v(z) = \frac{(wz)^{1-\gamma}}{(1-\gamma)(1-\beta)}$$

for the *log* utility case and respectively for the $\gamma \neq 1$ case.

and for the case of $\gamma \neq 1$, we get

$$\Delta = 1 - \left(\frac{1-\beta}{1-\beta \exp\left(-\gamma(1-\gamma)\frac{\sigma^2}{2}\right)} \right)^{\frac{1}{1-\gamma}}$$

Straightforward linearization of the welfare formula yields welfare costs in the fashion of the Lucas (1987) approximation[29]

$$\Delta \doteq \frac{\beta}{1-\beta}\gamma\frac{\sigma^2}{2}$$

It can be seen from this formula that the welfare effects of permanent shocks can grow arbitrarily large. Krebs (2007) discusses this point in detail and provides an extensive discussion on the relation between permanent income shocks and the costs of business cycles.

2.3.4 Volatility adjustment

In the discussion of the optimal consumption-saving decision we saw that agents optimally accumulate assets to get to the *target insurance ratio*. At the target ratio, agents keep assets constant and consume only their current income. In this situation, a share of their income is derived from capital and this share is not subject to income shocks. Hence, agents achieve consumption smoothing by financing part of consumption from other sources than labor income.[30] If we want to account for this effect, we have to scale income risk appropriately. To derive a scaling factor, we define $y_t^E := wz_t$ and $y_t^P := wz_t + ra_t$, and determine the conditional coefficient of variation for y_{t+1}^E and y_{t+1}^P given the current state. For the endowment economy, we get

$$\begin{aligned} \frac{\sqrt{var[y_{t+1}^E|y_t^E]}}{\mathbb{E}[y_{t+1}^E|y_t^E]} &= \frac{\sqrt{var[wz_{t+1}|z_t]}}{y_t^E} \\ &= \sqrt{var[\varepsilon_{t+1}]} \\ &= \sigma \end{aligned}$$

For the production economy, we make the simplifying assumption that asset holdings stay constant over time, i.e. $a_t = a_{t+1}$, corresponding to the idea that the agent lives at the *target*

[29]Lucas derived $\gamma\frac{\sigma^2}{2}$ as the welfare costs of business cycles for a representative agent with i.i.d. transitory shocks to consumption.

[30]Remember that in the calibration for the capital income share we followed Cooley and Prescott (1995) who explicitly account for consumer durables and government services.

insurance ratio.

$$\begin{aligned}\frac{\sqrt{var[y^P_{t+1}|y^P_t]}}{\mathbb{E}[y^P_{t+1}|y^P_t]} &= \frac{\sqrt{var[wz_{t+1}+ra_{t+1}|z_t,a_t]}}{y^P_t} \\ &= \frac{wz_t\sqrt{var[\varepsilon_{t+1}]}}{wz_t+ra_t} \\ &= \frac{wz_t}{wz_t+ra_t}\sigma\end{aligned}$$

and we see that the appropriate scaling factor $1-\zeta$ to get equal coefficients of variation for the two income streams is the labor income share in individual income

$$1-\zeta = \frac{wz_t}{wz_t+ra_t}$$

When we plug in mean values for z_t and a_t and use the equilibrium relations for r and w, the scaling factor becomes

$$1-\zeta = \frac{(1-\alpha)k^\alpha}{k^\alpha-\delta k}$$

and if we had $\delta = 0$, we would get

$$1-\zeta = 1-\alpha$$

Hence, at the mean, to get the same volatility in the endowment economy and in the production economy without depreciation, we have to scale the volatility according to the labor income share in output. For the benchmark economy with $\delta > 0$, the scaling factor at the mean becomes

$$\begin{aligned}1-\zeta &= \frac{1-\alpha}{1-\frac{\delta k}{k^\alpha}} \\ &= \frac{1-\alpha}{1-\phi}\end{aligned}$$

where ϕ denotes the investment-to-output ratio in equilibrium.

At this point, it is important to distinguish the labor share in income and the labor share in output because depreciation drives a wedge between the two shares. The scaling factor $1-\phi$ in the denominator accounts for the fact that capital income for the agent is net of depreciation. The aggregate capital share in output α is the gross share that goes as capital income, however, due to depreciation the net income share at the individual level is smaller and the investment adjusted income share accounts for this fact. With this expression for the scaling factor at hand, we finally note that both α and ϕ are known in the current model, because both are calibration targets, so that we know the appropriate scaling factor for the mean agent without having to solve the model numerically.

2.3.5 Welfare formula

We have shown that once we take the transition phase to the deterministic steady state into account, we can abstract from welfare costs due to endowment effects. Furthermore, because borrowing constraints are non-binding, no welfare losses arise due to missing borrowing possibilities. Welfare costs are therefore only caused by changes in income risk. The welfare costs in the endowment economy discussed in the last section are only based on consumption resp. income risk, hence, the welfare formula lends itself to be applied here if we appropriately account for the reduced consumption risk. This is done by augmenting the welfare formula by the scaling factor $1 - \zeta$ to account for the reduced consumption risk

$$\tilde{\Delta} = (1 - \zeta)^2 \frac{\beta}{1 - \beta} \gamma \frac{\sigma^2}{2}$$

If we apply this formula for our benchmark parameterization, we get $\tilde{\Delta} = 6.37\%$. If we use the (exact) non-linear formula, we get $\tilde{\Delta} = 6.17\%$.[31] The numerical approximation of the welfare effects that involves solving the whole model together with the transition dynamics has been $\Delta = 5.90\%$. This shows that for the mean agent, we get an accurate prediction for the welfare effects of market incompleteness. Furthermore, we have shown that the cross-sectional effects are close to the effects for the mean agent, the formula gives therefore also a good prediction for the welfare effects in the economy as a whole.[32]

This result shows that a simple back of the envelope calculation allows us to assess the welfare costs of market incompleteness in a model with permanent income shocks. The result shows further that we can still use a stylized consumption rule, i.e. a rule where agents consume only their current income, to assess the welfare costs of market incompleteness if we take into account the reduced consumption response to income shocks. In the appendix, we provide an extensive sensitivity analysis with respect to risk aversion, individual income risk, and the initial distribution to verify the good performance of the welfare approximation formula.

2.3.6 Social welfare

To see why the approximation formula for the mean agent also yields good predictions for the cross section, we take a closer look at the consumption-saving dynamics. For the derivation of

[31] This shows that the linearization slightly overstates the welfare effects due to concavity.

[32] In the next section, we explain why we get the good approximation of welfare effects also for the cross-section of agents.

the approximation formula, we use a capital income share that is constant over time. Intuitively, this is a good approximation at the *target insurance ratio* because the policy function yields contraction of the cash-at-hand ratio to this point. To see whether this approximation is only a locally valid approximation, we approximate the *speed of convergence* to the target insurance rate using the linear approximation of the policy function and abstract from stochastic effects, i.e. we set $\varepsilon_{t+1} = 1$ for all t. After rearranging terms we get

$$\left| \frac{\tilde{x}_{t+1} - \tilde{x}_t}{\tilde{x}_t - \bar{\tilde{x}}} \right| = |-\xi(1+r)| \approx \xi \tag{2.3}$$

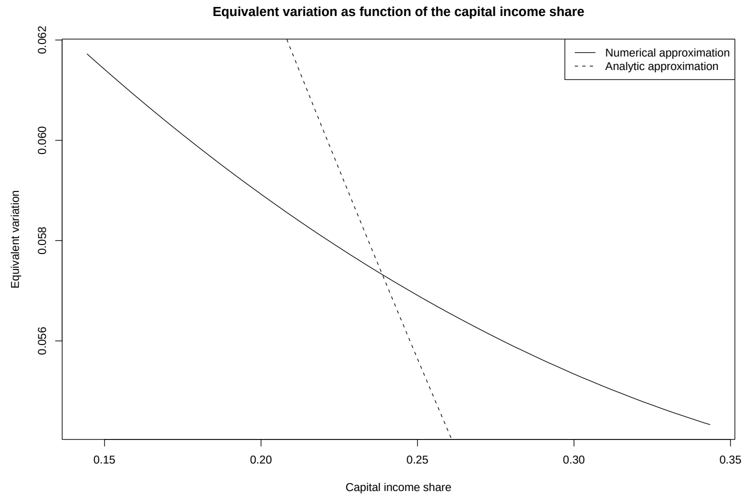
Numerically, the linear approximation of the policy function yields a quite accuracte fit if we set $\xi = \frac{\gamma+1}{\gamma}\frac{\sigma^2}{2}$. If we plug this into the formula for the speed of convergence in (2.3) and choose the parameterization for the benchmark model ($\sigma = 0.1$ and $\gamma = 1$), we get as speed of convergence $\xi \approx 0.01$. This means that only one precent of the remaining distance to the *target insurance ratio* is removed in each period, hence, a quite slow rate of convergence towards the *target insurance ratio*. For example, it would need approximately 69 periods to go half the way to the target insurance ratio from any initial cash-at-hand ratio and in the calibrated average life-time of 35 periods only about 30% of the distance to the target insurance ratio is removed. Hence, for initial cash-at-hand ratios that are in some sense *not too far* from the *target insurance ratio* the cash-at-hand ratio will not move much. It might, however, change significantly once we start *far away* from the *target insurance ratio*. For these cases, we should expect a worse prediction of the approximation formula, and indeed, this is the case as can be seen from figure 2.3. However, the approximation error is symmetric around the target ratio. such that the errors cancel out in the cross-section and we still get a good performance of the approximation formula for the mean equivalent variation in table 2.4. Furthermore, it is important to recall that the distribution of the ratio variable can be quite concentrated, as it is the case in our benchmark calibration, but the model can still feature a large disperion of income and wealth levels in the cross-section.

2.3.7 Changes in income risk

The welfare experiment always considers the case of a complete shut-down of income risk. However, we might also be interested in a partial shut-down or increases of income risk. In this section, we quantify the welfare costs of an increase in permanent income risk. This is particular important because the empirical literature on income risk provides broad evidence for an increase in permanent risk during the 70s up until the mid 80s of the 20th century.

We consider an increase of the standard deviation from $\sigma_1 = 0.10$ to $\sigma_2 = 0.135$. This increase

Figure 2.3: Analytic approximation and numerically determined welfare effects



Notes: Welfare effects given as equivalent variation in consumption of the deterministic economy including the transition path. The analytic approximation uses the formula derived in the main text. The numerically derived welfare effects are determined using the numerically determined value functions.

can be roughly taken as the empirically documented increase in income risk.[33] We calibrate the initial economy to match our calibration targets in table 2.1. We solve for the two stationary equilibria together with the transition path. We get a welfare loss of the increase in permanent income risk of $\Delta = 5.05\%$ for the mean agent. We use the proposed formula that is linear in permanent income risk to do a back of the envelope calculation to assess the welfare effects. We use the capital income share of the initial calibrated economy and get an approximate welfare loss of about 5.24% which gives us an accurate prediciton of the welfare loss associated with the increase in income risk. This shows that the simple welfare formula also delivers a reliable approximation of the welfare costs of market incompleteness in cases of a gradual increase in income risk.

This result suggests that the proposed formula is to the extend model independent that it only depends on the increase in consumption risk caused by an increase in permanent income risk. It might therefore also be applied to situations where the channel of insurance for income risk is not explicitly modelled but only the extend of the transmission from permanent income risk to consumption risk is known. In the current model, the consumption response is approximately equal to the labor income share and welfare effects can be derived without solving the model numerically. We discuss the consumption response to permanent income shocks in the next section and compare it to findings from other studies.

[33]See for example Meghir and Pistaferri (2004). We normalize the average of the time period $1970-1974$ to 0.10 and compared it to the average for the time period $1985-1989$ where we apply the same normalization factor.

2.4 Consumption response to income shocks

Recently, empirical researchers have studied the correlation between permanent income shocks and the consumption-saving decision (See Blundell et al. (2008), Jappelli et al. (2008)). Kaplan and Violante (2009) and Carroll (2009) have studied the quantitative predictions in partial equilibrium models where permanent income risk is calibrated to the estimates from the empirical literature. This paper is to our knowledge the first paper that studies the quantitative predictions in a general equilibrium framework. Imposing general equilibrium restrictions has important quantitative implications for the consumption-saving decision as discussed in chapter 1. In this section, we study the *insurance coefficient* (Blundell et al. (2008)) and the *marginal propensity to consume* out of permanent income as two commonly used measures to quantify the consumption response to an unexpected permanent income shock.

2.4.1 Insurance coefficient

The insurance coefficient as proposed by Blundell et al. (2008) measures the co-movement of consumption growth and income growth. We use their definition

$$\varphi := 1 - \frac{Cov(\Delta c_t, \varepsilon_t)}{Var(\varepsilon_t)}$$

where Δc_t is the consumption growth rate and ε_t is the income growth rate. Kaplan and Violante (2009) calculate the insurance coefficient in an OLG model with permanent income risk. They conclude that the model does not align with the empirical estimates for the insurance coefficient reported in Blundell et al. (2008). We calculate the insurance coefficient for our benchmark economy and get a mean insurance coefficient of 0.28. The empirical point estimate by Blundell et al. (2008) is 0.36 but refers to non-durable consumption. This shows that self-insurance can explain a substantial amount of insurance against permanent income shocks but it still gives leeway to other channels to explain the empirical estimates.[34]

To see how the insurance coefficient relates to our measure of self-insurance, the capital income

[34] The general equilibrium approach together with the calibration of the capital-to-output ratio imposes important discipline on this quantitative exercise. If we do a partial equilibrium experiment, where we keep all parameters like in the benchmark model and vary only the interest rate, we uncover a strong sensitivity of the quantitative results. For $r = 4.5763\%$ the insurance coefficient in the cross-section is 0.37 whereas for $r = 3.0263\%$ the insurance coefficient is 0.00. This strong reaction is mitigated if a life-cycle motive governs the consumption-saving decision. The second case corresponds to the cases studied in Deaton (1991) where agents run down assets to zero and stay borrowing constrained forever.

share, we rewrite the insurance coefficient as follows

$$\begin{aligned}
\varphi &= 1 - \frac{Cov(\Delta c_t, \varepsilon_t)}{Var(\varepsilon_t)} \\
(1-\varphi)\sigma_\varepsilon^2 &= Cov(\Delta c_t, \varepsilon_t) \\
(1-\varphi)\sigma_\varepsilon^2 &= \rho(\Delta c_t, \varepsilon_t)\sigma_{\Delta c_t}\sigma_\varepsilon \\
\frac{(1-\varphi)}{\rho(\Delta c_t, \varepsilon_t)}\sigma_\varepsilon &= \sigma_{\Delta c_t}
\end{aligned}$$

We see that the insurance coefficient has to be adjusted by the correlation coefficient of income and consumption growth to get the appropriate scaling factor to go from the standard deviation of income growth to the standard deviation of consumption growth.[35] For the capital income share we use $\hat{\alpha} = \frac{ra}{ra+wz}$ and get the following non-linear relation between the capital income share and the cash-at-hand to permanent labor income ratio

$$\hat{\alpha} = \frac{r(\tilde{x}_t - 1)}{r\tilde{x} + 1}$$

We combine the linear approximation to the optimal policy function and the law of motion for the cash-at-hand ratio, this yields

$$\tilde{c}_{t+1} = \frac{r(\tilde{x}_t - \tilde{c}_t)}{\varepsilon_{t+1}} + 1 + \xi\left(\frac{1+r}{\varepsilon_{t+1}}(\tilde{x}_t - \tilde{c}_t)\right) + \xi\varepsilon_{t+1}(1 - \bar{\tilde{x}})$$

and we can derive the consumption growth rate as

$$\Delta c_t = \varepsilon_{t+1}\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = r(\tilde{x}_t\tilde{c}_t^{-1} - 1) + \xi((1+r)(\tilde{x}_t\tilde{c}_t^{-1} - 1)) + \varepsilon(\tilde{c}_t^{-1} - (\tilde{x}_t\tilde{c}_t^{-1} - 1))$$

From this expression for the consumption growth rate, we can derive the relationship between the capital income share and the insurance coefficient

$$\begin{aligned}
\varphi &= 1 - \frac{Cov(\Delta c_t, \varepsilon_t)}{Var(\varepsilon_t)} \\
&= 1 - \frac{1 - \xi(\bar{\tilde{x}} - 1)}{(1-\hat{\alpha})^{-1} + \xi(\tilde{x}_t - \bar{\tilde{x}})} \\
&= \frac{\hat{\alpha} + (1-\hat{\alpha})\xi(\tilde{x}_t - 1)}{1 + (1-\hat{\alpha})\xi(\tilde{x}_t - \bar{\tilde{x}})}
\end{aligned}$$

We see that the insurance coefficient is always larger than the capital income share. The difference is governed by the factor ξ that captures the adjustment towards the *target insurance*

[35] If the consumption policy function were linear, then the correlation would be one and the insurance coefficient alone would yield the scaling factor.

ratio. The numerical results allow us to calculate the respective statistics for the mean agent. These can be found in table 2.3.

Table 2.3: Consumption insurance measures

φ	0.244
$\rho(\Delta c_t, \varepsilon_t)$	0.99999
$\frac{(1-\varphi)}{\rho(\Delta c_t, \varepsilon_t)}$	0.756
$1 - \hat{\alpha}$	0.802

Notes: The insurance coefficient φ is determined at the mean endowments and the numerically approximated policy function is used. The scaling factor for income risk in the third row determines the share of income risk that translates into consumption risk as derived in the main text. The labor share is determined for the mean agent and is used in the analytic approximation for the welfare effects to measure the transmission of income risk to consumption risk.

We see that the insurance coefficient at the mean is smaller than the mean insurance coefficient (0.28) suggesting a convex relationship over the (reduced) state space, and we see further, that the correlation between income growth and consumption growth is very close to 1 suggesting an almost linear policy function.[36] If we predict the welfare effects using the capital income share on the one hand and the insurance coefficient on the other hand, we get the approximations of the welfare effects given in table 2.4.

We slighty overestimate the welfare effect if use the scaling factor based on the capital income share, and we underestimate the welfare effect if we use the scaling factor based on the insurance coefficient. However, the capital income share has the advantage that it can be easily derived from the calibration targets, whereas the policy function is needed to derive the insurance coefficient for the mean agent because it does not coincide with the mean insurance coefficient obtained in empirical studies.

It is furthermore important to recognize that, as we show in the sensitivity analysis, the insurance coefficient is increasing in income risk, however, the welfare costs of market incompleteness

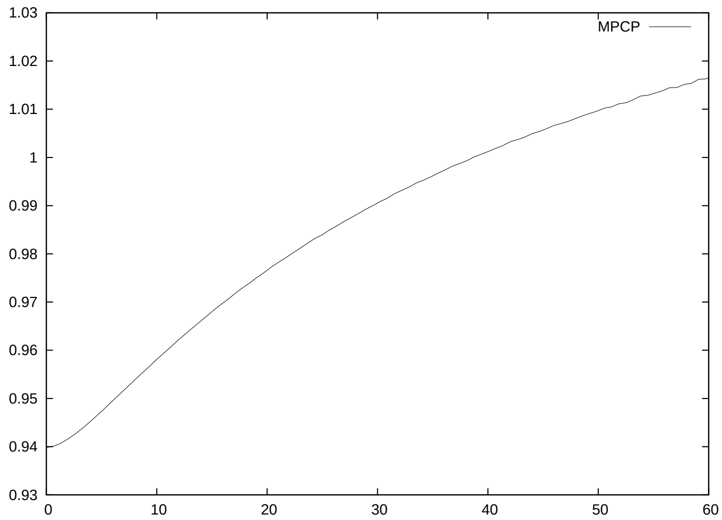
[36] The correlation has been computed using a linear approximation of the policy function such that the reported correlation is an upper bound, however, we tried other interpolation schemes and the correlation always stayed above 0.99999.

Table 2.4: Welfare effects using different scaling factors

	Income share	Insurance coefficient	No approximation
mean agent (nonlinear)	6.17%	5.49%	5.90%
approximation error	0.27%	0.41%	–
mean agent (linearized)	6.37%	5.65%	5.90%
approximation error	0.47%	0.25%	–

Notes: Approximation of the welfare effects using the analytic approximation formula and different scaling factors to go from the variance of income risk to the variance of consumption risk risk.

Figure 2.4: Marginal propensity to consume



Notes: The marginal propensity to consume ($\frac{\partial c(a,z)}{\partial z}$) out of permanent income. Asset levels are fixed and permanent income changes marginally. The numerically approximated policy function is used.

increase at the same time. The insurance coefficient alone can therefore not be used to assess the welfare costs of market incompleteness but can even be misleading.

2.4.2 Marginal propensity to consume

Finally, we look at the marginal propensity to consume out of permanent income as an additional measure for the consumption response to income shocks. We derive it as the derivative of the optimal policy function with respect to permanent income. The derivative is determined numerically using the optimal policy function.[37] In figure 2.4, we plot the derivative for a fixed productivity level and different asset levels.

We see that the marginal propensity is below one for low asset values and it is above one for high asset values. The minimum is at zero asset holdings and it is roughly equal to 0.94. This means that a permanent increase in income results in an increase of consumption that is about 94% of the increase in income. This shows that for a given productivity level and low asset holdings agents are willing to undertake a significant saving effort in reaction to a marginal increase in permanent income. The asset accumulation increases the consumption response up to a one-to-one relationship and we see that for high asset values it is even larger than one-for-one. This happens as soon as the agent is above the *target insurance ratio* and she wants to decumulate asset holdings. Carroll (2009) calculates the average marginal propensity to consume for permanent income and finds values between 0.75 and 0.92 depending on the

[37] We keep asset holdings constant and change productivity marginally.

calibration. His model is a partial equilibrium model and contains also transitory shocks. To get to the marginal propensity to consume out of permanent income, he takes the average over the realizations of the transitory shock. Our results that are derived under general equilibrium restrictions suggest a higher marginal reaction to income shocks.

2.5 Conclusions

In this paper, we show that asset trade in a model with permanent income shocks can cause a substantial reduction in the welfare costs of market incompleteness. Although, welfare costs are large, self-insurance through asset trade provides an effective channel to insulate consumption risk from labor income risk. We calculate for an Aiyagari-style economy with permanent income shocks the optimal consumption-saving policy and the equilibrium allocation. Based on these results, we propose a stylized consumption rule that captures the key dynamics of the asset accumulation decision. From this intuitive approximation of the policy function, we derive a scaling factor to individual income risk that, on the one hand, provides a measure for the effectiveness of self-insurance, and on the other hand, allows us to approximate the welfare costs of market incompleteness in closed form. The scaling factor coincides with the labor share in total income and accounts for the fact that capital income is not exposed to labor income shocks. Models that do not account for this channel of self-insurance, respectively, other sources of income to finance consumption are very likely to overstate the welfare costs associated with income risk and missing insurance markets.

We show that our approximation formula for the welfare effects also works for a partial increase in income risk. To this end, we provide a little quantitative example where we quantify the welfare loss of the empirically documented increase in labor market risk starting in the 1980s. Finally, we discuss the model's prediction for other partial insurance measures that have been proposed in the literature.

Appendix

B.1 Welfare consequences of transitory income risk

In this section, we examine the effect of additional transitory i.i.d. shocks. The agent receives in every period an i.i.d. transitory shock η_t with $\mathbb{E}[\eta_t] = 1$ for all t. The budget constraint of the agent becomes

$$c_t = (1 + r)a_t + wz_t\eta_t - a_{t+1}$$

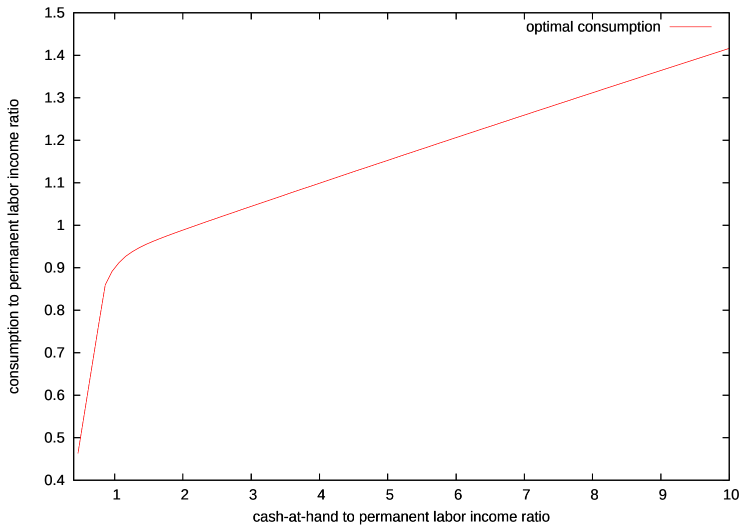
and the law of motion on the reduced state space is

$$\tilde{x}_{t+1} = \frac{(1 + r)}{\varepsilon_{t+1}} (\tilde{x}_t - \tilde{c}_t) + \eta_t$$

i.e. we only normalize by permanent income wz_t. We choose $\sigma_\eta = 0.2$, a number that is consistent with Kaplan and Violante (2009) who choose 0.22. In contrast to the results in the main part of the paper borrowing constraints become now binding for combinations of low asset holdings and bad transitory shocks. This can be seen in figure B.5 where we plot the optimal policy function on the reduced state space.

If we solve for the stationary equilibrium and the transition to the deterministic steady state, we get welfare costs of market incompleteness for the average agent that are $\Delta = 5.95\%$ and the average equivalent variation in the cross-section is 5.89%. This result shows that the welfare effect of uninsurable transitory shocks is negligible compared to the effect of uninsurable permanent income shocks, because adding transitory risk increases welfare costs only by 0.05% whereas the effect of the permanent shocks is 5.90%. These costs are larger by two orders of magnitude. This verifies that agents can almost perfectly self-insure against transitory i.i.d. shocks. A fact that is also reflected in the *insurance coefficient* for transitory shocks that is 0.96, whereas for permanent shocks it is 0.28.

Figure B.5: Optimal consumption with transitory risk



Notes: Optimal consumption in the case where also transitory income shocks are present. Optimal consumption is expressed relative to permanent income ($\tilde{c}(\tilde{x})$). The state variable $\tilde{x}$ is the cash at hand to permanent labor income ratio.

B.2 Sensitivity analysis

We perform a sensitivity analysis of the welfare effects and the model's cross-sectional inequality along three dimensions: income risk (σ_ε), risk aversion (γ), and the parameters of the initial distribution ($\sigma_{z0}, \sigma_{a0}, \rho$).

B.2.1 Income risk

Table B.5: Sensitivity of welfare effects with respect to income risk

σ	Δ	$\bar{\Delta}$	φ	$\hat{\Delta}$	$\tilde{\Delta}$
0.05	1.71%	1.70%	0.22	1.75%	1.73%
0.075	3.61%	3.58%	0.25	3.77%	3.70%
0.10	**5.90%**	**5.83%**	**0.28**	**6.37%**	**6.17%**
0.125	8.40%	8.26%	0.32	9.40%	8.97%
0.15	10.95%	10.73%	0.35	12.74%	11.96%

Notes: Welfare effects and insurance coefficient for deviations in income risk from the benchmark economy (in bold). Δ is the equivalent variation for the mean agent, $\bar{\Delta}$ is the average equivalent variation, φ is the average insurance coefficient, $\hat{\Delta}$ is the approximated equivalent variation using the linearized formula, and $\tilde{\Delta}$ is the approximated equivalent variation using the non-linear formula.

Table B.6: Sensitivity of inequality measures with respect to income risk

	Gini			log variance	
σ	assets	income	consumption	income	consumption
0.05	0.22	0.27	0.25	0.24	0.21
0.075	0.22	0.31	0.28	0.35	0.27
0.1	**0.24**	**0.37**	**0.33**	**0.52**	**0.36**
0.125	0.26	0.42	0.37	0.75	0.47
0.15	0.29	0.47	0.41	1.05	0.61

Notes: Changes in inequality measures after deviations in income risk from the benchmark economy (in bold).

B.2.2 Risk aversion

Table B.7: Sensitivity of welfare effects with respect to risk aversion

γ	Δ	$\bar{\Delta}$	φ	$\hat{\Delta}$	$\tilde{\Delta}$
1	**5.90%**	**5.83%**	**0.28**	**6.37%**	**6.17%**
2	9.70%	9.72%	0.30	10.41%	10.44%
3	12.27%	12.37%	0.31	12.40%	13.42%

Notes: Welfare effects and insurance coefficient for deviations in risk aversion from the benchmark economy (in bold). Δ is the equivalent variation for the mean agent, $\bar{\Delta}$ is the average equivalent variation, φ is the average insurance coefficient, $\hat{\Delta}$ is the approximated equivalent variation using the linearized formula, and $\tilde{\Delta}$ is the approximated equivalent variation using the non-linear formula.

Table B.8: Sensitivity of inequality measures with respect to risk aversion

	Gini			log variance	
γ	assets	income	consumption	income	consumption
1	**0.24**	**0.37**	**0.33**	**0.52**	**0.36**
2	0.25	0.36*	0.32	0.52	0.37
3	0.25	0.36*	0.33	0.51*	0.37

Notes: Changes in inequality measures after deviations in risk aversion from the benchmark economy (in bold). The values with a star are lower resp. higher due to simulation noise, theoretically, they should coincide with the values for the benchmark economy.

B.2.3 Initial distribution

Table B.9: Sensitivity of welfare effects with respect to the initial distribution

σ_{z0}	σ_{a0}	ρ	Δ	$\bar{\Delta}$	φ	$\hat{\Delta}$	$\tilde{\Delta}$
0.3873	**0.3873**	**1.0**	**5.90%**	**5.83%**	**0.28**	**6.37%**	**6.17%**
0.3873	0.3873	0.0	5.90%	5.84%	0.29	6.36%	6.17%
0.3873	0.0	0.0	5.91%	5.81%	0.29	6.37%	6.17%
0.3873	0.5477	1.0	5.90%	5.87%	0.27	6.37%	6.17%
0.0	0.0	–	5.90%	5.83%	0.28	6.37%	6.17%
0.3873	0.3873	0.5	5.91%	5.84%	0.29	6.37%	6.17%

Notes: Welfare effects and insurance coefficient for deviations in the initial distribution from the benchmark economy (in bold). Δ is the equivalent variation for the mean agent, $\bar{\Delta}$ is the average equivalent variation, φ is the average insurance coefficient, $\hat{\Delta}$ is the approximated equivalent variation using the linearized formula, and $\tilde{\Delta}$ is the approximated equivalent variation using the non-linear formula.

Table B.10: Sensitivity of inequality measures with respect to the initial distribution

			Gini			log variance	
σ_{z0}	σ_{a0}	ρ	assets	income	consumption	income	consumption
0.3873	**0.3873**	**1.0**	**0.24**	**0.37**	**0.33**	**0.52**	**0.36**
0.3873	0.3873	0.0	0.21	0.36*	0.30	0.51*	0.30
0.3873	0.0	0.0	0.10	0.37	0.30	0.52	0.29
0.3873	0.5477	1.0	0.30	0.37	0.34	0.52	0.39
0.0	0.0	–	0.08	0.30	0.24	0.37	0.21
0.3873	0.3873	0.5	0.22	0.37	0.31	0.52	0.33

Notes: Inequality measures for deviations in the initial distribution from the benchmark economy (in bold). The values with a star are lower due to simulation noise, theoretically, they should coincide with the values for the benchmark economy.

B.3 Linear policy function

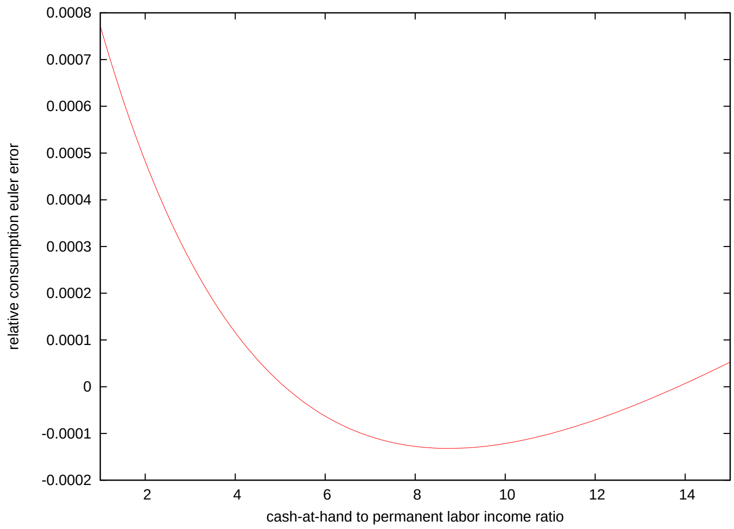
In the main part of the paper, we proposed the following linear approximation to the optimal policy function

$$\tilde{c}(\tilde{x}) = \frac{r}{1+r}\tilde{x} + \frac{1}{1+r} + \frac{\gamma+1}{\gamma}\frac{\sigma^2}{2}(\tilde{x} - \bar{\tilde{x}})$$

to check how well this approximation describes the optimal policy function, we plot the relative euler error in consumption that is a standard measure to assess the goodness of an approximated policy function. Figure B.6 shows the plot. The maximum is at $\tilde{x} = 1$, i.e. $a = 0$, but it is still smaller than $8 * 10^{-4}$ what we consider to be still an appropriate approximation[38]. If we calculate the average error in the cross-section from the linear approximation, we get $3 * 10^{-5}$ which shows that the larger approximation error at the boundary does only affect few agents.

[38]The approximation error becomes also larger in the upper part of the state space. For $\tilde{x} = 500$, it is $4 * 10^{-3}$.

Figure B.6: Euler error of the linear approximation



Notes: Relative euler error of the linear approximation to the optimal policy function as described in the main text.

B.4 Computational appendix

B.4.1 Agent's problem

To solve for the optimal consumption policy of the agent, we implement a slightly modified *policy function iteration* compared to the algorithm in chapter 1. The approach significatly increases computational speed but lacks a theoretical footing unlike the *policy function iteration* approach for which chapter 1 establishes convergence.

Since the stochastic labor income process is non-stationary, we can not approximate it by a finite state Markov chain using standard procedures. Hence, we use Gauss-Hermite quadrature to evaluate the expectations. We choose 7 integration nodes for the permanent as well as for the transitory shocks to get an appropriate degree of accuracy in the integration procedure.

To approximate the policy function, we set up an equally spaced grid on the one dimensional state space. We try different number of grid points and find that 5000 seems to be a well-suited choice to achieve a high degree of accuracy. We choose the grid points in the range $[\underline{\eta}, 500]$, where $\underline{\eta}$ is the smallest of the quadrature nodes for the transitory shock in the productivity space. If there are no transitory shocks, then there is a natural lower bound of the state space at 1. The upper bound of the state space is chosen in an ad-hoc fashion, but we try a wide range of values and find that the value is sufficiently large to not affect the equilibrium outcome for all parameter constellations considered. Since we find that the policy function is almost linear, we use linear interpolation to evaluate the policy function in-between the grid points. The stopping criterion for the convergence of the policy function is set to $\varepsilon_C := 1.0e-4$. The

problem is solved for a given aggregate capital-to-labor ratio implying r and w.
Here is an outline of the algorithm

1. Define a set of N equally spaced grid points on $[\tilde{x}_{min}, \tilde{x}_{max}]$ and make an initial guess for the policy function $\hat{c}_0(\hat{x}_i)$ at every grid point $\hat{x}_i$ for $i = 1, \ldots, N$.

2. Derive the permanent $\{\hat{\varepsilon}_j\}_{j=1}^S$ and transitory $\{\hat{\eta}_j\}_{j=1}^S$ shocks from the Gauss-Hermite integration nodes.

3. For every $\hat{x}_i$ calculate the right hand side of the Euler equation and apply the inverse marginal utility function, to get an update on the policy function at this node.

$$\hat{c}_{t+1}(\hat{x}_i) = \min \left[\hat{x}_i,\ \left(\beta(1-\theta)(1+r)\mathbb{E}\left[\varepsilon^{-\gamma}(\hat{c}_t(\hat{x}_i'))^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}\right]$$

where we use that for $\hat{\varepsilon}_s$ and $\hat{\eta}_h$

$$\hat{x}_i' = \hat{\eta}_h + \frac{1+r}{\hat{\varepsilon}_s}(\hat{x}_t - \tilde{c}_t)$$

and the expectation operator is evaluated as

$$\mathbb{E}\left[\varepsilon^{-\gamma}(\tilde{c}_t(\tilde{x}'))^{-\gamma}\right] = \sum_{s=1}^{S}\sum_{h=1}^{S} \hat{\varepsilon}_s^{-\gamma}(\tilde{c}_t(\hat{x}_i'))^{-\gamma}\omega_{\varepsilon,s}\omega_{\eta,h}$$

where $\omega_{\varepsilon,s}$ and $\omega_{\eta,h}$ denote the appropriate Gauss-Hermite integration weights.

4. Check convergence $\|\tilde{c}_{t+1}(\tilde{x}) - \tilde{c}_t(\tilde{x})\|_2 < \varepsilon_C$, with a stopping criterion ε_C sufficiently small. Stop if policy function has converged, otherwise keep iterating.

Instead of the modified iterating approach, we could also use a numerical rootfinder to solve the Euler equation for $\tilde{c}_t$ given $\hat{x}_t$. This corresponds to the approach as outlined in chapter 1. We run this approach, too, and find that it is as accurate as the modified approach but much slower in application.

B.4.2 Finding equilibrium prices

The algorithm for finding equilibrium prices is taken from Aiyagari (1994). The algorithm is a simple bisection approach to market clearing.
The consumer's problem is solved for given prices. Using the optimal policy function and the law of motion for the productivity state we simulate the model. This is done for a large set of consumers and periods. We choose here 50000 individuals and 5000 time periods. From this

simulation we derive the capital supply at given prices. This is done by averaging over the last 3000 periods in the simulation. We derive the aggregate labor supply, too, to get the aggregate capital-to-labor ratio. Considering the capital-to-labor ratio and averaging over 3000 periods is aimed at reducing simulation noise.
The demand curve can be derived from the firm's first order conditions analytically. If the supply at current prices exceeds the demand, the interest rate is lowered, otherwise it is increased. This is done using a bisection approach.
The bisection is initialized with $r_{0,max} = \frac{1}{\beta} - 1$ and $r_{0,min}$ at some arbitrary level such that we can be sure that this is below the equilibrium value. Chapter 1 derives a lower bound for the case without transitory risk.
After the bisection step we start over and solve the agent's problem again given updated prices. We iterate on this procedure until convergence, i.e. capital demand and supply coincide.
Outline of the algorithm:

1. Initialize the asset and productivity distribution. a_0 and z_0, where a_0 and z_0 are N dimensional vectors. N being the number of individuals in the cross section. We label all individuals by an index $i \in 1, 2, \dots, N$.

2. Draw transitory and permanent shocks from the appropriate distributions[39]. Draw the survival shock from a standard uniform distribution $\tau \sim \mathcal{U}[0, 1]$.

3. Dervive for all i next period's values. If $\tau > \theta$ then $a_{t+1,i} = a_{t,i}(1 + r) + \eta_{t,i} w \exp(z_{t,i}) - \tilde{c}(\tilde{x}_{t,i}) w \exp(z_{t,i})$ and $z_{t+1,i} = z_{t,i} + \varepsilon_{t+1,i}$, otherwise $a_{t+1,i}$ is drawn according to the implemented bequest scheme and $z_{t+1,i} = 0$.

4. For $t \geq 2000$ calculate $\bar{K} = \frac{1}{N} \sum_{i=1}^{N} a_{t,i}$ and $\bar{L} = \frac{1}{N} \sum_{i=1}^{N} exp(z_{t,i})$.

5. Form the (time) average over the $\frac{\bar{K}}{\bar{L}}$ and compare it to the capital-to-labor demand implied by the current interest rate. If $\frac{\bar{K}}{\bar{L}}$ exceeds the implied demand then reduce the interest rate, otherwise increase the interest rate. This is done using the bisection procedure.

6. Check convergence of the interest rate. If it has not converged, solve the consumer's problem with the new interest rate and simulate again.

B.4.3 Calculate the transition dynamics

We fix the transition phase to take place over T periods. We choose for our calculations $T = 150$. For the final period T we impose that the economy has reached its steady state.

[39]We simulate the productivity in the *log* productivity space.

We initialize the algorithm by guessing a transition path from r_0 to r_T. From this we derive the implied transition for the wage using the equilibrium relationship for the wage and the interest rate. Given these price sequence, we solve by using backward iteration for the sequence of optimal policy functions $\{c_t(x)\}_{t=0}^T$ along the transition.

Using the optimal policy, we simulate a large cross-section of agents along the transition path starting from the stationary distribution of the initial economy. To reduce simulation noise we replicate the initial cross-section 10 times such that we simulate $500,000$ agents along the transition path.

From the simulation we derive a sequence of aggregate capital supplies $\{K_t\}_{t=0}^T$ for the given the price sequence $\{r_t, w_t\}_{t=0,s}^T$. The s denotes the number of updates on this price sequence, it is therefore zero after the initialization. This sequence of aggregate capital $\{K_t\}_{t=0}^T$ implies a new sequence of prices $\{r_t, w_t\}_{t=0,s'}^T$. We update our guess for the price sequence along the transition by forming a convex combination with combination weight a of the price sequences[40] s and s'

$$\{r_t, w_t\}_{t=1,s+1}^{T-1} = a\{r_t, w_t\}_{t=1,s}^{T-1} + (1-a)\{r_t, w_t\}_{t=1,s'}^{T-1}$$

Note that we only update the guess for the transition period, i.e. $t = 1, 2, \dots, T-1$. After updating the price sequence, we check convergence of the price sequence, i.e.

$$\|\{r_t, w_t\}_{t=0,s+1}^T - \{r_t, w_t\}_{t=0,s}^T\|_2 < \varepsilon_{r,w}$$

where $\varepsilon_{r,w}$ is an appropriately chosen convergence criterion. If the price sequence has not yet converged, we solve the agent's problem using the new price sequence and iterate on this procedure until convergence occurs.

B.4.4 Accuracy

For the numeric policy functions our main concern is the accuracy of the results. Since we use a very dense grid over the state space for the approximation of the optimal policy function, we can achieve a highly accurate approximation of the true policy function, where the accuracy is measured by the relative error on the Euler equation as introduced in Judd (1992)[41]. For ratio variables the relative error is

$$e = 1 - \tilde{c}_t^{-1}\left(\beta(1+r)\mathbb{E}\left[\varepsilon_{t+1}^{-\gamma}\tilde{c}_{t+1}^{-\gamma}\right]\right)^{-\frac{1}{\gamma}}$$

[40] In our calculations we use a very *conservative* updating rule with $a = 0.99$. We thank Alexander Ludwig for helping us with this problem.

[41] We report take the absolute value of the error for all statistics.

We choose the state space for the cash-at-hand to labor income ratio to be $[1.0, 500.0]$. The errors we report apply to an 5% extended state space at the upper end, so that $\tilde{x}_{max} = 525$. We do so to show that the quality of the approximation does not drop rapidely when we have to do extrapolations during the calculations. For the benchmark economy we get the minimum $1.93e-11$, the median $1.91e-10$, the mean $1.45e-08$, and the maximum relative error $1.82e-06$.

We use the relative error because of its nice interpretation. The error can be interpreted as the relative error that occurs in the consumption decision of an agent due to the fact that she relies on an approximate policy instead of the true policy function. An error of 0.01 means that she does an error of 1€ for every 100€ spent. The errors here are always below $1.82e-06$, i.e. the agent makes at most an error of 1.82€ for every $1,000,000$€ spent. This shows that we have found a quite accurate approximation to the true policy function. Especially, because the mean and the median are smaller by two orders of magnitude.

Furthermore, since we consider the complete state space, we can conclude that in equilibrium there are no binding borrowing constraints, because if constraints are binding the Euler equation holds as a strict inequality.

Chapter 3

Labor market rigidity and the transmission of business cycle shocks

with Philip Jung

3.1 Introduction

Compared to the US the German labor market is characterized by substantially lower average hiring and firing rates[1]. Institutional differences, in particular the employment protection legislation and the influence of unions in the wage setting process, have been pointed out as causal for the lower transition rates. Moreover, Petrongolo and Pissarides (2009) study empirically the contribution of hirings and firings to unemployment volatility and argue that in countries like France with stricter employment protection legislation *'it is not surprising that the employment-unemployment transition contributes less to cyclical volatility'* (pp.11). Similarly, though on theoretical grounds, Veracierto (2008) shows that within a model of job reallocation firing taxes *'lower the response of the economy to aggregate productivity changes'* (pp.3).

This paper studies empirically and theoretically whether low average hiring and firing rates can indeed be associated with a lower contribution of firings to unemployment volatility and a lower response of the unemployment rate to business cycle shocks. We document that the evidence for Germany and the US suggests the opposite relationship, namely that a more rigid labor market is associated with more fluctuations over the business cycle rather than less. In Germany, firing rates are compared to the US lower by a factor of 4 and hiring rates by a factor

[1] We use the term *hiring* for unemployment-to-employment transitions and the term *firing* for employment-to-unemployment transitions.

of 5, but the unemployment rate is 1.2 times, the firing rate 2.5 times, and the hiring rate as volatile as the respective US counterparts. These volatility differences translate for Germany into a 30% stronger reaction of unemployment rates to business cycle shocks of the same size and firings that contribute $60-70\%$ to unemployment volatlity whereas for the US the opposite is true and hirings account for $60-70\%$. A similar finding holds for earnings, where we show that, if anything, German earnings are more flexible than US earnings, and wage rigidity can therefore not account for the observed differences.
To explain these empirical facts, we develop an extended version of the standard search and matching model featuring endogenous firings, search on the job, and match heterogeneity. In this model, we show analytically that lower hiring and firing rates are in general inversely related to business cycle volatility. This means that labor market policies that induce a decline in transition rates increase unemployment volatility, yield a higher contribution of firings to unemployment volatility, and moreover, induce a substantial increase in the persistence of shocks. A calibrated version of our model generates a reaction of the unemployment rate to a business cycle shock that is five years after the shock still two times larger in Germany than in the US. This pattern is consistent with the empirical observation after the large oil price shocks in the eighties.
Thereby, our results add theoretical insights to the empirical discussion on *'shocks vs. institutions'* in Blanchard and Wolfers (2000). Our model shows how institutional difference leading to low transition rates do not only amplify the transmission of shocks but also increase the persistence of the reaction. It is exactly this interplay that makes rigid labor markets so vulnerable to business cycle shocks.

On empirical grounds, the paper fills a gap on the contribution of the *'ins'* and *'outs'* in Europe and provides a detailed analysis of labor market flows and earning dynamics for Germany.[2] We show that the stylized labor market facts as stated in Shimer (2005) for the US can also be found for Germany. However, two crucial differences arise. On the one hand, we observe substantially lower transition rates, but on the other hand, a substantially higher firing rate volatility. Extending the methodology of Fujita and Ramey (2007) to a three state decomposition adding flows from non-employment, we show that in Germany firings are twice as important in explaining unemployment fluctuations than hirings.
A second empirical contribution of this paper is to uncover an important dimension of labor market heterogeneity. We show that the bulk of worker flows results from low tenured and

[2] There are two other studies on worker flows in Germany. Bachmann (2005) uses a different concept to measure worker flows and focuses on the dynamics of annual transition rates. Some selected results are consistent with our findings. Gartner et al. (2009) consider quarterly transition rates and do not control for non-employment, tenure, and earnings. The aggregation to quarterly transition rates makes their results not comparable to our findings.

badly paid matches. Good matches, i.e. matches that are long lasting and relatively better paid, are associated with lower firing and quitting rates. This match heterogeneity provides an empirical motivation for both search on the job and endogenous firings in labor market models. To complete the picture, we also look at differences in worker flows across education groups and sex.

Besides the employment protection legislation, unions or, more generally, the wage setting mechanism has been identified as a prime candidate in explaining the unemployment volatility puzzle originally recognized by Shimer (2005), Hall (2005) and Costain and Reiter (2005).[3] As a third empirical contribution we show that rigidity of this kind is likely not at the root of the cross-country differences, confirming results in Pissarides (2009) who finds that the co-movement of wages with the business cycle might even be higher in Europe than in the US. We provide strong support for this claim for Germany. We apply different approaches proposed in the literature to control for composition bias in wage dynamics but our robust finding across all methods is that German earnings are not rigid and have an elasticity between $0.6 - 0.8$ with respect to productivity for all types of workers.[4] A similar finding for the US has been established by Haefke et al. (2007).

The theoretical contribution of the paper is to account for the above stylized facts within the context of an extended version of the standard search and matching model, featuring endogenous firings, as suggested in den Haan et al. (2000), search on the job, as recently explored in Fujita and Ramey (2007), as well as heterogeneity across matches, as discussed in Menzio and Shi (2009). Our model allows us to provide a simple closed form solution up to a first order approximation for special cases.

We show that endogenous firings have compared to exogenous firings no impact on the hiring rate volatility and can therefore not resolve the basic volatility puzzle. However, the endogeneity of firings is crucial to explain the empirically observed large contribution of firings in the unemployment volatility. Adding search on the job does quantitatively not help in explaining a significant fraction of the unemployment volatility.[5] Yet, it helps to reconcile the model and the data by matching the *Beveridge curve* confirming numerical results in Ramey (2008). In contrast to the results in Pries (2008), we find that type differences and the associated composition effects across workers have only a weak impact on aggregate volatilities. However, the

[3]Resolutions typically rely on arguments that make wages react only weakly to aggregate conditions inducing a strong surplus for firms to hire in booms. The proposed changes to the benchmark Nash bargaining solution were to change the bargaining set as in Hall (2008), inducing countercyclical bargaining power (Shimer (2005)), using optimal contracts with risk averse agents (Rudanko (2009)), or using staggered wage contracts (Gertler and Trigari (2005)).

[4]Peng and Siebert (2007) using GSOEP data, though limited by the sample size, also provide evidence that wages appear to be fairly flexible in Germany.

[5]At least for Germany, though we find a mildly larger impact for the US.

introduction of heterogeneity across job types generates positive incentives to search, accounts for observed type differences in average firing rates, and delivers a large average surplus of a match. Type differences paired with search on the job address therefore at least partly the critique raised to the small surplus calibration of Hagedorn and Manovskii (2008).

We show using a Kalman filtering strategy that once the model is calibrated at the right macroeconomic elasticities, it predicts the entire time-series pattern of labor market dynamics very well. We perform an impulse-response analysis of the model and find that the implied persistence to shocks turns out to be dramatically different across countries. Five years after a shock hit the economy, the deviation of the unemployment rate from its long-run trend will still be twice as large in Germany as in the US. The large differences in the persistence of shocks do not stem from differences in the wage elasticity but can be traced to the different reaction of firings over the cycle. While a single institution alone cannot be held responsible for all cross-country differences our findings suggest that in particular differences in the bargaining power and the hiring and firing costs across countries can explain the observed differences in the firing rate volatility, the low average transition rates, and the high unemployment persistence.

We proceed in 4 steps. Section 3.2 describes the data and documents stylized facts about labor market transitions, decomposes the unemployment volatility, examines the effect of tenure, and studies the cyclical behavior of earnings. Section 3.3 describes and solves an augmented search and matching model. In section 3.4, we provide a calibration for our model that jointly matches Germany and the US and perform the impulse-response analysis. Section 3.5 concludes. The appendix provides more detailed information on different subgroups males, females, education or other observable features of the data.

3.2 Data description and aggregate dynamics

Our dataset is the IAB employment panel that is a 2% representative subsample taken from the German social security and unemployment records for the period from $1975 - 2004$. The sample contains employees that are covered by the compulsory German social security system, it excludes self-employed and civil servants ('Beamte'). Still, it covers about 80% of Germany's labor force. Since the East German labor market was subject to additional regulations and restructuring after the reunification, we exclude all persons with employment spells in East Germany from our sample.[6] We observe the entire employment history of each worker on a daily basis. The income reported at one spell is the average daily income of an individual during the employment spell. We do not observe hours worked but observe whether the person is full-

[6]We do a first step sample selection where we remove very few individuals with missing observations. Details can be found in the appendix.

time, part-time, or from 1999 on in marginal employment.[7] We impute earnings above the social security threshold following Gartner (2005), adjust for the change in income reporting in 1983 following Fitzenberger (1999), and adjust the education variables following Fitzenberger et al. (2006). Our basic time-period will be one month. Other studies using the IAB-panel study transitions at lower frequency (see Gartner et al. (2009) and Bachmann (2005)). Using a monthly frequency allows us, among other things, to reduce the time-aggregation bias.
Aggregate data are taken from the statistic office ('Statistische Bundesamt'). We use nominal GDP and convert it to real GDP by the CPI deflator from the Bundesbank. We deflate nominal earnings in the IAB sample using again the same CPI deflator. After 1991, we only observe GDP for the unified Germany and we control for the structural break. Productivity measures are obtained by dividing through total employment or total hours worked, as is done by the statistical office. This measure is rather noisy and does not correspond to the BLS productivity measure for the US who use a more disaggregate procedure, but still suffers from aggregation problems highlighted when discussing earnings below. Further details are relegated to the appendix.

3.2.1 Basic Properties

The stylized labor market facts for Germany are highlighted in table 3.1 and refer to all workers. In the appendix, we give the flows separated by sex and education. For a better comparison we also present the corresponding US statistics in table 3.2.
We find cyclical patterns across the two countries that are similar along many dimensions. In particular, we find for Germany that while aggregate output is approximately as cyclical as in the US, aggregate unemployment rates and vacancy rates[8] are more volatile in Germany. Firing rates (EU) are highly countercyclical. Jobfinding rates (UE) are procyclical in both countries but are considerably more correlated with the cycle in the US than in Germany.[9] Quitting rates, defined as on the job transitions to a new establishment, are highly procyclical

[7] We control for transitions into part-time or from part-time to full-time. We mainly report aggregate statistics. All transitions into different subclasses are available upon request.

[8] Our vacancy measure is fairly crude given that it includes only open positions reported to the Bundesagentur fuer Arbeit. Most job offers will go through internal firm markets as well as newspaper adds etc., so neither the scale nor the volatility should be over-interpreted. However, the correlation structure across the two countries is almost identical as well as the broad picture that vacancy are substantially more volatile than output.

[9] It is important to notice that the correlation structure in almost all labor market variables is considerably more pronounced when we look at a broad aggregate measure, GDP per capita, instead of a productivity measure like output per person or per hour. Productivity measured as output per employed or per hour will be a problematic concept in our framework when viewed, within the model, as an exogenous TFP shock. Productivity will suffer from the same composition effects Haefke et al. (2007) highlight for wages and which we will extensively discuss below. However, for Germany due to the reunification the bias might be particularly severe, and the HP-filter is particularly problematic. We will typically rely on a broader measure of economic activity like GDP per capita, which seems less affected.

Table 3.1: German GDP and productivity, employment, and labor market flows $Jan1980 - Sep2004$

	Mean	Std	Rel. Std	Corr (GDP)	Corr (GDP p. Emp.)	Autocorr
GDP		0.024	1	1	0.7809	0.9533
GDP per Emp.		0.0164	0.6836	0.7809	1	0.9246
GDP per Hour		0.0187	0.7808	0.7979	0.9534	0.9608
U-rate (official)	0.0837	0.1808	7.535	−0.7629	−0.4448	0.9794
Vacancies		0.3337	13.9	0.818	0.5556	0.9777
IAB median earnings		0.0168	0.6997	0.8447	0.6764	0.8605
IAB U-rate	0.0758	0.1694	7.059	−0.7222	−0.3231	0.9734
IAB E-rate	0.9242	0.0119	0.4969	0.6409	0.1319	0.9728
Firm exit	0.0239	0.0549	2.288	0.4719	0.2262	0.7532
Empl. exit	0.0152	0.0382	1.592	−0.4284	−0.2031	0.5096
EU	0.0053	0.1479	6.163	−0.8043	−0.501	0.9
EN	0.01	0.0633	2.637	0.5493	0.4008	0.7789
UE	0.0622	0.1034	4.31	0.4157	0.0728	0.7894
UN	0.0488	0.1024	4.268	0.4672	0.5309	0.7978
NE*	0.0649*	0.178	7.418	0.326	−0.0558	0.8511
NU*	0.0234*	0.1596	6.651	−0.2098	−0.1126	0.8984
Quits	0.0086	0.158	6.585	0.6528	0.327	0.9189

Notes: All data are in logs and are HP-filtered with $\lambda = 100,000$. GDP data is nominal GDP per capita from the statistic office deflated by the CPI, taken from the Bundesbank. Employment and total hours worked are also taken from the statistics office. IAB data are quarterly averages of monthly data. All IAB data are authors' calculations. Firm Exit is defined as the sum of EU+EN+Quits. Employment Exit is defined as EU+EN. Quits are defined as job-job transitions between two consecutive dates and a change in the firm counter as defined in the IAB-data. The star at the non-employment flows indicate that the denominator, that is the state of non-employed workers is measured with problems given that we do not have the corresponding universe of searching non-employed. We partially control for this by dropping (early)-retired and only look at workers that eventually will return to the labor market in our sample period. The (log)-volatility measures might be less affected by the problem.

in Germany. For the US, we do not have equivalent data, but the analysis in Nagypal (2005) suggests that this also holds for the US. Separation rates from the firm's perspective (the sum of quits, EN, and EU) are procyclical implying that the behavior of quits and separations into non-employment dominate the behavior of firings. Given that both rates have counteracting correlation signs overall separation is rather acyclical. We lack a precise counterpart of this variable for the US, given that we do not observe quits on the job directly. Employment exit rates (the sum of EU and EN) are countercyclical both in the US and in Germany. Employment to non-employment rates are procyclical in both countries and are mirroring the behavior of quits on the job, suggesting that, if anything, they reflect quitting behavior. Median earnings obtained from the IAB data are highly procyclical.

There are two fundamental differences across the two countries. First, average transition rates in Germany are substantially lower than in the US. The average jobfinding rate in Germany is smaller by a factor of 5 and the firing rate by a factor of 4. Second, firing rates are roughly 2.3 times as volatile in Germany as in the US. Relative to GDP firing rates are 2.5 times and the

Table 3.2: US GDP and productivity, employment, and labor market flows $Jan1980 - Sep2004$

	Mean	Std	Rel. Std	Corr (GDP)	Corr (GDP p. Emp.)	Autocorr
GDP		0.0263	1	1	0.4443	0.9309
GDP per Emp.		0.0140	0.5307	0.4443	1	0.8487
GDP per hour		0.0142	0.5385	0.1448	0.8883	0.8769
Earnings (BLS)		0.0177	0.6739	0.4231	0.6182	0.9427
U-rate	0.0626	0.1505	5.7224	−0.8904	−0.0272	0.9579
E-rate		0.0143	0.5420	0.8580	−0.0035	0.9576
Vacancies		0.2044	7.7719	0.8457	0.0553	0.9629
Empl. exit	0.0477	0.0372	1.4156	−0.2438	−0.1192	0.3425
EU	0.0203	0.0653	2.4818	−0.7166	−0.3759	0.5083
EN	0.0274	0.0458	1.7413	0.4420	0.2583	0.4418
UE	0.3069	0.1123	4.2705	0.8152	−0.0715	0.8943
UN	0.2658	0.0911	3.4629	0.7276	−0.0477	0.8756
NE	0.0424	0.0592	2.2512	0.6277	0.2285	0.5752
NU	0.0357	0.0713	2.7114	−0.5544	−0.1496	0.6997

Notes: US output data are taken from the NIPA and are deflated by the GDP deflator, productivity and unemployment rate data are taken from the BLS, vacancy postings are taken measured by the Help wanted index, and the labor market transition probabilities are taken from Shimer (2005). All data are in logs and are HP-filtererd with $\lambda = 100,000$.

unemployment rate 1.2 times as volatile but hiring rates are equally volatile. These differences translate into the finding of the next section that German unemployment volatility is mainly driven by variations in firings, explaining between $60 - 70\%$ of the unemployment volatility while in the US unemployment volatility is dominated by the behavior of hirings and firings account only for $30 - 40\%$. We will now make this statement quantitatively precise.

3.2.2 Unemployment volatility decomposition

Petrongolo and Pissarides (2009) analyze the contribution of job in- and outflow rates to the fluctuations in unemployment for UK, France, and Spain. Fujita and Ramey (2007) do an analysis for the US. The analysis in both papers is based on a first-order approximation around trend unemployment but the detrending methods and the considered labor market flows differ. The analysis in Petrongolo and Pissarides (2009) is based on a first difference filter allowing for four aggregate transition rates whereas Fujita and Ramey (2007) use the HP-Filter and a two state decomposition. Fujita and Ramey show that the first difference filter is typically very sensitive to high-frequency fluctuations. To address the importance of firings and jobfindings in explaining unemployment volatility, we extend the methodology proposed in Fujita and Ramey (2007) but allow for a three states with six transition rates. We describe briefly the decomposition in Fujita and Ramey (2007) and our extension. An extensive sensitivity analysis with respect to different methods, time periods, and group selection can be found in the appendix.

To derive the contribution rates we taken an approximation around trend unemployment

$$
\begin{aligned}
u_t &\approx \frac{\Pi_{EU,t}}{\Pi_{EU,t}+\Pi_{UE,t}} \\
\log\left(\frac{u_t}{\bar{u}_t}\right) &= (1-\bar{u}_t)\log\left(\frac{\Pi_{UE,t}}{\bar{\Pi}_{UE,t}}\right) - (1-\bar{u}_t)\log\left(\frac{\Pi_{EU,t}}{\bar{\Pi}_{EU,t}}\right) + \epsilon_t \\
du_t &= dUE_t + dEU_t + \epsilon_t
\end{aligned}
$$

where $\Pi_{EU,t}$ denotes the firing probability while $\Pi_{UE,t}$ is the hiring probability and a bar denotes the trend component of the respective variable. $\log(u_t/\bar{u}_t)$ measures the relative deviation of the unemployment rate from its trend. Fujita and Ramey (2007) show that the variance of $\ln(u_t/\bar{u}_t)$ can then be decomposed such that $1 = \beta_{\pi_{ue}} + \beta_{\pi_{eu}} + \beta_\epsilon$ where $\beta_x = \frac{cov(du_t, d\Pi_x)}{var(du_t)}$. The decomposition allows us to obtain two separate components (and an error term) for the importance of the respective series in explaining the cyclical variation of the unemployment rate. Using an equivalent steady state approximation for the three state case and defining weights $\alpha := \frac{\bar{\Pi}_{NU}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}$ and $\lambda_{ij} := (1-\bar{u})\frac{\bar{\Pi}_{ij}}{\bar{\Pi}_u}$, as well as the (weighted) average of separation and hiring rates $\bar{\Pi}_u := \bar{\Pi}_{EU} + \frac{\bar{\Pi}_{NU}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}\bar{\Pi}_{EN}$ and $\bar{\Pi}_e := \bar{\Pi}_{UE} + \frac{\bar{\Pi}_{UN}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}\bar{\Pi}_{NE}$, we obtain an extended decomposition

$$
\begin{aligned}
\log\left(\frac{u_t}{\bar{u}}\right) &= \log\left(\frac{\Pi_{EU,t}}{\bar{\Pi}_{EU}}\right)\lambda_{EU} - \log\left(\frac{\Pi_{UE,t}}{\bar{\Pi}_{UE}}\right)\lambda_{UE} \\
&\quad + \log\left(\frac{\Pi_{EN,t}}{\bar{\Pi}_{EN}}\right)\alpha\lambda_{EN} - \log\left(\frac{\Pi_{NE,t}}{\bar{\Pi}_{NE}}\right)(1-\alpha)(\lambda_{UE}+\lambda_{UN}-\lambda_{EU}) \\
&\quad + \log\left(\frac{\Pi_{NU,t}}{\bar{\Pi}_{NU}}\right)\alpha(\lambda_{EU}+\lambda_{EN}-\lambda_{UE}) - \log\left(\frac{\Pi_{UN,t}}{\bar{\Pi}_{UN}}\right)(1-\alpha)\lambda_{UN} + \varepsilon_t \\
du_t &= dEU_t + dUE_t + dEN_t + dNE_t + dNU_t + dUN_t + \varepsilon_t
\end{aligned}
$$

Using again $\beta_x = \frac{cov(du_t, d\Pi_x)}{var(du_t)}$ a similar covariance decomposition as in Fujita and Ramey (2007) of the form $1 = \sum_{i=1}^{n}\beta_i + \varepsilon_t$ applies.[10] Table 3.3 summarizes our finding based on the two-state and three state decomposition.

The way of detrending is not innocent for many datasets given that the steady state approximation is not necessarily very accurate during certain time periods. However, for Germany our results are not driven by the detrending method used. We obtain the same decomposition with

[10] The formula is similar to the first difference filter obtained in Petrongolo and Pissarides (2009), though they essentially lump the non-employment rates $dEN_t + dUN_t$ and the corresponding inflow rate into $dNE_t + dNU_t$ together. In fact the non-employment flows are hard to interpret in their decomposition. It is important to notice that the decomposition does not rely on knowing the state of non-employed workers, which is not available for Germany but only the (gross) flows are needed. A derivation is available upon request.

Table 3.3: Unemployment decomposition

Country	Data	EU	UE	NE	EN	NU	UN	ε
Germany	IAB	0.6073	0.3898					0.0030
	IAB	0.4186	0.2498	0.2020	−0.0470	0.0678	0.1122	−0.0020
US	Shimer	0.3260	0.6763					−0.0022
	Fujita/Ramey	0.3837	0.6185					−0.0022
	Shimer	0.2013	0.4855	0.0884	−0.0378	0.1039	0.1516	0.0072

Notes: Data is HP-filtered ($\lambda = 100,000$) for the period $1980q1 - 2004q4$. For Germany the transition rates are for all male and female workers. The US data is obtained from Shimer (2005) and Fujita and Ramey (2007)

a first difference filter.[11]

Based on a two state decomposition the contribution of firing rates account, depending on the sample periods used, for $60-70\%$ of the volatility while in the US it accounts for $30-40\%$. The robust finding, using a three state decomposition, indicates that German firing rates contribute twice as much as jobfinding rates to the unemployment volatility while in the US the opposite is true. Firing and jobfinding rates taken together account in both countries for around 70% of the unemployment volatility possibly justifying the focus on a two-state decomposition. The left panel of figure 3.1 visualizes the tight connection of firings und unemployment in Germany by plotting the HP-filtered firing rate against the cyclical component of the unemployment rate. It is evident that firings lead the unemployment rate by one quarter but is otherwise almost perfectly correlated with the unemployment rate. The right panel shows the tight connection between quits on the job and hirings, suggesting a common matching technology.

So far, we have analyzed aggregate worker flows, however, the aggregate picture masks important differences in the characteristics of workers with respect to observable characteristics, in particular tenure on the job. We now turn to a discussion of these disaggregated facts.

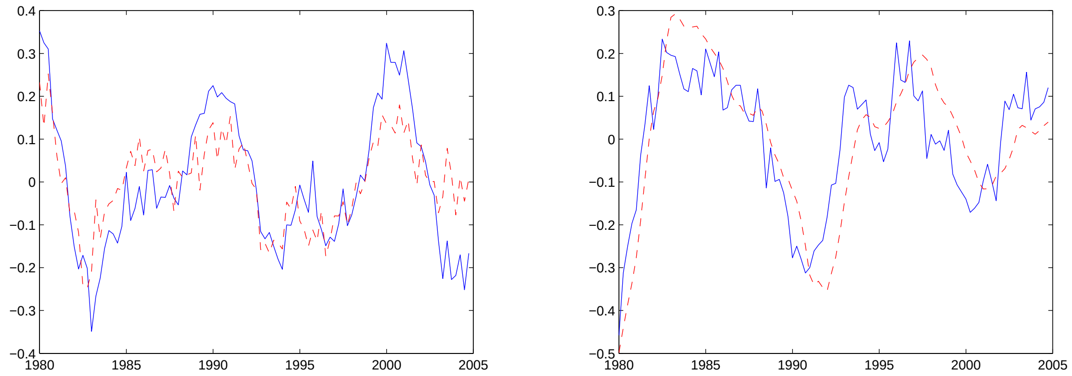
3.2.3 Disaggregation of firings, quits, and jobfindings by tenure

The analysis of aggregate flows between employment, unemployment, and non-employment abstracted from heterogeneity within these flows and their composition. In the following analysis, we focus on one special dimension of heterogeneity by distinguishing labor market flows by the duration in the previous labor market state. For simplicity, we call this duration from now on *tenure*. We break the analysis further down and distinguish labor market flows also by sex and education. The analysis shows that tenure is not only a widely unexplored dimension of heterogeneity but also of primary importance when it comes to understanding labor market

[11] The appendix provides a sensitivity check with respect to the first difference filter of Petrongolo and Pissarides (2009) and also gives the results for different education groups, different sample periods and separated by sex.

Figure 3.1: Labor market cyclicality

(a) Firing rate and unemployment rate (b) Quits and jobfinding rate



Notes: Left Panel: The blue solid line reports the HP-filtered firing rate. The red dotted line reports the HP-filtered unemployment rate. Right Panel: The blue solid line reports the HP-filtered quitting rate. The red dotted line reports the HP-filtered jobfinding rate.

flows.

To document the important role of tenure for hiring, firing, and quitting rates, we construct four tenure cells and assign labor market flows to one of the cells according to the time spent in the initial state of the labor market flow. For transitions out of employment we only count the time with the current employer. In 1975, we do not observe tenure of a particular worker, and therefore, we start in 1980 to overcome the truncation problem. Given that our maximum tenure class is five years and above, we can assign from 1980 on the transition to the correct tenure cell.

Table 3.4 reports the basic statistics for firing, quitting, and jobfinding rates disaggregated to the tenure cells. The table highlights the strong impact of tenure on mean transition rates. The firing risk drops by an order of magnitude from 1.8% for low tenured workers to 0.15% for high tenured workers. The quitting probability collapses by the same order of magnitude from 1.8% for low tenured workers to 0.36% for high tenured workers. The jobfinding probability shows the same pattern decreasing from 9.3% for short unemployment spells to only 2.0% for long unemployment spells.

The *relative share* in table 3.4 measures the part of the transitions that originates from the particular tenure cell. It shows the same pattern across tenure cells as the transition rates. We see that 70% of all firings and 57% of all quits fall into the class of workers having tenure of less than 2 years.[12] For jobfindings, we get that over 85% of all jobfindings accrue to persons

[12] We do a sensitivity analysis with respect to very short spells in the appendix to rule out high frequency noise of unstable jobs. The results show that most jobs survive beyond the threshold of 6 month and results are robust.

Table 3.4: Tenure on the job

Quits	< 365 days	365 – 730 days	730 – 1825 days	> 1825 days	*overall* days
mean	0.0183	0.0110	0.0079	0.0036	0.0081
std	0.1193	0.1575	0.1749	0.1481	0.1620
rel. share	0.4165	0.1504	0.2175	0.2156	
rel. earnings	0.9018	0.9308	0.9248	0.9197	
corr (GDP)	0.5595	0.5568	0.6053	0.5149	0.6345
corr (Productivity)	0.2545	0.3261	0.3104	0.3580	0.3312
Firings	< 365 days	365 – 730 days	730 – 1825 days	> 1825 days	*overall* days
mean	0.0176	0.0066	0.0035	0.0015	0.0054
std	0.1936	0.1726	0.2283	0.2302	0.1500
rel. share	0.5860	0.1344	0.1454	0.1341	
rel. earnings	0.8549	0.8208	0.8046	0.8125	
corr (GDP)	-0.7727	-0.7308	-0.7195	-0.5574	-0.7616
corr (Productivity)	-0.4650	-0.4644	-0.4213	-0.2546	-0.4739
Jobfindings	< 180 days	180 – 365 days	365 – 730 days	> 730 days	*overall* days
mean	0.0934	0.0431	0.0301	0.0204	0.0614
std	0.1156	0.1122	0.1370	0.1730	0.1019
rel. share	0.7137	0.1480	0.0821	0.0562	
corr (GDP)	0.3442	0.0444	0.4166	0.4970	0.3919
corr (Productivity)	0.0532	-0.1251	0.1757	0.1249	0.0485

Notes: The data is for full-time employed males and females for the period $Jan1980 - Sep2004$. The columns contain the bounds of the different tenure groups. The tenure groups are formed for the labor market state before the transition and are given in days.
All statistics are computed conditional on being in the respective labor market state and tenure group. *mean* is the average transition probability of the respective labor market transition. *std* is the relative deviation of the transition rate over time. *rel. share* is the average share of transitions falling in this tenure group relative to all transitions. *rel. earnings* are the average relative earnings of all persons with a transition relative to the average earnings in the current labor market and tenure group. *corr* are the respective correlations of the transition rate with GDP per capita respectively per employed as our business cycle measures.

that have been unemployed for less that one year, however, the major share of persons find a job already within six month (71%).

When it comes to cyclical fluctuations, we see that the correlation and the standard deviation are similar across tenure cells for all transitions.

Finally, to learn more about the job quality of destructed jobs, we look at the median earnings ratio of destructed to continued jobs (*relative earnings*). Our data shows that destructed jobs come from the lower end of the distribution. The discount is $15 - 20\%$ for jobs destructed due to firings and $7 - 10\%$ for jobs destructed due to quits.

We interpret this result as giving support to match heterogeneity in the labor market. We think of the following situation: A firm posts an open position and an unemployed worker looks for open positions. They meet but realize immediately that the worker does not fit the new job but since the alternative for the firm would be to leave the position open and for the worker to stay unemployed, they bargain and share the small surplus of the match. This constitutes what we call a bad match, and the worker will be searching for a new job somewhere else hoping that the new task will better match her skills such that a good match can be formed. Furthermore, the firm is more likely to fire the worker because the surplus of the match is small. Taken together, this kind of match heterogeneity explains the higher destruction rates of low tenured jobs and the earnings discount on these jobs relative to the peer group of continued matches. Although the data does not rule out other mechanisms to generate the observed pattern, we opt for match heterogeneity in our model because we find it to be most convincing yet tractable enough to incorporate it into a business cycle model.

Of course, the above measures might be affected by other composition effects. In the appendix, we report the same statistics for males and females and control for different education levels. Although the general pattern remains unchanged, some results are worthwhile to mention. First, jobfinding is substantially lower for low skilled compared to medium and high skilled workers. Second, differences in unemployment rates across medium and high-skilled workers are driven mainly by differences in average firing rates, not by differences in the jobfinding rates. The earnings discount, expressed relative to the peer tenure-education group, is particularly large for high skilled workers.

3.2.4 Earnings

In a recent survey article Pissarides (2009) discusses the empirical evidence on wage rigidity for the US and Europe. He concludes that the available evidence suggests a stronger co-movement of wages with the business cycle in Europe than in the US. We provide strong support for this claim for Germany, arguing that at least rigid earnings[13] are not at the root of the cross-country differences highlighted above.

Table 3.1 shows that German median earnings are tightly connected to aggregate productivity measures. However, the cyclicality of aggregate statistics can be substantially biased due to composition effects in the labor force as highlighted in Solon et al. (1994) and extensively discussed in Haefke et al. (2007).

Several approaches have been proposed to control for this composition bias. Solon et al. (1994) use a group selection procedure to fix the group of individuals to avoid changes in the composition over time. The long panel dimension and the high quality of our income data allows us

[13] The IAB data, although superior to many other data sets for labor market transitions has the disadvantage of lacking information on hours worked. It only contains information on the employment status (full-time, part-time, marginal employment) of an individual. However, this limitation is not severe because hours worked per person employed does not vary much with the cycle. Other studies for the US have provided evidence that earnings and wage cyclicality are quantitatively close (see Haefke et al. (2007)). We will therefore in the following repeatedly refer to studies that considered wages instead of earnings.

to do the same, however, with a substantially larger cross-section. We identify in our dataset ongoing job relations that do not only exist on a year-to-year basis but over the whole sample period. The large cross-section of our dataset allows us to focus on this particular homogeneous subpanel of workers, namely workers who had a job in 1975 and were continuously full-time employed until 2004 at the same firm. That is, for this non-representative group, we ensure that no quit and no firing happened during their entire work experience nor any non-employment spell.[14] For this group we only have earning information at annual frequency, and in contrast to part of the business cycle literature, we have to move to annual frequency. However, the annual frequency might actually be the more natural frequency given that bonus and special payments are typically not paid out quarterly. Given that, at least in models with risk-neutral agents, the precise timing of the payment is indetermined, we believe that an annual frequency offers some advantages over a quarterly analysis. Although the group of continuously employed workers is highly selective, it allows us to examine the earnings dynamics of very stable jobs. The selection procedure addresses therefore also concerns regarding job quality over the cycle raised by Gertler and Triagari (2005).

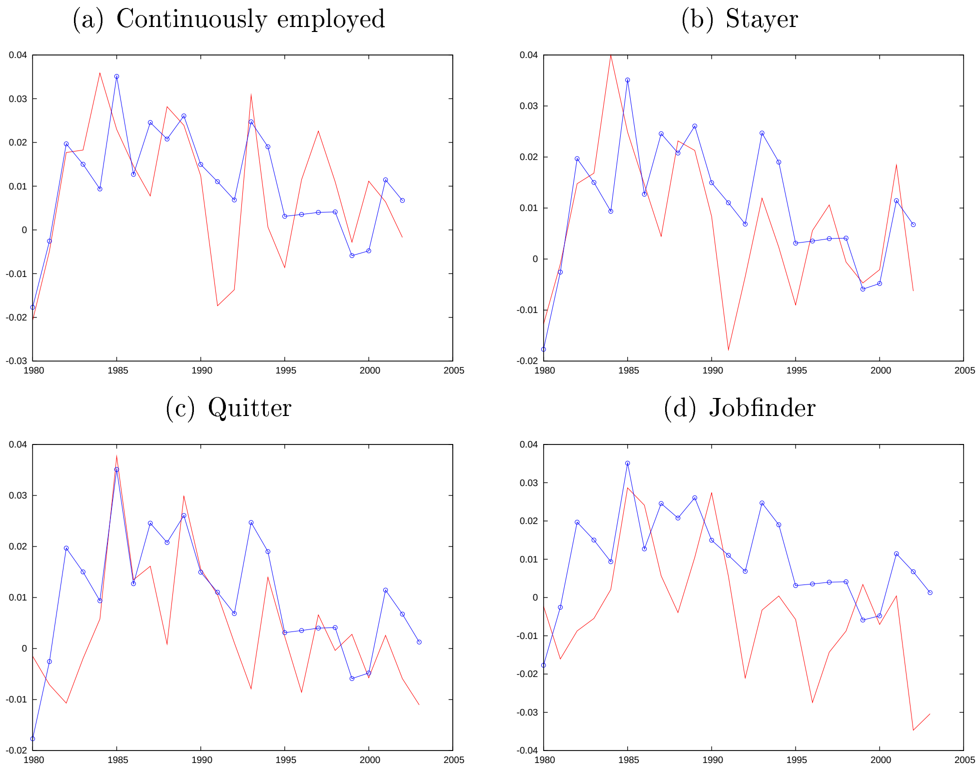
Starting with Bils (1985) researchers have estimated individual wage growth equations using first differences along the panel dimension to control for individual specific fixed effects. The approach might be restrictive if only a short panel dimension is available. In particular, last earnings of unemployed workers might not exist or are unobserved. For quitting workers this problem does not exist. Our long panel dimension allows us to keep track of last earnings of unemployed workers which we use as a proxy for unobserved earnings in the regressions. We construct a sample comprising all spells with certain labor market transitions, e.g. quits. For this sample we regress individual earnings growth on the particular labor market event on several individual control variables and the growth of the respective business cycle statistic. The labor market events are grouped by years and individual controls are a fourth order polynomial in potential labor market experience, dummies for sex, three education groups, and for foreigners. We also include a time-trend. Aggregating at annual frequency allows us to abstract from adjusting for seasonality in the data. We run ordinary least squares (OLS) and least absolute deviations (LAD) regressions to check the sensitivity of our results with respect to outliers.

Although, the panel dimension of our dataset allows us to overcome missing pre-employment earnings for jobfinder, there might still be concern regarding this approach. To overcome potential concerns, we follow Haefke et al. (2007) who propose a wage index construction. They propose to control for observable characteristics like age, sex, education, and experience and to focus on the behavior of the residual. We follow their procedure and construct earnings

[14] The group still consists of approximately 6,126 workers and is therefore large enough to provide reasonable estimates.

indices for jobfinder, quitter, persons who stayed at the same firm throughout the year (stayer), and for the group of continuously employed workers described above. We plot the cyclicality of the earnings index together with our business cycle measure in figure 3.2. Table 3.5 summarizes the estimation results. The details of the estimation procedures and an extensive sensitivity check can be found in the appendix.

Figure 3.2: Earnings index cyclicality



Notes: Earnings index (blue dotted-line) cyclicality ($1st$ difference filter) for full-time workers (male and female), business cycle measure GDP per employed (red line). Time period is $Jan1980 - Sep2004$.

Table 3.5: Earnings elasticity

	Quitter	Jobfinder	Stayer	Cont. employed
Index	0.5909	0.6328	0.6651	0.7440
(Std. error)	(0.1353)	(0.1945)	(0.1502)	(0.1702)
Growth	0.3302	0.8089	0.6714	0.6214
(Std. error)	(0.1264)	(0.2493)	(0.1400)	(0.1629)
Correlation	0.5764	0.4651	0.5860	0.5811

Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). *Index* refers to the earnings index using the first difference filter. *Correlation* refers to the correlation coefficient of the earnings index and the business cycle measure. *Growth* refers to the estimation in first difference using OLS. The business cycle measure is GDP per employed.

We see that earnings and productivity are tightly connected. For continuously employed the correlation between the two series is around 0.6 and the elasticity estimate is beteen $0.62-0.74$. This finding is robust for all other groups and across methods. Jobfinder and quitter have an elasticity of 0.6 when we look at the earnings index. Using individual growth rates we find a higher elasticity of 0.8 for jobfinder, and a lower elasticity for quitter, likely due to outliers. Using an LAD robustness check supports this view. There, we find an elasticity of around 0.6. The appendix provides detailed estimates for different education groups, sex, sample periods, and filtering methods all confirming that the earnings elasticity on productivity (or other aggregate measures) is between $0.6-0.8$.

3.3 Model

The empirical analysis has highlighted important features of the German labor market. Most importantly, (*i*) the cyclical variation in the firing rate, (*ii*) the importance of job-to-job transitions, and (*iii*) the heterogeneity of transition probabilities across tenure classes. To account for these empirical observations, we present a stylized search model featuring (*i*) endogenous firings, (*ii*) a basic search on the job mechanism, and (*iii*) we allow for type differences to capture heterogeneity in match quality. The model is simple enough to work out the basic mechanisms in closed form, yet rich enough to capture the essence of labor market fluctuations over the business cycle.

Setup : Time is discrete. There is a measure of size one of workers in society. Workers and firms are risk-neutral. Workers can be employed in two types of jobs $z \in [g, b]$, good or bad, or are unemployed. At the beginning of each period, the aggregate state of the economy is given by the triple $(a, l_b, l_g) = s \in S = A \times L \times L$. The first element is the aggregate exogenous productivity state $a \in A = \mathbb{R}$ following a Markov process. The second and third component $l_z \in L = [0, 1]$ denote the measure of workers that are employed in a good respectively bad job. Letting u denote the measure of unemployed, we have the accounting identity $l_g + l_b + u = 1$. When firms open a position they randomly draw a job type. With probability π_g the new job is good and with probability $\pi_b = 1 - \pi_g$ it is bad. At the beginning of the period, workers who are currently in a match relation of a particular type bargain jointly and efficiently about the wage and the separation decision for next period. If the bargaining is successful, they produce output according to the linear production technology $y_z = AB_z$ where the aggregate technology A is assumed to evolve exogenously and common to all matches. The individual types are differentiated by the assumption that $B_g > B_b$.[15]

The decision to search on the job is taken as exogenous and random, reflecting idiosyncratic

[15]We normalize such that $\frac{B_g+B_b}{2} = 1$.

utility attached to the particular match. However, the probability of receiving an offer is still endogenous and related to the aggregate matching function.[16] At the end of the period, after production has taken place, workers in bad matches might decide to search for a different job. We assume that they search with exogenous probability π_s and receive an outside offer from a competing firm with endogenous probability π_{ee}. They accept outside job offers for sure, given that they will obtain an expected wage gain. We assume that workers in good matches do not search for new jobs because their expected wage gain is zero.[17]

If, at the end of the period, the worker has decided not to search or has not received an offer, the firm receives an idiosyncratic cost shock ϵ_i, where ϵ_i is an i.i.d. random shock logistically distributed with mean zero and variance $\frac{\pi^2}{3}\psi_z^2$. The firm has to pay the costs only if it wishes to continue the production process. The costs are sunk after the current period and will not be relevant for any future decision. The assumption of a logistic distribution allows us to obtain closed form solutions and is done for convenience (see Jung (2008) for details). Let $\bar{\omega}$ denote the threshold for the continuation costs. The threshold level will be part of the bargaining set and will be efficiently bargained about by workers and firms. All matches with cost realizations above the threshold will decide to dissolve the match. If the match is dissolved, the firm has to pay a firing tax τ to the government[18] and the worker becomes unemployed. An unemployed worker searches for a job and is matched in a matching market governed by a standard Cobb-Douglas matching function. Unemployed workers are matched with probability π_{ue} and become employed, with probability $(1 - \pi_{ue})$ they remain unemployed and keep on searching. While unemployed they receive unemployment benefits $b < 1$.

Firm's surplus: Consider a worker-firm pair at the beginning of the period. The firm discounts the future, as does the agent, with a constant discount factor β. For given wages $w_z : S \to \mathbb{R}$ and cut-off strategies $\bar{\omega} : S \to \mathbb{R}$ the firm's surplus follows the recursive formulation

$$J(S, B_z) = AB_z - w_z(S) + (1 - \pi_{ee}(S)\pi_s)\Big((1 - \pi_{eu,z}(S))\beta\mathbb{E}\left[J(S, B_z)\right] - \pi_{eu,z}(S)\tau + \Psi_z(S)\Big)$$

[16] It is straightforward to make the search decision also a discrete choice and endogenous. Though it helps in making aggregate quits more volatile, we decided to keep it exogenous in this version for simplicity. An older working paper version had an endogenous search decision, also modeled as a discrete choice. The qualitative results are unaffected, but the derivations become more complex without adding new insights.

[17] Again, this assumption can be relaxed considerably without changing the main mechanism. See Menzio and Shi (2009) for a richer model along these lines.

[18] Note that τ is expressed as a firing tax, or a reorganization cost and does not include severance payments. In our framework, severance payments are efficiently bargained away and would have no effects on the equilibrium outcomes. The government transfers all income lump sum back to the worker, so under risk-neutrality, there is no need to specify formally governmental behavior.

The firing probability $\pi_{eu,z} : S \to \mathbb{R}$ and the option value[19] $\Psi_z : S \to \mathbb{R}$ follow directly from the assumption of a logistically distributed random variable

$$\begin{aligned}
\Psi_z(S) &= -\psi_z\Big((1-\pi_{eu,z}(S))\log(1-\pi_{eu,z}(S)) + \pi_{eu,z}(S)\log(\pi_{eu,z}(S))\Big) \\
\pi_{eu,z}(S) &= \left(1+\exp\left(\frac{\bar{\omega}(S)}{\psi_z}\right)\right)^{-1}
\end{aligned}$$

The quitting probability is given by $\pi_{ee}(S)\pi_s$ and is zero in the case of a good match $(z = g)$.

Worker's surplus: The value flows of the different types of employed workers $V_{e,z} : S \to \mathbb{R}$ and unemployed workers $V_u : S \to \mathbb{R}$ are given by

$$\begin{aligned}
V_{e,b}(S) &= w_b(S) + \pi_{eu,b}\beta\mathbb{E}\left[V_u(S')\right] \\
&\quad + (1-\pi_{eu,b})\left((1-\pi_{ee}(S)\pi_s)\beta\mathbb{E}\left[V_{e,b}(S')\right] + \pi_{ee}(S)\pi_s\left(\sum_z \pi_z\beta\mathbb{E}\left[V_{e,z}(S')\right]\right)\right) \\
V_{e,g}(S) &= w_g + (1-\pi_{eu,g}(S))\beta\mathbb{E}\left[V_{e,g}(S')\right] + \pi_{eu,g}\beta\mathbb{E}\left[V_u(S')\right] \\
V_u(S) &= b + \pi_{ue}(S)\left(\sum_z \pi_z\beta\mathbb{E}\left[V_{e,z}(S')\right]\right) + (1-\pi_{ue}(S))\beta\mathbb{E}\left[V_u(S')\right]
\end{aligned}$$

and the worker's surplus becomes

$$\Delta_z = V_{e,z}(S) - V_u(S)$$

Matching: New matches are formed by a standard Cobb-Douglas matching technology that links the measure of searching workers to the measure of vacancies v. The measure of searching workers is the sum of unemployed workers and the fraction of workers searching on the job. We denote the resulting matches by m and $\varkappa$ denotes a scaling parameter of the matching function.

$$m = \varkappa v^{1-\varrho}(u + \pi_s l_b)^{\varrho}$$

Labor market tightness is given as the ratio of vacancies to searching workers $x := \frac{v}{u+\pi_s l_b}$. The

[19] The term Ψ_z captures the option value of having the choice to continue the match and is always positive. The reason is that although the idiosyncratic shock has an unconditional mean of zero the manager only continues if the continuation value is positive. The payoff resembles the payoff profile of an option and is therefore increasing in the variance ψ of the shock.

probability of a searching worker to find a new job is

$$\pi_{ue} = \frac{m}{u + \pi_s l_b} = \varkappa x^{1-\varrho}$$

and the probability that a firm fills its vacancy is given by

$$\pi_{ve} = \frac{m}{v} = \varkappa x^{-\varrho}$$

Free entry: To determine the number of vacancies posted, we impose a standard free entry condition. In equilibrium, the cost to post a vacancy κ must equal the expected profits of a match

$$\kappa = \pi_{ve} \sum_z \pi_z \beta \mathbb{E}\left[J_z(S', B_z)\right]$$

Bargaining: We assume standard Nash-bargaining jointly over wages and separation decisions. The outcome of the bargaining process is characterized by

$$(w_z, \bar{\omega}) \in \arg\max_{w_z, \bar{\omega}} \mu \log\left(\Delta_{z,t}\right) + (1-\mu)\log(J_z)$$

where μ denotes the bargaining power of the worker. First order conditions deliver

$$\begin{aligned} \bar{\omega}(S) &= \beta \mathbb{E}\left[\Delta_z(S') + J_z(S')\right] + \tau \\ \frac{\mu}{1-\mu} &= \frac{\Delta_z(S)}{J_z(S)} \end{aligned}$$

Law of Motion: The law of motion for the state variables is given by

$$\begin{aligned} l'_g &= l_g(1 - \pi_{eu,g}) + l_b \pi_s \pi_{ee} \pi_g + u \pi_{ue} \pi_g \\ l'_b &= l_b(1 - \pi_{eu,b} - \pi_s \pi_{ee} \pi_g) + u \pi_{ue} \pi_g \\ u' &= u(1 - \pi_{ue}) + l_g \pi_{eu,g} + l_b(1 - \pi_s \pi_{ee}) \pi_{eu,b} \end{aligned}$$

Technology evolves exogenously according to

$$\begin{aligned} A &= \exp(a) \\ a' &= \rho a + \eta' \end{aligned}$$

where ρ denotes the autocorrelation coefficient.

3.3.1 Basic Results

To understand the basic mechanisms, we consider first the special case of homogenous types, i.e. $\pi_g = 0$. With homogeneous matches search on the job does not deliver a wage gain. If we set $\pi_s = 0$, the model nests the standard model without search on the job. All choices in the model are then functions of the total surplus $H := J + \Delta$. Proposition 1 summarizes the properties of the basic model up to a first order approximation around the deterministic steady state. We use $\bar{x}$ to denote the steady state of variable x and use $\hat{x}$ to denote the deviation from the steady state, i.e. $\hat{x} := x - \bar{x}$.

Proposition 1. *Up to a first order approximation, the dynamics of the model are only functions of the business cycle shock a*

$$\hat{H} \approx \sigma_H a \qquad \hat{\pi}_{eu} \approx \sigma_{eu} a \qquad \hat{x} \approx \sigma_x a \qquad \hat{\pi}_{ue} \approx \sigma_{ue} a \qquad \hat{w} \approx \sigma_w a$$

and coefficients are given by

Surplus:

$$\begin{aligned}\sigma_H \;=\; & \Bigg(1 - \beta\rho(1 - \bar{\pi}_{ue} - \bar{\pi}_{eu} + \pi_s \bar{\pi}_{ue}\bar{\pi}_{eu}) + \bar{\pi}_{ue}\beta\rho\left(\frac{\mu - \varrho}{\varrho}\right) \\ & + \pi_s \bar{\pi}_{ue}\beta\rho\left(\frac{1-\mu}{\varrho} + \frac{1-\varrho}{\varrho}\psi\frac{\log(1-\bar{\pi}_{eu})}{\bar{H}}\right)\Bigg)^{-1} \end{aligned} \tag{3.1}$$

Firing:

$$\frac{\sigma_{eu}}{\bar{\pi}_{eu}} = -(1 - \bar{\pi}_{eu})\frac{\rho\beta}{\psi}\sigma_H \tag{3.2}$$

Tightness:

$$\frac{\sigma_x}{\bar{x}} = \frac{\rho}{\varrho}\frac{\sigma_H}{\bar{H}} \tag{3.3}$$

Jobfinding:

$$\frac{\sigma_{ue}}{\bar{\pi}_{ue}} = (1 - \varrho)\frac{\sigma_x}{\bar{x}} \tag{3.4}$$

Wage setting:

$$\begin{aligned}\sigma_w \;=\; & \mu\sigma_H\Bigg(1 - \beta\rho(1 - \bar{\pi}_{ue} - \bar{\pi}_{eu} + \pi_s\bar{\pi}_{ue}\bar{\pi}_{eu}) \\ & + \beta\rho\bar{\pi}_{ue}(1 - \pi_s\bar{\pi}_{eu})\frac{1-\varrho}{\varrho} - \bar{\pi}_{eu}(1-\bar{\pi}_{eu})(1 - \pi_s\bar{\pi}_{ue})\beta\frac{\bar{H}}{\psi}\Bigg)\end{aligned} \tag{3.5}$$

Volatility of the unemployment rate:

$$\begin{aligned} var(\hat{u}) &= \frac{z_2^2}{1-z_1^2}\,\frac{1+\rho z_1}{1-\rho z_1} var(a) \\ z_1 &= 1-\bar{\pi}_{ue}-\bar{\pi}_{eu}+\pi_s\bar{\pi}_{ue}\bar{\pi}_{eu} \\ z_2 &= \sigma_{eu}(1-\pi_s\bar{\pi}_{ue})(1-\bar{u})-\sigma_{ue}(\pi_s\bar{\pi}_{eu}+\bar{u}(1-\pi_s\bar{\pi}_{eu})) \end{aligned} \tag{3.6}$$

Beveridge curve:

$$\frac{Cov(\hat{v},\hat{u})}{\bar{v}\bar{u}} = \left(\frac{\sigma_x}{\bar{x}}\frac{\rho(1-z_1^2)}{z_2(1+\rho z_1)} + \frac{1-\pi_s}{\bar{u}(1-\pi_s)+\pi_s}\right)\frac{var(\hat{u})}{\bar{u}} \tag{3.7}$$

The proof is straightforward and therefore omitted. As the labelling suggests, the absolute values of the coefficients coincide with the standard deviation of the respective variable relative to the standard deviation of the productivity process. Throughout the paper, we focus on standard deviations of log rates rather than on the standard deviations of absolute rates, and to ease the exposition, we use $\tilde{\sigma}_x$ to denote the log standard deviation of variable x, i.e. we define

$$\tilde{\sigma}_x := \frac{\sigma_x}{\bar{x}}$$

3.3.2 Discussion

Proposition 1 provides analytic expressions for the firing and hiring volatility as well as the implied expressions for the unemployment rate volatility and the *Beveridge curve.* This allows us to discuss the effects of observed cross-country differences on differences in labor market volatilities and the implications for labor market institutions. Within the model, we capture labor market institutions by six parameters summarizing in a reduced form important differences across countries. (i) The bargaining power μ captures the influence of unions in the wage setting process, (ii) vacancy posting costs κ relate to rigidities in the firm entry process, (iii) firing restrictions τ are used as a summary measure of employment protection, (iv) the contact rate π_s parameterizes the willingness of searching on the job, (v) the outside option b directly relates to the competitiveness of the labor market, and (vi) the idiosyncratic firm shock variance ψ measures wage compression by parameterizing the number of workers living around the cut-off value.

Firing Volatility

Formula (3.2) provides an expression for the firing volatility. To simplify the analysis, we drop quantitatively negligile terms and set $1-\bar{\pi}_{eu}\approx 1$, $\bar{\pi}_{eu}\pi_s\approx 0$, and $\beta\rho\approx 1$. Using this simplification, we derive the following expression

$$\begin{aligned}
\frac{\tilde{\sigma}_{eu}^{Ger}}{\tilde{\sigma}_{eu}^{US}} &= \frac{\psi^{US}}{\psi^{Ger}} \frac{\sigma_H^{Ger}}{\sigma_H^{US}} \\
&= \frac{\psi^{US}}{\psi^{Ger}} \frac{\varrho\bar{\pi}_{eu}^{US} + (\mu^{US} + \pi_s(1-\mu^{US}))\bar{\pi}_{ue}^{US}}{\varrho\bar{\pi}_{eu}^{Ger} + (\mu^{Ger} + \pi_s(1-\mu^{Ger}))\bar{\pi}_{ue}^{Ger}}
\end{aligned} \tag{3.8}$$

We see that if we ignore search on the job ($\pi_s = 0$) and assume that both countries face the same idiosyncratic productivity shock processes ($\psi^{US} = \psi^{Ger}$), then the countries can not operate at the efficiency point of the *Hosios condition* ($\varrho = \mu$) because this would imply

$$\frac{\tilde{\sigma}_{eu}^{Ger}}{\tilde{\sigma}_{eu}^{US}} = \frac{\bar{\pi}_{eu}^{US} + \bar{\pi}_{ue}^{US}}{\bar{\pi}_{eu}^{Ger} + \bar{\pi}_{ue}^{Ger}} \approx 5 > 2.5$$

However, equation (3.8) makes transparent that the differences in average transition rates map into substantial differences in firing volatilities across countries. The intuition for this result is simple: Consider a positive business cycle shock in both countries. The same shock increases the surplus of an employed worker in Germany more than for her US counterpart ($\sigma_H^{Ger} > \sigma_H^{US}$). To see this, look at a currently unemployed worker. In Germany, it takes her much longer to find a new job, and she participates therefore later from the booming conditions. This implies a stronger increase of the value of having a job in Germany than in the US, and the total surplus of a match increases also more in Germany. In return, firings decline more in Germany and we get a higher firing rate volatility. This argument is generic for most simple search and matching frameworks and it establishes the inverse relationship between transition rates and volatilities that we document empirically in section 3.2.

In fact, lower hiring and firing rates alone would generate too much firing volatility and it must be the case that either the bargaining power in Germany is higher ($\mu^{Ger} > \mu^{US}$) to dampen the surplus reaction or the idiosyncratic shock variance in the US is lower ($\psi^{Ger} > \psi^{US}$). A higher bargaining power has some empirical support[20] while explanations relying on a shock variance is unattractive. In our model, the idiosyncratic shock variance would map into cross-sectional wage inequality. Recent country studies by Fuchs-Schündeln et al. (2009) and Heathcote et al. (2009) for the US and Germany provide cross-country comparable inequality measures that show that the US wage inequality exceeds German inequality by far.

[20] The *OECD Employment database* (see www.oecd.org for details) reports a union density, i.e. the share of workers affiliated to a trade union, for Germany of 35% in 1975 and 22% in 2004, whereas for the US the respective numbers are as low as 22% in 1975 and only 12% in 2004.

Hiring Volatility

Using equation (3.4) and $1 - \pi_{eu} \approx 1$, we can derive the following expression

$$\frac{\tilde{\sigma}_{ue}^{Ger}}{\tilde{\sigma}_{ue}^{Ger}} = \frac{\psi^{Ger}}{\psi^{US}} \frac{\bar{H}^{US}}{\bar{H}^{Ger}} \frac{\tilde{\sigma}_{eu}^{Ger}}{\tilde{\sigma}_{eu}^{US}}$$

If we plug in the expression for the relationship between $\tilde{\sigma}_{eu}^{Ger}$ and $\tilde{\sigma}_{eu}^{US}$, we see that matching the cross country differences requires

$$\frac{\bar{H}^{Ger}}{\bar{H}^{US}} \approx 2.5 \frac{\psi^{Ger}}{\psi^{US}}$$

Hence, to explain cross country differences we either need a larger steady state surplus in Germany ($\bar{H}^{Ger} > \bar{H}^{US}$) or lower idiosyncratic volatilities in Germany ($\psi^{Ger} < \psi^{US}$). A higher surplus in Germany would, ceteris paribus, lead to the puzzling observation that one should expect higher mean jobfinding rates in Germany given that jobfinding rates are proportional to the surplus ($\bar{\pi}_{ue} \propto \bar{H}$). Looking at the data, we see that the average jobfinding rates in Germany are lower by a factor of 5. Hence, a lower idiosyncratic shock variance, i.e. higher wage compression, in Germany is likely the more relevant channel to align the model with the data. Empirically, the higher idiosyncratic shock variance for the US receives support as discussed above.

Note that we still need a small surplus for newly hired workers to match the hiring rate volatility. Neither of the newly introduced features alters the basic problem along this dimension. In particular, the surplus equation (3.1) shows that a model with endogenous firings generates an identical surplus response as a model with exogenous firings up to a first order approximation.

Unemployment volatility

Equation (3.6) delivers a model-based closed form approximation of the unemployment volatility which is exclusively based on the linearization of the unemployment flow equation around the steady state. The formula has the property shared by many simple search frameworks that the volatility of hirings and firings are only functions of the productivity state and perfectly negative correlated.[21] Furthermore, all empirical counterparts of the variables in this equation can be directly read off from tables 3.1 and 3.2 in section 3.2. We use this fact to identify the main drivers behind unemployment volatility by conducting a series of four comparative statics experiments. The first three experiments focus on the effect of hirings and firings, and the fourth experiment examines the effect of on the job search on the unemployment volatility. This

[21] This property holds in much more general circumstances and is characterized formally in Menzio and Shi (2009) who use a substantially richer model of directed search on the job.

analysis complements and extends the empirical decomposition of the unemployment volatility in section 3.2.2.

Table 3.6: Unemployment volatility in the benchmark model

	$\bar{\pi}_{eu}$	$\bar{\pi}_{ue}$	$\tilde{\sigma}_{eu}$	$\tilde{\sigma}_{ue}$	$\pi_s\bar{\pi}_{ee}$	$\bar{u}$	$\tilde{\sigma}_u$	$\tilde{s}_u$	$1-\tilde{\sigma}_u/\tilde{s}_u$
Germany	0.0053	0.062	−6.20	4.30	0	0.079	8.60	7.06	−22%
US	0.020	0.31	−2.48	4.27	0	0.062	6.18	5.72	−8%

Notes: The table reports in columns $1-5$ the data equivalents to the respective model variables, and column 6 gives the model prediction for the mean unemployment rate. Column 7 reports the relative standard deviation of the unemployment rate to the standard deviation of the productivity process in the model ($\tilde{\sigma}_u$). Column 8 reports the empirical counterpart of this number ($\tilde{s}_u$). The last column gives the residual unexplained volatility of the model relative to the data. We use an autocorrelation coefficient of $\rho = 0.975$ in line with our estimates.

Before turning to our experiments, we check the validity of the approximation. In table 3.6, we compare the model's prediction for the unemployment volatility to its empirical counterpart from tables 3.1 and 3.2. Although the model generates slightly too much unemployment volatility compared to the data, the fit is quite accurate. Hence, the model captures the main mechanisms behind the unemployment volatility in the data. We take this model as our benchmark to see how much of the benchmark volatility can be attributed to the different sources. We summarize the findings of the experiments in table 3.7.

Table 3.7: Model-based unemployment decomposition experiments

	$\bar{\pi}_{eu}$	$\bar{\pi}_{ue}$	$\tilde{\sigma}_{eu}$	$\tilde{\sigma}_{ue}$	$\pi_s\bar{\pi}_{ee}$	$\bar{u}$	$\tilde{\sigma}_u$	$\tilde{\sigma}_u^*$	$1-\tilde{\sigma}_u/\tilde{\sigma}_u^*$
Experiment 1: *Role of separations*									
Germany	0.0053	0.062	**0**	4.30	0	0.079	3.40	8.60	60%
US	0.02	0.31	**0**	4.27	0	0.062	3.90	6.18	37%
Experiment 2: *Role of means*									
Germany	**0.02**	**0.31**	−6.20	4.08	0	**0.062**	10.03	8.60	−17%
US	**0.0053**	**0.062**	−2.48	4.27	0	**0.079**	5.33	6.18	14%
Experiment 3: *Role of standard deviations*									
Germany	0.0053	0.062	**−2.48**	**4.27**	0	0.079	5.33	8.60	38%
US	0.020	0.31	**−6.20**	**4.30**	0	0.062	9.96	6.18	−61%
Experiment 4: *Role of quits*									
Germany	0.0053	0.062	−6.20	4.30	**0.01**	0.079	9.01	8.60	−5%
US	0.02	0.31	−2.48	4.27	**0.01**	0.062	6.76	6.18	−9%
US	0.02	0.31	−2.48	4.27	**0.02**	0.062	7.32	6.18	−18%

Notes: The table reports in columns $1-5$ the data equivalents to the respective model variables, and column 6 gives the model prediction for the mean unemployment rate. Column 7 reports the relative standard deviation of the unemployment rate to the standard deviation of the productivity process after the comparative statics experiment ($\tilde{\sigma}_u$). Column 8 reports the equivalent of this number for the benchmark model ($\tilde{\sigma}_u^*$). The last column gives the residual unexplained volatility in the model relative to benchmark model.

In experiment 1, we reproduce —within the model— the empirical thought experiment by Shimer (2005). We set firing volatilities to zero and compare the predicted unemployment volatilities of the model with constant firing rates to our benchmark model. The results align very well with our empirical estimates from section 3.2.2. We find that for Germany firings are more important than hirings, and that quantitatively firing volatility explains around 60% of the unconditional standard deviation of the unemployment rate. For the US, we find that the firing volatility accounts for 37% of the benchmark unemployment volatility, a result that again aligns well with the empirical contribution rates derived before.

In experiment 2, we ask how much of the unemployment volatility can be attributed to differences in mean hiring and firing rates but we keep volatilities constant and only exchange the US and German mean rates. We see that the impact of the differences in the mean rates is around 15% in absolute value, and hence, rather small.

In experiment 3, we perform the same experiment for hiring and firing volatilities. This time, we hold mean rates constant and focus on the effect of differences in volatilities. We see that the sole impact of differences in volatilities is very large. For Germany, the model generates an unemployment volatility that falls 38% short of the benchmark economy, whereas for the US the unemployment volatility is 61% too high compared to the benchmark model.

Experiments 2 and 3 taken together document that countries might differ substantially in their transition rates, but as long as the volatilities are similar, the aggregate volatility will be similar. It is important to recall that during these experiments we kept the volatilities constant while changing the means. However, we showed before that as soon as we impose equilibrium restrictions, this ceteris paribus assumption would clearly be violated.

Finally, experiment 4 examines how much the introduction of on the job search changes the unemployment volatility. For Germany, we use an average quit rate of 1%, in line with our empirical estimate. We find that the contribution of quits is very small, leading to a 5% higher unemployment volatility compared to the benchmark model. For the US, we lack an exact empirical counterpart for the quit rate, however, the impact might be considerably higher (18% increase) if we are willing to assume a monthly quitting probability of 2% which is in line with some estimates from the literature (see Nagypal (2005)). In our simulation experiments, we found that this effect enhances the ability of the model in generating more unemployment volatility, in line with findings of Menzio and Shi (2009), but it is quantitatively small.

Beveridge curve

Search on the job plays an important role (in both countries) in explaining within the model the *Beveridge curve*. With search on the job there are, as Equation (3.7) shows, two counteracting forces at work. The first term captures the negative covariance between unemployment and productivity (noticing that $z_2 < 0$). The second positive term captures the effect that in a recession many workers lose their jobs and increase the pool of searching workers. The increase in the search pool makes it relatively cheap for firms to find new workers given that their matching probability increases. They start to post more vacancies in times of high unemployment rates, inducing a positive correlation between vacancies and unemployment. This effect might well dominate and destroy the *Beveridge curve*. Search on the job can mitigate the problem. In the limit, if all workers are searching ($\pi_s = 1$), the positive term disappears completely. We find that, quantitatively, search on the job indeed restores the *Beveridge curve*.[22]

We showed how differences in mean transition rates and labor market institutions interact in explaining the observed cross-country differences in volatilities. In the next step, we calibrate our model including match heterogeneity and study the implications of the observed differences on the transmission of business cycle shocks to the labor market.

3.4 Calibration

Our basic time period is one month and we aggregate to quarterly rates when simulating the model. We target for both countries a discount rate of annualized 4% and set the matching elasticity to the linearity point ($\varrho = \frac{1}{2}$) in line with recent estimates by Petrongolo and Pissarides (2001).

Means: There are three differences in the average rates across countries we want to match. These are the average jobfinding rates that differ by a factor of 5, the average firing rates that differ by a factor of 4, and the average quit rates that differ by a factor of 2. This imposes the following cross-country restrictions

$$\frac{\bar{\pi}_{ue}^{US}}{\bar{\pi}_{ue}^{Ger}} \approx 5 \qquad \frac{\bar{\pi}_{eu}^{US}}{\bar{\pi}_{eu}^{Ger}} \approx 4 \qquad \frac{\pi_s^{US}\bar{\pi}_{ee}^{US}}{\pi_s^{Ger}\bar{\pi}_{ee}^{Ger}} \approx 2$$

Standard deviations: There are two facts about the second moments across countries we want to match. These are the standard deviation of the firing rate and the jobfinding rate. Relative to the business cycle, the standard deviation of firings differ by a factor of 2.5 but hirings are equally volatile across countries.

$$\frac{\tilde{\sigma}_{eu}^{Ger}}{\tilde{\sigma}_{eu}^{US}} \approx 2.5 \qquad \frac{\tilde{\sigma}_{ue}^{Ger}}{\tilde{\sigma}_{ue}^{US}} \approx 1$$

Wage elasticity: Finally, we want our model to be consistent with our estimates on the wage

[22] See Ramey (2008) for a similar finding.

elasticities. We focus on a wage elasticity of $\tilde{\sigma}_w = 0.8$ for both countries. This number is in line with our upper bound estimates for Germany and the estimate of Haefke et al. (2007) for jobfinder in the US.[23] A common elasticity estimate implies that our findings will not be driven by differences in wage rigidity but can be traced to institutional differences.

The introduction of heterogenous types does not alter the main mechanisms outlined above. As explained in the empirical analysis, we target the differences in the mean firing rates across tenure groups as a proxy for match heterogeneity to pin down the additional parameters. For the US, we lack a precise empirical counterpart for the effect of tenure, however, results in Menzio and Shi (2009) suggest that a similar pattern as observed for Germany also holds for the US. In table 3.8, we summarize our numerical results and our calibration strategy, which is otherwise standard given the targets outlined above.

Table 3.8: Calibration

Parameter	*heterogeneous types*		*homogeneous types*		*Target (Ger, US)*	*Source*
	Germany	*US*	*Germany*	*US*		
β	0.997		0.997		Annual real rate of 4%	
$\varkappa$	0.198		0.198		Normalization	
ϱ	0.5		0.5		Matching elasticity	Petronglo
ρ	0.975		0.975		Kalman estimates	Solow Residual
B_b	0.975		1		Quit premium 5%	Data
B_g	1.025				Normalization	
π_g	0.12	0.03	0		$\bar{\pi}_{eu,b} = (0.017, 0.04)$	Data
π_s	0.58	0.14	0.128	0.065	Mean quits $(0.008, 0.02)$	Data
ψ_b	0.61	0.73	0.80	0.90	$\bar{\pi}_{eu} = (0.005, 0.02)$	Data
ψ_g	1.64	1.76			$\bar{\pi}_{eu,g} = (0.002, 0.002)$	Data
κ	0.23	0.03	0.27	0.05	$\bar{\pi}_{ue} = (0.0622, 0.3069)$	Data
τ	2.3	2.0	3.0	2.9	Rel. Std π_{eu}	see table 3.7
b	0.96	0.93	0.95	0.94	Rel. Std π_{ue}	see table 3.7
μ	0.68	0.47	0.63	0.38	Wage elasticity $(0.8, 0.8)$	Data

Notes: This table documents our chosen parameters. We allow six parameters to differ across countries to target the six differences identified in the main text. In the case of heterogenous agents, we allow for differences in π_g and ψ to additionally target the differences across tenure groups using information for Germany.

3.4.1 Results

There exist an equilibrium that matches jointly all targets in both countries. The model demands a small surplus from opening a position, implying a very high outside option b.[24] Yet, due to search on the job and the time it takes to find a good job, the average surplus

[23] It turns out, quantitatively, that the particular number chosen has almost no bearing on the results. With the exception that it has to be smaller one, of course.

[24] Hall and Milgrom (2007) provide a rational by reinterpreting the outside option.

in the society is substantial, and amounts to roughly 60% of annual income in Germany and 40% of annual income in the US. The surplus of a bad job is small and amounts to roughly one monthly income. An important difference across countries lies in the substantially higher bargaining power ($\mu^{Ger} > \mu^{US}$) and in the costs to open a position ($\kappa^{Ger} > \kappa^{US}$). Firing costs are calibrated to be slightly higher in Germany ($\tau^{Ger} > \tau^{US}$). The exogenous search rate must be much higher in Germany given that Germans have to search considerably more often to find a better job due to the lower contact rates ($\pi_S^{Ger} > \pi_S^{US}$). Finally, idiosyncratic productivity risk is found to be higher in the US ($\psi^{US} > \psi^{Ger}$).

Given that we used up the two volatilities in our calibration and therefore lost an important metric of success, we evaluate the performance of the model by studying its predictive power. To this end, we estimate for both countries the underlying TFP process using a Kalman filter on GDP growth. We feed the estimated process into the model and predict all endogenous variables applying an HP-filter ($\lambda = 100,000$) to the resulting time-series.[25] Figures 3.3 and 3.4 graphically illustrates the success of the model. Table 3.9 reports the standard deviations of the estimated series as well as the correlation between predicted and actual values as a measure of fit.[26]

Table 3.9: Summary Statistics

	Germany			US		
Name	Std (Data)	Std (Model)	Corr	Std (Data)	Std (Model)	Corr
URate	0.18	0.198	0.87	0.150	0.16	0.89
Productivity	0.016	0.014	0.47	0.014	0.02	0.48
Wage income	0.015	0.011	0.78	0.018*	0.017*	0.29*
Quits (NE)*	0.20	0.10	0.60	0.059*	0.127*	0.48*
Vacancies*	0.33	0.16*	0.46*	0.20	0.17	0.80
	calibrated moments					
Firings	**0.15**	**0.15**	0.71	**0.06**	**0.06**	0.71
Jobfinding	**0.10**	**0.10**	0.59	**0.11**	**0.11**	0.72

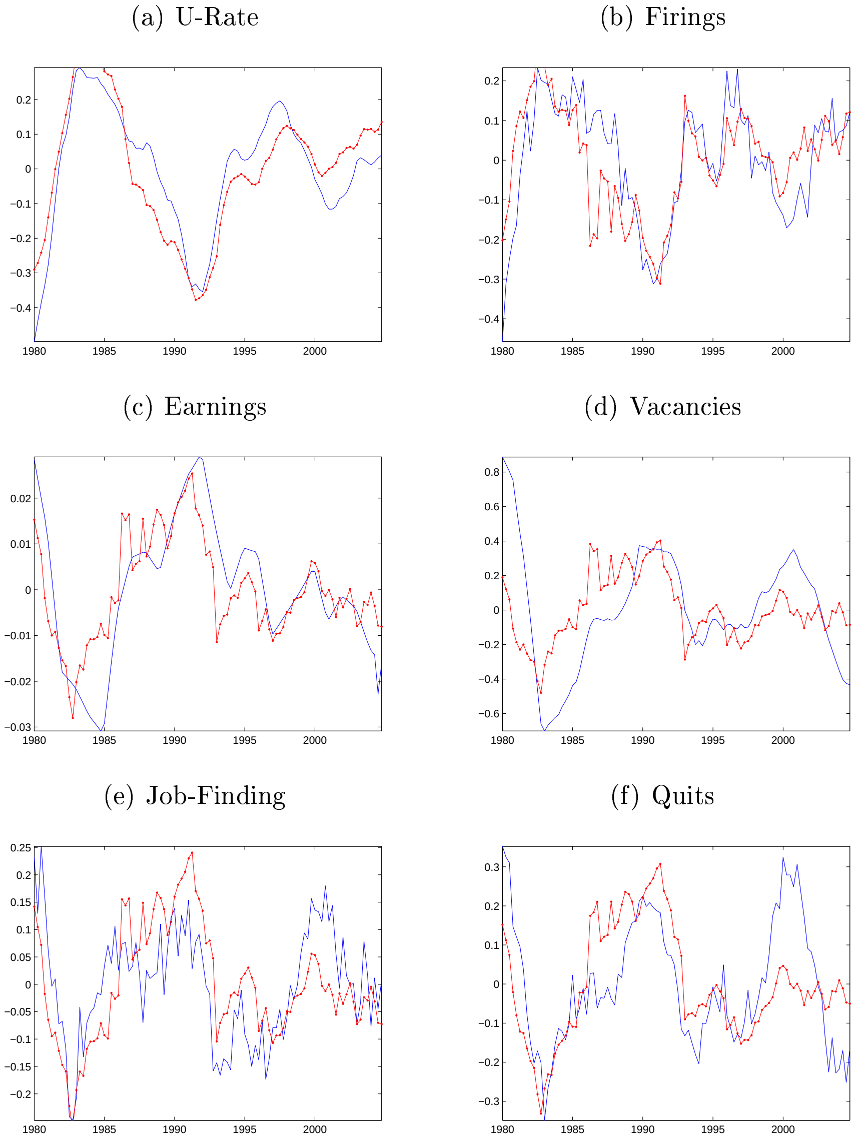
Notes: The table reports summary statistics for all endogenous variables predicted by the model and compares them to the data. Corr refers to the correlation between the actual and the predicted data. The star indicates that for quits in the US we do not have corresponding data and proxy by the NE flows (see Nagypal (2005)). However, the proxy might capture very distinct phenomena and should be interpreted with care. Calibrated moments are in bold.

The model reproduces the time series pattern of the unemployment rate almost perfectly and captures firing dynamics very well in both countries. German median earnings obtained from the microdata are also fitted almost perfectly, while the model fails to match the BLS earnings

[25] When applying a Bandpass-filter, the findings are very similar.

[26] We only report results for the heterogenous agent case, though the basic fit of the model is not much affected when focussing on the homogenous agent case.

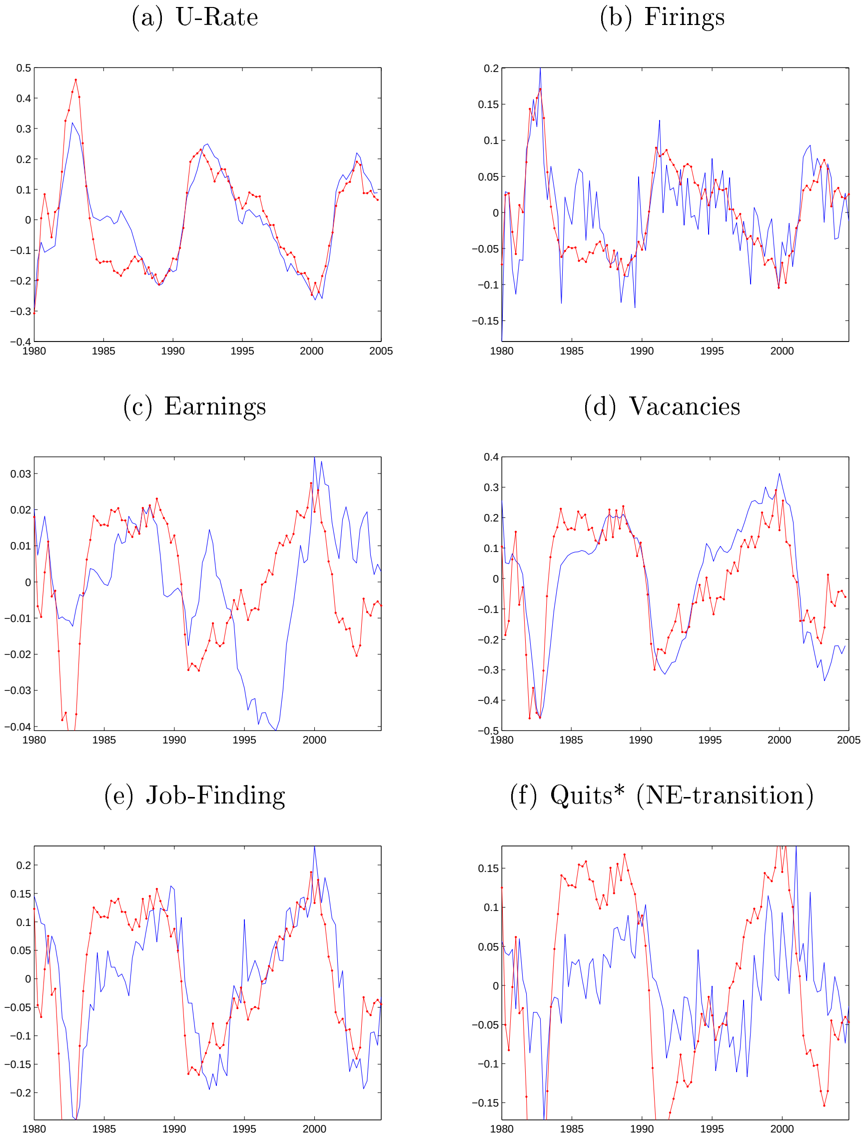
Figure 3.3: Prediction for Germany



Notes: The figure plots the model predictions (red dotted lines) and the data (blue solid line). The prediction is based on a technology process obtained from a Kalman filter on GDP growth. Model and data are in logs and are HP-filtered with $\lambda = 100,000$. Earnings for Germany refer to median earnings obtained from the micro-data. Vacancies are open position obtained from the *Bundesagentur fuer Arbeit* and do not correspondent to the universe of all open positions.

series. Given that the aggregate earnings series for the US is not very reliable and faces the same composition effects as discussed in section 3.2.4, it is not clear whether the mismatch is exclusively a model problem or partly a data problem as well. The model captures the correlation structure of vacancies, quits, and jobfinding rates well but underestimates the volatility for the vacancy proxy in Germany. The data on open positions for Germany does only capture a small universe of all open positions, so clearly the model and data are measuring different things. The quit correlation is captured well, but the standard deviation is off considerably.

Figure 3.4: Prediction for the US



Notes: The figure plots the model predictions (red dotted lines) and the data (blue solid line). The prediction is based on a technology process obtained from a Kalman filter on GDP growth. Model and data are in logs and are HP-filtered with $\lambda = 100,000$. Earnings for the US are from the Bureau of Labor statistics. Labor market transitions are taken from Shimer (2005).

An easy fix would be a procyclical search probability which would be the outcome of almost all models based on endogenous search effort. The correlation between vacancies and unemployment for the US is -0.8 in the model, showing that quits on the job indeed help recovering the *Beveridge curve*. For Germany, we do slightly worse given that the correlation is -0.6, but we still reproduce the negative *Beveridge curve* relation fairly well.

The model is driven by one contemporaneous shock hitting the demand for labor while the data likely requires a richer shock structure to capture some of the autocorrelated deviations and measurement error. However, the basic model driven by one shock seems to capture the main forces in the labor market fairly well. Furthermore, the simple and stylized model can reconcile labor market dynamics across countries relying only on differences in institutional parameters.

3.4.2 Transmission of shocks

We have demonstrated that our model reproduces the right macro-elasticities with respect to aggregate shocks for important labor market dimensions. In this section, we use the model to inform us about the transmission of business cycle shocks into the labor market. In particular, we examine how the German labor market reacts to business cycle shocks compared to the US market. As Figure 3.5 makes clear, the impulse-response functions for a large shock (-5%) are substantially different across countries.

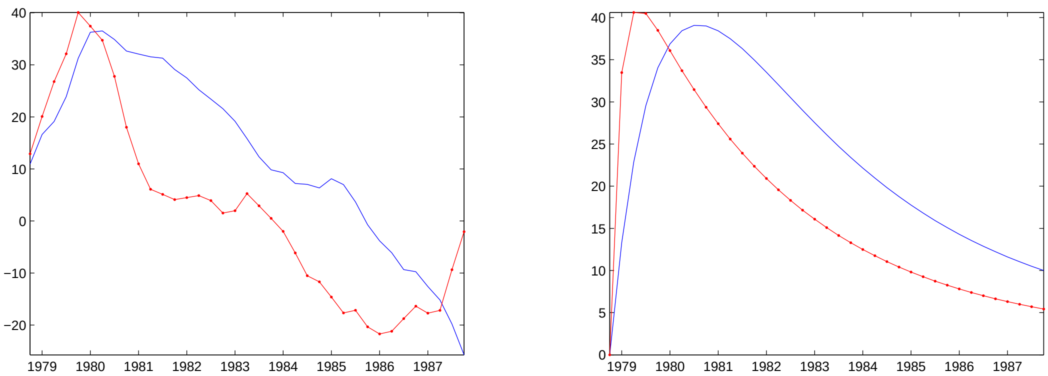
Figure 3.5: Impulse response functions

Notes: The figure plots the impulse response functions for the US (red dotted lines) and Germany (blue solid line). Good type and bad type employed refer to the measure of good and bad matches.

The model predicts, on impact, a stronger increase in unemployment rates measured in percentage deviation from the respective long-run rates in Germany. The differences are not generated by differences in the reaction of wages. Despite the lower bargaining power in the US the wage reaction was targeted to be the same across the two countries, and is confirmed in figure 3.5(e). The difference is also not due to the hiring margin, given that the job-finding rate reacts very similar (see figure 3.5(d)). The assumption of heterogenous job types does not drive the results either, in fact, we obtain the same picture for the aggregate rates when focussing on the homogenous agent case. The composition effects do not have a strong impact on the aggregate reaction, but interesting differences can be stated. At our calibrated parameters, more bad matches will be destroyed in Germany compared to the US. Over time (roughly after two years) there will be more workers in bad jobs compared to the long-run steady state in Germany. These workers will over time move into good jobs. However, this process takes time and the number of good jobs will be below the long-run average for a substantial amount of time. Correspondingly, there might be considerable risk involved for good workers when being fired, loosing a substantial fraction of their surplus during the periods of unemployment and search for a good job.

Figure 3.6: Rescaled impulse response functions

(a) Unemployment rate (data) - HP-detrended   (b) Unemployment rate (model)



Notes: Right panel: Impulse response functions for unemployment rates in the US (red dotted lines) and Germany (blue solid line) rescaled to match the initial shock. The shock magnitude for the US is 5% and for Germany 3.72%. Left panel: HP-filtered percentage deviation of the actual data. The red dotted line are the US, the blue solid line is Germany.

In fact, to have a similar response on impact the shock would have to be only 3.72% in Germany, implying an impact that is approximately 30% larger in Germany. The right panel of figure 3.6 shows the re-scaled impulse response function, where we scale the German shock such that both countries face the same peak unemployment rate. In the left panel of figure 3.6, we plot the behavior of the actual unemployment rate in Germany and the US after the large oil price shock at the beginning of the eighties.

The plots show that the striking difference across countries is the differences in the persistence of the shock. According to our model the US will recover fairly quickly, while Germany will likely suffer much longer. After 20 quarters (or five years) we see that the German unemployment rate will still be 25% away from its long-run average while the US is only 12% away. When we compare the model to the data, we observe the same pattern. Although Germany came from considerably lower unemployment level, it appears to be the case that the shock showed much more persistence in Germany while it leveled off much more quickly in the US.

This experiment in our realistically calibrated model shows that shocks can be the driver of substantially higher unemployment rates for a long time but that the reason for the persistence is not the shock by itself but the interplay with the more rigid labor market institutions. It is therefore the coexistence of low transition rates, high volatilities, and long persistence that makes rigid labor markets so vulnerable to business cycle shocks.

3.5 Conclusion

In this paper we document that the German and the US labor market share many similarities in their dynamics over the business cycle. We find that many of the stylized facts for the US as stated in Shimer (2005) do also hold for Germany, however, two crucial differences arise: (*i*) lower average transition rates and (*ii*) a higher firing volatility that constitutes the major driving force behind unemployment volatility.

We show that these differences across country matter in a quantitative sense. Shocks in Germany are considerably amplified (+30%) and are substantially more persistent than in the US (+25% after five years). The volatility differences are not rooted in different wage reactions across country. If anything, the earnings elasticity in Germany is at least as high as in the US. Instead we find that differences in labor market institutions leading to lower average transition rates in Germany are responsible for the large amplification of business cycle shocks. Viewed through the lens of a search and matching framework, no mean-variance trade-off between higher unemployment rates on the one hand and lower business cycle volatility on the other hand exists. This raises fear for the future given the large shock currently hitting the labor market in both countries. The relatively modest effect on unemployment rates we witnessed so far in Germany might partly be due to a reaction of policy, substantially subsidizing layoffs and preventing a boost in firings. Whether this policy reaction is indeed an optimal choice will be studied in our future research.

Appendix

C.1 Data

C.1.1 Data description

The data is taken from the IAB regional files that cover the period January 1975 to December 2004. The data consists of employment records of workers that have at least for one day been employed in a job under mandatory social security. The dataset comprises a 2% representative subsample of workers drawn from these records. Once an individual has been put into the sample, the full employment history of this individual during the sampling period is observed. The employment history consists of employment spells that are subject to mandatory social security and unemployment spells where social security benefits have been paid. The sample does therefore not contain spells in public service (*Beamte*), self-employment, and periods of non-employment. We describe below in detail how we control for these periods by constructing artifical spells. Still, the data covers about 80% of the German workforce.

C.1.2 Sampling period and sample selection

Due to measurement problems in unemployment during the years 1977 and 1978 we use the first 5 years ($1975 - 1979$) only as a pre-sample and start our main analysis in 1980.

In a first step sample selection, we drop all individuals where the East-West information is missing ($2,787$ individuals dropped) or information regarding the current job[27] ($14,490$ individuals dropped). Furthermore, we drop homeworkers ('Heimarbeiter') from the sample ($7,315$ individuals dropped). This results in a dropping rate of 1.81% for the whole sample, and leaves us with a sample of employment histories for $1,336,357$ individuals. After the German reunification the data contains employment histories with spells that are located in East Germany. Since the East German labor market was subject to additional regulations and restructuring after the reunification, we exclude in a second step all persons with employment spells in the

[27] stib information missing.

East from our sample. This leaves us with a final sample of $1,087,555$ employment records. From these records we drop all marginal employment spells to avoid mismeasurement because marginal employment spells are only reported for the last five years of the sample period.

C.1.3 Construction of monthly employment histories

The employment history is given as a collection of employment spells on a daily basis. A new spell can either occur due to administrative reasons of the social security system or changes within a given firm, due to a quit to a new firm, the begin of an unemployment or a non-employment spell. Regularly, individuals have periods of parallel employment in the sample. This is reported as multiple spells. For every spell, we observe whether it is a full-time, part-time, or marginal employment.

If persons have parallel spells in their employment history, we consider only what we call *primary spells*. The idea is to consider the employment spell that generates the most income and occupies the most working time of an individual. To identify the *primary spell*, we apply a hierarchical selection procedure. If a person is at the same time full-time and part-time employed, we label him or her as full-time employed and drop the part-time spells, if a person has two part-time employments, we follow the ordering in the dataset that applies a hierarchical ordering based on income and part-time status over parallel spells, finally, if a person has employment and unemployment spells at the same time, we label the employment spells as primary to be consistent with the procedure in the next step of determining the employment status.[28]

Our basic time-period will be one month. We adopt the ILO timing convention to measure the employment status of a person in a given month. For each month we determine the Monday of the second week in the month and take the week starting from this Monday as our reference week. We look at all spells that overlap with this week. If only one spell overlaps, then this spell determines the labor market status. If several spells overlap, we use a hierarchical ordering of spells where a full-time employment spell dominates part-time spells and any employment spell beats unemployment or non-employment spells. From this classification of monthly employment states, we construct time-series at monthly frequency. By tracking the employment histories through time, we can generate additional labor market statistics like tenure on the current job and can construct the sample of continuously employed workers. To check whether a person stays with the same employer, we use the establishment number of the employment spells. A transition of a person between establishments but within the same firm is then also counted as a quit. The definition of who is counted as unemployed follows from the content of the dataset.

[28] This problem only arises with marginal employment and can therefore be disregarded for the analysis in this paper.

A person is unemployed if she receives unemployment benefits or other benefits on the basis of the Social Security Code III ('Sozialgestzbuch III'). We can not follow the ILO definition that is based on interview questions on job search because this is unobservable in our sample.
We label inactive employment that is reported in the dataset as non-employment. These spells are periods of sustained employment relationships but that are currently inactive, i.e. the worker does not work and no income is paid. Examples for these periods are maternity leave, long periods of illness, sabbaticals. We construct additional non-employment spells as residual spells in the dataset. The additional spells are included if a person is not observed in the sample for some time period between two spells. To deal with persons entering the sample or dropping out of the sample, we introduce additional labor market states that we label *labor market entry* and *retirement*. The labor market entry state is an artificial state that we add before the first employment state. The retirement state is an artificial state at the end of the labor market history. We assign it to persons that are of age 55 or older when they have their last observed spell. The retirement state is by construction an absorbing state. Persons that are below 55 and have no future spells in the sample are labeled as *other employment* and are no longer considered after the transition into this non-employment state, i.e. they do not generate transitions out of non-employment. Persons that are below 55 but have future spells are labelled as *out of the labor force*. The labor market entry, the reported spells of inactivity, and the *out of the labor force* spells constitute the pool from which all non-employment transitions originate. Table C.10 gives an overview over the different non-employment states in our analysis

Table C.10: Description of non-employment states

status	definition
retirement	age ≥ 55, no further spells
other employment	age < 55, no further spells
labor market entry	before 1^{st} spell of labor market history
out of the labor force	age < 55, further spells
inactive	in data

C.1.4 Measurement error

For variables regarding the job status, the income paid, or the duration of the job the data contains virtually no measurement error because it is taken from the social security and unemployment records that are used to determine social security contributions and benefits. The personal characteristics that we observe with every spell like year of birth, education, industry, and location of the employer may, however, contain measurement error. Fitzenberger et al. (2006) point out that the education variable may be subject to higher measurement error and provide imputation and correction rules for this variable. We adopt their imputation and correction procedure and determine the highest attained education level of an individual over the employment history to group persons into education classes.

C.1.5 Earnings

The income reported at one spell is the average daily income of an individual during the employment spell[29]. We do not observe hours worked but observe whether the person is full-time, part-time, or from 1999 on in marginal employment. We use income of the *primary spell* for the analysis in this paper.

C.1.6 Imputation and correction for structural breaks

Income in the sample is top-censored at the upper contribution limit ('Beitragsbemessungsgrenze') of the German social security system, and bottom censored at the marginal employment contribution level ('Geringfuegigkeitsgrenze'). For some of steps of the analysis we need an uncensored income distribution. For these steps we impute income above and below the two censoring points using the method proposed in Gartner (2005). The imputation uses a censored regression together with the log-normality assumption for income to impute the censored observations. For details see Gartner (2005).

Starting 1984 the income data also includes overtime and bonus payments. We correct for this structural break using the method proposed in Fitznberger (1999). His procedure leaves the median and all observations below the median unchanged and corrects income observations only above the median. The approach is based on measuring the excess growth of the upper income quantiles between 1983 and 1984. For details see Fitznberger (1999).

C.1.7 Aggregate data

Aggregate data are taken from the statistic office ('Statistische Bundesamt'). We use nominal GDP and convert it to real GDP by the CPI deflator from the Bundesbank. We deflate nominal income in the sample using the same CPI deflator. Productivity measures are obtained by dividing through total employment or total hours worked, as is done by the statistical office. This measure is rather noisy and does not correspond to the BLS productivity measure for the US that uses a more disaggregate procedure, but still suffers from aggregation problems highlighted when discussing the cyclical properties of income. After 1991, we only observe GDP

[29] The working period is not adjust for weekends or holidays.

for the unified Germany. We use the X-12 ARIMA method to align the series in the fourth quarter of the year 1991 to avoid jumping behavior of the series.

C.1.8 Seasonal adjustment

All data that is generated based on our own calculations is seasonally adjusted at monthy frequency using the X-12 ARIMA method. We also perform the default outlier correction implemented in X-12 ARIMA.

C.2 Sensitivity

C.2.1 Transitions by education and sex for all workers

Table C.11: Labor market flows $Jan1980 - Sep2004$ for workers by sex

	Mean	Std	Rel. Std	Corr (GDP)	Corr (GDP p. Emp.)	Autocorr
	Males					
Firm exit	0.0236	0.0561	2.34	0.3245	0.1559	0.6724
Empl. exit	0.0148	0.0517	2.154	−0.5201	−0.2935	0.6238
EU	0.0056	0.1812	7.553	−0.8073	−0.5159	0.9034
EN	0.0092	0.0743	3.095	0.5293	0.371	0.7894
UE	0.0679	0.1172	4.883	0.3615	0.0558	0.7273
UN	0.0444	0.1138	4.742	0.4964	0.5952	0.7937
NE*	0.0784	0.1762	7.342	0.3535	0.0081	0.8115
NU*	0.033	0.1552	6.468	−0.3826	−0.2136	0.8782
Quits	0.0087	0.1589	6.622	0.6118	0.3306	0.8931
	Females					
Firm exit	0.0243	0.0595	2.478	0.6099	0.3027	0.8287
Empl. exit	0.0158	0.0339	1.412	−0.0571	0.0899	0.3581
EU	0.0048	0.1024	4.266	−0.7474	−0.4361	0.8356
EN	0.011	0.0588	2.451	0.5065	0.4059	0.6953
UE	0.0542	0.1051	4.381	0.5897	0.2254	0.8364
UN	0.0556	0.0935	3.898	0.2948	0.3222	0.6992
NE*	0.0551	0.1846	7.694	0.2917	−0.1165	0.8724
NU*	0.0163	0.177	7.377	0.0147	0.0142	0.8845
Quits	0.0085	0.1601	6.671	0.6877	0.3126	0.9352

Notes: All data are in logs and are HP-filterer with $\lambda = 100,000$. The rates are quarterly averages of monthly data. Firm exit is defined as the sum of EU+EN+Quits. Employment exit is defined as EU+EN. Quits are defined as job-job transitions between two consecutive dates and a change in the firm counter as defined in the IAB-data. All IAB-rates are authors' calculations. The star at the non-employment flows indicate that the denominator, that is the state of non-employed workers is measured with problems given that we do not have the corresponding universe of searching non-employed. We partially control for this by dropping early retired and only look at workers that eventually will return to the labor market in our sample period. The log volatility measures might be less affected by the problem.

Table C.12: Labor market flows $Jan1980 - Sep2004$ for workers by education

	Mean	Std	Rel. Std	Corr (GDP)	Corr (GDP p. Emp.)	Autocorr
	Low education					
Firm exit	0.0245	0.0761	3.172	0.3526	0.2753	0.7477
Empl. exit	0.0191	0.0692	2.884	0.0635	0.1612	0.7223
EU	0.0053	0.136	5.668	−0.5839	−0.2139	0.8261
EN	0.0138	0.0894	3.727	0.397	0.2962	0.7887
UE	0.034	0.1474	6.144	0.4107	0.1008	0.7695
UN	0.0524	0.1195	4.98	0.2481	0.4766	0.7164
NE*	0.0824	0.2267	9.449	0.3888	0.0036	0.8912
NU*	0.0325	0.1838	7.66	0.0781	0.1788	0.8856
Quits	0.0054	0.1963	8.181	0.5502	0.3009	0.8743
	Medium education					
Firm exit	0.0236	0.0521	2.173	0.4796	0.1997	0.75
Empl. exit	0.0147	0.0379	1.581	−0.5814	−0.3431	0.5495
EU	0.0054	0.1578	6.576	−0.8261	−0.538	0.9073
EN	0.0093	0.0617	2.57	0.6093	0.4296	0.7736
UE	0.0684	0.1012	4.219	0.4295	0.0814	0.7709
UN	0.0475	0.1049	4.37	0.515	0.5321	0.7967
NE	0.0637	0.1674	6.977	0.374	−0.0221	0.8495
NU	0.0248	0.157	6.545	−0.2729	−0.1626	0.8907
Quits	0.0089	0.1569	6.54	0.6681	0.3309	0.9165
	High education					
Firm exit	0.0262	0.0921	3.839	0.3217	0.2251	0.7204
Empl. exit	0.0154	0.0892	3.718	0.019	0.1454	0.4976
EU	0.004	0.1236	5.153	−0.527	−0.2329	0.7488
EN	0.0114	0.1266	5.274	0.1753	0.1889	0.5525
UE	0.0664	0.1201	5.006	0.5075	0.172	0.8066
UN	0.0544	0.0895	3.729	0.2258	0.279	0.5942
NE	0.0538	0.1917	7.99	0.1019	−0.1795	0.7512
NU	0.0105	0.1868	7.783	−0.1652	−0.1582	0.7272
Quits	0.0108	0.1438	5.992	0.5027	0.2486	0.8784

Notes: All data are in logs and are HP-filterer with $\lambda = 100,000$. The rates are quarterly averages of monthly data. Firm exit is defined as the sum of EU+EN+Quits. Employment exit is defined as EU+EN. Quits are defined as job-job transitions between two consecutive dates and a change in the firm counter as defined in the IAB-data. All IAB-rates are authors calculations. The star at the non-employment flows indicate that the denominator, that is the state of non-employed workers is measured with problems given that we do not have the corresponding universe of searching non-employed. We partially control for this by dropping early retired and only look at workers that eventually will return to the labor market in our sample period. The log volatility measures might be less affected by the problem.

C.2.2 Unemployment decomposition

We perform the unemployment decomposition for different subgroups. We use the decomposition based on the HP-Filter ($\lambda = 100,000$)

Table C.13: Unemployment decomposition for different subgroups

Sample	Data	EU	UE	NE	EN	NU	UN	ε
Men	IAB	0.6391	0.3580					0.0029
	IAB	0.4517	0.2545	0.1707	−0.0433	0.0833	0.0851	−0.0010
Women	IAB	0.4831	0.5137					0.0032
	IAB	0.3261	0.2854	0.2573	−0.0460	0.0359	0.1449	−0.0037
Low education	IAB	0.4740	0.5244					0.0016
	IAB	0.2806	0.2719	0.3174	−0.0362	0.0810	0.0868	−0.0015
Medium education	IAB	0.6340	0.3627					0.0033
	IAB	0.4438	0.2422	0.1822	−0.0472	0.0720	0.1093	−0.0024
High education	IAB	0.5165	0.4830					0.0030
	IAB	0.3682	0.2652	0.2142	−0.0166	0.0654	0.1028	0.0008

Notes: Contribution of labor market transitions to unemployment fluctuations. Data is HP-filtered ($\lambda = 100,000$) for the period $1980q1 - 2004q3$. For Germany the transition rates are for all male and female workers. The US data is obtained from Shimer and Fujita/Ramey.

We perform the decomposition of unemployment fluctuations based on a first difference filter as derived in Petrongolo and Pissarides (2009). The first difference filter for the case including non-employment does not allow to separate the contributions of EN and NU flows and the contribution of UN and NE flows. The columns are therefore reordered in this table. We report the decomposition using the HP-filter ($\lambda = 100,000$) as given in the main text for comparison.

Table C.14: Unemployment decomposition for different filters

Country	Data	EU	UE	EN	NU	UN	NE	ε
Germany	IAB (Δ)	0.6353	0.3647					0.0000
	IAB (Δ)	0.3610	0.2131	0.2625		0.1634		−0.0000
	IAB (HP)	0.4186	0.2498	−0.0469	0.0677	0.1122	0.2020	−0.0020
US	Shimer (Δ)	0.6434	0.3566					−0.0000
	Fujita/Ramey (Δ)	0.5174	0.4826					−0.0000
	Shimer (Δ)	0.4010	0.3054	0.1207		0.1729		−0.0000
	Shimer (HP)	0.2013	0.4855	−0.0378	0.1039	0.1516	0.0884	0.0072

Notes: Contribution of labor market transitions to unemployment fluctuations. Data is detrended by a first difference filter (Δ) for the period $1980q1 - 2004q3$. The row labelled (HP) contains the numbers for the decomposition using the HP-filter ($\lambda = 100,000$). For Germany the transition rates are for all male and female workers. The US data is obtained from Shimer and Fujita/Ramey.

Table C.15: Unemployment decomposition for the period 1977 – 2004

Country	Data	EU	UE	NE	EN	NU	UN	ε
Germany	IAB	0.6600	0.3371					0.0029
	IAB	0.4486	0.2014	0.1323	−0.0481	0.1423	0.1238	−0.0004
US	Shimer	0.3677	0.6361					−0.0038
	Fujita/Ramey	0.4054	0.5986					−0.0040
	Shimer	0.2316	0.4702	0.0853	−0.0403	0.0957	0.1499	0.0076

Notes: Contribution of labor market transitions to unemployment fluctuations. Data is HP-filtered ($\lambda = 100,000$) for the period $1977q1 - 2004q3$. For Germany the transition rates are for all male and female workers. The US data is obtained from Shimer and Fujita/Ramey.

Table C.16: Unemployment decomposition before and after the German reunification

Period	Data	EU	UE	NE	EN	NU	UN	ε
$1980q1 - 1991q4$	IAB	0.6188	0.3766					0.0046
	IAB	0.4585	0.2403	0.2080	−0.0796	−0.0309	0.2066	−0.0029
$1992q1 - 2004q4$	IAB	0.5855	0.4116					0.0029
	IAB	0.3678	0.2374	0.1862	−0.0362	0.1825	0.0586	0.0036

Notes: Contribution of labor market transitions to unemployment fluctuations before and after the German reunification. Data is HP-filtered ($\lambda = 100,000$). The transition rates are for all male and female workers.

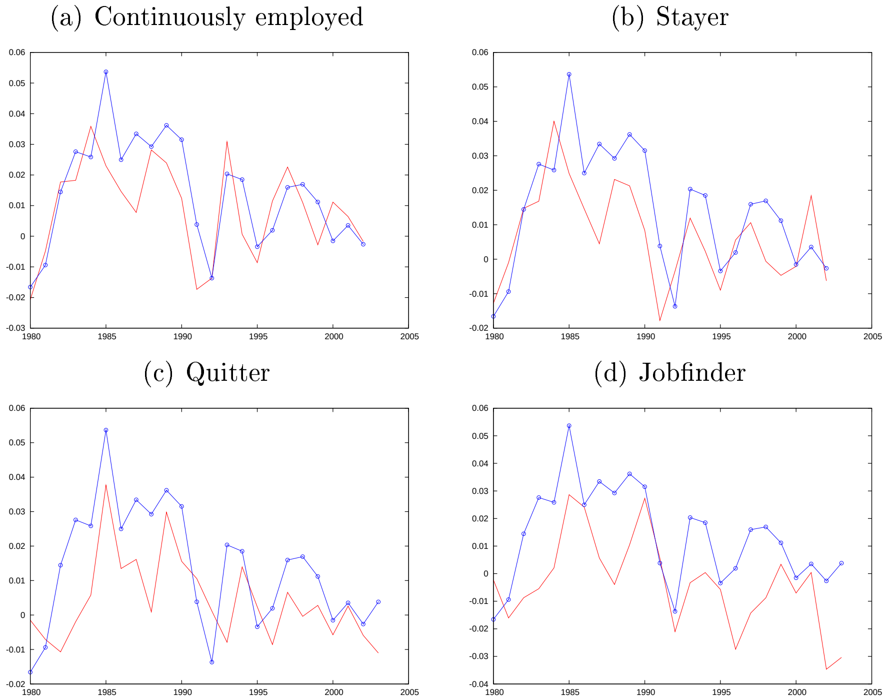
Table C.17: Unemployment decomposition for full-time employed workers

Sample	Data	EU	UE	NE	EN	NU	UN	ε
Full-time	IAB	0.6073	0.3898					0.0030
	IAB	0.4181	0.2494	0.2018	−0.0469	0.0677	0.1120	−0.0020

Notes: Contribution of labor market transitions to unemployment fluctuations if only full-time employment is considered. Data is HP-filtered ($\lambda = 100,000$). The transition rates are for male and female workers.

C.2.3 Earnings

Figure C.7: Earnings cyclicality and GDP per capita



Notes: Earnings index cyclicality ($1st$ difference filter) for full-time workers (male and female), business cycle measure GDP per capita. Time period is $Jan1980 - Sep2004$.

Table C.18: Earnings cyclicality using GDP per capita

	Quitter	Jobfinder	Stayer	Cont. employed
Index	0.4841	0.6231	0.5414	0.6436
std error	(0.0854)	(0.1163)	(0.0943)	(0.1006)
Correlation	0.6874	0.6668	0.6982	0.7358
Growth	0.4588	0.8160	0.6759	0.6491
std error	(0.0534)	(0.1297)	(0.0817)	(0.0918)

Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). *Index* refers to the earnings index using the first difference filter. *Correlation* refers to the correlation coefficient of the earnings index and the business cycle measure. *Growth* refers to the estimation in first difference using OLS. standard errors are clustered by time periods. The business cycle measure is GDP per capita. Time period is $Jan1980 - Sep2004$.

Table C.19: Earnings cyclicality for the period $1977 - 2004$

	Quitter	Jobfinder	Stayer	Cont. employed
Index	0.6091	0.6111	0.7221	0.7478
std error	(0.1336)	(0.2060)	(0.1304)	(0.1497)
Correlation	0.5620	0.4045	0.6398	0.6005
Growth	0.3292	0.7854	0.8036	0.6858
std error	(0.1104)	(0.2251)	(0.1342)	(0.1373)

Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). *Index* refers to the earnings index using the first difference filter. *Correlation* refers to the correlation coefficient of the earnings index and the business cycle measure. *Growth* refers to the estimation in first difference using OLS. The business cycle measure is GDP per employed. Time period is $Jan1977 - Sep2004$.

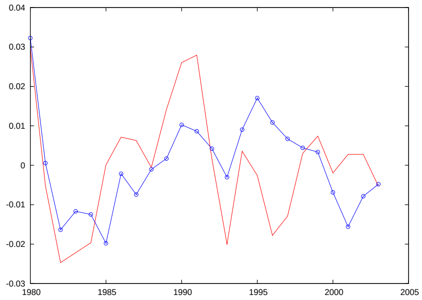
Table C.20: Earnings cyclicality (HP filtered)

	Quitter	Jobfinder	Stayer	Cont. employed
$Index(p.cap.)$	0.5420	0.6101	0.5387	0.6416
std error	(0.0819)	(0.1147)	(0.0878)	(0.0911)
Correlation	0.8036	0.7357	0.7878	0.8264
$Index(p.empl.)$	0.7454	0.7244	0.6100	0.6633
std error	(0.1686)	(0.2369)	(0.1956)	(0.2259)
Correlation	0.6700	0.5295	0.5452	0.5221

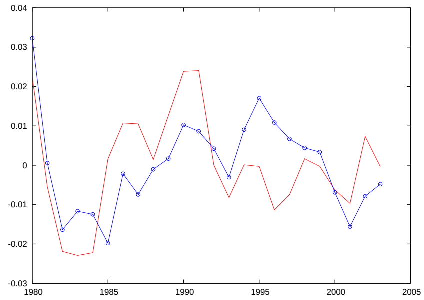
Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). *Index p.cap.* refers to the earnings index using the HP-filter ($\lambda = 100,000$) and GDP per capita as business cycle measure and *Index p.empl.* refers to the earnings index using the HP-filter ($\lambda = 100,000$) and GDP per employed as business cycle measure. *Correlation* refers to the correlation coefficient of the earnings index and the business cycle measure. Time period is $Jan1980 - Sep2004$.

Figure C.8: Earnings cyclicality (HP filtered) and GDP per employed

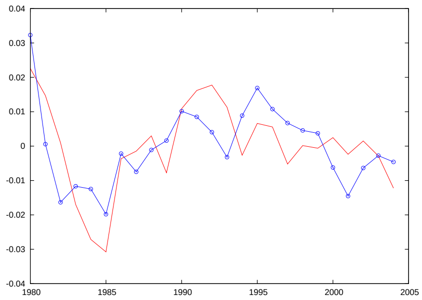
(a) Continuously employed



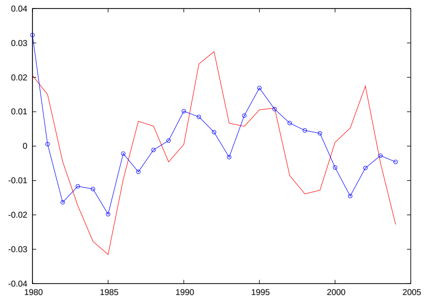
(b) Stayer



(c) Quitter



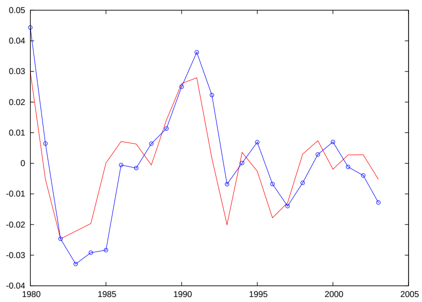
(d) Jobfinder



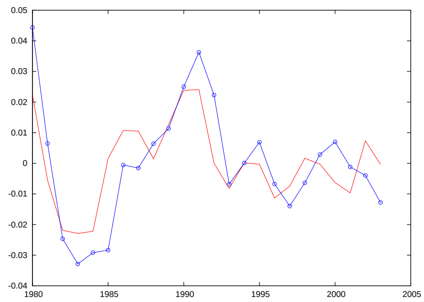
Notes: Earnings index cyclicality (HP filter, $\lambda = 100,000$) for full-time workers (male and female), business cycle measure GDP per employed. Time period is $Jan1980 - Sep2004$.

Figure C.9: Earnings cyclicality (HP filtered) and GDP per capita

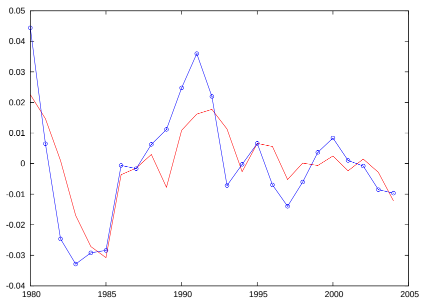
(a) Continuously employed



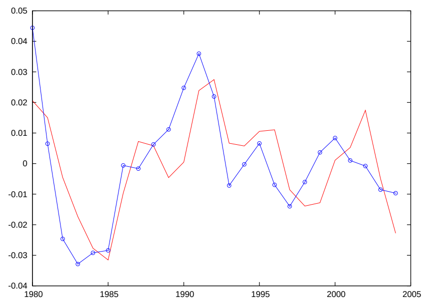
(b) Stayer



(c) Quitter



(d) Jobfinder



Notes: Earnings index cyclicality (HP filter, $\lambda = 100,000$) for full-time workers (male and female), business cycle measure GDP per capita. Time period is $Jan1980 - Sep2004$.

Table C.21: Earnings cyclicality (LAD estimation)

	Quitter	Jobfinder	Stayer	Cont. employed
$Growth(p.cap.)$	0.5181	0.6455	0.5895	0.5702
std error	(0.0633)	(0.1452)	(0.0870)	(0.1097)
$Growth(p.empl.)$	0.4870	0.6751	0.6389	0.6056
std error	(0.1263)	(0.2446)	(0.1613)	(0.1840)

Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). *Growth p.cap.* refers to the estimation in first difference using a LAD regression and GDP per capita as business cycle measure. *Growth p.empl.* refers to the estimation in first difference using a LAD regression and GDP per employed as business cycle measure. Standard errors are bootstrapped with 100 repetitions and clustered by time periods. Time period is $Jan1980 - Sep2004$.

Table C.22: Earnings cyclicality for full-time employed workers

	Quitter	Jobfinder
$Growth(p.cap.)$	0.5277	0.7709
std error	(0.0604)	(0.0928)
$Growth(p.empl.)$	0.3875	0.7164
std error	(0.1394)	(0.2125)

Notes: Annual earnings cyclicality for full-time employed workers (male and female, all education groups). Sample is restricted to unemployed that are unemployed less than 360 days and employed that are employed for at least 180 days.*Growth p.cap.* refers to the estimation in first difference using OLS and GDP per capita as business cycle measure. *Growth p.empl.* refers to the estimation in first difference using a OLS and GDP per employed as business cycle measure. Standard errors are clustered by time periods. Time period is $Jan1980 - Sep2004$.

C.2.4 Tenure

The table reports tenure transition rates for EN flows. The NE transition rates are not reported because of mismeasurement of the number of non-employmed workers.

Table C.23: Tenure statistics for non-employment flows

EN	< 365 days	$365 - 730$ days	$730 - 1825$ days	> 1825 days	*overall* days
mean	0.0194	0.0058	0.0041	0.0018	0.0060
std	0.0996	0.1291	0.1511	0.2302	0.1334
rel. share	0.5959	0.1073	0.1539	0.1429	
rel. earnings	0.7224	0.7724	0.7769	0.8125	
corr (per capita)	0.2003	0.2022	0.2588		0.4464
corr (per empl.)	-0.4650	-0.4644	-0.4213	-0.2546	-0.4739
av. obs.	$1,202$	218	312	286	$2,017$

Notes: Tenure statistics for non-employment flows (*out of the labor force* state) for the period $Jan1980-Sep2004$. The columns contain the bounds of the different tenure groups. The tenure groups are formed for the labor market state before the transition and are given in days. All statistics are computed conditional on being in the respective labor market state and tenure group. *mean* is the average transition probability of the respective labor market transition. *std* is the relative deviation of the transition rate over time. *rel. share* is the average share of transitions falling in this tenure group relative to all transitions. *rel. earnings* are the average relative earnings of all persons with a transition relative to the average earnings in the current labor market and tenure group. *corr* are the respective correlations of the transition rate with GDP per capita respectively per employed as our business cycle measures. *av. obs* are the average number of transitions per month from the respective labor market and tenure group.

Table C.24: Sensitivity of the relative share with respect to short spells

Jobfindings

0	0.7137	0.148	0.0821	0.0562
15	0.6791	0.1658	0.092	0.0631
30	0.6447	0.1838	0.1018	0.0696

Quits

0	0.4165	0.1504	0.2175	0.2156
60	0.3432	0.1694	0.245	0.2424
90	0.3041	0.1796	0.2596	0.2567
100	0.2985	0.181	0.2617	0.2587
120	0.274	0.1874	0.2709	0.2677

Firings

0	0.586	0.1344	0.1454	0.1341
60	0.5257	0.154	0.1667	0.1537
90	0.4887	0.1661	0.1796	0.1657
100	0.4823	0.1681	0.1819	0.1677
120	0.4609	0.1751	0.1894	0.1746

Notes: The first column contains the minimum tenure in days in the initial state for the transition to be counted. The next four columns contain the share of transitions in the respective tenure class given the restriction. The first row contains the benchmark case without selection that is reported in the main part of the paper.

Table C.25: Transition rates between labor market states for full-time employed workers (males and females, $Jan1980 - Sep2004$)

Low skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0126	0.0067	0.0043	0.0024	0.0048
std	0.1915	0.2428	0.2436	0.1976	0.1997
rel. share	0.4204	0.1227	0.1642	0.2927	
rel. earnings	0.8849	0.8689	0.9012	0.9217	
corr (per capita)	0.3803	0.2687	0.4351	0.2804	0.5170
corr (per empl.)	0.1286	0.2192	0.2383	0.2870	0.3142
av. obs.	72.9433	21.2435	28.0935	49.4463	171.7266
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0557	0.0258	0.0175	0.0093	0.0335
std	0.1684	0.1836	0.2076	0.3042	0.1466
rel. share	0.6719	0.1616	0.0991	0.0674	
corr (per capita)	0.2697	0.3168	0.3168	0.4203	0.3734
corr (per empl.)	-0.0185	0.0745	0.0857	0.1339	0.0477
av. obs.	129.0431	30.7276	18.6991	12.8358	191.3055
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0207	0.0081	0.0035	0.0020	0.0057
std	0.1763	0.2121	0.2828	0.2815	0.1418
rel. share	0.5651	0.1205	0.1114	0.2030	
rel. earnings	0.9297	0.8349	0.8345	0.8568	
corr (per capita)	-0.6787	-0.5440	-0.5159	-0.3415	-0.5588
corr (per empl.)	-0.3776	-0.2961	-0.2626	-0.0555	-0.2500
av. obs.	115.9110	25.0279	24.0253	42.4594	207.4235
Medium skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0193	0.0112	0.0079	0.0037	0.0083
std	0.1214	0.1581	0.1750	0.1476	0.1612
rel. share	0.4238	0.1476	0.2137	0.2150	
rel. earnings	0.9068	0.9244	0.9045	0.9002	
corr (per capita)	0.5802	0.5916	0.6284	0.5321	0.6474
corr (per empl.)	0.2666	0.3502	0.3290	0.3634	0.3330
av. obs.	938.8651	327.4333	469.2897	471.4502	2207.0383
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.1009	0.0464	0.0325	0.0234	0.0674
std	0.1111	0.1098	0.1384	0.1792	0.0992
rel. share	0.7224	0.1451	0.0782	0.0543	
corr (per capita)	0.3694	-0.0064	0.4245	0.4965	0.4053
corr (per empl.)	0.0773	-0.1551	0.1915	0.1141	0.0572
av. obs.	1197.9836	243.3641	131.5879	89.4922	1662.4279
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0187	0.0068	0.0037	0.0015	0.0057
std	0.2015	0.1837	0.2367	0.2308	0.1598
rel. share	0.5947	0.1315	0.1466	0.1272	
rel. earnings	0.8736	0.8394	0.8134	0.8196	
corr (per capita)	-0.7845	-0.7386	-0.7437	-0.6018	-0.7830
corr (per empl.)	-0.4778	-0.4745	-0.4526	-0.3034	-0.5066
av. obs.	879.9904	194.8828	220.8215	190.2625	1485.9571
High skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0161	0.0125	0.0103	0.0051	0.0100
std	0.1070	0.1753	0.1664	0.1857	0.1432
rel. share	0.3651	0.1845	0.2666	0.1838	
rel. earnings	0.8356	0.9298	0.9516	0.9592	
corr (per capita)	0.3295	0.3083	0.3868	0.3862	0.4722
corr (per empl.)	0.1879	0.1406	0.1694	0.1981	0.2280
av. obs.	125.1352	65.1689	93.2625	65.7016	349.2682
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0896	0.0501	0.0414	0.0368	0.0661
std	0.1268	0.1473	0.1881	0.3145	0.1194
rel. share	0.6664	0.1636	0.1056	0.0644	
corr (per capita)	0.4534	0.1301	0.2985	0.2938	0.4783
corr (per empl.)	0.1538	-0.0125	0.1278	0.1930	0.1536
av. obs.	94.1415	23.5517	15.4566	9.6481	142.7979
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0084	0.0046	0.0026	0.0010	0.0036
std	0.1743	0.1796	0.2375	0.2588	0.1252
rel. share	0.5236	0.1907	0.1867	0.0991	
rel. earnings	0.7190	0.7218	0.7013	0.7567	
corr (per capita)	-0.5051	-0.2938	-0.4658	-0.2228	-0.4523
corr (per empl.)	-0.2119	-0.1883	-0.2304	0.0583	-0.1883
av. obs.	62.2201	23.4874	22.9498	12.5574	121.2147

Notes: The data is for full-time employed males and females for the period $Jan1980 - Sep2004$. The columns contain the bounds of the different tenure groups. The tenure groups are formed for the labor market state before the transition and are given in days. All statistics are computed conditional on being in the respective labor market state and tenure group. *mean* is the average transition probability of the respective labor market transition. *std* is the relative deviation of the transition rate over time. *rel. share* is the average share of transitions falling in this tenure group relative to all transitions. *rel. earnings* are the average relative earnings of all persons with a transition relative to the average earnings in the current labor market and tenure group. *corr* are the respective correlations of the transition rate with GDP per capita respectively per employed as our business cycle measures. *av. obs* are the average number of transitions per month from the respective labor market and tenure group.

Table C.26: Transition rates between labor market states for full-time employed workers (males, $Jan1980 - Sep2004$)

			Low skilled		
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0139	0.0075	0.0048	0.0026	0.0052
std	0.2230	0.2662	0.2868	0.2192	0.2153
rel. share	0.4326	0.1202	0.1554	0.2918	
rel. earnings	0.8627	0.8543	0.9018	0.9221	
corr (per capita)	0.4247	0.2761	0.4030	0.2111	0.5237
corr (per empl.)	0.1791	0.2280	0.1978	0.2778	0.3490
av. obs.	47.8621	13.2773	16.9000	31.1387	109.1781
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0649	0.0295	0.0199	0.0091	0.0383
std	0.1882	0.2355	0.2805	0.3561	0.1661
rel. share	0.6829	0.1516	0.1000	0.0655	
corr (per capita)	0.2663	0.3124	0.2151	0.2457	0.3633
corr (per empl.)	0.0185	0.0609	0.0248	0.0207	0.0839
av. obs.	81.1979	17.8317	11.7300	7.8381	118.5978
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0232	0.0087	0.0036	0.0018	0.0060
std	0.2073	0.2910	0.3765	0.3563	0.1703
rel. share	0.6016	0.1166	0.0974	0.1844	
rel. earnings	0.8874	0.8218	0.8465	0.8935	
corr (per capita)	-0.6882	-0.5508	-0.4234	-0.3514	-0.5984
corr (per empl.)	-0.4304	-0.3177	-0.1980	-0.0745	-0.2793
av. obs.	75.9306	15.0722	12.9324	23.4219	127.3572
			Medium skilled		
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0208	0.0118	0.0079	0.0037	0.0085
std	0.1277	0.1580	0.1721	0.1471	0.1624
rel. share	0.4341	0.1418	0.1966	0.2274	
rel. earnings	0.8915	0.9170	0.9127	0.8974	
corr (per capita)	0.5791	0.5484	0.5665	0.4810	0.6018
corr (per empl.)	0.2781	0.3503	0.3167	0.3645	0.3256
av. obs.	647.1872	211.5719	290.4672	334.5967	1483.8230
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.1108	0.0495	0.0321	0.0219	0.0718
std	0.1306	0.1373	0.1658	0.1872	0.1111
rel. share	0.7372	0.1356	0.0752	0.0520	
corr (per capita)	0.3058	-0.0359	0.3567	0.4378	0.3410
corr (per empl.)	0.0518	-0.1899	0.0922	0.0681	0.0219
av. obs.	815.0900	151.7632	83.8491	56.4856	1107.1878
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0212	0.0072	0.0038	0.0014	0.0060
std	0.2211	0.3863	0.2747	0.2844	0.1877
rel. share	0.6192	0.1238	0.1345	0.1225	
rel. earnings	0.8590	0.8435	0.8277	0.8458	
corr (per capita)	-0.7940	-0.3993	-0.7192	-0.5472	-0.7833
corr (per empl.)	-0.5095	-0.2512	-0.4274	-0.2547	-0.5107
av. obs.	638.3019	127.5673	141.2551	127.5865	1034.7107
			High skilled		
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0159	0.0128	0.0106	0.0052	0.0098
std	0.1123	0.1799	0.1600	0.1913	0.1392
rel. share	0.3342	0.1815	0.2753	0.2089	
rel. earnings	0.8508	0.9325	0.9434	0.9787	
corr (per capita)	0.1991	0.1808	0.3211	0.3557	0.3917
corr (per empl.)	0.1275	0.0630	0.1551	0.1953	0.1910
av. obs.	84.7555	47.5185	71.0302	55.1714	258.4756
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0888	0.0483	0.0392	0.0324	0.0631
std	0.1495	0.1948	0.2171	0.3761	0.1323
rel. share	0.6627	0.1611	0.1115	0.0647	
corr (per capita)	0.3735	-0.0115	0.2979	0.2346	0.3769
corr (per empl.)	0.0640	-0.0862	0.1272	0.1934	0.0510
av. obs.	53.5241	13.2736	9.3044	5.5077	81.6099
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0075	0.0039	0.0023	0.0009	0.0030
std	0.2034	0.2056	0.2443	0.3211	0.1525
rel. share	0.5127	0.1793	0.1947	0.1133	
rel. earnings	0.7093	0.7202	0.7029	0.8008	
corr (per capita)	-0.4846	-0.2830	-0.3987	-0.2396	-0.4338
corr (per empl.)	-0.1894	-0.1144	-0.1618	0.0362	-0.1365
av. obs.	38.7696	14.1463	15.4448	9.3647	77.7254

Notes: The data is for full-time employed males for the period $Jan1980 - Sep2004$. The columns contain the bounds of the different tenure groups. The tenure groups are formed for the labor market state before the transition and are given in days. All statistics are computed conditional on being in the respective labor market state and tenure group. *mean* is the average transition probability of the respective labor market transition. *std* is the relative deviation of the transition rate over time. *rel. share* is the average share of transitions falling in this tenure group relative to all transitions. *rel. earnings* are the average relative earnings of all persons with a transition relative to the average earnings in the current labor market and tenure group. *corr* are the respective correlations of the transition rate with GDP per capita respectively per employed as our business cycle measures. *av. obs* are the average number of transitions per month from the respective labor market and tenure group.

Table C.27: Transition rates between labor market states for full-time employed workers (females, $Jan1980 - Sep2004$)

Low skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0107	0.0057	0.0039	0.0022	0.0042
std	0.1949	0.2655	0.2885	0.2900	0.2055
rel. share	0.4014	0.1285	0.1802	0.2899	
rel. earnings	0.8930	0.8506	0.8737	0.9028	
corr (per capita)	0.1991	0.1295	0.3606	0.1867	0.4182
corr (per empl.)	-0.0023	0.0641	0.1960	0.1664	0.2088
av. obs.	25.3265	7.9807	11.3525	18.3500	63.0097
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0449	0.0222	0.0149	0.0095	0.0281
std	0.1709	0.1942	0.2791	0.4515	0.1548
rel. share	0.6561	0.1770	0.0965	0.0704	
corr (per capita)	0.3515	0.2352	0.3170	0.3653	0.4123
corr (per empl.)	0.0183	0.0934	0.1340	0.1583	0.0570
av. obs.	48.3157	12.9190	6.9710	4.9977	73.2033
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0175	0.0073	0.0035	0.0021	0.0053
std	0.1629	0.2253	0.2794	0.2450	0.1211
rel. share	0.5124	0.1260	0.1309	0.2307	
rel. earnings	0.9668	0.8552	0.8356	0.8552	
corr (per capita)	-0.5042	-0.2932	-0.4202	-0.2790	-0.3822
corr (per empl.)	-0.2191	-0.1159	-0.2179	-0.0456	-0.1681
av. obs.	40.3859	9.9306	11.1931	19.1049	80.6144
Medium skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0167	0.0102	0.0078	0.0036	0.0081
std	0.1163	0.1701	0.1879	0.1674	0.1673
rel. share	0.4033	0.1591	0.2500	0.1876	
rel. earnings	0.9145	0.8986	0.8612	0.8651	
corr (per capita)	0.5771	0.6228	0.6710	0.5760	0.7008
corr (per empl.)	0.2204	0.3345	0.3129	0.3307	0.3266
av. obs.	291.4141	115.1827	178.2881	134.6110	719.4960
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0847	0.0421	0.0331	0.0268	0.0598
std	0.1039	0.1041	0.1352	0.2025	0.1002
rel. share	0.6937	0.1641	0.0838	0.0584	
corr (per capita)	0.6008	0.1807	0.4064	0.5198	0.5895
corr (per empl.)	0.2369	0.0628	0.3134	0.2044	0.2252
av. obs.	384.7245	91.7228	47.7336	33.0633	557.2441
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0142	0.0061	0.0034	0.0017	0.0051
std	0.1565	0.1479	0.1903	0.1505	0.1079
rel. share	0.5383	0.1491	0.1738	0.1388	
rel. earnings	0.8704	0.8024	0.7703	0.7456	
corr (per capita)	-0.7070	-0.6740	-0.7305	-0.6986	-0.7345
corr (per empl.)	-0.3471	-0.4101	-0.4497	-0.4517	-0.4614
av. obs.	241.6299	67.2226	79.0972	62.7242	450.6738
High skilled					
Quits	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0165	0.0117	0.0095	0.0046	0.0107
std	0.1420	0.2098	0.2537	0.2998	0.1742
rel. share	0.4611	0.1936	0.2402	0.1051	
rel. earnings	0.8632	0.9171	0.9504	0.8436	
corr (per capita)	0.4274	0.5563	0.4946	0.3078	0.5874
corr (per empl.)	0.2204	0.3195	0.2462	0.1641	0.2764
av. obs.	40.0849	17.8844	22.5298	10.5418	91.0409
Jobfindings	< 180 days	180 − 365 days	365 − 730 days	> 730 days	*overall* days
mean	0.0907	0.0528	0.0453	0.0487	0.0707
std	0.1353	0.1999	0.2507	0.6424	0.1278
rel. share	0.6726	0.1663	0.0973	0.0637	
corr (per capita)	0.4734	0.2526	0.0969	0.1539	0.5210
corr (per empl.)	0.2738	0.1177	0.0325	0.0811	0.2913
av. obs.	40.5999	10.2701	6.1552	4.1436	61.1688
Firings	< 365 days	365 − 730 days	730 − 1825 days	> 1825 days	*overall* days
mean	0.0103	0.0065	0.0034	0.0015	0.0055
std	0.1733	0.2257	0.2826	0.3652	0.1306
rel. share	0.5454	0.2121	0.1720	0.0705	
rel. earnings	0.8114	0.8240	0.7712	0.6307	
corr (per capita)	-0.4134	-0.2501	-0.4152	-0.0378	-0.3596
corr (per empl.)	-0.2082	-0.2540	-0.2747	0.0312	-0.2780
av. obs.	23.5057	9.3485	7.5014	3.1862	43.5418

Notes: The data is for full-time employed females for the period $Jan1980 - Sep2004$. The columns contain the bounds of the different tenure groups. The tenure groups are formed for the labor market state before the transition and are given in days. All statistics are computed conditional on being in the respective labor market state and tenure group. *mean* is the average transition probability of the respective labor market transition. *std* is the relative deviation of the transition rate over time. *rel. share* is the average share of transitions falling in this tenure group relative to all transitions. *rel. earnings* are the average relative earnings of all persons with a transition relative to the average earnings in the current labor market and tenure group. *corr* are the respective correlations of the transition rate with GDP per capita respectively per employed as our business cycle measures. *av. obs* are the average number of transitions per month from the respective labor market and tenure group.

C.3 Unemployment decomposition

HP filter, Fujita and Ramey (2007)

Denote the unemployment rate in period t by u_t and denote by $\bar{u}_t$ the Hp-filtered trend component of the unemployment rate. Following Shimer (2005) the unemployment rate and the trend unemployment rate can be approximated by

$$u_t = \frac{s_t}{s_t + f_t} \qquad \bar{u}_t = \frac{\bar{s}_t}{\bar{s}_t + \bar{f}_t}$$

where s_t is the job separation hazard rate and f_t is the job finding hazard rate from the continuous time setting. These rates coincide with the probabilities for small values of s_t and f_t, and again, $\bar{s}_t$ and $\bar{f}_t$ denote their trend counterparts obtained from the HP-filter.

We rearrange terms to get

$$(1 - u_t)s_t - u_t f_t = 0 \qquad (1 - \bar{u}_t)\bar{s}_t - \bar{u}_t \bar{f}_t = 0$$

We linearize around the trend equation and get

$$(1 - u_t)s_t - u_t f_t = (1 - \bar{u}_t)\bar{s}_t - \bar{u}_t \bar{f}_t - (\bar{s}_t + \bar{f}_t)(u_t - \bar{u}_t) + (1 - \bar{u}_t)(s_t - \bar{s}_t) - \bar{u}_t(f_t - \bar{f}_t)$$

using the log linearization $\frac{x_t - \bar{x}}{\bar{x}} \approx \log\left(\frac{x_t}{\bar{x}}\right)$, we get

$$\begin{aligned}
&(1 - u_t)s_t - u_t f_t - (1 - \bar{u}_t)\bar{s}_t + \bar{u}_t \bar{f}_t = \\
&\qquad -(\bar{s}_t + \bar{f}_t)\bar{u}_t \log\left(\frac{u_t}{\bar{u}_t}\right) + (1 - \bar{u}_t)\bar{s}_t \log\left(\frac{s_t}{\bar{s}_t}\right) - \bar{u}_t \bar{f}_t \log\left(\frac{f_t}{\bar{f}_t}\right) \\
&\frac{(1 - u_t)s_t - u_t f_t - (1 - \bar{u}_t)\bar{s}_t + \bar{u}_t \bar{f}_t}{\bar{s}_t + \bar{f}_t} = \\
&\qquad -\bar{u}_t \log\left(\frac{u_t}{\bar{u}_t}\right) + (1 - \bar{u}_t)\bar{u}_t \log\left(\frac{s_t}{\bar{s}_t}\right) - \bar{u}_t(1 - \bar{u}_t) \log\left(\frac{f_t}{\bar{f}_t}\right) \\
&\underbrace{\frac{(1 - u_t)s_t - u_t f_t - (1 - \bar{u}_t)\bar{s}_t + \bar{u}_t \bar{f}_t}{\bar{s}_t}}_{=:-\varepsilon_t} = -\log\left(\frac{u_t}{\bar{u}_t}\right) + (1 - \bar{u}_t)\log\left(\frac{s_t}{\bar{s}_t}\right) - (1 - \bar{u}_t)\log\left(\frac{f_t}{\bar{f}_t}\right) \\
&\log\left(\frac{u_t}{\bar{u}_t}\right) = (1 - \bar{u}_t)\log\left(\frac{s_t}{\bar{s}_t}\right) - (1 - \bar{u}_t)\log\left(\frac{f_t}{\bar{f}_t}\right) + \varepsilon_t
\end{aligned}$$

a similar expression can be derived using a first difference filter as we will show below.

We define

$$du_t := \log\left(\frac{u_t}{\bar{u}_t}\right) \qquad ds_t := (1-\bar{u}_t)\log\left(\frac{s_t}{\bar{s}_t}\right) \qquad df_t := -(1-\bar{u}_t)\log\left(\frac{f_t}{\bar{f}_t}\right)$$

If we use these definitions the expression above can be compactly written as

$$du_t = ds_t + df_t + \varepsilon_t$$

We apply the variance operator to both sides of this equation and obtain

$$var(du_t) = var(ds_t) + var(df_t) + var(\varepsilon_t) + 2cov(ds_t, df_t) + 2cov(ds_t, \varepsilon_t) + 2cov(df_t, \varepsilon_t)$$

Denote by μ_j the mean of all variables $j = \{u, f, s, \varepsilon\}$. We can now derive the following result

$$\begin{aligned}
du_t &= ds_t + df_t + \varepsilon_t \\
(du_t - \mu_u)(dj_t - \mu_j) &= (dj_t - \mu_j)ds_t + (dj_t - \mu_j)df_t + (dj_t - \mu_j)\varepsilon_t - (dj_t - \mu_j)\mu_u \\
(du_t - \mu_u)(dj_t - \mu_j) &= (dj_t - \mu_j)ds_t + (dj_t - \mu_j)df_t + (dj_t - \mu_j)\varepsilon_t - (dj_t - \mu_j)(\mu_s + \mu_f + \mu_\varepsilon) \\
(du_t - \mu_u)(dj_t - \mu_j) &= (dj_t - \mu_j)(ds_t - \mu_s) + (dj_t - \mu_j)(df_t - \mu_f) + (dj_t - \mu_j)(\varepsilon_t - \mu_\varepsilon)
\end{aligned}$$

Taking expectations on both sides yields

$$cov(du_t, dj_t) = cov(dj_t, ds_t) + cov(dj_t, df_t) + cov(dj_t, \varepsilon_t)$$

If we use this relationship for $j = \{f, s, \varepsilon\}$, we obtain

$$\begin{aligned}
cov(du_t, ds_t) + cov(du_t, df_t) + cov(du_t, \varepsilon_t) &= var(ds_t) + var(df_t) + var(\varepsilon_t) \\
&\quad +2cov(ds_t, df_t) + 2cov(ds_t, \varepsilon_t) + 2cov(df_t, \varepsilon_t)
\end{aligned}$$

Plugging this back into the expression for $var(du_t)$ yields

$$var(du_t) = cov(du_t, ds_t) + cov(du_t, df_t) + cov(du_t, \varepsilon_t)$$

Deviding by $var(du_t)$ yields the Fujita and Ramey decomposition formula

$$\frac{cov(du_t, ds_t)}{var(du_t)} + \frac{cov(du_t, df_t)}{var(du_t)} + \frac{cov(du_t, \varepsilon_t)}{var(du_t)} = \beta_s + \beta_f + \beta_e = 1$$

$1st$ difference filter, Petrongolo and Pissardides (2009)

Again, we use the approximation formula by Shimer (2005) to describe unemployment and rearrange terms to get

$$u_t = \frac{s_t}{s_t + f_t} \qquad u_{t-1} = \frac{s_{t-1}}{s_{t-1} + f_{t-1}}$$

Subtract the two equations from each other and add the zero term $u_t(f_{t-1}+s_{t-1})-u_t(f_{t-1}+s_{t-1})$ to get

$$\begin{aligned}
(1-u_t)s_t - u_t f_t - (1-u_{t-1})s_{t-1} + u_{t-1}f_{t-1} + u_t(f_{t-1}+s_{t-1}) - u_t(f_{t-1}+s_{t-1}) &= 0 \\
u_{t-1}(f_{t-1}+s_{t-1}) - u_t(f_{t-1}+s_{t-1}) + (1-u_t)s_t - u_t f_t - (1-u_t)s_{t-1} + u_t f_{t-1} &= 0 \\
(u_{t-1}-u_t)(f_{t-1}+s_{t-1}) + (1-u_t)s_t - (1-u_t)s_{t-1} - u_t f_t + u_t f_{t-1} &= 0 \\
-\Delta u_t(f_{t-1}+s_{t-1}) + (1-u_t)\Delta s_t - u_t \Delta f_t &= 0 \\
(1-u_t)\frac{s_{t-1}}{f_{t-1}+s_{t-1}}\frac{\Delta s_t}{s_{t-1}} - u_t\frac{f_{t-1}}{f_{t-1}+s_{t-1}}\frac{\Delta f_t}{f_{t-1}} &= \Delta u_t \\
(1-u_t)u_{t-1}\frac{\Delta s_t}{s_{t-1}} - u_t(1-u_{t-1})\frac{\Delta f_t}{f_{t-1}} &= \Delta u_t
\end{aligned}$$

We define

$$ds_t := \Delta u_t \qquad ds_t := (1-u_t)u_{t-1}\frac{\Delta s_t}{s_{t-1}} \qquad df_t := -u_t(1-u_{t-1})\frac{\Delta f_t}{f_{t-1}}$$

and the decomposition can be compactly written as before as

$$ds_t + df_t = du_t$$

If we use the unemployment rate from the data for the decomposition, then the decomposition formula has to be augmented by an extra error term as in the case for the HP filter. However, if we use the unemployment rate constructed using Shimer's formula there will be no approximation error and the decomposition is exact such that all variations can be attributed to separations and jobfindings from the decomposition formula.

$1st$ difference filter, Petrongolo and Pissarides (2009)

Denote the three states by E (employment), U (unemployment), and N (non-employment), and denote by $\Pi_{ij,t}$ the transition probability from state i to state j in period t. The steady state flow condition is

$$\begin{aligned}\Pi_{EU,t}E_t + \Pi_{NU,t}N_t &= (\Pi_{UE,t} + \Pi_{UN,t})U_t \\ \Pi_{UE,t}U_t + \Pi_{NE,t}N_t &= (\Pi_{EU,t} + \Pi_{EN,t})E_t\end{aligned}$$

Substitution N_t out of the flow equations and rearranging terms yields

$$u_t = \frac{U_t}{U_t + E_t} = \frac{\Pi_{EU,t} + \frac{\Pi_{NU,t}}{\Pi_{NE,t}+\Pi_{NU,t}}\Pi_{EN,t}}{\Pi_{UE,t} + \Pi_{EU,t} + \frac{\Pi_{NE,t}}{\Pi_{NE,t}+\Pi_{NU,t}}\Pi_{UN,t} + \frac{\Pi_{NU,t}}{\Pi_{NE,t}+\Pi_{NU,t}}\Pi_{EN,t}}$$

Define $\hat{s}_t$ and $\hat{f}_t$

$$\hat{s}_t := \Pi_{EU,t} + \frac{\Pi_{NU,t}}{\Pi_{NE,t} + \Pi_{NU,t}}\Pi_{EN,t} \qquad \hat{f}_t := \Pi_{UE,t} + \frac{\Pi_{NE,t}}{\Pi_{NE,t} + \Pi_{NU,t}}\Pi_{UN,t}$$

going through the same steps as for the two states case yields

$$\Delta u_t = (1 - u_t)u_{t-1}\frac{\Delta\hat{s}_t}{\hat{s}_{t-1}} - u_t(1 - u_{t-1})\frac{\Delta\hat{f}_t}{\hat{f}_{t-1}}$$

Define the components measuring the NU and UN contributions

$$\begin{aligned}NU : \Delta n_{1,t} &:= \frac{\Pi_{NU,t}}{\Pi_{NE,t} + \Pi_{NU,t}}\Pi_{EN,t} - \frac{\Pi_{NU,t-1}}{\Pi_{NE,t-1} + \Pi_{NU,t-1}}\Pi_{EN,t-1} \\ UN : \Delta n_{2,t} &:= \frac{\Pi_{NE,t}}{\Pi_{NE,t} + \Pi_{NU,t}}\Pi_{UN,t} - \frac{\Pi_{NE,t-1}}{\Pi_{NE,t-1} + \Pi_{NU,t-1}}\Pi_{UN,t-1}\end{aligned}$$

Rewrite the expression for Δu_t as follows

$$\begin{aligned}\Delta u_t &= (1 - u_t)u_{t-1}\frac{\Delta\Pi_{EU,t}}{\hat{s}_{t-1}} - u_t(1 - u_{t-1})\frac{\Delta\Pi_{UE,t}}{\hat{f}_{t-1}} + (1 - u_t)u_{t-1}\frac{\Delta n_{1,t}}{\hat{s}_{t-1}} - u_t(1 - u_{t-1})\frac{\Delta n_{2,t}}{\hat{f}_{t-1}} \\ du_t &= ds_t + df_t + dn_{1,t} + dn_{2,t}\end{aligned}$$

and the contribution rates can be derived as for the two state case

$$\beta_s = \frac{cov(du_t, ds_t)}{var(du_t)} \qquad \beta_f = \frac{cov(du_t, df_t)}{var(du_t)} \qquad \beta_{n,1} = \frac{cov(du_t, dn_{1,t})}{var(du_t)} \qquad \beta_{n,2} = \frac{cov(du_t, dn_{2,t})}{var(du_t)}$$

If we use the data unemployment rate instead of the constructed one using the Shimer formula, then there is also a contribution factor β_ε originating from the approximation error.

Recognize that the decomposition is unaffected by a mismeasurement of the pool of non-employmened. Although, the formula contains the transition rates, the transition rates can be replace by flows that are measured correctly because the level effect of the flows will cancel out.

HP filter

In this section, we extend the Fujita and Ramey (2007) approach to a three state environment as in the Petrongolo and Pissarides (2009) framework.

From the steady state flow equation, we can derive a steady state unemployment rate

$$
\begin{aligned}
u_t &= \frac{\Pi_{EU,t}\Pi_{NE,t} + \Pi_{EU,t}\Pi_{NU,t} + \Pi_{NU,t}\Pi_{EN,t}}{\Pi_{UE,t}\Pi_{NE,t} + \Pi_{UE,t}\Pi_{NU,t} + \Pi_{EU,t}\Pi_{NE,t} + \Pi_{EU,t}\Pi_{NU,t} + \Pi_{UN,t}\Pi_{NE,t} + \Pi_{NU,t}\Pi_{EN,t}} \\
\bar{u} &= \frac{\bar{\Pi}_{EU}\bar{\Pi}_{NE} + \bar{\Pi}_{EU}\bar{\Pi}_{NU} + \bar{\Pi}_{NU}\bar{\Pi}_{EN}}{\bar{\Pi}_{UE}\bar{\Pi}_{NE} + \bar{\Pi}_{UE}\bar{\Pi}_{NU} + \bar{\Pi}_{EU}\bar{\Pi}_{NE} + \bar{\Pi}_{EU}\bar{\Pi}_{NU} + \bar{\Pi}_{UN}\bar{\Pi}_{NE} + \bar{\Pi}_{NU}\bar{\Pi}_{EN}}
\end{aligned}
$$

where again the second expression contains the trend components from the HP filter. To ease notation, we define

$$
\Omega := \bar{\Pi}_{UE}\bar{\Pi}_{NE} + \bar{\Pi}_{UE}\bar{\Pi}_{NU} + \bar{\Pi}_{NE}\bar{\Pi}_{UN} + \bar{\Pi}_{EU}\bar{\Pi}_{NE} + \bar{\Pi}_{EU}\bar{\Pi}_{NU} + \bar{\Pi}_{NU}\bar{\Pi}_{EN}
$$

Rearranging terms yields

$$
\begin{aligned}
& u_t \left(\Pi_{UE,t}\Pi_{NE,t} + \Pi_{UE,t}\Pi_{NU,t} + \Pi_{NE,t}\Pi_{UN,t}\right) \\
& \qquad -(1-u_t)\left(\Pi_{EU,t}\Pi_{NE,t} + \Pi_{EU,t}\Pi_{NU,t} + \Pi_{NU,t}\Pi_{EN,t}\right) = 0 \\
& \bar{u}\left(\bar{\Pi}_{UE}\bar{\Pi}_{NE} + \bar{\Pi}_{UE}\bar{\Pi}_{NU} + \bar{\Pi}_{NE}\bar{\Pi}_{UN}\right) \\
& \qquad -(1-\bar{u})\left(\bar{\Pi}_{EU}\bar{\Pi}_{NE} + \bar{\Pi}_{EU}\bar{\Pi}_{NU} + \bar{\Pi}_{NU}\bar{\Pi}_{EN}\right) = 0
\end{aligned}
$$

We linearized around the trend component

$$
\begin{aligned}
& (u_t - \bar{u})\Omega + (\Pi_{UE,t} - \bar{\Pi}_{UE})\left(\bar{\Pi}_{NE}\bar{u} + \bar{\Pi}_{NU}\bar{u}\right) \\
+ \quad & (\Pi_{NE,t} - \bar{\Pi}_{NE})\left(\bar{\Pi}_{UE}\bar{u} + \bar{\Pi}_{UN}\bar{u} - (1-\bar{u})\bar{\Pi}_{EU}\right) \\
+ \quad & (\Pi_{NU,t} - \bar{\Pi}_{NU})\left(\bar{\Pi}_{UE}\bar{u} - (1-\bar{u})\bar{\Pi}_{EU} - (1-\bar{u})\bar{\Pi}_{EN}\right) \\
+ \quad & (\Pi_{UN,t} - \bar{\Pi}_{UN})\left(\bar{\Pi}_{NE}\bar{u}\right) \\
+ \quad & (\Pi_{EU,t} - \bar{\Pi}_{EU})\left(-(1-\bar{u})\bar{\Pi}_{NE} - (1-\bar{u})\bar{\Pi}_{NU}\right) \\
+ \quad & (\Pi_{EN,t} - \bar{\Pi}_{EN})\left(-(1-\bar{u})\bar{\Pi}_{NU}\right) + \hat{\varepsilon}_t \\
= \quad & 0
\end{aligned}
$$

using log linearization yields

$$
\begin{aligned}
&\log\left(\frac{u_t}{\bar{u}}\right)\Omega + \bar{\Pi}_{UE}\log\left(\frac{\Pi_{UE,t}}{\bar{\Pi}_{UE}}\right)\left(\bar{\Pi}_{NE}+\bar{\Pi}_{NU}\right)\\
+&\ \bar{\Pi}_{NE}\log\left(\frac{\Pi_{NE,t}}{\bar{\Pi}_{NE}}\right)\left(\bar{\Pi}_{UE}+\bar{\Pi}_{UN}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{EU}\right)\\
+&\ \bar{\Pi}_{NU}\log\left(\frac{\Pi_{NU,t}}{\bar{\Pi}_{NU}}\right)\left(\bar{\Pi}_{UE}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{EU}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{EN}\right)\\
+&\ \bar{\Pi}_{UN}\log\left(\frac{\Pi_{UN,t}}{\bar{\Pi}_{UN}}\right)\bar{\Pi}_{NE}\\
+&\ \bar{\Pi}_{EU}\log\left(\frac{\Pi_{EU,t}}{\bar{\Pi}_{EU}}\right)\left(-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NE}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NU}\right)\\
+&\ \bar{\Pi}_{EN}\log\left(\frac{\Pi_{EN,t}}{\bar{\Pi}_{EN}}\right)\left(-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NU}\right)+\frac{\hat{\varepsilon}_t}{\bar{u}}\\
=&\ 0
\end{aligned}
$$

We get the following decomposition

$$
\begin{aligned}
&\log\left(\frac{u_t}{\bar{u}}\right) = -\log\left(\frac{\Pi_{UE,t}}{\bar{\Pi}_{UE}}\right)\frac{\bar{\Pi}_{UE}\bar{\Pi}_{NE}+\bar{\Pi}_{UE}\bar{\Pi}_{NU}}{\Omega}\\
-&\ \log\left(\frac{\Pi_{NE,t}}{\bar{\Pi}_{NE}}\right)\frac{\bar{\Pi}_{NE}\bar{\Pi}_{UE}+\bar{\Pi}_{NE}\bar{\Pi}_{UN}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NE}\bar{\Pi}_{EU}}{\Omega}\\
-&\ \log\left(\frac{\Pi_{NU,t}}{\bar{\Pi}_{NU}}\right)\frac{\bar{\Pi}_{NU}\bar{\Pi}_{UE}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NU}\bar{\Pi}_{EU}-\frac{(1-\bar{u})}{\bar{u}}\bar{\Pi}_{NU}\bar{\Pi}_{EN}}{\Omega}\\
-&\ \log\left(\frac{\Pi_{UN,t}}{\bar{\Pi}_{UN}}\right)\frac{\bar{\Pi}_{UN}\bar{\Pi}_{NE}}{\Omega}\\
+&\ \log\left(\frac{\Pi_{EU,t}}{\bar{\Pi}_{EU}}\right)\frac{(1-\bar{u})}{\bar{u}}\frac{\bar{\Pi}_{EU}\bar{\Pi}_{NE}+\bar{\Pi}_{EU}\bar{\Pi}_{NU}}{\Omega}\\
+&\ \log\left(\frac{\Pi_{EN,t}}{\bar{\Pi}_{EN}}\right)\frac{(1-\bar{u})}{\bar{u}}\frac{\bar{\Pi}_{EN}\bar{\Pi}_{NU}}{\Omega}+\varepsilon_t
\end{aligned}
$$

and define

$$
\begin{aligned}
&\bar{\Pi}_u := \bar{\Pi}_{EU}+\frac{\bar{\Pi}_{NU}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}\bar{\Pi}_{EN} \qquad \bar{\Pi}_e := \bar{\Pi}_{UE}+\frac{\bar{\Pi}_{UN}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}\bar{\Pi}_{NE}\\
&\lambda_{EU} := (1-\bar{u})\frac{\bar{\Pi}_{EU}}{\bar{\Pi}_u} \qquad \lambda_{UE} := (1-\bar{u})\frac{\bar{\Pi}_{UE}}{\bar{\Pi}_e}\\
&\lambda_{EN} := (1-\bar{u})\frac{\bar{\Pi}_{EN}}{\bar{\Pi}_u} \qquad \lambda_{UN} := (1-\bar{u})\frac{\bar{\Pi}_{UN}}{\bar{\Pi}_e}\\
&\alpha := \frac{\bar{\Pi}_{NU}}{\bar{\Pi}_{NE}+\bar{\Pi}_{NU}}
\end{aligned}
$$

Using these definitions, we get

$$\begin{aligned}
\log\left(\frac{u_t}{\bar{u}}\right) &= \log\left(\frac{\Pi_{EU,t}}{\bar{\Pi}_{EU}}\right)\lambda_{EU} - \log\left(\frac{\Pi_{UE,t}}{\bar{\Pi}_{UE}}\right)\lambda_{UE} \\
&+ \log\left(\frac{\Pi_{EN,t}}{\bar{\Pi}_{EN}}\right)\alpha\lambda_{EN} - \log\left(\frac{\Pi_{NE,t}}{\bar{\Pi}_{NE}}\right)(1-\alpha)(\lambda_{UE}+\lambda_{UN}-\lambda_{EU}) \\
&+ \log\left(\frac{\Pi_{NU,t}}{\bar{\Pi}_{NU}}\right)\alpha(\lambda_{EU}+\lambda_{EN}-\lambda_{UE}) - \log\left(\frac{\Pi_{UN,t}}{\bar{\Pi}_{UN}}\right)(1-\alpha)\lambda_{UN} + \varepsilon_t \\
du &= dEU + dUE + dEN + dNE + dNU + dUN + \varepsilon_t
\end{aligned}$$

The covariance decomposition as for the case of two variables generalizes to the case of n variables

$$\begin{aligned}
du &= \sum_{i=1}^{n} di \\
(du-\mu_u)(dj-\mu_j) &= \left(\sum_{i=1}^{n} di\right)(dj-\mu_j) - \mu_u(dj-\mu_j) \\
(du-\mu_u)(dj-\mu_j) &= \left(\sum_{i=1}^{n} di\right)(dj-\mu_j) - \left(\sum_{i=1}^{n} \mu_i\right)(dj-\mu_j) \\
(du-\mu_u)(dj-\mu_j) &= \left(\sum_{i=1}^{n}(di-\mu_i)\right)(dj-\mu_j)
\end{aligned}$$

and we obtain the covariance decomposition

$$cov(du, dj) = \sum_{i=1}^{n} cov(di, dj)$$

The generalized formula for the variance of the unemployment rate reads

$$\begin{aligned}
var(du) &= \sum_{i=1}^{n} var(di) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{N} cov(di, dj) \\
var(du) &= \sum_{i=1}^{n}\sum_{j=1}^{n} cov(di, dj)
\end{aligned}$$

Plugging in the expression we just derived for $\sum_{i=1}^{n} cov(di, dj)$ yields

$$
\begin{aligned}
var(du) &= \sum_{i=1}^{n} cov(du, di) \\
1 &= \sum_{i=1}^{n} \frac{cov(du, di)}{var(du)} \\
1 &= \sum_{i=1}^{n} \beta_i
\end{aligned}
$$

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Lebenslauf

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Eidesstattliche Erklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig angefertigt und mich keiner anderen als der in ihr angegebenen Hilfsmittel bedient zu haben. Insbesondere sind sämtliche Zitate aus anderen Quellen als solche gekennzeichnet und mit Quellenangaben versehen. Alle Arbeiten mit Koautoren wurden ausdrücklich als solche gekennzeichnet.

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